Chapter 2

Notation

Let $\mathbb{Q}_p \subseteq L \subset \mathbb{C}_p$ be a field of finite degree d over $\mathbb{Q}_p$, o_L the ring of integers of L, $\pi_L \in o_L$ a fixed prime element, $k_L = o_L/\pi_L o_L$ the residue field, $q := |k_L|$, e the absolute ramification index of L and $G_L := \mathrm{Gal}(\overline{L}/L)$. We always use the absolute value $|\ |$ on $\mathbb{C}_p$ which is normalized by $|\pi_L| = q^{-1}$. We *warn* the reader, though, that we will repeatedly use the references [8, 33, 49, 68, 69, 74, 76] in which the absolute value is normalized differently from this paper by $|p| = p^{-1}$. Our absolute value is the dth power of the one in these references. The transcription of certain formulae to our convention will usually be done silently.

Let further $\mathrm{Rep}_L(G_L)$ denote the category of finite-dimensional L-vector spaces equipped with a continuous linear G_L-action.

We fix a Lubin–Tate formal o_L-module $\mathrm{LT} = \mathrm{LT}_{\pi_L}$ over o_L corresponding to the prime element π_L. We always identify LT with the open unit disk around zero, which gives us a global coordinate Z on LT. The o_L-action then is given by formal power series $[a](Z) \in o_L[\![Z]\!]$. For simplicity the formal group law will be denoted by $+_{\mathrm{LT}}$.

The power series $\frac{\partial(X +_{\mathrm{LT}} Y)}{\partial Y}_{|(X,Y)=(Z,0)}$ is a unit in $o_L[\![Z]\!]$ and we let $g_{\mathrm{LT}}(Z)$ denote its inverse. Then $g_{\mathrm{LT}}(Z)dZ$ is, up to scalars, the unique invariant differential form on LT (cf. [37, §5.8]). We also let

$$\log_{\mathrm{LT}}(Z) = Z + \cdots \tag{2.1}$$

denote the unique formal power series in $L[\![Z]\!]$ whose formal derivative is g_{LT}. This $\log_{\mathrm{LT}}$ is the logarithm of LT (cf. [47, §8.6]). In particular, $g_{\mathrm{LT}}dZ = d\log_{\mathrm{LT}}$. The invariant derivation ∂_{inv} corresponding to the form $d\log_{\mathrm{LT}}$ is determined by

$$f'dZ = df = \partial_{\mathrm{inv}}(f)d\log_{\mathrm{LT}} = \partial_{\mathrm{inv}}(f)g_{\mathrm{LT}}dZ$$

and hence is given by

$$\partial_{\mathrm{inv}}(f) = g_{\mathrm{LT}}^{-1} f'. \tag{2.2}$$

For any $a \in o_L$ we have

$$\log_{\mathrm{LT}}\big([a](Z)\big) = a \cdot \log_{\mathrm{LT}} \quad \text{and hence} \quad a g_{\mathrm{LT}}(Z) = g_{\mathrm{LT}}\big([a](Z)\big) \cdot [a]'(Z) \tag{2.3}$$

(cf. [47, 8.6 Lemma 2]).

Let T_π be the Tate module of LT. Then T_π is a free o_L-module of rank one, say with generator $\eta = (\eta_n)$, and the action of G_L on T_π is given by a continuous character $\chi_{\mathrm{LT}} : G_L \to o_L^\times$. Let T'_π denote the Tate module of the p-divisible group Cartier dual to LT with period Ω (depending on the choice of a generator of T'_π), which again is a free o_L-module of rank one. The Galois action on $T'_\pi \cong T^*_\pi(1)$ is given by the continuous character $\tau := \chi_{\mathrm{cyc}} \cdot \chi_{\mathrm{LT}}^{-1}$, where χ_{cyc} is the cyclotomic character.

For $n \geq 0$ we let L_n/L denote the extension (in $\mathbb{C}_p$) generated by the π_L^n-torsion points of LT, and we put $L_\infty := \bigcup_n L_n$. The extension L_∞/L is Galois. We let $\Gamma_L := \mathrm{Gal}(L_\infty/L)$ and $H_L := \mathrm{Gal}(\overline{L}/L_\infty)$. The Lubin–Tate character χ_{LT} induces an isomorphism $\Gamma_L \xrightarrow{\cong} o_L^\times$.

Henceforth, we use the same notation as in [80]. In particular, the ring endomorphisms induced by sending Z to $[\pi_L](Z)$ are called φ_L where applicable, e.g., for the ring $\mathcal{A}_L$ defined to be the π_L-adic completion of $o_L[\![Z]\!][Z^{-1}]$ or $\mathcal{B}_L := \mathcal{A}_L[\pi_L^{-1}]$ which denotes the field of fractions of $\mathcal{A}_L$. Recall that we also have introduced the unique additive endomorphism ψ_L of $\mathcal{B}_L$ (and then $\mathcal{A}_L$) which satisfies

$$\varphi_L \circ \psi_L = \pi_L^{-1} \cdot \mathrm{trace}_{\mathcal{B}_L/\varphi_L(\mathcal{B}_L)}.$$

Moreover, projection formula

$$\psi_L\big(\varphi_L(f_1)f_2\big) = f_1\psi_L(f_2) \quad \text{for any } f_i \in \mathcal{B}_L$$

and the formula

$$\psi_L \circ \varphi_L = \frac{q}{\pi_L} \cdot \mathrm{id}$$

hold. An étale (φ_L, Γ_L)-module M comes with a Frobenius operator φ_M semilinear with respect to φ_L and an induced operator denoted by ψ_M.

Let $\widetilde{\mathbf{E}}^+ := \varprojlim o_{\mathbb{C}_p}/p o_{\mathbb{C}_p}$ with the transition maps being given by the Frobenius $\varphi(a) = a^p$. We may also identify $\widetilde{\mathbf{E}}^+$ with $\varprojlim o_{\mathbb{C}_p}/\pi_L o_{\mathbb{C}_p}$ with the transition maps being given by the q-Frobenius $\varphi_q(a) = a^q$. Recall that $\widetilde{\mathbf{E}}^+$ is a complete valuation ring with residue field $\overline{\mathbb{F}_p}$ and its field of fractions $\widetilde{\mathbf{E}} = \varprojlim \mathbb{C}_p$ being algebraically closed of characteristic p. Let $\mathfrak{m}_{\widetilde{\mathbf{E}}}$ denote the maximal ideal in $\widetilde{\mathbf{E}}^+$.

The q-Frobenius φ_q first extends by functoriality to the rings of the Witt vectors $W(\widetilde{\mathbf{E}}^+) \subseteq W(\widetilde{\mathbf{E}})$ and then o_L-linearly to

$$W(\widetilde{\mathbf{E}}^+)_L := W(\widetilde{\mathbf{E}}^+) \otimes_{o_{L_0}} o_L \subseteq W(\widetilde{\mathbf{E}})_L := W(\widetilde{\mathbf{E}}) \otimes_{o_{L_0}} o_L,$$

where L_0 is the maximal unramified subextension of L. The Galois group G_L obviously acts on $\widetilde{\mathbf{E}}$ and $W(\widetilde{\mathbf{E}})_L$ by automorphisms commuting with φ_q. This G_L-action is continuous for the weak topology on $W(\widetilde{\mathbf{E}})_L$ (cf. [73, Lem. 1.5.3]).

Sometimes we omit the index q, L, or M from the Frobenius operator, but we always write φ_p when dealing with the p-Frobenius.

The evaluation of the global coordinate Z of LT at π_L-power torsion points induces a map (not a homomorphism of abelian groups) $\iota : T_\pi \to \widetilde{\mathbf{E}}^+$. Namely, if $t = (z_n)_{n\geq 1} \in T_\pi$ with $[\pi_L](z_{n+1}) = z_n$ and $[\pi_L](z_1) = 0$, then $z_{n+1}^q \equiv z_n \bmod \pi_L$ and hence $\iota(t) := (z_n \bmod \pi_L)_n \in \widetilde{\mathbf{E}}^+$. As before, we fix an o_L-generator η of T_π

and put $\omega := \iota(\eta)$. Then there exists a (unique) lift $\omega_{\mathrm{LT}} \in W(\widetilde{\mathbf{E}}^+)_L$ of ω satisfying the following (cf. [80, Lem. 4.1]):

(i) if $\eta' = a\eta$ with $a \in o_L^\times$ denotes another generator of T_π, then $\omega'_{\mathrm{LT}} = [a](\omega_{\mathrm{LT}})$ is the corresponding lift;

(ii) $\varphi_q(\omega_{\mathrm{LT}}) = [\pi_L](\omega_{\mathrm{LT}})$;

(iii) $\sigma(\omega_{\mathrm{LT}}) = [\chi_{\mathrm{LT}}(\sigma)](\omega_{\mathrm{LT}})$ for any $\sigma \in G_L$.