Chapter 3

Wach modules à la Kisin–Ren

3.1 Wach modules

In this section, we recall the theory of Wach modules à la Kisin–Ren [44] (with the simplifying assumption that – in their notation – $K = L$, $m = 1$, etc.).

By sending Z to $\omega_{\mathrm{LT}} \in W(\widetilde{\mathbf{E}}^+)_L$ we obtain a G_L-equivariant, Frobenius compatible embedding of rings

$$o_L[\![Z]\!] \to W(\widetilde{\mathbf{E}}^+)_L,$$

the image of which we call $\mathbf{A}_L^+$; it is a subring of $\mathbf{A}_L$ (the image of $\mathcal{A}_L$ in $W(\widetilde{\mathbf{E}})_L$). The latter ring is a complete discrete valuation ring with prime element π_L and residue field of the image $\mathbf{E}_L$ of $k_L((Z)) \hookrightarrow \widetilde{\mathbf{E}}$ sending Z to $\omega := \omega_{\mathrm{LT}} \bmod \pi_L$. We form the maximal integral unramified extension ($=$ strict henselization) $\mathbf{A}_L^{nr}$ of $\mathbf{A}_L$ inside $W(\widetilde{\mathbf{E}})_L$. Its p-adic completion $\mathbf{A}$ still is contained in $W(\widetilde{\mathbf{E}})_L$. Note that $\mathbf{A}$ is a complete discrete valuation ring with prime element π_L and residue field of the separable algebraic closure $\mathbf{E}_L^{\mathrm{sep}}$ of $\mathbf{E}_L$ in $\widetilde{\mathbf{E}}$. By the functoriality properties of strict henselizations, the q-Frobenius φ_q preserves $\mathbf{A}$. According to [44, Lem. 1.4], the G_L-action on $W(\widetilde{\mathbf{E}})_L$ respects $\mathbf{A}$ and induces an isomorphism

$$H_L = \ker(\chi_{\mathrm{LT}}) \xrightarrow{\cong} \mathrm{Aut}^{\mathrm{cont}}(\mathbf{A}/\mathbf{A}_L).$$

We set $\mathbf{A}^+ := \mathbf{A} \cap W(\widetilde{\mathbf{E}}^+)_L$.

Set $Q := \frac{[\pi_L](\omega_{\mathrm{LT}})}{\omega_{\mathrm{LT}}} \in \mathbf{A}_L^+$, which satisfies by definition $\varphi_L(\omega_{\mathrm{LT}}) = Q \cdot \omega_{\mathrm{LT}}$.

Following [44] we write $\mathcal{O} = \mathcal{O}_L(\mathbf{B})$ for the ring of rigid analytic functions on the open unit disk $\mathbf{B}$ over L, or equivalently the ring of power series in Z over L converging in $\mathbf{B}$. Via sending ω_{LT} to Z we view $\mathbf{A}_L^+$ as a subring of $\mathcal{O}$. We denote by $\mathrm{Mod}_{\mathcal{O}}^{\varphi_L,\Gamma_L,\mathrm{an}}$ the category consisting of finitely generated free $\mathcal{O}$-modules $\mathcal{M}$ together with the following data:

(i) an isomorphism $1 \otimes \varphi_{\mathcal{M}} : (\varphi_L^* \mathcal{M})[\frac{1}{Q}] \cong \mathcal{M}[\frac{1}{Q}]$,[1]

(ii) a semilinear Γ_L-action on $\mathcal{M}$, commuting with $\varphi_{\mathcal{M}}$ and such that the induced action on $D(\mathcal{M}) := \mathcal{M}/\omega_{\mathrm{LT}}\mathcal{M}$ is trivial.

We note that since $\mathcal{M}/\omega_{\mathrm{LT}}\mathcal{M} = \mathcal{M}[\frac{1}{Q}]/\omega_{\mathrm{LT}}\mathcal{M}[\frac{1}{Q}]$, the map $\varphi_{\mathcal{M}}$ induces an L-linear endomorphism of $D(\mathcal{M})$, which we denote by $\varphi_{D(\mathcal{M})}$. As a consequence of (i), in fact, it is an automorphism.

[1] By $\varphi_L^* \mathcal{M}$ we understand the module $\mathcal{O} \otimes_{\mathcal{O},\varphi_L} \mathcal{M}$.

The Γ_L-action on $\mathcal{M}$ is differentiable (cf. [8, Lem. 3.4.13]),[2] and the corresponding derived action of $\mathrm{Lie}(\Gamma_L)$ is L-bilinear (cf. [8, Rem. 3.4.15]).[3]

Similarly, we denote by $\mathrm{Mod}^{\varphi_L,\Gamma_L,\mathrm{an}}_{\mathbf{A}_L^+}$ the category consisting of finitely generated free $\mathbf{A}_L^+$-modules N together with the following data:

(i) an isomorphism $1 \otimes \varphi_N : (\varphi_L^* N)[\frac{1}{Q}] \cong N[\frac{1}{Q}]$,[4]

(ii) a semilinear Γ_L-action on N, commuting with φ_N and such that the induced action on $N/\omega_{\mathrm{LT}} N$ is trivial.

The map φ_N induces an L-linear automorphism of $D(N) := N[\frac{1}{p}]/\omega_{\mathrm{LT}} N[\frac{1}{p}]$ which we denote by $\varphi_{D(N)}$.

Obviously, we have the base extension functor

$$\mathcal{O} \otimes_{\mathbf{A}_L^+} - : \mathrm{Mod}^{\varphi_L,\Gamma_L,\mathrm{an}}_{\mathbf{A}_L^+} \to \mathrm{Mod}^{\varphi_L,\Gamma_L,\mathrm{an}}_{\mathcal{O}}.$$

It satisfies

$$D(\mathcal{O} \otimes_{\mathbf{A}_L^+} N) = D(N). \tag{3.1}$$

We write $\mathrm{Mod}^{\varphi_L,\Gamma_L,0}_{\mathcal{O}}$ for the full subcategory of $\mathrm{Mod}^{\varphi_L,\Gamma_L,\mathrm{an}}_{\mathcal{O}}$ consisting of all $\mathcal{M}$ such that $\mathcal{R} \otimes_{\mathcal{O}} \mathcal{M}$ is pure of slope 0. Here $\mathcal{R}$ denotes the Robba ring, which will be recalled in detail in Section 4.3.1.

By $\mathrm{Mod}^{F,\varphi_q}_L$ we denote the category of finite-dimensional L-vector spaces D equipped with an L-linear automorphism $\varphi_q : D \xrightarrow{\cong} D$ and a decreasing, separated, and exhaustive filtration, indexed by $\mathbb{Z}$, by L-subspaces. In $\mathrm{Mod}^{F,\varphi_q}_L$ we have the full subcategory $\mathrm{Mod}^{F,\varphi_q,\mathrm{wa}}_L$ of weakly admissible objects (cf. [44, (2.3.2)]). For D in $\mathrm{Mod}^{F,\varphi_q,\mathrm{wa}}_L$ let

$$V_L(D) := (B_{\mathrm{cris},L} \otimes_L D)^{\varphi_q = 1} \cap \mathrm{Fil}^0(B_{\mathrm{dR}} \otimes_L D),$$

where, as usual, $B_{\mathrm{cris},L} := B_{\mathrm{cris}} \otimes_{L_0} L$. In order to formulate the crystalline comparison theorem in this context, we also consider the category $\mathrm{Mod}^{F,\varphi}_{L_0 \otimes_{\mathbb{Q}_p} L}$ of finitely generated free $L_0 \otimes_{\mathbb{Q}_p} L$-modules $\mathfrak{D}$ equipped with a $(\varphi_p \otimes \mathrm{id})$-linear automorphism $\varphi : \mathfrak{D} \xrightarrow{\cong} \mathfrak{D}$ as well as a decreasing, separated, and exhaustive filtration on $\mathfrak{D}_L := \mathfrak{D} \otimes_{L_0} L$, indexed by $\mathbb{Z}$, by $L \otimes_{\mathbb{Q}_p} L$-submodules. For $\mathfrak{D}$ in $\mathrm{Mod}^{F,\varphi}_{L_0 \otimes_{\mathbb{Q}_p} L}$

[2] Note that the statements in loc. cit. are all over the character variety; but by the introduction to §3.4 they are also valid over the open unit ball – with even easier proofs.

[3] In [44], being L-analytic is an extra condition in the definition of $\mathrm{Mod}^{\varphi_L,\Gamma_L,\mathrm{an}}_{\mathcal{O}}$, which by this remark is automatically satisfied, whence the corresponding categories with and without the superscript "an" in [44] coincide!

[4] By $\varphi_L^* N$ we understand the module $\mathbf{A}_L^+ \otimes_{\mathbf{A}_L^+,\varphi_L} N$, and formally φ_N is a map from N to $N[\frac{1}{Q}]$.

we define, as usual,

$$V(\mathfrak{D}) := (B_{\mathrm{cris}} \otimes_{L_0} \mathfrak{D})^{\varphi=1} \cap \mathrm{Fil}^0(B_{\mathrm{dR}} \otimes_L \mathfrak{D}_L).$$

Let $\mathrm{Rep}_{o_L,f}(G_L)$ denote the category of finitely generated free o_L-modules with a continuous linear G_L-action and $\mathrm{Rep}^{\mathrm{cris,an}}_{o_L,f}(G_L)$ the full subcategory of those T which are free over o_L and such that the representation $V := L \otimes_{o_L} T$ is crystalline and *analytic*, i.e., satisfying that, if $D_{\mathrm{dR}}(V) := (V \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}})^{G_L}$, the filtration on $D_{\mathrm{dR}}(V)_{\mathfrak{m}}$ is trivial for each maximal ideal $\mathfrak{m}$ of $L \otimes_{\mathbb{Q}_p} L$ which does not correspond to the identity $\mathrm{id} : L \to L$. Correspondingly, we let $\mathrm{Rep}^{\mathrm{cris}}_L(G_L)$, resp., $\mathrm{Rep}^{\mathrm{cris,an}}_L(G_L)$, denote the category of continuous G_L-representations in finite-dimensional L-vector spaces which are crystalline, resp., crystalline and analytic. The base extension functor $L \otimes_{o_L} -$ induces an equivalence of categories

$$\mathrm{Rep}^{\mathrm{cris,an}}_{o_L,f}(G_L) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \xrightarrow{\simeq} \mathrm{Rep}^{\mathrm{cris,an}}_L(G_L).$$

Here applying $\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$ to a $\mathbb{Z}_p$-linear category means applying this functor to the Hom-modules. For V in $\mathrm{Rep}^{\mathrm{cris,an}}_L(G_L)$ we set $D_{\mathrm{cris},L}(V) := (B_{\mathrm{cris},L} \otimes_L V)^{G_L} = (B_{\mathrm{cris}} \otimes_{L_0} V)^{G_L}$ and $D_{\mathrm{cris}}(V) := (B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} V)^{G_L}$. The usual crystalline comparison theorem says that D_{cris} and V are equivalences of categories between $\mathrm{Rep}^{\mathrm{cris}}_L(G_L)$ and the subcategory of weakly admissible objects in $\mathrm{Mod}^{F,\varphi}_{L_0 \otimes_{\mathbb{Q}_p} L}$.

Lemma 3.1.1 ([78, Lem. 5.3] and subsequent discussion, or [44, Cor. 3.3.1]). *There is a fully faithful $\otimes$-functor*

$$\begin{aligned} \sim : \mathrm{Mod}^{F,\varphi_q}_L &\to \mathrm{Mod}^{F,\varphi}_{L_0 \otimes_{\mathbb{Q}_p} L} \\ D &\mapsto \widetilde{D} := L_0 \otimes_{\mathbb{Q}_p} D, \end{aligned}$$

whose essential image consists of all analytic *objects, i.e., those for which the filtration on the non-identity components is trivial. A quasi-inverse functor from the essential image is given by sending $\mathfrak{D}$ to the base extension $L \otimes_{L_0 \otimes_{\mathbb{Q}_p} L} \mathfrak{D}$ for the multiplication map $L_0 \otimes_{\mathbb{Q}_p} L \to L$.*

Lemma 3.1.1 implies that

$$D_{\mathrm{cris},L}(V)^{\sim} \cong D_{\mathrm{cris}}(V) \quad \text{for any } V \text{ in } \mathrm{Rep}^{\mathrm{cris,an}}_L(G_L).$$

We denote by $\mathfrak{M}^{\mathrm{et}}(\mathbf{A}_L)$ the category of étale (φ_q, Γ_L)-modules over $\mathbf{A}_L$ (cf. [80, Def. 3.7]) and by $\mathfrak{M}^{\mathrm{et}}_f(\mathbf{A}_L)$ the full subcategory consisting of those objects, which are finitely generated free as $\mathbf{A}_L$-module. For M in $\mathfrak{M}^{\mathrm{et}}_f(\mathbf{A}_L)$, resp., for T in $\mathrm{Rep}_{o_L,f}(G_L)$, we put $V(M) := (\mathbf{A} \otimes_{\mathbf{A}_L} M)^{\varphi_q \otimes \varphi_M = 1}$, resp., $D_{\mathrm{LT}}(T) := (\mathbf{A} \otimes_{o_L} T)^{\ker(\chi_{\mathrm{LT}})}$.

Having defined all of the relevant categories (and most of the functors) we now contemplate the following diagram of functors:

$$
\begin{tikzcd}
 & \mathfrak{M}^{\mathrm{et}}_f(\mathbf{A}_L) \arrow[r, shift left, "V", "\simeq"'] & \mathrm{Rep}_{o_L,f}(G_L) \arrow[l, shift left, "D_{\mathrm{LT}}"] \\
\mathrm{Mod}^{\varphi_L,\Gamma_L,\mathrm{an}}_{\mathbf{A}_L^+} \arrow[ur, "\mathbf{A}_L\otimes_{\mathbf{A}_L^+}-"] \arrow[rr, "\simeq"] \arrow[d] \arrow[dd, bend right=60, "\mathcal{O}\otimes_{\mathbf{A}_L^+}-"'] & & \mathrm{Rep}^{\mathrm{cris,an}}_{o_L,f}(G_L) \arrow[u, "\subseteq"'] \arrow[dd, "L\otimes_{o_L}-"] \\
\mathrm{Mod}^{\varphi_L,\Gamma_L,\mathrm{an}}_{\mathbf{A}_L^+}\otimes_{\mathbb{Z}_p}\mathbb{Q}_p \arrow[d, "\simeq"'] & & \\
\mathrm{Mod}^{\varphi_L,\Gamma_L,0}_{\mathcal{O}} \arrow[r, shift left, "D", "\simeq"'] \arrow[d, "\subseteq"'] & \mathrm{Mod}^{F,\varphi_q,\mathrm{wa}}_L \arrow[l, shift left, "\mathcal{M}"] \arrow[r, shift left, "V_L", "\simeq"'] \arrow[d, "\subseteq"'] & \mathrm{Rep}^{\mathrm{cris,an}}_L(G_L) \arrow[l, shift left, "D_{\mathrm{cris},L}"] \\
\mathrm{Mod}^{\varphi_L,\Gamma_L,\mathrm{an}}_{\mathcal{O}} \arrow[r, shift left, "D", "\simeq"'] & \mathrm{Mod}^{F,\varphi_q}_L . \arrow[l, shift left, "\mathcal{M}"]
\end{tikzcd}
\tag{3.2}
$$

The arrows without decoration are the obvious natural ones. The following pairs of functors are quasi-inverse $\otimes$-equivalences of $\otimes$-categories:

- (D_{LT}, V) by [44, Thm. 1.6];
- $(D_{\mathrm{cris},L}, V_L)$ by combining the crystalline comparison theorem (cf. [31, Rem. 3.6.7]) and Lemma 3.1.1;
- $(D, \mathcal{M})$ by [44, Prop. 2.2.6] (or [8, Thm. 3.4.16]) and [44, Cor. 2.4.4], to which we also refer for the definition of the functor $\mathcal{M}$.

In particular, all functors in the above diagram are $\otimes$-functors. The second arrow in the left column, resp., the left arrow in the second horizontal row, is an equivalence of categories by [44, Cor. 2.4.2], resp., by [44, Cor. 3.3.8]. The lower square and the upper quadrangle are commutative for trivial reasons.

Remark 3.1.2. We list a few additional properties of these functors.

(i) For any M in $\mathfrak{M}^{\mathrm{et}}_f(\mathbf{A}_L)$ the inclusion $V(M) \subseteq \mathbf{A} \otimes_{\mathbf{A}_L} M$ extends to an isomorphism

$$\mathbf{A} \otimes_{o_L} V(M) \xrightarrow{\cong} \mathbf{A} \otimes_{\mathbf{A}_L} M, \tag{3.3}$$

which is compatible with the φ_q- and Γ_L-actions on both sides.

(ii) The functors D_{LT}, V, and $V(\mathbf{A}_L \otimes_{\mathbf{A}_L^+} -)$ respect exact sequences (of abelian groups).

(iii) (Cf. [8, Prop. 3.4.14].) For any object $\mathcal{M}$ in $\mathrm{Mod}_{\mathcal{O}}^{\varphi_L,\Gamma_L,\mathrm{an}}$ the projection map $\mathcal{M}[\frac{\omega_{\mathrm{LT}}}{t_{\mathrm{LT}}}] \to D(\mathcal{M})$ restricts to an isomorphism $\mathcal{M}[\frac{\omega_{\mathrm{LT}}}{t_{\mathrm{LT}}}]^{\Gamma_L} \xrightarrow{\cong} D(\mathcal{M})$ such that the diagram

$$\begin{tikzcd} \mathcal{M}[\frac{\omega_{\mathrm{LT}}}{t_{\mathrm{LT}}}]^{\Gamma_L} \arrow[r, "\cong"] \arrow[d, "\varphi_{\mathcal{M}}"'] & D(\mathcal{M}) \arrow[d, "\varphi_{D(\mathcal{M})}"] \\ \mathcal{M}[\frac{\omega_{\mathrm{LT}}}{t_{\mathrm{LT}}}]^{\Gamma_L} \arrow[r, "\cong"] & D(\mathcal{M}) \end{tikzcd}$$

is commutative; moreover, we have

$$\mathcal{M}\Big[\frac{\omega_{\mathrm{LT}}}{t_{\mathrm{LT}}}\Big] = \mathcal{O}\Big[\frac{\omega_{\mathrm{LT}}}{t_{\mathrm{LT}}}\Big] \otimes_L \mathcal{M}\Big[\frac{\omega_{\mathrm{LT}}}{t_{\mathrm{LT}}}\Big]^{\Gamma_L} \cong \mathcal{O}\Big[\frac{\omega_{\mathrm{LT}}}{t_{\mathrm{LT}}}\Big] \otimes_L D(\mathcal{M}).$$

Now we recall that A_{cris} is the p-adic completion of a divided power envelope of $W(\widetilde{\mathbf{E}}^+)$ (cf. [62, §1.4.1]) and let $A_{\mathrm{cris},L} := A_{\mathrm{cris}} \otimes_{o_{L_0}} o_L$. The inclusion $W(\widetilde{\mathbf{E}}^+) \subseteq A_{\mathrm{cris}}$ induces an embedding $\mathbf{A}_L^+ \subseteq W(\widetilde{\mathbf{E}}^+)_L \subseteq A_{\mathrm{cris},L}$.

We observe that $t_{\mathrm{LT}} = \log_{\mathrm{LT}}(\omega_{\mathrm{LT}})$ belongs to $B^\times_{\mathrm{cris},L}$. Indeed, by [22, §III.2] we know that $\varphi_p(B_{\max}) \subseteq B_{\mathrm{cris}} \subseteq B_{\max}$, whence we obtain

$$\varphi_q(B_{\max} \otimes_{L_0} L) \subseteq B_{\mathrm{cris},L} \subseteq B_{\max} \otimes_{L_0} L,$$

where the definition of $B_{\max}$ can be found in loc. cit. By [24, Prop. 9.10, Lem. 9.17, §9.7], t_{LT} and ω_{LT} are invertible in $B_{\max,L} \subseteq B_{\max} \otimes_{L_0} L$ (this reference assumes that the power series $[\pi_L](Z)$ is a polynomial. But, by some additional convergence considerations, the results can be seen to hold in general (cf. [73, §2.1] for more details)). Hence, by the above inclusions and using that $\varphi_q(t_{\mathrm{LT}}) = \pi_L t_{\mathrm{LT}}$, we see that t_{LT} is a unit in $B_{\mathrm{cris},L}$. In particular, we have an inclusion $A_{\mathrm{cris},L}[\frac{1}{\pi_L}, \frac{1}{t_{\mathrm{LT}}}] \subseteq B_{\mathrm{cris},L}$. Moreover, since $\varphi_q(\omega_{\mathrm{LT}}) = Q\omega_{\mathrm{LT}}$ is invertible in $\varphi_q(B_{\max} \otimes_{L_0} L)$, the elements ω_{LT} and Q are units in $B_{\mathrm{cris},L}$ as well. In particular, we have an inclusion

$$\mathbf{A}^+\Big[\frac{1}{\omega_{\mathrm{LT}}}\Big] \subseteq B_{\mathrm{cris},L}. \tag{3.4}$$

Next we shall recall in Lemma 3.1.4 below that the above inclusion $\mathbf{A}_L^+ \subseteq A_{\mathrm{cris},L}$ extends to a (continuous) ring homomorphism

$$\mathcal{O} \to A_{\mathrm{cris},L}\Big[\frac{1}{\pi_L}\Big] \subseteq B_{\mathrm{cris},L}. \tag{3.5}$$

For $\alpha \in \widetilde{\mathbf{E}}^+ \cong \operatorname{proj\,lim}_n o_{\mathbb{C}_p}$ (cf. [73, Lem. 1.4.5]) we denote by $\alpha^{(0)}$ as usual its zero-component.

Lemma 3.1.3. *The following diagram of o_{L_0}-modules is commutative:*

$$\begin{tikzcd} 0 \arrow[r] & J \arrow[r] \arrow[d] & W(\widetilde{\mathbf{E}}^+) \arrow[r, "\Theta"] \arrow[d, "u"] & o_{\mathbb{C}_p} \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \ker(\Theta_L) \arrow[r] & W(\widetilde{\mathbf{E}}^+)_L \arrow[r, "\Theta_L"] & o_{\mathbb{C}_p} \arrow[r] & 0, \end{tikzcd} \tag{3.6}$$

where

$$\begin{aligned} J &:= \ker(\Theta), \\ \Theta &: \sum_{n\geq 0} [\alpha_n] p^n \mapsto \sum_{n\geq 0} \alpha_n^{(0)} p^n, \quad \textit{and similarly} \\ \Theta_L &: \sum_{n\geq 0} [\alpha_n] \pi_L^n \mapsto \sum_{n\geq 0} \alpha_n^{(0)} \pi_L^n, \end{aligned}$$

while u *denotes the canonical map as defined in* [30, Lem. 1.2.3], *which sends Teichmüller lifts* $[\alpha]$ *with respect to* $W(\widetilde{\mathbf{E}}^+)$ *to the Teichmüller lift* $[\alpha]$ *with respect to* $W(\widetilde{\mathbf{E}}^+)_L$.

Proof. First of all, we recall from [73, Lem. 1.6.1] that Θ and Θ_L are continuous and show that also u is continuous, each time with respect to the weak topology. Let V_p and V_{π_L} denote the Verschiebung on $W(\widetilde{\mathbf{E}}^+)$ and $W(\widetilde{\mathbf{E}}^+)_L$, respectively.

A fundamental system of open neighborhoods in these rings consists of

$$U_{\mathfrak{a},m} := \big\{(b_0, b_1, \ldots) \in W(\widetilde{\mathbf{E}}^+) \mid b_0, \ldots, b_{m-1} \in \mathfrak{a}\big\} = \sum_{i=0}^{m-1} V_p^i\big([\mathfrak{a}]\big) + p^m W(\widetilde{\mathbf{E}}^+)$$

and similarly $U^L_{\mathfrak{a},m} := \{(b_0, b_1, \ldots) \in W(\widetilde{\mathbf{E}}^+)_L | b_0, \ldots, b_{m-1} \in \mathfrak{a}\}$ for open ideals $\mathfrak{a}$ of $\widetilde{\mathbf{E}}^+$ and $m \geq 0$; see §1.5 in loc. cit. By o_{L_0}-linearity, we see that $u(p^m W(\widetilde{\mathbf{E}}^+)) \subseteq p^m W(\widetilde{\mathbf{E}}^+)_L$. Using the relation

$$u(V_p x) = \tfrac{p}{\pi_L} V_{\pi_L}\big(u(F^{f-1}x)\big)$$

from [30, Lem. 1.2.3][5] one easily concludes that

$$u\big(V_p^i([b])\big) = \big(\tfrac{p}{\pi_L}\big)^i V_{\pi_L}^i\big([b^{p^{i(f-1)}}]\big),$$

whence $u(U_{\mathfrak{a},m}) \subseteq U^L_{\mathfrak{a},m}$ and the continuity of u follows.

Since the commutativity is clear on Teichmüller lifts and on p by o_{L_0}-linearity, which generate a dense ideal, the result follows by continuity. ■

The following lemma generalizes parts from [62, Prop. 1.5.2].

Lemma 3.1.4. *Sending* $f = \sum_{n\geq 0} a_n Z^n$ *to* $f(\omega_{\mathrm{LT}})$ *induces a continuous map*

$$\mathcal{O} \to A_{\mathrm{cris},L}\big[\tfrac{1}{\pi_L}\big],$$

where the source carries the Fréchet topology while the target is a topological o_{L_0}*-module, of which the topology is uniquely determined by requiring that* $A_{\mathrm{cris},L}$ *is open; i.e., the system* $p^m A_{\mathrm{cris},L}$ *with* $m \geq 0$ *forms a basis of open neighborhoods of* 0.

[5] Note the typos in loc. cit., where $u(V_p x) = \frac{p}{\pi_L} V_{\pi_L}(F^{f-1} u(x))$ is stated. Moreover, one has the relation $u(F^f x) = F u(x)$.

Proof. First of all, the relation $J^p \subseteq pA_{\mathrm{cris}}$ from [62, §1.4.1, bottom of p. 96] (note that $J^p \subseteq W_p(R)$ regarding the notation in loc. cit. for the last object) implies easily by flat base change

$$J_L^p \subseteq pA_{\mathrm{cris},L} \tag{3.7}$$

with $J_L := J \otimes_{o_{L_0}} o_L$. By [73, Lem. 2.1.12] we know that ω_{LT} belongs to $\ker(\Theta_L)$. Now we claim that there exists a natural number r' such that $\omega_{\mathrm{LT}}^{r'}$ lies in $W_1 := J_L + pW(\widetilde{\mathbf{E}}^+)_L$, whence for $r := pr'$ we have $\omega_{\mathrm{LT}}^r \in W_p$ with $W_m := W_1^m$ for all $m \geq 0$. To this aim, note that diagram (3.6) induces the following commutative diagram with exact lines:

$$\begin{tikzcd}
0 \arrow[r] & W_1 \arrow[r] \arrow[d] & W(\widetilde{\mathbf{E}}^+)\otimes_{o_{L_0}} o_L \arrow[r, "\Theta"] \arrow[d, "\cong"'] & (o_{\mathbb{C}_p}\otimes_{o_{L_0}} o_L)/p(o_{\mathbb{C}_p}\otimes_{o_{L_0}} o_L) \arrow[r] \arrow[d, "\mu"'] & 0 \\
0 \arrow[r] & \ker(\Theta_L)+pW(\widetilde{\mathbf{E}}^+)_L \arrow[r] & W(\widetilde{\mathbf{E}}^+)_L \arrow[r, "\Theta_L"] & o_{\mathbb{C}_p}/po_{\mathbb{C}_p} \arrow[r] & 0,
\end{tikzcd}$$

where the map μ is induced by sending $a \otimes b$ to ab and a reference for the middle vertical isomorphism is [73, Prop. 1.1.26]. By the snake lemma the cokernel of the left vertical map is isomorphic to

$$\begin{aligned}
\ker(\mu) &\subseteq \ker\big((o_{\mathbb{C}_p} \otimes_{o_{L_0}} o_L)/p(o_{\mathbb{C}_p} \otimes_{o_{L_0}} o_L) \to \bar{k}\big) \\
&= \ker(o_{\mathbb{C}_p}/po_{\mathbb{C}_p} \otimes_k o_L/po_L \to o_{\mathbb{C}_p}/\mathfrak{m}_{\mathbb{C}_p} \otimes_k o_L/\pi_L o_L) \\
&= \mathfrak{m}_{\mathbb{C}_p} \otimes_k o_L/po_L + o_{\mathbb{C}_p}/po_{\mathbb{C}_p} \otimes_k \pi_L o_L/po_L
\end{aligned}$$

and thus consists of nilpotent elements, whence the claim follows. Here $\mathfrak{m}_{\mathbb{C}_p}$ denotes the maximal ideal of $o_{\mathbb{C}_p}$.

Now let $f = \sum_{n\geq 0} a_n Z^n$ satisfy that $|a_n|\rho^n$ tends to zero for all $\rho < 1$. Writing $n = q_n r + r_n$ with $0 \leq r_n < r$, we have

$$a_n \omega_{\mathrm{LT}}^n = a_n \omega_{\mathrm{LT}}^{r_n} (\omega_{\mathrm{LT}}^r)^{q_n} \in a_n W_{pq_n} \subseteq a_n p^{q_n} A_{\mathrm{cris},L},$$

where the last inclusion follows from (3.7). But $|a_n p^{q_n}| \leq |a_n|_p p^{1-\frac{n}{r}}$ tends to 0 for $n \to \infty$. Thus, the series $\sum_{n\geq 0} a_n \omega_{\mathrm{LT}}^n$ converges in $A_{\mathrm{cris},L}[\frac{1}{\pi_L}]$.

Moreover, since one has $\sup|a_n p^{-1+\frac{n}{r}}| \leq p\|f\|_\rho$ for the usual norms $\| \ \|_\rho$ if $1 > \rho > p^{-\frac{1}{r}}$, we obtain for any m that

$$\big\{f \mid \|f\|_\rho < p^{-m-1}\big\} \subseteq \big\{f \in \mathcal{O} \mid f(\omega_{\mathrm{LT}}) \in p^m A_{\mathrm{cris},L}\big\},$$

whence the latter set, which is the preimage of $p^m A_{\mathrm{cris},L}$, is open. This implies continuity. ∎

Lemma 3.1.5. *The big square in the middle of diagram* (3.2) *is a commutative square of ⊗-functors (up to a natural isomorphism of ⊗-functors).*

Proof. We have to establish a natural isomorphism

$$L \otimes_{o_L} V(\mathbf{A}_L \otimes_{\mathbf{A}_L^+} N) \cong V_L\big(D(\mathcal{O} \otimes_{\mathbf{A}_L^+} N)\big) \quad \text{for any } N \text{ in } \mathrm{Mod}_{\mathbf{A}_L^+}^{\varphi_L,\Gamma_L,\mathrm{an}}. \tag{3.8}$$

In fact, we shall prove the dual statement, i.e., using (3.1), that

$$\big(L \otimes_{o_L} V(\mathbf{A}_L \otimes_{\mathbf{A}_L^+} N)\big)^* \cong V_L\big(D(N)\big)^*, \tag{3.9}$$

where * indicates the L-dual. From the canonical isomorphisms

$$\begin{aligned}\mathrm{Hom}_{\mathbf{A}_L,\varphi_q}(M,\mathbf{A}) &\cong \mathrm{Hom}_{\mathbf{A},\varphi_q}(\mathbf{A} \otimes_{\mathbf{A}_L} M, \mathbf{A})\\ &\cong \mathrm{Hom}_{\mathbf{A},\varphi_q}\big(\mathbf{A} \otimes_{o_L} V(M), \mathbf{A}\big)\\ &\cong \mathrm{Hom}_{o_L}\big(V(M), \mathbf{A}^{\varphi_q=1}\big)\\ &\cong \mathrm{Hom}_{o_L}\big(V(M), o_L\big),\end{aligned}$$

where we used (3.3) for the second isomorphism and write M for $\mathbf{A}_L \otimes_{\mathbf{A}_L^+} N$, we conclude that the left-hand side of (3.9) is canonically isomorphic to $\mathrm{Hom}_{\mathbf{A}_L,\varphi_q}(\mathbf{A}_L \otimes_{\mathbf{A}_L^+} N, \mathbf{A}) \otimes_{o_L} L$. Let $\mathbf{A}^+ := \mathbf{A} \cap W(\widetilde{\mathbf{E}}^+)_L$. On the one hand, by [44, Lem. 3.2.1], base extension induces an isomorphism

$$\mathrm{Hom}_{\mathbf{A}_L^+,\varphi_q}\big(N, \mathbf{A}^+[\tfrac{1}{\omega_{\mathrm{LT}}}]\big) \xrightarrow{\cong} \mathrm{Hom}_{\mathbf{A}_L,\varphi_q}(\mathbf{A}_L \otimes_{\mathbf{A}_L^+} N, \mathbf{A}).$$

On the other hand, in [44, Prop. 3.2.3], Kisin and Ren construct a natural isomorphism

$$\mathrm{Hom}_{\mathbf{A}_L^+,\varphi_q}\big(N, \mathbf{A}^+[\tfrac{1}{\omega_{\mathrm{LT}}}]\big) \otimes_{o_L} L \xrightarrow{\cong} \mathrm{Hom}_{L,\varphi_q,\mathrm{Fil}}\big((N/\omega_{\mathrm{LT}}N)[\tfrac{1}{p}], B_{\mathrm{cris},L}\big).^{6} \tag{3.10}$$

Therefore, the left-hand side of (3.9) becomes naturally isomorphic to

$$\mathrm{Hom}_{L,\varphi_q,\mathrm{Fil}}\big(D(N), B_{\mathrm{cris},L}\big) \cong V_L\big(D(N)^*\big), \tag{3.11}$$

where the last isomorphism is straightforward. Thus, the proof of (3.9) is reduced to the canonical identity

$$V_L\big(D(N)^*\big) \cong V_L\big(D(N)\big)^*. \tag{3.12}$$

This can be proved in the same way as in [31, Rem. 3.4.5 (iii), Rem. 3.6.7]: Since V_L is a rigid $\otimes$-functor, it preserves inner Hom-objects, in particular duals.

In order to see that (3.8) is compatible with tensor products, note that base change, taking L-duals or applying comparison isomorphisms, is $\otimes$-compatible. Thus the claim is reduced to the tensor compatibility of the isomorphism (3.10), the construction of which we therefore recall from [44]. It is induced by a natural map

$$\mathrm{Hom}_{\mathbf{A}_L^+}\big(N, \mathbf{A}^+[\tfrac{1}{\omega_{\mathrm{LT}}}]\big) \otimes_{o_L} L \to \mathrm{Hom}_L\big((N/\omega_{\mathrm{LT}}N)[\tfrac{1}{p}], B_{\mathrm{cris},L}\big)$$

[6] $\otimes_{o_L} L$ is missing in the reference!

which comes about as follows. Let $f \in \mathrm{Hom}_{\mathbf{A}_L^+}(N, \mathbf{A}^+[\frac{1}{\omega_{\mathrm{LT}}}])$. By composing f with the inclusion (3.4) we obtain $f_1 : N \to B_{\mathrm{cris},L}$. By base extension to $\mathcal{O}$ via (3.5) and then localization in Q the map f_1 gives rise to a map $f_2 : (\mathcal{O} \otimes_{\mathbf{A}_L^+} N)[\frac{1}{Q}] \to B_{\mathrm{cris},L}$. We precompose this one with the isomorphism $1 \otimes \varphi_N$ to obtain

$$f_3 : (\mathcal{O} \otimes_{\mathbf{A}_L^+,\varphi_L} N)\big[\tfrac{1}{Q}\big] \xrightarrow{\cong} (\mathcal{O} \otimes_{\mathbf{A}_L^+} N)\big[\tfrac{1}{Q}\big] \to B_{\mathrm{cris},L}.$$

Now we observe the inclusions

$$\begin{aligned}(\mathcal{O} \otimes_{\mathbf{A}_L^+,\varphi_L} N)\big[\tfrac{1}{Q}\big] \subseteq (\mathcal{O} \otimes_{\mathbf{A}_L^+,\varphi_L} N)\big[\tfrac{\omega_{\mathrm{LT}}}{t_{\mathrm{LT}}}\big] &\supseteq (\mathcal{O} \otimes_{\mathbf{A}_L^+,\varphi_L} N)\big[\varphi_L\big(\tfrac{\omega_{\mathrm{LT}}}{t_{\mathrm{LT}}}\big)\big] \\ &= \mathcal{O} \otimes_{\mathcal{O},\varphi_L} \big((\mathcal{O} \otimes_{\mathbf{A}_L^+} N)\big[\tfrac{\omega_{\mathrm{LT}}}{t_{\mathrm{LT}}}\big]\big).\end{aligned}$$

They only differ by elements which are invertible in $B_{\mathrm{cris},L}$. Therefore giving the map f_3 is equivalent to giving a map $f_4 : \mathcal{O} \otimes_{\mathcal{O},\varphi_L} ((\mathcal{O} \otimes_{\mathbf{A}_L^+} N)[\frac{\omega_{\mathrm{LT}}}{t_{\mathrm{LT}}}]) \to B_{\mathrm{cris},L}$. Finally, we use Remark 3.1.2 (iii) which gives the map

$$\xi : (N/\omega_{\mathrm{LT}} N)\big[\tfrac{1}{p}\big] \underset{\mathrm{pr}}{\xleftarrow{\cong}} \big((\mathcal{O} \otimes_{\mathbf{A}_L^+} N)\big[\tfrac{\omega_{\mathrm{LT}}}{t_{\mathrm{LT}}}\big]\big)^{\Gamma_L} \xrightarrow{\subseteq} (\mathcal{O} \otimes_{\mathbf{A}_L^+} N)\big[\tfrac{\omega_{\mathrm{LT}}}{t_{\mathrm{LT}}}\big].$$

By precomposing f_4 with $1 \otimes \xi$ we at last arrive at a map $f_5 : (N/\omega_{\mathrm{LT}} N)[\frac{1}{p}] \to B_{\mathrm{cris},L}$. From this description the compatibility with tensor products is easily checked. ∎

Suppose that N is in $\mathrm{Mod}_{\mathbf{A}_L^+}^{\varphi_L,\Gamma_L,\mathrm{an}}$ and put $T := V(\mathbf{A}_L \otimes_{\mathbf{A}_L^+} N)$ in $\mathrm{Rep}_{o_L,f}^{\mathrm{cris,an}}(G_L)$. Then, by Remark 3.1.2 (iii) and Lemma 3.1.5, we have a natural isomorphism of $\otimes$-functors

$$\mathrm{comp} : \mathcal{O}\big[\tfrac{\omega_{\mathrm{LT}}}{t_{\mathrm{LT}}}\big] \otimes_{\mathbf{A}_L^+} N \xrightarrow{\cong} \mathcal{O}\big[\tfrac{\omega_{\mathrm{LT}}}{t}{}_{\mathrm{LT}}\big] \otimes_L D_{\mathrm{cris},L}(L \otimes_{o_L} T) \tag{3.13}$$

which is compatible with the diagonal φ's on both sides.

In the proof of [44, Cor. 3.3.8], it is shown that, for any T in $\mathrm{Rep}_{o_L,f}^{\mathrm{cris,an}}(G_L)$, there exists an $\mathbf{A}_L^+$-submodule $\mathfrak{M} \subseteq D_{\mathrm{LT}}(T)$ which

(N1) lies in $\mathrm{Mod}_{\mathbf{A}_L^+}^{\varphi_L,\Gamma_L,\mathrm{an}}$ with $\varphi_{\mathfrak{M}}$ and the Γ_L-action on $\mathfrak{M}$ being induced by the (φ_q, Γ_L)-structure of $D_{\mathrm{LT}}(T)$ and

(N2) satisfies $\mathbf{A}_L \otimes_{\mathbf{A}_L^+} \mathfrak{M} = D_{\mathrm{LT}}(T)$.

Note that property (N2) implies that $\mathfrak{M}$ is *p-saturated in* $D_{\mathrm{LT}}(T)$, i.e., $\mathfrak{M}[\frac{1}{p}] \cap D_{\mathrm{LT}}(T) = \mathfrak{M}$, since $\mathbf{A}_L^+$ is obviously p-saturated in $\mathbf{A}_L$.

We once and for all pick such an $N(T) := \mathfrak{M}$. This defines a functor

$$N : \mathrm{Rep}_{o_L,f}^{\mathrm{cris,an}}(G_L) \to \mathrm{Mod}_{\mathbf{A}_L^+}^{\varphi_L,\Gamma_L,\mathrm{an}}$$

which is quasi-inverse to the undecorated horizontal arrow in the above big diagram (3.2). Note that N is in a unique way a $\otimes$-functor by [65, §I.4.4.2.1].

Remark 3.1.6. For T in $\mathrm{Rep}^{\mathrm{cris,an}}_{o_L,f}(G_L)$ and $N := N(T)$ in $\mathrm{Mod}^{\varphi_L,\Gamma_L,\mathrm{an}}_{\mathbf{A}_L^+}$ we have the following:

(i) If $V := L \otimes_{o_L} T$ is a positive (the Hodge–Tate weights are non-positive, i.e., $gr^j D_{\mathrm{dR}}(V) \neq 0$ implies that $j \geq 0$) analytic crystalline representation, then N is stable under φ_N.

(ii) If the Hodge–Tate weights of $L \otimes_{o_L} T$ are all ≥ 0, then we have

$$N \subseteq \mathbf{A}_L^+ \cdot \varphi_N(N),$$

where the latter means the $\mathbf{A}_L^+$-span generated by $\varphi_N(N)$.

Proof. The corresponding assertions for $\mathcal{M} := \mathcal{O} \otimes_{\mathbf{A}_L^+} N$ are contained in [8, Cor. 3.4.9]. Let $n_1, \ldots, n_d$ be an $\mathbf{A}_L^+$-basis of N.

For (i) we have to show that $\varphi_N(n_j) \in N$ for any $1 \leq j \leq d$. Writing $\varphi_N(n_j) = \sum_{i=1}^d f_{ij} n_i$ we know from the definition of the category $\mathrm{Mod}^{\varphi_L,\Gamma_L,\mathrm{an}}_{\mathbf{A}_L^+}$ that $f_{ij} \in \mathbf{A}_L^+[\frac{1}{Q}]$ and from the above observation that $f_{ij} \in \mathcal{O}$. This reduces us to showing the inclusion $o_L[\![Z]\!][\frac{1}{Q}] \cap \mathcal{O} \subseteq o_L[\![Z]\!]$. Suppose therefore that $Q^r h = f$ for some $r \geq 1$, $h \in \mathcal{O}$, and $f \in o_L[\![Z]\!]$. The finitely many zeros of $Q \in o_L[\![Z]\!]$, which are the non-zero π_L-torsion points of the Lubin–Tate formal group, all lie in the open unit disk. By Weierstrass preparation it follows that Q must divide f already in $o_L[\![Z]\!]$. Hence, $h \in o_L[\![Z]\!]$.

For (ii) we have to show that $n_j = \sum_{i=1}^d f_{ij} \varphi_N(n_i)$, for any $1 \leq j \leq d$, with $f_{ij} \in \mathbf{A}_L^+$. For the same reasons as in the proof of (1) we have

$$n_j = \sum_{i=1}^d f'_{ij} \varphi_N(n_i) = \sum_{i=1}^d f''_{ij} \varphi_N(n_i)$$

with $f'_{ij} \in \mathbf{A}_L^+[\frac{1}{Q}]$ and $f''_{ij} \in \mathcal{O}$. Then $\sum_{i=1}^d (f'_{ij} - f''_{ij}) \varphi_N(n_i) = 0$. But, again by the definition of the category $\mathrm{Mod}^{\varphi_L,\Gamma_L,\mathrm{an}}_{\mathbf{A}_L^+}$, the $\varphi_N(n_i)$ are linearly independent over $\mathbf{A}_L^+[\frac{1}{Q}]$ and hence over $\mathcal{O}[\frac{1}{Q}]$. It follows that $f'_{ij} = f''_{ij} \in \mathbf{A}_L^+$. ∎

First, we further investigate any T in $\mathrm{Rep}^{\mathrm{cris,an}}_{o_L,f}(G_L)$ whose *Hodge–Tate weights are all* ≤ 0, i.e., which is positive. For this purpose we need the ring $\mathbf{A}^+ = \mathbf{A} \cap W(\widetilde{\mathbf{E}}^+)_L$. One has the following general fact.

Lemma 3.1.7. *Let F be any non-archimedean valued field which contains $o_L/\pi_L o_L$, and let o_F denote its ring of integers; we have the following:*

(i) *Let $\alpha \in W(F)_L$ be any element; if the $W(o_F)_L$-submodule of $W(F)_L$ generated by $\{\varphi_q^i(\alpha)\}_{i \geq 0}$ is finitely generated, then $\alpha \in W(o_F)_L$.*

(ii) *Let X be a finitely generated free o_L-module, and let M be a finitely generated $W(o_F)_L$-submodule of $W(F)_L \otimes_{o_L} X$; if M is $\varphi_q \otimes \mathrm{id}$-invariant, then $M \subseteq W(o_F)_L \otimes_{o_L} X$.*

Proof. (i) This is a simple explicit calculation as given, for example, in the proof of [23, Lem. III.5]. (ii) This is a straightforward consequence of (i). ∎

Proposition 3.1.8. *For positive T in $\mathrm{Rep}^{\mathrm{cris,an}}_{o_L,f}(G_L)$ we have*

$$N(T) \subseteq D^+_{\mathrm{LT}}(T) := (\mathbf{A}^+ \otimes_{o_L} T)^{\ker(\chi_{\mathrm{LT}})},$$

and $N(T)$ is p-saturated in $D^+_{\mathrm{LT}}(T)$.

Proof. By Remark 3.1.6 (i) the $\mathbf{A}_L^+$-submodule $N(T)$ of $W(\widetilde{\mathbf{E}})_L \otimes_{o_L} T$ is $\varphi_q \otimes \mathrm{id}$-invariant (and finitely generated). Hence, we may apply Lemma 3.1.7 (ii) to $M := W(\widetilde{\mathbf{E}}^+)_L \cdot N(T)$ and obtain that

$$N(T) \subseteq \big(W(\widetilde{\mathbf{E}}^+)_L \otimes_{o_L} T\big) \cap (\mathbf{A} \otimes_{o_L} T)^{\ker(\chi_{\mathrm{LT}})} = D^+_{\mathrm{LT}}(T).$$

Since $N(T)$ is even p-saturated in $D_{\mathrm{LT}}(T)$, the same holds with respect to the smaller $D^+_{\mathrm{LT}}(T)$. ∎

Corollary 3.1.9. *For positive T in $\mathrm{Rep}^{\mathrm{cris,an}}_{o_L,f}(G_L)$ the $\mathbf{A}_L^+$-module $D^+_{\mathrm{LT}}(T)$ is free of the same rank as $N(T)$.*

Proof. By the argument in the proof of [23, Lem. III.3] the $\mathbf{A}_L^+$-module $D^+_{\mathrm{LT}}(T)$ is always free of a rank less than or equal to the rank of $N(T)$. The equality of the ranks in the positive case then is a consequence of Proposition 3.1.8. ∎

Next we relate $N(T)$ to the construction in [5, Prop. II.1.1].

Proposition 3.1.10. *Suppose that T in $\mathrm{Rep}^{\mathrm{cris,an}}_{o_L,f}(G_L)$ is positive. For $N := N(T)$ we then have the following:*

(i) *N is the unique $\mathbf{A}_L^+$-submodule of $D_{\mathrm{LT}}(T)$ which satisfies (N1) and (N2).*

(ii) *N is also the unique $\mathbf{A}_L^+$-submodule of $D^+_{\mathrm{LT}}(T)$ which satisfies the following:*

   (a) *N is free of rank equal to the rank of $D^+_{\mathrm{LT}}(T)$;*

   (b) *N is Γ_L-invariant;*

   (c) *the induced Γ_L-action on $N/\omega_{\mathrm{LT}}N$ is trivial;*

   (d) *$\omega^r_{\mathrm{LT}} D^+_{\mathrm{LT}}(T) \subseteq N$ for some $r \geq 0$.*

Proof. Let $P = P(\mathbf{A}_L^+)$ denote the set of height one prime ideals of $\mathbf{A}_L^+$. It contains the prime ideal $\mathfrak{p}_0 := (\omega_{\mathrm{LT}})$.

Step 1. We show the existence of a unique $\mathbf{A}_L^+$-submodule N' of $D^+_{\mathrm{LT}}(T)$ which satisfies (a)–(d), and we show that this N' is φ_q-invariant.

Existence. We begin by observing that the $\mathbf{A}_L^+$-submodule $N := N(T)$ of $D^+_{\mathrm{LT}}(T)$ has the properties (a), (b), and (c), but possibly not (d). In particular, the quotient

$D^+_{\mathrm{LT}}(T)/N$ is an $\mathbf{A}^+_L$-torsion module. Hence, the localizations $N_{\mathfrak{p}} = D^+_{\mathrm{LT}}(T)_{\mathfrak{p}}$ coincide for all but finitely many $\mathfrak{p} \in P$. By [16, VII.4.3 Thm. 3] there exists a unique intermediate $\mathbf{A}^+_L$-module $N \subseteq N' \subseteq D^+_{\mathrm{LT}}(T)$ which is finitely generated and reflexive and such that $N'_{\mathfrak{p}_0} = N_{\mathfrak{p}_0}$ and $N'_{\mathfrak{p}} = D^+_{\mathrm{LT}}(T)_{\mathfrak{p}}$ for any $\mathfrak{p} \in P \setminus \{\mathfrak{p}_0\}$. Since $\mathbf{A}^+_L$ is a two-dimensional regular local ring, the finitely generated reflexive module N' is actually free and then, of course, must have the same rank as N and $D^+_{\mathrm{LT}}(T)$. We also have $N' = \bigcap_{\mathfrak{p}} N'_{\mathfrak{p}} = N_{\mathfrak{p}_0} \cap \bigcap_{\mathfrak{p} \neq \mathfrak{p}_0} D^+_{\mathrm{LT}}(T)_{\mathfrak{p}}$. Since $\mathfrak{p}_0$ is preserved by $\varphi_{D^+_{\mathrm{LT}}(T)}$ and Γ_L, it follows that N' is $\varphi_{D^+_{\mathrm{LT}}(T)}$- and Γ_L-invariant. Next the identities

$$L \otimes_{o_L} N/\omega_{\mathrm{LT}} N = N_{\mathfrak{p}_0}/\omega_{\mathrm{LT}} N_{\mathfrak{p}_0} = N'_{\mathfrak{p}_0}/\omega_{\mathrm{LT}} N'_{\mathfrak{p}_0} = L \otimes_{o_L} N'/\omega_{\mathrm{LT}} N' \supseteq N'/\omega_{\mathrm{LT}} N'$$

show that the induced Γ_L-action on $N'/\omega_{\mathrm{LT}} N'$ is trivial. By using [16, VII.4.4 Thm. 5] we obtain, for some $m_1, \ldots, m_d \geq 0$, a homomorphism of $\mathbf{A}^+_L$-modules $D^+_{\mathrm{LT}}(T)/N' \to \bigoplus_{i=1}^d \mathbf{A}^+_L/\mathfrak{p}_0^{m_i}\mathbf{A}^+_L$ whose kernel is finite. Any finite $\mathbf{A}^+_L$-module is annihilated by a power of the maximal ideal in $\mathbf{A}^+_L$. We see that $D^+_{\mathrm{LT}}(T)/N'$ is annihilated by a power of $\mathfrak{p}_0$, which proves (d).

Uniqueness. Observing that

$$\gamma(\omega_{\mathrm{LT}}) = \big[\chi_{\mathrm{LT}}(\gamma)\big](\omega_{\mathrm{LT}}) \quad \text{for any } \gamma \in \Gamma_L$$

(cf. [73, Lem. 2.1.15]), this is exactly the same computation as in the uniqueness part of the proof of [5, Prop. II.1.1].

Step 2. We show that N' is p-saturated in $D^+_{\mathrm{LT}}(T)$. By construction we have $(N')_{(\pi_L)} = D^+_{\mathrm{LT}}(T)_{(\pi_L)}$. This implies that the p-torsion in the quotient $D^+_{\mathrm{LT}}(T)/N'$ is finite. On the other hand, both modules, N' and $D^+_{\mathrm{LT}}(T)$, are free of the same rank. Hence, the finitely generated $\mathbf{A}^+_L$-module $D^+_{\mathrm{LT}}(T)/N'$ has projective dimension ≤ 1 and therefore has no non-zero finite submodule (cf. [59, Prop. 5.5.3 (iv)]).

Step 3. We show that $N' = N$. Since both, N and N', are p-saturated in $D^+_{\mathrm{LT}}(T)$, it suffices to show that the free $\mathbf{B}^+_L$-modules $N(V) := N[\frac{1}{p}]$ and $N'(V) := N'[\frac{1}{p}]$ over the principal ideal domain $\mathbf{B}^+_L := \mathbf{A}^+_L[\frac{1}{p}]$ coincide. As they are both Γ_L-invariant, so is the annihilator ideal

$$I := \mathrm{ann}_{\mathbf{B}^+_L}\big(N'(V)/N(V)\big).$$

Hence, by a standard argument as in [5, Lem. 1.3.2], the ideal I is generated by an element f of the form $\omega_{\mathrm{LT}}^{\alpha_0} \prod_{n\geq 1}^{s} \varphi_L^{n-1}(Q)^{\alpha_n}$ with certain $\alpha_n \geq 0$, $0 \leq n \leq s$, for some (minimal) $s \geq 0$. Since $N(V)_{(\omega_{\mathrm{LT}})} = N'(V)_{(\omega_{\mathrm{LT}})}$ by the construction of N', it follows that $\alpha_0 = 0$. Assuming that $M := N'(V)/N(V) \neq 0$, we conclude that $s \geq 1$ (with $\alpha_s \geq 1$), i.e., that, with $\mathfrak{p}_n := (\varphi_L^{n-1}(Q))$, we have $M_{\mathfrak{p}_s} \neq 0$ while $M_{\mathfrak{p}_{s+1}} = 0$. We claim that $(\varphi_L^* M)_{\mathfrak{p}_{s+1}} \neq 0$. First, note that we have an exact sequence

$$0 \to (\mathbf{B}^+_L)^d \xrightarrow{A} (\mathbf{B}^+_L)^d \to M \to 0,$$

with f dividing $\det(A) \in \mathbf{B}_L^+ \setminus (\mathbf{B}_L^+)^\times_{\mathfrak{p}_s}$, which induces an exact sequence

$$0 \to (\mathbf{B}_L^+)^d \xrightarrow{\varphi_L(A)} (\mathbf{B}_L^+)^d \to \varphi_L^* M \to 0.$$

Since $\varphi_L(f) = \prod_{n\geq 2}^{s+1} \varphi_L^{n-1}(Q)^{\alpha_{n-1}}$ divides $\det(\varphi_L(A))$, we conclude that $\det(\varphi_L(A))$ belongs to $\mathfrak{p}_{s+1}$ which implies the claim.

Now consider the following diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & (\varphi_L^* N(V))[\frac{1}{Q}] \arrow[r, "\cong"] \arrow[d, hook] & N(V)[\frac{1}{Q}] \arrow[r] \arrow[d, hook] & 0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & (\varphi_L^* N'(V))[\frac{1}{Q}] \arrow[r] & N'(V)[\frac{1}{Q}] \arrow[r] & C \arrow[r] & 0.
\end{tikzcd}$$

The upper isomorphism comes from the definition of the category $\mathrm{Mod}_{\mathbf{A}_L^+}^{\varphi_L,\Gamma_L,\mathrm{an}}$ in which N lies. The map

$$(\varphi_L^* N'(V))[\tfrac{1}{Q}] \to N'(V)[\tfrac{1}{Q}]$$

is injective since $\varphi_L^* N' \to N'$ is the restriction of the isomorphism

$$\varphi_L^* D_{\mathrm{LT}}(T) \xrightarrow{\cong} D_{\mathrm{LT}}(T).$$

By the snake lemma and as $Q \notin \mathfrak{p}_{s+1}$ we obtain an injection

$$0 \neq (\varphi_L^* M)_{\mathfrak{p}_{s+1}} \hookrightarrow M_{\mathfrak{p}_{s+1}} = 0,$$

which is a contradiction. Thus, $M = 0$ as had to be shown. ■

Remark 3.1.11. The following are two examples:

(i) We have $N(o_L(\chi_{\mathrm{LT}}^{-1})) = \omega_{\mathrm{LT}} \mathbf{A}_L^+ \otimes_{o_L} o_L \eta^{\otimes -1}$ and $N(o_L) = \mathbf{A}_L^+$ (recall that η denotes our fixed generator of T_π).

(ii) Let $o_L(\chi) = o_L t_0$ with $\chi : G_L \to o_L^\times$ unramified. Then there exists an $a \in W(\bar{k}_L)_L^\times$ with $\sigma a = \chi^{-1}(\sigma) a$ for all $\sigma \in G_L$ by Remark 3.2.4;[7] in particular,

$$N\big(o_L(\chi)\big) = D_{\mathrm{LT}}^+\big(o_L(\chi)\big) = \mathbf{A}_L^+ n_0 \quad \text{for } n_0 = a \otimes t_0,$$

where Γ_L fixes n_0 and $\varphi_{N(o_L(\chi))}(n_0) = c n_0$ with $c := \frac{\varphi_L(a)}{a} \in o_L^\times$.

Proof. Each case belongs to a positive representation T: in all cases the right-hand side of the equality satisfies the properties characterizing $N(T)$ in Proposition 3.1.10 (ii) (cf. [73, Lem. 2.1.15]). ■

[7] Since π_L has trivial G_L-action, the period a there can be normalized such that it becomes a unit in $W(\bar{k}_L)_L$.

Lemma 3.1.12. *For any $T \in \operatorname{Rep}^{\mathrm{cris,an}}_{o_L,f}(G_L)$ we have the following:*

- *$N(T)$ is the unique $\mathbf{A}_L^+$-submodule of $D_{\mathrm{LT}}(T)$ which satisfies* (N1) *and* (N2)*;*
- *$N(T(\chi_{\mathrm{LT}}^{-r})) \cong \omega_{\mathrm{LT}}^r N(T) \otimes_{o_L} o_L\eta^{\otimes -r}$.*

Proof. First, we choose $r \geq 0$ such that $T(\chi_{\mathrm{LT}}^{-r})$ is positive. Sending N to

$$\omega_{\mathrm{LT}}^r N(T) \otimes_{o_L} o_L\eta^{\otimes -r} \subseteq D_{\mathrm{LT}}(T) \otimes_{o_L} o_L\eta^{\otimes -r}$$

viewed in $D_{\mathrm{LT}}(T) \otimes_{o_L} o_L\eta^{\otimes -r} \cong D_{\mathrm{LT}}(T(\chi_{\mathrm{LT}}^{-r}))$ sets up a bijection between the $\mathbf{A}_L^+$-submodules of $D_{\mathrm{LT}}(T)$ and $D_{\mathrm{LT}}(T(\chi_{\mathrm{LT}}^{-r}))$, respectively. One checks that N satisfies (N1) and (N2) if and only if its image does. Hence, (i) and (ii) (for such r) are a consequence of Proposition 3.1.10 (i). That (ii) holds in general follows from the obvious transitivity property of the above bijections. ∎

Proposition 3.1.13. *Let T be in $\operatorname{Rep}^{\mathrm{cris,an}}_{o_L,f}(G_L)$ of o_L-rank d and such that $V = L \otimes_{o_L} T$ is positive with Hodge–Tate weights $-r = -r_d \leq \cdots \leq -r_1 \leq 0$. Taking* (3.13) *as an identification, we then have*

$$\left(\tfrac{t_{\mathrm{LT}}}{\omega_{\mathrm{LT}}}\right)^r \mathcal{O} \otimes_{\mathbf{A}_L^+} N(T) \subseteq \mathcal{O} \otimes_L D_{\mathrm{cris},L}(L \otimes_{o_L} T) \subseteq \mathcal{O} \otimes_{\mathbf{A}_L^+} N(T) \tag{3.14}$$

with elementary divisors

$$\left[\mathcal{O} \otimes_{\mathbf{A}_L^+} N(T) : \mathcal{O} \otimes_L D_{\mathrm{cris},L}(L \otimes_{o_L} T)\right] = \left[\left(\tfrac{t_{\mathrm{LT}}}{\omega_{\mathrm{LT}}}\right)^{r_1} : \cdots : \left(\tfrac{t_{\mathrm{LT}}}{\omega_{\mathrm{LT}}}\right)^{r_d}\right].$$

Proof. We abbreviate $D := D_{\mathrm{cris},L}(V)$. By the definition of the functor $\mathcal{M}$ in [44] we have

$$\mathcal{O} \otimes_L D \subseteq \mathcal{M}(D) \subseteq \left(\tfrac{t_{\mathrm{LT}}}{\omega_{\mathrm{LT}}}\right)^{-r} \mathcal{O} \otimes_L D. \tag{3.15}$$

On the other hand, the commutativity of the big diagram before Remark 3.1.2 says that $\mathcal{M}(D) \cong \mathcal{O} \otimes_{\mathbf{A}_L^+} N(T)$. This implies the inclusions (3.14).

Concerning the second part of the assertion, we first of all note that although $\mathcal{O}$ is only a Bézout domain, it does satisfy the elementary divisor theorem (cf. [75, proof of Prop. 4.4]). We may equivalently determine the elementary divisors of the $\mathcal{O}$-module $\mathcal{M}(D)/(\mathcal{O} \otimes_L D)$. The countable set $\mathbb{S}$ of zeros of the function $\frac{t_{\mathrm{LT}}}{\omega_{\mathrm{LT}}} \in \mathcal{O}$ coincides with the set of non-zero torsion points of our Lubin–Tate formal group, each occurring with multiplicity one. The first part of the assertion implies that the $\mathcal{O}$-module $\mathcal{M}(D)/(\mathcal{O} \otimes_L D)$ is supported on $\mathbb{S}$. Let $\mathcal{M}_z(D)$, resp., $\mathcal{O}_z$, denote the stalk in $z \in \mathbb{S}$ of the coherent sheaf on the open unit disk $\mathbf{B}$ defined by $\mathcal{M}(D)$, resp., $\mathcal{O}$. The argument in the proof of [8, Prop. 1.1.10] then shows that we have

$$\mathcal{M}(D)/(\mathcal{O} \otimes_L D) = \prod_{z \in \mathbb{S}} \mathcal{M}_z(D)/(\mathcal{O}_z \otimes_L D).$$

The ring $\mathcal{O}_z$ is a discrete valuation ring with maximal ideal $\mathfrak{m}_z$ generated by $\frac{t_{\mathrm{LT}}}{\omega_{\mathrm{LT}}}$. We consider on its field of fractions $\mathrm{Fr}(\mathcal{O}_z)$ the $\mathfrak{m}_z$-adic filtration and then on $\mathrm{Fr}(\mathcal{O}_z) \otimes_L D$ the tensor product filtration. By [43, Lem. 1.2.1 (2)] (or [8, Lem. 3.4.4]) we have

$$\mathcal{M}_z(D) \cong \mathrm{Fil}^0(\mathrm{Fr}(\mathcal{O}_z) \otimes_L D) \quad \text{for any } z \in \mathbb{S},$$

and this isomorphism preserves $\mathcal{O}_z \otimes_L D$. At this point we let $0 \leq s_1 < \cdots < s_m < r$ denote the jumps of the filtration $\mathrm{Fil}^\bullet D$, i.e., the r_j but without repetition. We write

$$D = D_1 \oplus \cdots \oplus D_m \quad \text{such that} \quad \mathrm{Fil}^{s_i} D = D_i \oplus \cdots \oplus D_m.$$

For the following computation let, for notational simplicity, R denote any L-algebra which is a discrete valuation ring with maximal ideal $\mathfrak{m}$. We compute

$$\begin{aligned}
\mathrm{Fil}^0\big(\mathrm{Fr}(R) \otimes_L D\big) &= \sum_{j \in \mathbb{Z}} \mathfrak{m}^{-j} \otimes_L \mathrm{Fil}^j D = \sum_{j=0}^{r} \mathfrak{m}^{-j} \otimes_L \mathrm{Fil}^j D \\
&= \sum_{i=1}^{m} \mathfrak{m}^{-s_i} \otimes_L \mathrm{Fil}^{s_i} D = \sum_{i=1}^{m} \sum_{j=i}^{m} \mathfrak{m}^{-s_i} \otimes_L D_j \\
&= \sum_{j=1}^{m} \Big(\sum_{i=1}^{j} \mathfrak{m}^{-s_i} \Big) \otimes_L D_j = \sum_{j=1}^{m} \mathfrak{m}^{-s_j} \otimes_L D_j .
\end{aligned}$$

Hence, we obtain

$$\mathrm{Fil}^0\big(\mathrm{Fr}(R) \otimes_L D\big)/(R \otimes_L D) = \bigoplus_{j=1}^{m} \mathfrak{m}^{-s_j}/R \otimes_L D_j \cong \bigoplus_{j=1}^{m} R/\mathfrak{m}^{s_j} \otimes_L D_j .$$

By combining all of the above, we finally arrive at

$$\begin{aligned}
\mathcal{M}(D)/(\mathcal{O} \otimes_L D) &= \prod_{z \in \mathbb{S}} \mathcal{M}_z(D)/(\mathcal{O}_z \otimes_L D) \\
&\cong \prod_{z \in \mathbb{S}} \mathrm{Fil}^0\big(\mathrm{Fr}(\mathcal{O}_z) \otimes_L D\big)/(\mathcal{O}_z \otimes_L D) \\
&\cong \prod_{z \in \mathbb{S}} \Big(\bigoplus_{j=1}^{m} \mathcal{O}_z/\mathfrak{m}_z^{s_j} \otimes_L D_j \Big) \\
&= \bigoplus_{j=1}^{m} \Big(\prod_{z \in \mathbb{S}} \big(\mathcal{O}_z/(\tfrac{t_{\mathrm{LT}}}{\omega_{\mathrm{LT}}})^{s_j} \mathcal{O}_z \otimes_L D_j\big) \Big) \\
&= \bigoplus_{j=1}^{m} \Big(\prod_{z \in \mathbb{S}} \mathcal{O}_z/(\tfrac{t_{\mathrm{LT}}}{\omega_{\mathrm{LT}}})^{s_j} \mathcal{O}_z \Big) \otimes_L D_j \\
&= \bigoplus_{j=1}^{m} \mathcal{O}/(\tfrac{t_{\mathrm{LT}}}{\omega_{\mathrm{LT}}})^{s_j} \mathcal{O} \otimes_L D_j .
\end{aligned}$$

■

For a first application of this result we recall the comparison isomorphism

$$\begin{tikzcd} N(V) \arrow[d, "\subseteq"'] & & \\ N(V)[\frac{1}{Q}] \arrow[r, "\subseteq"] & \mathcal{O}[\frac{\omega_{\mathrm{LT}}}{t_{\mathrm{LT}}}] \otimes_{\mathbf{A}_L^+} N(V) \arrow[r, "\mathrm{comp}", "\cong"'] & \mathcal{O}[\frac{\omega_{\mathrm{LT}}}{t_{\mathrm{LT}}}] \otimes_L D_{\mathrm{cris},L}(V) \end{tikzcd} \tag{3.16}$$

for any T in $\mathrm{Rep}^{\mathrm{cris,an}}_{o_L,f}(G_L)$, $V := L \otimes_{o_L} T$, and $N(V) := N(T)[\frac{1}{\pi_L}]$. The left horizontal inclusion comes from the fact that $\frac{t_{\mathrm{LT}}}{\omega_{\mathrm{LT}}}$ is a multiple of Q in $\mathcal{O}$. In particular, we have the commutative diagram

$$\begin{tikzcd} & N(V) \arrow[dl, "\varphi_{N(V)}"'] \arrow[d, "\varphi_{N(V)}"] \arrow[r, "\mathrm{comp}"] & \mathcal{O}[\frac{\omega_{\mathrm{LT}}}{t_{\mathrm{LT}}}] \otimes_L D_{\mathrm{cris},L}(V) \arrow[d, "\varphi_L \otimes \varphi_{\mathrm{cris}}"] \\ N^{(\varphi)}(V) \arrow[r, "\subseteq"] & N(V)[\frac{1}{Q}] \arrow[r, "\mathrm{comp}"] & \mathcal{O}[\frac{\omega_{\mathrm{LT}}}{t_{\mathrm{LT}}}] \otimes_L D_{\mathrm{cris},L}(V), \end{tikzcd}$$

where we let $N^{(\varphi)}(V)$ be the $\mathbf{A}_L^+$-submodule of $N(V)[\frac{1}{Q}]$ generated by the image of $N(V)$ under $\varphi_{N(V)}$ and where φ_{cris} denotes the q-Frobenius on $D_{\mathrm{cris},L}(V)$. We note that since Q is invertible in $\mathbf{A}_L$, $N^{(\varphi)}(V)$ can also be viewed as the $\mathbf{A}_L^+$-submodule of $D_{\mathrm{LT}}(V) = \mathbf{A}_L \otimes_{\mathbf{A}_L^+} N(V)$ generated by the image of $N(V)$ under $\varphi_{D_{\mathrm{LT}}(V)}$. From this one easily deduces (use the projection formula for the ψ-operator) that the map $\psi_{D_{\mathrm{LT}}(V)}$ on $D_{\mathrm{LT}}(V)$ restricts to an operator

$$\psi_{N(V)} : N^{(\varphi)}(V) \to N(V).$$

Corollary 3.1.14. *Assume that the Hodge–Tate weights of V are all in $[0, r]$. Then we have*

$$\begin{aligned} \mathrm{comp}\big(N(V)\big) &\subseteq \mathcal{O} \otimes_L D_{\mathrm{cris},L}(V), \\ \mathrm{comp}\big(N^{(\varphi)}(V)\big) &\subseteq \mathcal{O} \otimes_L D_{\mathrm{cris},L}(V), \\ \mathrm{comp}\big(N^{(\varphi)}(V)^{\psi_{N(V)}=0}\big) &\subseteq \mathcal{O}^{\psi_L=0} \otimes_L D_{\mathrm{cris},L}(V). \end{aligned} \tag{3.17}$$

Proof. Apply Proposition 3.1.13 to $T(\chi_{\mathrm{LT}}^{-r})$, and then divide the resulting (left) inclusion in (3.14) by t_{LT}^r and tensor with $o_L(\chi_{\mathrm{LT}}^r)$. This gives the first inclusion by Lemma 3.1.12 upon noting that

$$t_{\mathrm{LT}}^r D_{\mathrm{cris},L}(L \otimes_{o_L} T) \otimes_L L\eta^{\otimes -r} = D_{\mathrm{cris},L}\big(L \otimes_{o_L} T(\chi_{\mathrm{LT}}^{-r})\big).$$

The second inclusion easily derives from the first by using that the map comp is compatible with the φ's.

For the third inclusion we consider any element $x = \sum_i f_i \varphi_{N(V)}(x_i) \in N^{(\varphi)}(V)$, with $f_i \in \mathbf{A}_L^+$ and $x_i \in N(V)$, such that $\psi_{N(V)}(x) = \sum_i \psi_L(f_i)x_i = 0$. We choose an L-basis $e_1, \ldots, e_m$ of $D_{\mathrm{cris},L}(V)$ and write $\mathrm{comp}(x_i) = \sum_j f_{ij} \otimes e_j$ with $f_{ij} \in \mathcal{O}$. Then

$$0 = \mathrm{comp}\big(\psi_{N(V)}(x)\big) = \sum_i \psi_L(f_i)\mathrm{comp}(x_i) = \sum_i \sum_j \psi_L(f_i) f_{ij} \otimes e_j$$

and it follows that

$$\psi_L\Big(\sum_i f_i\varphi_L(f_{ij})\Big) = \sum_i \psi_L(f_i) f_{ij} = 0,$$

i.e., that $\sum_i f_i\varphi_L(f_{ij}) \in \mathcal{O}^{\psi_L=0}$. On the other hand, we compute

$$\begin{aligned} \mathrm{comp}(x) &= \sum_i f_i \varphi_{N(V)}(x_i) = \sum_i f_i(\varphi_L \otimes \varphi_{\mathrm{cris}})\big(\mathrm{comp}(x_i)\big) \\ &= \sum_i \sum_j f_i\big(\varphi_L(f_{ij}) \otimes \varphi_{\mathrm{cris}}(e_j)\big) \\ &= \sum_j \Big(\sum_i f_i\varphi_L(f_{ij})\Big) \otimes \varphi_{\mathrm{cris}}(e_j). \end{aligned} \qquad \blacksquare$$

Corollary 3.1.15. *In the situation of Proposition 3.1.13, we have*

$$D_{\mathrm{cris},L}(V) \cong \big(\mathcal{O} \otimes_{\mathbf{A}_L^+} N(T)\big)^{\Gamma_L}.$$

Proof. We set $\mathcal{M} := \mathcal{O} \otimes_{\mathbf{A}_L^+} N(T)$ and identify the modules $D(\mathcal{M})$ and $D_{\mathrm{cris},L}(V)$ by using Lemma 3.1.5 and (3.14). The proof of [44, Prop. 2.2.6] combined with Remark 3.1.2 (iii) implies the commutativity of the following diagram:

$$\begin{tikzcd} D(\mathcal{M}) \arrow[r, hook, "\text{incl.}"'] \arrow[dr, hook, "\text{incl.}"'] & \mathcal{O}\big[\frac{\omega_{\mathrm{LT}}}{t_{\mathrm{LT}}}\big] \otimes_L D(\mathcal{M}) \arrow[r, "\xi", "\cong"'] & \mathcal{M}\big[\frac{\omega_{\mathrm{LT}}}{t_{\mathrm{LT}}}\big] \\ & \mathcal{M}\big(D(\mathcal{M})\big) \arrow[u, hook, "\text{incl.}"'] \arrow[r, "\cong"] & \mathcal{M}, \arrow[u, hook, "\text{incl.}"'] \end{tikzcd}$$

in which the right vertical map is the canonical inclusion while the left vertical map stems from the definition of the functor $\mathcal{M}$ as in (3.15) (which also implies the commutativity of the left triangle). Taking Γ_L-invariants and using the fact that the upper line induces the isomorphism $D(\mathcal{M}) \cong \mathcal{M}[\frac{\omega_{\mathrm{LT}}}{t_{\mathrm{LT}}}]^{\Gamma_L}$ in Remark 3.1.2 (iii), the result follows. $\blacksquare$

Corollary 3.1.16. *In the situation of Proposition 3.1.13, we have $Q^r N(V) \subseteq N^{(\varphi)}(V)$.*

Proof. In the present situation, $\varphi_{N(V)} : N(V) \to N(V)$ is a semilinear endomorphism of $N(V)$ by Remark 3.1.6 (i). Then

$$\varphi_{\mathcal{O}\otimes_{\mathbf{A}_L^+} N(V)} = \varphi_L \otimes \varphi_{N(V)} : \mathcal{O} \otimes_{\mathbf{A}_L^+} N(V) \to \mathcal{O} \otimes_{\mathbf{A}_L^+} N(V)$$

is an endomorphism as well. The corresponding linearized maps are

$$\begin{aligned} \varphi_{N(V)}^{\mathrm{lin}} : \mathbf{A}_L^+ \otimes_{\mathbf{A}_L^+,\varphi_L} N(V) &\xrightarrow{\cong} N^{(\varphi)}(V) \subseteq N(V) \\ f \otimes x &\mapsto f\varphi_{N(V)}(x) \end{aligned}$$

and

$$\varphi^{\mathrm{lin}}_{\mathcal{O}\otimes_{\mathbf{A}_L^+} N(V)} = \mathrm{id}_{\mathcal{O}} \otimes \varphi^{\mathrm{lin}}_{N(V)} : \mathcal{O} \otimes_{\mathbf{A}_L^+,\varphi_L} N(V)$$
$$= \mathcal{O} \otimes_{\mathbf{A}_L^+} \big(\mathbf{A}_L^+ \otimes_{\mathbf{A}_L^+,\varphi_L} N(V)\big) \to \mathcal{O} \otimes_{\mathbf{A}_L^+} N(V).$$

Since $\mathcal{O}$ is flat over $\mathbf{A}_L^+[\frac{1}{\pi_L}]$, it follows that

$$\mathcal{O} \otimes_{\mathbf{A}_L^+} N(V)/\operatorname{im}\big(\varphi^{\mathrm{lin}}_{\mathcal{O}\otimes_{\mathbf{A}_L^+} N(V)}\big) = \mathcal{O} \otimes_{\mathbf{A}_L^+} \big(N(V)/N^{(\varphi)}(V)\big).$$

But $\mathcal{O}$ is even faithfully flat over $\mathbf{A}_L^+[\frac{1}{\pi_L}]$. Hence, the natural map

$$N(V)/N^{(\varphi)}(V) \to \mathcal{O} \otimes_{\mathbf{A}_L^+} N(V)/\operatorname{im}\big(\varphi^{\mathrm{lin}}_{\mathcal{O}\otimes_{\mathbf{A}_L^+} N(V)}\big)$$

is injective. This reduces us to proving that

$$Q^r\big(\mathcal{O} \otimes_{\mathbf{A}_L^+} N(V)\big) \subseteq \operatorname{im}\big(\varphi^{\mathrm{lin}}_{\mathcal{O}\otimes_{\mathbf{A}_L^+} N(V)}\big).$$

As for any object in the category $\mathrm{Mod}_{\mathcal{O}}^{\varphi_L,\gamma_L,\mathrm{an}}$, we do have

$$Q^h\big(\mathcal{O} \otimes_{\mathbf{A}_L^+} N(V)\big) \subseteq \operatorname{im}\big(\varphi^{\mathrm{lin}}_{\mathcal{O}\otimes_{\mathbf{A}_L^+} N(V)}\big)$$

for some sufficiently big integer h. On the other hand, (3.14) says that

$$\big(\tfrac{t_{\mathrm{LT}}}{\omega_{\mathrm{LT}}}\big)^r \mathrm{comp}\big(\mathcal{O} \otimes_{\mathbf{A}_L^+} N(V)\big) \subseteq \mathcal{O} \otimes_L D_{\mathrm{cris},L}(V) \subseteq \mathrm{comp}\big(\mathcal{O} \otimes_{\mathbf{A}_L^+} N(V)\big).$$

Since φ_{cris} is bijective, we can sharpen the right-hand inclusion to

$$\mathcal{O} \otimes_L D_{\mathrm{cris},L}(V) \subseteq \mathrm{comp}\big(\operatorname{im}(\varphi^{\mathrm{lin}}_{\mathcal{O}\otimes_{\mathbf{A}_L^+} N(V)})\big).$$

It follows that

$$\big(\tfrac{t_{\mathrm{LT}}}{\omega_{\mathrm{LT}}}\big)^r \big(\mathcal{O} \otimes_{\mathbf{A}_L^+} N(V)\big) \subseteq \operatorname{im}\big(\varphi^{\mathrm{lin}}_{\mathcal{O}\otimes_{\mathbf{A}_L^+} N(V)}\big).$$

Since the greatest common divisor of Q^h and $(\frac{t_{\mathrm{LT}}}{\omega_{\mathrm{LT}}})^r$ is $Q^{\min(h,r)}$, we finally obtain that

$$Q^r\big(\mathcal{O} \otimes_{\mathbf{A}_L^+} N(V)\big) \subseteq \operatorname{im}\big(\varphi^{\mathrm{lin}}_{\mathcal{O}\otimes_{\mathbf{A}_L^+} N(V)}\big). \qquad \blacksquare$$

Corollary 3.1.17. *In the situation of Proposition 3.1.13, we have, with regard to an $\mathbf{A}_L^+$-basis of $N := N(T)$ and with $s := \sum_{i=1}^d r_i$, that*

$$\det\big(\varphi_N : N(T) \to N(T)\big) = \det\big(\varphi_{N(V)} : N(V) \to N(V)\big) = Q^s$$

up to an element in $o_L^\times \cdot (\varphi_L - 1)((\mathbf{A}_L^+)^\times)$.

Proof. Note first that N is φ_N-stable by Remark 3.1.6 (i). Moreover, the determinant of φ_N acting on $N(V)$ equals the determinant of $\varphi_L \otimes \varphi_N$ acting on $\mathcal{O} \otimes_{\mathbf{A}_L^+} N(T)$ since we can take for both an $\mathbf{A}_L^+$-basis of $N(T)$. Since $\varphi_L(\frac{t_{\mathrm{LT}}}{\omega_{\mathrm{LT}}}) = \frac{\pi_L}{Q}\frac{t_{\mathrm{LT}}}{\omega_{\mathrm{LT}}}$, by Proposition 3.1.13, the latter determinant equals $(\frac{\pi_L}{Q})^{-s}$ multiplied by the determinant of $\varphi_L \otimes \mathrm{Frob}$ acting on $\mathcal{O} \otimes_L D_{\mathrm{cris},L}(V)$. The latter is equal to the determinant of Frob on $D_{\mathrm{cris},L}(V)$, which is π_L^s up to a unit in o_L since the filtered Frobenius module $D_{\mathrm{cris},L}(V)$ is weakly admissible. This shows the claim up to an element in $o_L^\times \cdot (\varphi_L - 1)(\mathcal{O}^\times)$. But $\mathcal{O}^\times = \pi_L^{\mathbb{Z}} \times (\mathbf{A}_L^+)^\times$ by [48, (4.8)]. Hence, $(\varphi_L - 1)(\mathcal{O}^\times) = (\varphi_L - 1)((\mathbf{A}_L^+)^\times)$. ∎

3.2 The determinant of the crystalline comparison isomorphism

Let T be any object in $\mathrm{Rep}_{o_L,f}^{\mathrm{cris,an}}(G_L)$ of o_L-rank d and such that $V = L \otimes_{o_L} T$ has Hodge–Tate weights $-r = -r_d \leq \cdots \leq -r_1$; we set $s := \sum_{i=1}^d r_i$, $N := N(T)$ and $\mathcal{M} = \mathcal{O} \otimes N$. Consider the integral lattice

$$\mathcal{D} := \mathcal{D}(T) \subseteq D_{\mathrm{cris},L}(V)$$

which is defined as the image of $N/\omega_{\mathrm{LT}}N \subseteq D(N)$ under the natural isomorphisms $D(N) \cong D(\mathcal{M}) \cong D_{\mathrm{cris},L}(V)$ arising from Lemma 3.1.5 and (3.1). Then with $N(-)$ also $\mathcal{D}(-)$ is a $\otimes$-functor. The aim of this subsection is to prove the following result.

Proposition 3.2.1. *With regard to bases of T and $\mathcal{D}$, the determinant of the crystalline comparison isomorphism*

$$B_{\mathrm{cris},L} \otimes_L V \cong B_{\mathrm{cris},L} \otimes_L D_{\mathrm{cris},L}(V)$$

belongs to $t_{\mathrm{LT}}^s W(\bar{k}_L)_L^\times$.

We write $\bigwedge V$ for the highest exterior power of V over L.

Remark 3.2.2. If V is L-analytic (Hodge–Tate, crystalline), then so is $\bigwedge V$.

Since $D_{\mathrm{cris},L}$ is a tensor functor, we are mainly reduced to consider characters $\rho : G_L \to L^\times$, for which we denote by V_ρ its representation space.

Remark 3.2.3. Let V_ρ be Hodge–Tate. We then have the following:

(i) The character ρ coincides on an open subgroup of the inertia group I_L of G_L with

$$\prod_{\sigma \in \Sigma_L} \sigma^{-1} \circ \chi_{\sigma L,\mathrm{LT}}^{n_\sigma},$$

for some integers n_σ, where Σ_L denotes the set of embeddings of L into $\overline{L}$ and $\chi_{\sigma L,\mathrm{LT}}$ is the Lubin–Tate character for σL and $\sigma(\pi_L)$.

(ii) If, in addition, V_ρ is L-analytic, then ρ coincides on an open subgroup of the inertia group I_L with χ_{LT}^n for some integer n.

Proof. This follows from [81, III.A4 Prop. 4 as well as III.A5 Thm. 2 and its corollary]. ∎

Remark 3.2.4. Let ρ be a crystalline (hence, Hodge–Tate) and L-analytic character. We then have the following:

(i) If ρ factorizes through $\mathrm{Gal}(L'/L)$ for some discretely valued Galois extension L' of L, then the determinant of the crystalline comparison isomorphism for V_ρ belongs to $(W(\bar{k}_L)_L[\frac{1}{p}])^\times$ (with respect to arbitrary bases of V and $D_{\mathrm{cris},L}(V)$).

(ii) If ρ has Hodge–Tate weight $-s$, then the determinant of the crystalline comparison isomorphism for V_ρ lies in $t_{\mathrm{LT}}^s(W(\bar{k}_L)_L[\frac{1}{p}])^\times$.

(iii) ρ is of the form $\chi_{\mathrm{LT}}^n\chi^{un}$ with an integer n and an unramified character χ^{un}.[8]

Proof. We shall write K_0 for the maximal absolutely unramified subextension of K, any algebraic extension of $\mathbb{Q}_p$. Taking $G_{L'}$-invariants of the comparison isomorphism shows that the latter is already defined over

$$B_{\mathrm{cris},L}^{G_{L'}} = (L \otimes_{L_0} B_{\mathrm{cris}})^{G_{L'}} = L \otimes_{L_0} (B_{\mathrm{cris}})^{G_{L'}} = L \otimes_{L_0} \widehat{L_0'} \subseteq W(\bar{k}_L)_L\left[\tfrac{1}{p}\right],$$

whence (i). Using Remark 3.2.3 (ii) and applying (i) to $\rho\chi_{\mathrm{LT}}^{-n}$ gives (ii). By the same argument it suffices to prove (iii) in the case of Hodge–Tate weight 0. Then its period lies in the completion of the maximal unramified extension of L by (i), whence the claim that ρ is unramified follows, as the inertia subgroup of G_L must act trivially. ∎

By Proposition 3.1.8 we have

$$N(T) \subseteq D_{\mathrm{LT}}^+(T) \subseteq \mathbf{A}^+ \otimes_{o_L} T$$

if T is positive. Using (N2) and the isomorphism

$$\mathbf{A} \otimes_{\mathbf{A}_L} D_{\mathrm{LT}}(T) \cong \mathbf{A} \otimes_{o_L} T,$$

we obtain a canonical injection

$$\mathbf{A}^+ \otimes_{\mathbf{A}_L^+} N(T) \hookrightarrow \mathbf{A}^+ \otimes_{o_L} T. \tag{3.18}$$

Proposition 3.2.5. *If T is positive, then the determinant of* (3.18) *with respect to the bases of $N(T)$ and T is contained in $\omega_{\mathrm{LT}}^s(\mathbf{A}_L^+)^\times \cdot W(\bar{k}_L)_L^\times$.*

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[8] Also the converse statement is true: Indeed, any unramified character is locally algebraic (by definition, see [81]), whence HT. By [81, A3 Prop. 3, A5 Thm. 2] the Hodge–Tate weights are all zero, whence ρ is L-analytic. It is crystalline as it is admissible; i.e., there is a period in $\mathbb{C}_p^\times$, which in this case needs to lie in the fixed field under inertia, which is contained in B_{cris}.

Proof. Let $M \in M_d(\mathbf{A}^+)$ be the matrix of a basis of $N(T)$ with respect to a basis of T and $P \in M_d(\mathbf{A}_L^+)$ the matrix of φ_L with respect to the same basis of $N(T)$. Then we have $\varphi_L(M) = MP$. By Corollary 3.1.17 we have $\det(P) = Q^s\varphi_L(f)f^{-1}u$ for some $f \in (\mathbf{A}_L^+)^\times$ and $u \in o_L^\times$. But $Q = \varphi_L(\omega_{\mathrm{LT}})\omega_{\mathrm{LT}}^{-1}$. We deduce that

$$\varphi_L\big(\det(M)\big) = \varphi_L(\omega_{\mathrm{LT}}^s f)(\omega_{\mathrm{LT}}^s f)^{-1}u\det(M),$$

i.e., that

$$(\omega_{\mathrm{LT}}^s fa)^{-1}\det(M) \in \mathbf{A}^{\varphi_L=1} = o_L$$

with $a \in W(\bar{k}_L)_L^\times$ such that $\varphi_L(a)/a = u$. It follows that

$$\det(M) \in \omega_{\mathrm{LT}}^s o_L(\mathbf{A}_L^+)^\times \cdot W(\bar{k}_L)_L^\times.$$

But we also have $\det(M) \in \mathbf{A}^\times$. Hence, we finally obtain[9]

$$\det(M) \in \omega_{\mathrm{LT}}^s o_L(\mathbf{A}_L^+)^\times \cdot W(\bar{k}_L)_L^\times \cap \mathbf{A}^\times = \omega_{\mathrm{LT}}^s(\mathbf{A}_L^+)^\times \cdot W(\bar{k}_L)_L^\times.$$ ∎

Remark 3.2.6. For $T = o_L(\chi)$ with unramified χ as in Remark 3.1.11 the map (3.18) maps the basis n_0 to $a \otimes t_0$.

Lemma 3.2.7. *If T is positive, then we have the following:*

(i) $\mathcal{O} \otimes_{o_L} \mathcal{D}(T) = \mathcal{O} \otimes_L D_{\mathrm{cris},L}(V) \subseteq \mathrm{comp}(\mathcal{O} \otimes_{\mathbf{A}_L^+} N(T))$;

(ii) *the determinant of the inclusion in* (i) *with respect to the bases of $\mathcal{D}(T)$ and $N(T)$ belongs to $(\frac{t_{\mathrm{LT}}}{\omega_{\mathrm{LT}}})^s(\mathbf{A}_L^+)^\times$;*

(iii) *for $T = o_L(\chi)$ with unramified χ as in Remark 3.1.11,* $\mathrm{comp}(n_0) = \varphi_L(a) \otimes t_0 = ca \otimes t_0 \in D_{\mathrm{cris},L}(V)$ *with $c = \frac{\varphi_L(a)}{a} \in o_L^\times$; in particular, the element $a \otimes t_0$ is a basis of $\mathcal{D}(T)$.*

Proof. By construction the comparison isomorphism (3.13) is of the form

$$\mathrm{comp} = \mathrm{id}_{\mathcal{O}[\frac{\omega_{\mathrm{LT}}}{t_{\mathrm{LT}}}]} \otimes_L \mathrm{comp}_0$$

with

$$\mathrm{comp}_0 : \big(\mathcal{O} \otimes_{\mathbf{A}_L^+} N[\tfrac{\omega_{\mathrm{LT}}}{t_{\mathrm{LT}}}]\big)^{\Gamma_L} \xrightarrow[\mathrm{pr}]{\cong} N/\omega_{\mathrm{LT}}N\big[\tfrac{1}{p}\big] = D(N) \xrightarrow{\cong} D_{\mathrm{cris},L}(V)$$

the right-hand arrow being the natural isomorphism from Lemma 3.1.5. For positive T we know in addition from the proof of Corollary 3.1.15 that

$$\big(\mathcal{O} \otimes_{\mathbf{A}_L^+} N\big)^{\Gamma_L} = \big(\mathcal{O} \otimes_{\mathbf{A}_L^+} N\big[\tfrac{\omega_{\mathrm{LT}}}{t_{\mathrm{LT}}}\big]\big)^{\Gamma_L}.$$

We deduce that

$$\mathrm{comp}\big(\mathcal{O} \otimes_{\mathbf{A}_L^+} N\big) \supseteq \mathcal{O} \otimes_L \mathrm{comp}_0\big(\big(\mathcal{O} \otimes_{\mathbf{A}_L^+} N\big)^{\Gamma_L}\big) = \mathcal{O} \otimes_L D_{\mathrm{cris},L}(V).$$

[9]The argument also shows that $\omega_{\mathrm{LT}}^s\mathbf{A}^+ \otimes_{o_L} T \subseteq \mathbf{A}^+ \otimes_{\mathbf{A}_L} N(T) \subseteq \mathbf{A}^+ \otimes_{o_L} T$. But the left inclusion does even hold with r instead of s (cf. [83] following [5] in the cyclotomic case).

By Proposition 3.1.13 we know that the determinant in (ii) is of the form $(\frac{t_{\mathrm{LT}}}{\omega_{\mathrm{LT}}})^s f(\omega_{\mathrm{LT}})$ with $f(\omega_{\mathrm{LT}}) \in \mathcal{O}^\times$. On the other hand, if we base change the inclusion in (i) to $L = \mathcal{O}/\omega_{\mathrm{LT}}\mathcal{O}$, then we obtain the base change from o_L to L of the isomorphism $\mathcal{D} \cong N/\omega_{\mathrm{LT}}N$. By our choice of bases the determinant of the latter lies in $o_L^\times$. Since evaluation in zero maps $(\frac{t_{\mathrm{LT}}}{\omega_{\mathrm{LT}}})^s f(\omega_{\mathrm{LT}})$ to $f(0)$, it follows that $f(0)$ belongs to $o_L^\times$ and hence (cf. [48, (4.8)]) that $f(\omega_{\mathrm{LT}})$ belongs to $(\mathbf{A}_L^+)^\times$.

Now we prove (iii): By the above description of comp_0 we have to show that the image $\bar{n}_0 \in D(N(T))$ of n_0 is mapped to $ca \otimes t_0$ under the natural isomorphism from Lemma 3.1.5. Since under the crystalline comparison isomorphisms these elements are sent to $a \otimes (a^{-1} \otimes \bar{n}_0) \in B_{\mathrm{cris},L} \otimes_L V_L(D(N))$ and $ca \otimes t_0 \in B_{\mathrm{cris},L} \otimes_{o_L} T$, respectively, it suffices to show that the map (3.8) sends $a^{-1} \otimes n_0 \in L \otimes_{o_L} V(\mathbf{A}_L \otimes_{\mathbf{A}_L^+} N)$ (which corresponds to t_0 under the canonical isomorphism $T \cong V(\mathbf{A}_L \otimes_{\mathbf{A}_L^+} N)$) to $(ca)^{-1} \otimes \bar{n}_0 \in V_L(D(N))$. Dualizing, this is equivalent to the claim that the map (3.9) sends the dual basis $\delta_{a^{-1} \otimes n_0} \in (L \otimes_{o_L} V(M))^*$ of $a^{-1} \otimes n_0$ to $\delta_{(ca)^{-1} \otimes \bar{n}_0} \in V_L(D(N))^*$. Note that the isomorphism

$$\begin{aligned}\big(L \otimes_{o_L} V(M)\big)^* &\cong L \otimes_{o_L} \mathrm{Hom}_{\mathbf{A}_L,\varphi_q}(\mathbf{A}_L \otimes_{\mathbf{A}_L^+} N, \mathbf{A}) \\ &\cong L \otimes_{o_L} \mathrm{Hom}_{\mathbf{A}_L^+,\varphi_q}\big(N, \mathbf{A}^+\big[\tfrac{1}{\omega_{\mathrm{LT}}}\big]\big)\end{aligned}$$

sends $\delta_{a^{-1} \otimes n_0}$ to $a\delta_{n_0}$. Thus, it suffices to show that the map (3.10) sends $a\delta_{n_0}$ to $ca\delta_{\bar{n}_0}$ in $\mathrm{Hom}_{L,\varphi_q,\mathrm{Fil}}((N/\omega_{\mathrm{LT}}N)[\frac{1}{p}], B_{\mathrm{cris},L})$ since the latter corresponds under (3.11) to $ca \otimes \delta_{\bar{n}_0} \in V_L(D(N)^*)$ which in turn corresponds to $\delta_{(ca)^{-1} \otimes \bar{n}_0}$ under (3.12).

If $f = a\delta_{n_0}$, which is the map which sends n_0 to a, then – in the notation of the proof of Lemma 3.1.5 – f_1 and f_2 share this property, while f_3 (and hence f_4) sends $c^{-1}n_0$ to a because $\varphi_N(c^{-1}n_0) = c^{-1}\varphi_L(a)a^{-1}n_0 = n_0$. Then f_5 sends $c^{-1}\bar{n}_0$ to a because $\xi(c^{-1}\bar{n}_0) = c^{-1}n_0$. Altogether this means that $a\delta_{n_0}$ is mapped to $\varphi_L(a)\delta_{n_0} = ca\delta_{\bar{n}_0}$ as claimed. ∎

Proof of Proposition 3.2.1. The functor $D_{\mathrm{cris},L}(-)$ on crystalline Galois representations is a $\otimes$-functor and commutes with exterior powers, and the crystalline comparison isomorphism is compatible with tensor products and exterior powers. The analogous facts hold for the functor $N(-)$ and hence for the functor $\mathcal{D}(-)$ (by base change). The case of the functor $N(-)$ reduces, by using the properties (N1) and (N2) in Lemma 3.1.12 (i), to the case of the functor $D_{\mathrm{LT}}(-)$. Here the properties can easily be seen by the comparison isomorphism (3.3).

Upon replacing T by its highest exterior power we may and do assume that the o_L-module T has rank 1. In addition, by twisting T if necessary with a power of χ_{LT} we may and do assume that T is positive with $s = 0$, i.e., unramified Remark 3.2.4. In this case it is clear that – using the notation of Lemma 3.2.7 (iii) – the crystalline comparison isomorphism sends t_0 to $a \otimes t_0$. Since the latter is also a basis of $\mathcal{D}(T)$ by the same lemma, the proposition follows. ∎

3.3 Non-negative Hodge–Tate weights

Now assume that for T in $\mathrm{Rep}^{\mathrm{cris,an}}_{o_L,f}(G_L)$ the *Hodge–Tate weights are all* ≥ 0 and set $N := N(T)$. By [80, Rem. 3.2 (i)–(ii)] the map ψ_L preserves $\mathbf{A}_L^+$. It follows that $\psi_{D_{\mathrm{LT}}(T)}$ maps $\mathbf{A}_L^+ \cdot \varphi_N(N)$ – and hence N by Remark 3.1.6 (i) (2) – into N. The following lemmata generalize those of [4, Appx. A].

Lemma 3.3.1. *For $m \geq 1$, there exists $Q_m \in o_L[\![Z]\!]$ such that*

$$\psi_L\left(\frac{1}{\omega_{\mathrm{LT}}^m}\right) = \frac{\pi_L^{m-1} + \omega_{\mathrm{LT}} Q_m(\omega_{\mathrm{LT}})}{\omega_{\mathrm{LT}}^m}.$$

Proof. According to the paragraph after Remark 2.1 combined with Remark 3.2 (ii) in [80], we have that

$$h(\omega_{\mathrm{LT}}) := \omega_{\mathrm{LT}}^m \psi_L\left(\frac{1}{\omega_{\mathrm{LT}}^m}\right) = \psi_L\left(\frac{[\pi_L]^m}{\omega_{\mathrm{LT}}^m}\right) \in \mathbf{A}_L^+.$$

Obviously, there exists $Q_m \in o_L[\![Z]\!]$ such that

$$h(\omega_{\mathrm{LT}}) - h(0) = \omega_{\mathrm{LT}} Q_m(\omega_{\mathrm{LT}}).$$

Thus, the claim follows from

$$\begin{aligned} h(0) &= \varphi_L\big(h(\omega_{\mathrm{LT}})\big)_{|\omega_{\mathrm{LT}}=0} \\ &= \varphi_L \circ \psi_L\left(\frac{[\pi_L]^m}{\omega_{\mathrm{LT}}^m}\right)_{|\omega_{\mathrm{LT}}=0} \\ &= \pi_L^{-1} \sum_{a \in \mathrm{LT}_1} \left(\frac{[\pi_L]^m (a +_{\mathrm{LT}} \omega_{\mathrm{LT}})}{(a +_{\mathrm{LT}} \omega_{\mathrm{LT}})^m}\right)_{|\omega_{\mathrm{LT}}=0} \\ &= \pi_L^{-1} \sum_{a \in \mathrm{LT}_1} \left(\frac{[\pi_L](\omega_{\mathrm{LT}})}{a +_{\mathrm{LT}} \omega_{\mathrm{LT}}}\right)^m_{|\omega_{\mathrm{LT}}=0} \\ &= \pi_L^{m-1} \end{aligned}$$

because $\left(\frac{[\pi_L](\omega_{\mathrm{LT}})}{a +_{\mathrm{LT}} \omega_{\mathrm{LT}}}\right)_{|\omega_{\mathrm{LT}}=0} = \pi_L$ for $a = 0$ and $= 0$ otherwise. ∎

Lemma 3.3.2. *We have*

$$\psi_{D_{\mathrm{LT}}(T)}\big(\pi_L D_{\mathrm{LT}}(T) + \omega_{\mathrm{LT}}^{-1} N(T)\big) \subseteq \pi_L D_{\mathrm{LT}}(T) + \omega_{\mathrm{LT}}^{-1} N(T)$$

and, for $k \geq 1$,

$$\psi_{D_{\mathrm{LT}}(T)}\big(\pi_L D_{\mathrm{LT}}(T) + \omega_{\mathrm{LT}}^{-(k+1)} N(T)\big) \subseteq \pi_L D_{\mathrm{LT}}(T) + \omega_{\mathrm{LT}}^{-k} N(T).$$

Proof. By Remark 3.1.6 (ii), we can write any $x \in N(T)$ in the form $x = \sum a_i \varphi_N(x_i)$ with $a_i \in \mathbf{A}_L^+$ and $x_i \in N(T)$. Therefore, $\psi_{D_{\mathrm{LT}}(T)}(\omega_{\mathrm{LT}}^{-(k+1)} x) = \sum \psi_L(\omega_{\mathrm{LT}}^{-(k+1)} a_i) x_i$ by the projection formula. Since ψ_L preserves $\mathbf{A}_L^+$ and is o_L-linear, we conclude by Lemma 3.3.1 that $\psi_L(\omega_{\mathrm{LT}}^{-(k+1)} a_i)$ belongs to $\pi_L \mathbf{A}_L + \omega_{\mathrm{LT}}^{-k} \mathbf{A}_L^+$, whenever $k \geq 1$, from which the second claim follows as $\psi_{D_{\mathrm{LT}}(T)}(\pi_L D_{\mathrm{LT}}(T)) \subseteq \pi_L D_{\mathrm{LT}}(T)$ by o_L-linearity of $\psi_{D_{\mathrm{LT}}(T)}$. For $k = 0$ finally, $\psi_L(\omega_{\mathrm{LT}}^{-1} a_i)$ belongs to $\omega_{\mathrm{LT}}^{-1} \mathbf{A}_L^+$, from which the first claim follows. ∎

Lemma 3.3.3. *If $k \geq 1$ and $x \in D_{\mathrm{LT}}(T)$ satisfies $\psi_{D_{\mathrm{LT}}(T)}(x) - x \in \pi_L D_{\mathrm{LT}}(T) + \omega_{\mathrm{LT}}^{-k} N(T)$, then x belongs to $\pi_L D_{\mathrm{LT}}(T) + \omega_{\mathrm{LT}}^{-k} N(T)$.*

Proof. Since $D_{\mathrm{LT}}(T)/\pi_L D_{\mathrm{LT}}(T)$ is a finitely generated (free) $k_L((\omega_{\mathrm{LT}}))$-module, there exists an integer $m \geq 0$ such that $x \in \pi_L D_{\mathrm{LT}}(T) + \omega_{\mathrm{LT}}^{-m} N(T)$; let l denote the smallest among them. Assume that $l > k$. Then Lemma 3.3.2 shows that

$$\psi_{D_{\mathrm{LT}}(T)}(x) \in \pi_L D_{\mathrm{LT}}(T) + \omega_{\mathrm{LT}}^{-(l-1)} N(T).$$

Hence, the element $\psi_{D_{\mathrm{LT}}(T)}(x) - x$ would belong to $\pi_L D_{\mathrm{LT}}(T) + \omega_{\mathrm{LT}}^{-l} N(T)$ but not to $(\pi_L D_{\mathrm{LT}}(T) + \omega_{\mathrm{LT}}^{-(l-1)} N(T))$, a contradiction to our assumption. It follows that $l \leq k$, and we are done. ∎

Lemma 3.3.4. *It holds that $D_{\mathrm{LT}}(T)^{\psi_{D_{\mathrm{LT}}(T)}=1} \subseteq \omega_{\mathrm{LT}}^{-1} N(T)$, i.e.,*

$$D_{\mathrm{LT}}(T)^{\psi_{D_{\mathrm{LT}}(T)}=1} = \left(\omega_{\mathrm{LT}}^{-1} N(T)\right)^{\psi_{D_{\mathrm{LT}}(T)}=1}.$$

Proof. By induction on $k \geq 1$ we will show that $D_{\mathrm{LT}}(T)^{\psi_{D_{\mathrm{LT}}(T)}=1} \subseteq \pi_L^k D_{\mathrm{LT}}(T) + \omega_{\mathrm{LT}}^{-1} N(T)$, i.e., writing $x = \pi_L^k y_k + n_k \in D_{\mathrm{LT}}(T)^{\psi_{D_{\mathrm{LT}}(T)}=1}$, the sequence n_k will π_L-adically converge in $\omega_{\mathrm{LT}}^{-1} N(T)$ with limit x.

In order to show the claim, assume $x \in D_{\mathrm{LT}}(T)^{\psi_{D_{\mathrm{LT}}(T)}=1}$. As in the previous proof, there exists some minimal integer $m \geq 0$ such that $x \in \pi_L D_{\mathrm{LT}}(T) + \omega_{\mathrm{LT}}^{-m} N(T)$. Then $m = 1$ and we are done since otherwise Lemma 3.3.3 implies that m can be decreased by 1. This proves the claim for $k = 1$.

By our induction hypothesis we can write $x \in D_{\mathrm{LT}}(T)^{\psi_{D_{\mathrm{LT}}(T)}=1}$ as $x = \pi_L^k y + n$ with $y \in D_{\mathrm{LT}}(T)$ and $n \in \omega_{\mathrm{LT}}^{-1} N(T)$. The equation $\psi_{D_{\mathrm{LT}}(T)}(x) = x$ implies that $\psi_{D_{\mathrm{LT}}(T)}(n) - n = \pi_L^k(\psi_{D_{\mathrm{LT}}(T)}(y) - y)$. In the proof of Lemma 3.3.2 we have seen that $\psi_{D_{\mathrm{LT}}(T)}(n) - n \in \omega_{\mathrm{LT}}^{-1} N(T)$. Note that $\pi_L^k D_{\mathrm{LT}}(T) \cap \omega_{\mathrm{LT}}^{-1} N(T) = \pi_L^k \omega_{\mathrm{LT}}^{-1} N(T)$ because $\mathbf{A}_L/\omega_{\mathrm{LT}}^{-1} \mathbf{A}_L^+$ has no π_L-torsion. Therefore, $\psi_{D_{\mathrm{LT}}(T)}(y) - y \in \omega_{\mathrm{LT}}^{-1} N(T)$, whence y, by Lemma 3.3.3, belongs to $\pi_L D_{\mathrm{LT}}(T) + \omega_{\mathrm{LT}}^{-1} N(T)$ so that we can write $x = \pi_L^k(\pi_L y' + n') + n = \pi_L^{k+1} y' + (\pi_L^k n' + n)$ as desired. ∎

Set $V := T \otimes_{o_L} L$.

Lemma 3.3.5. *If $D_{\mathrm{cris},L}(V)^{\varphi_q=1} \neq 0$, then V has the trivial representation L as quotient; i.e., the co-invariants V_{G_L} are non-trivial.*

Proof. Let $W = V^*$ be the L-dual of V. Then, by [80, (51)] we have

$$(V_{G_L})^* \cong H^0(L, W) \cong D_{\mathrm{cris},L}(W)^{\varphi_q=1} \cap (B_{\mathrm{dR}}^+ \otimes_L W)^{G_L} = D_{\mathrm{cris},L}(W)^{\varphi_q=1} \neq 0$$

because

$$(B_{\mathrm{dR}}^+ \otimes_L W)^{G_L} = (B_{\mathrm{dR}} \otimes_L W)^{G_L} \supseteq D_{\mathrm{cris},L}(W)$$

since the Hodge–Tate weights of W are ≤ 0. ∎

Lemma 3.3.6. *If V does not have any quotient isomorphic to the trivial representation L, then $D_{\mathrm{LT}}(T)^{\psi_{D_{\mathrm{LT}}(T)}=1} \subseteq N(T)$, i.e.,*

$$D_{\mathrm{LT}}(T)^{\psi_{D_{\mathrm{LT}}(T)}=1} = N(T)^{\psi_{D_{\mathrm{LT}}(T)}=1}.$$

Proof. Because of Lemma 3.3.4, it suffices to show that $(\omega_{\mathrm{LT}}^{-1} N(T))^{\psi_{D_{\mathrm{LT}}(T)}=1} \subseteq N(T)$. Let $e_1, \ldots, e_d$ be a basis of $N := N(T)$ over $\mathbf{A}_L^+$. Then, by Remark 3.1.6 (ii) there exist $\beta_{ij} = \sum_{\ell \geq 0} \beta_{ij,\ell}\omega_{\mathrm{LT}}^{\ell} \in \mathbf{A}_L^+$ such that $e_i = \sum_{j=1}^d \beta_{ij}\varphi_N(e_j)$. Now assume that $\omega_{\mathrm{LT}}^{-1} n = \sum_{i=1}^d \alpha_i e_i = \sum_{i,j} \alpha_i \beta_{ij} \varphi_N(e_j)$ belongs to $(\omega_{\mathrm{LT}}^{-1} N)^{\psi_{D_{\mathrm{LT}}(T)}=1}$ with $\alpha_i = \sum_{\ell \geq -1} \alpha_{i,\ell}\omega_{\mathrm{LT}}^{\ell} \in \omega_{\mathrm{LT}}^{-1}\mathbf{A}_L^+$. By the projection formula this implies, for $1 \leq j \leq d$,

$$\alpha_j = \psi_L\Big(\sum_{i=1}^d \alpha_i \beta_{ij}\Big) \equiv \omega_{\mathrm{LT}}^{-1} \sum_{i=1}^d \alpha_{i,-1}\beta_{ij,0} \mod \mathbf{A}_L^+$$

because $\psi_L(\omega_{\mathrm{LT}}^{-1}) \equiv \omega_{\mathrm{LT}}^{-1} \mod \mathbf{A}_L^+$ by Lemma 3.3.1, whence

$$\varphi_L(\omega_{\mathrm{LT}})\varphi_L(\alpha_j) \equiv \sum_{i=1}^d \alpha_{i,-1}\beta_{ij,0} \mod \omega_{\mathrm{LT}}\mathbf{A}_L^+.$$

It follows from the definition of β_{ij} that

$$\begin{aligned} \varphi_N(n) &= \sum_j \varphi_L(\omega_{\mathrm{LT}})\varphi_L(\alpha_j)\varphi_N(e_j) \\ &\equiv \sum_{j,i} \alpha_{i,-1}\beta_{ij,0}\varphi_N(e_j) \\ &\equiv \sum_i \alpha_{i,-1} e_i \\ &\equiv n \mod \omega_{\mathrm{LT}} N, \end{aligned}$$

i.e., that $D_{\mathrm{cris},L}(V) \cong N/\omega_{\mathrm{LT}}N[\frac{1}{p}]$ (by (3.1) and Lemma 3.1.5) contains an eigenvector for φ_q with eigenvalue 1 if $\omega_{\mathrm{LT}}^{-1} n$ does not belong to N. Now the result follows from Lemma 3.3.5. ∎