Chapter 4

(φ_L, Γ_L)-modules over the Robba ring

4.1 Robba rings of character varieties

Throughout our coefficient field K is a complete intermediate extension $L \subseteq K \subseteq \mathbb{C}_p$. For any reduced affinoid variety $\mathfrak{Y}$ over $\mathbb{Q}_p$ of L we let $|\ |_{\mathfrak{Y}}$ denote the supremum norm on the affinoid algebra $\mathcal{O}_K(\mathfrak{Y})$ of K-valued holomorphic functions on $\mathfrak{Y}$. It is submultiplicative and defines the intrinsic Banach topology of this algebra.

4.1.1 The additive character variety and its Robba ring

Let $\mathbf{B}_1$ denote the rigid $\mathbb{Q}_p$-analytic open disk of radius one around the point $1 \in \mathbb{Q}_p$. The rigid analytic group variety

$$\mathfrak{X}_0 := \mathbf{B}_1 \otimes_{\mathbb{Z}_p} \operatorname{Hom}_{\mathbb{Z}_p}(o_L, \mathbb{Z}_p)$$

over $\mathbb{Q}_p$ (which non-canonically is a d-dimensional open unit polydisk) parametrizes the locally $\mathbb{Q}_p$-analytic characters of the additive group o_L: the point $z \otimes \beta$ is sent to the character $\chi_{z\otimes\beta}(a) := z^{\beta(a)}$. It is shown in [74, §2] that the rigid analytic group variety $\mathfrak{X}$ over L, which parametrizes the locally L-analytic characters of o_L, is the common zero set in $\mathfrak{X}_{0/L}$ of the functions

$$\sum_{j=1}^{d} z_j \otimes \beta_j \mapsto \sum_{j=1}^{d} \big(\beta_j(t_i) - t_i \cdot \beta_j(1)\big) \cdot \log(z_j)$$

for $1 \leq i \leq d$; here $t_1, \ldots, t_d$ is a $\mathbb{Z}_p$-basis of o_L and $\beta_1, \ldots, \beta_d$ is the corresponding dual basis. It is one dimensional, smooth, and connected. As a closed analytic subvariety of the Stein space $\mathfrak{X}_0$, the rigid variety $\mathfrak{X}$ is Stein as well.

For any $a \in o_L$ the map $b \mapsto ab$ on o_L is locally L-analytic. This induces an action of the multiplicative monoid $o_L \setminus \{0\}$ first on the $\mathbb{Z}_p$-module $\operatorname{Hom}_{\mathbb{Z}_p}(o_L, \mathbb{Z}_p)$ and then on the varieties $\mathfrak{X}_0$ and $\mathfrak{X}$. The latter actions further induce actions on the rings of K-valued holomorphic functions $\mathcal{O}_K(\mathfrak{X}_0) \twoheadrightarrow \mathcal{O}_K(\mathfrak{X})$, which we will denote by $(a, f) \mapsto a_*(f)$.

Furthermore, we have induced translation actions of $o_L \setminus \{0\}$ on the vector spaces $C^{an}_{\mathbb{Q}_p}(o_L, K)$, resp., $C^{an}(o_L, K)$, of K-valued locally $\mathbb{Q}_p$-analytic, resp., L-analytic, functions on o_L and then by duality on the spaces $D_{\mathbb{Q}_p}(o_L, K) \twoheadrightarrow D(o_L, K)$ of locally $\mathbb{Q}_p$-analytic and locally L-analytic distributions on o_L, respectively; they will be denoted by $(a, \lambda) \mapsto a_*(\lambda)$. By [74, Thm. 2.3] we have the Fourier isomorphism

$$D(o_L, K) \xrightarrow{\cong} \mathcal{O}_K(\mathfrak{X})$$
$$\lambda \mapsto F_\lambda(\chi) = \lambda(\chi).$$

One easily checks that this isomorphism is $o_L \setminus \{0\}$-equivariant. In the following, we will denote the endomorphism $(\pi_L)_*$ in all situations also by φ_L. The Fourier isomorphism maps the Dirac distribution δ_a, for any $a \in o_L$, to the evaluation function $\mathrm{ev}_a(\chi) := \chi(a)$.

The ψ-operator and the Mellin transform

Lemma 4.1.1. *The endomorphism φ_L makes $\mathcal{O}_K(\mathfrak{X})$ into a free module over itself of rank equal to the cardinality of $o_L/\pi_L o_L$; a basis is given by the functions ev_a for a running over a fixed system of representatives for the cosets in $o_L/\pi_L o_L$.*

Proof. This is most easily seen by using the Fourier isomorphism which reduces the claim to the corresponding statement about the distribution algebra $D(o_L, K)$. But here the ring homomorphism φ_L visibly induces an isomorphism between $D(o_L, K)$ and the subalgebra $D(\pi_L o_L, K)$ of $D(o_L, K)$. Let $R \subseteq o_L$ denote a set of representatives for the cosets in $o_L/\pi_L o_L$. Then the Dirac distributions $\{\delta_a\}_{a \in R}$ form a basis of $D(o_L, K)$ as a $D(\pi_L o_L, K)$-module. ∎

Lemma 4.1.2. *The $o_L^\times$-action on $D(o_L, K) \cong \mathcal{O}_K(\mathfrak{X})$ extends naturally to a (jointly) continuous $D(o_L^\times, K)$-module structure.*

Proof. In a first step we consider the case $K = L$, so that K is spherically complete. By [75, Cor. 3.4] it suffices to show that $C^{an}(G, K)$ as an $o_L^\times$-representation is locally analytic. This means we have to establish that, for any $f \in C^{an}(G, K)$, the orbit map $a \mapsto a^*(f)$ on $o_L^\times$ is locally analytic. But this map is the image of the locally analytic function $(a, g) \mapsto f(ag)$ under the isomorphism $C^{an}(o_L^\times \times G, K) = C^{an}(o_L^\times, C^{an}(G, K))$ in [77, Lem. A.1].

Now let K be general. All tensor products in the following are understood to be formed with the projective tensor product topology. By the universal property of the latter the jointly continuous bilinear map $D(o_L^\times, L) \times \mathcal{O}_L(\mathfrak{X}) \to \mathcal{O}_L(\mathfrak{X})$ extends uniquely to a continuous linear map $D(o_L^\times, L) \widehat{\otimes}_L \mathcal{O}_L(\mathfrak{X}) \to \mathcal{O}_L(\mathfrak{X})$. This further extends to the right-hand map in the sequence of continuous K-linear maps

$$\big(K \widehat{\otimes}_L D(o_L^\times, L)\big) \widehat{\otimes}_K \big(K \widehat{\otimes}_L \mathcal{O}_L(\mathfrak{X})\big) \to K \widehat{\otimes}_L \big(D(o_L^\times, L) \widehat{\otimes}_L \mathcal{O}_L(\mathfrak{X})\big)$$
$$\to K \widehat{\otimes}_L \mathcal{O}_L(\mathfrak{X}).$$

The left-hand map is the obvious canonical one. We refer to [61, §10.6] for the basics on scalar extensions of locally convex vector spaces. The same reasoning as in the proof of [8, Prop. 2.5 (ii)] shows that $K \hat{\otimes}_L \mathcal{O}_L(\mathfrak{X}) = \mathcal{O}_K(\mathfrak{X})$. It remains to check that $K \hat{\otimes}_L D(o_L^\times, L) = D(o_L^\times, K)$ holds true as well. For any open subgroup $U \subseteq o_L^\times$ we have $D(o_L^\times, -) = \bigoplus_{a \in o_L^\times/U} \delta_a D(U, -)$. Hence, it suffices to check that $K \hat{\otimes}_L D(U, L) = D(U, K)$ for one appropriate U. But $o_L^\times$ contains such a subgroup U which is isomorphic to the additive group o_L so that $D(U, -) \cong D(o_L, -) \cong \mathcal{O}_-(\mathfrak{X})$. In this case we had established our claim already. ∎

The operator φ_L has a distinguished K-linear continuous left inverse ψ_L^D which is defined to be the dual of the map

$$C^{\mathrm{an}}(o_L, K) \to C^{\mathrm{an}}(o_L, K)$$
$$f \mapsto (\pi_L)_!(f)(a) := \begin{cases} f(\pi_L^{-1}a) & \text{if } a \in \pi_L o_L, \\ 0 & \text{otherwise,} \end{cases}$$

and then, via the Fourier transform, induces an operator $\psi_L^{\mathfrak{X}}$ on $\mathcal{O}_K(\mathfrak{X})$. One checks that for Dirac distributions we have

$$\psi_L^D(\delta_a) = \begin{cases} \delta_{\pi_L^{-1}a} & \text{if } a \in \pi_L o_L, \\ 0 & \text{otherwise.} \end{cases} \tag{4.1}$$

Together with Lemma 4.1.1 this implies the following lemma.

Lemma 4.1.3. *Let $R_0 \subseteq o_L$ be a set of representatives for the non-zero cosets in $o_L/\pi_L o_L$. Then*

$$\ker(\psi_L^{\mathfrak{X}}) = \bigoplus_{a \in R_0} \mathrm{ev}_a \cdot \varphi_L\big(\mathcal{O}_K(\mathfrak{X})\big).$$

We also recall the resulting projection formula

$$\psi_L^{\mathfrak{X}}\big(\varphi_L(F_1)F_2\big) = F_1 \psi_L^{\mathfrak{X}}(F_2) \quad \text{for any } F_1, F_2 \in \mathcal{O}_K(\mathfrak{X}).$$

Sometimes it will be useful to view $\psi_L^{\mathfrak{X}}$ as a normalized trace operator. Since $\mathcal{O}_K(\mathfrak{X})$ is a free module over $\varphi_L(\mathcal{O}_K(\mathfrak{X}))$ of rank q, we have the corresponding trace map

$$\mathrm{trace}_{\mathcal{O}_K(\mathfrak{X})/\varphi_L(\mathcal{O}_K(\mathfrak{X}))} : \mathcal{O}_K(\mathfrak{X}) \to \varphi_L\big(\mathcal{O}_K(\mathfrak{X})\big).$$

Remark 4.1.4. $\psi_L^{\mathfrak{X}} = \frac{1}{q}\varphi_L^{-1} \circ \mathrm{trace}_{\mathcal{O}_K(\mathfrak{X})/\varphi_L(\mathcal{O}_K(\mathfrak{X}))}$.

Proof. Since the functions ev_a generate a dense subspace in $\mathcal{O}_K(\mathfrak{X})$ (cf. [75, Lem. 3.1], the proof of which remains valid for general K by [61, Cor. 4.2.6 and Thm. 11.3.5]), it suffices, by the continuity of all operators involved, to check the asserted equality on the functions ev_a. As before we choose a set of representatives $R \subseteq o_L$ for the cosets $o_L/\pi_L o_L$, so that the functions ev_c, for $c \in R$, form a basis of $\mathcal{O}_K(\mathfrak{X})$ over $\varphi_L(\mathcal{O}_K(\mathfrak{X}))$. *Case 1*: Let $a \in o_L^\times$. Then $\psi_L^{\mathfrak{X}}(\mathrm{ev}_a) = 0$ by (4.1). On the other hand, $\mathrm{ev}_a \cdot \mathrm{ev}_c = \mathrm{ev}_{a+c} \in \mathrm{ev}_{c'} \cdot \varphi_L(\mathcal{O}_K(\mathfrak{X}))$ for some $c \neq c' \in R$. Hence, the matrix of multiplication by ev_a with regard to our choice of basis has only zero entries on the diagonal. This means that $\mathrm{trace}_{\mathcal{O}_K(\mathfrak{X})/\varphi_L(\mathcal{O}_K(\mathfrak{X}))}(\mathrm{ev}_a) = 0$. *Case 2*: Let $a \in \pi_L o_L$. Then $\psi_L^{\mathfrak{X}}(\mathrm{ev}_a) = \mathrm{ev}_{\pi_L^{-1}a}$. On the other hand, the matrix of multiplication by ev_a now is the diagonal matrix with constant entry $\mathrm{ev}_a = \varphi_L(\mathrm{ev}_{\pi_L^{-1}a})$. We see that $\frac{1}{q}\varphi_L^{-1}(\mathrm{trace}_{\mathcal{O}_K(\mathfrak{X})/\varphi_L(\mathcal{O}_K(\mathfrak{X}))}(\mathrm{ev}_a)) = \frac{1}{q}\varphi_L^{-1}(q\varphi_L(\mathrm{ev}_{\pi_L^{-1}a})) = \mathrm{ev}_{\pi_L^{-1}a}$. ∎

In order to establish a formula for the composition $\varphi_L \circ \psi_L^{\mathfrak{X}}$, we let $\mathfrak{X}[\pi_L] := \ker(\mathfrak{X} \xrightarrow{\pi_L^*} \mathfrak{X})$. Then $\mathfrak{X}[\pi_L](\mathbb{C}_p)$ is the character group of the finite group $o_L/\pi_L o_L$. The points in $\mathfrak{X}[\pi_L](\mathbb{C}_p)$ are defined over some finite extension K_1/K. For any $\zeta \in \mathfrak{X}[\pi_L](\mathbb{C}_p)$ we have the continuous translation operator

$$\begin{aligned}\mathcal{O}_{K_1}(\mathfrak{X}) &\to \mathcal{O}_{K_1}(\mathfrak{X})\\ F &\mapsto ({}_\zeta F)(\chi) := F(\chi\zeta).\end{aligned}$$

Proposition 4.1.5. *The following holds true:*

(i) *For any $F \in \mathcal{O}_{K_1}(\mathfrak{X})$ we have*

$$[o_L : \pi_L o_L] \cdot \varphi_L \circ \psi_L^{\mathfrak{X}}(F) = \sum_{\zeta \in \mathfrak{X}[\pi_L](\mathbb{C}_p)} {}_\zeta F.$$

(ii) $\varphi_L(\mathfrak{O}_K(\mathfrak{X})) = \{F \in \mathfrak{O}_K(\mathfrak{X}) : {}_\zeta F = F$ *for any* $\zeta \in \mathfrak{X}[\pi_L](\mathbb{C}_p)\}$.

Proof. (i) Again it suffices to consider any $F = \mathrm{ev}_a$. We compute

$$\begin{aligned}\Big(\sum_\zeta {}_\zeta \mathrm{ev}_a\Big)(\chi) &= \sum_\zeta \mathrm{ev}_a(\chi\zeta) = \chi(a)\sum_\zeta \zeta(a)\\ &= \begin{cases}[o_L : \pi_L o_L]\cdot\chi(a) & \text{if } a \in \pi_L o_L,\\ 0 & \text{otherwise}\end{cases}\\ &= \begin{cases}[o_L : \pi_L o_L]\cdot\mathrm{ev}_a(\chi) & \text{if } a \in \pi_L o_L,\\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

On the other hand,

$$\varphi_L\big(\psi_L^{\mathfrak{X}}(\mathrm{ev}_a)\big) = \varphi_L\left(\begin{cases}\mathrm{ev}_{\pi_L^{-1}a} & \text{if } a \in \pi_L o_L,\\ 0 & \text{otherwise}\end{cases}\right) = \begin{cases}\mathrm{ev}_a & \text{if } a \in \pi_L o_L,\\ 0 & \text{otherwise.}\end{cases}$$

(ii) If ${}_\zeta F = F$ for any $\zeta \in \mathfrak{X}[\pi_L](\mathbb{C}_p)$, then $\varphi_L(\psi_L^{\mathfrak{X}}(F)) = F$ by (i). On the other hand,

$$\big({}_\zeta\varphi_L(F)\big)(\chi) = \varphi_L(F)(\chi\zeta) = F\big(\pi_L^*(\chi)\pi_L^*(\zeta)\big) = F\big(\pi_L^*(\chi)\big) = \varphi_L(F)(\chi). \quad ∎$$

We have observed in the above proof that the functions ev_a, for $a \in o_L$, generate a dense subspace of $\mathcal{O}_K(\mathfrak{X})$. Considering the topological decomposition

$$\begin{aligned} \mathcal{O}_K(\mathfrak{X}) &= \varphi_L\big(\mathcal{O}_K(\mathfrak{X})\big) \oplus \mathcal{O}_K(\mathfrak{X})^{\psi_L^{\mathfrak{X}}=0}, \\ F &= \varphi_L\big(\psi_L^{\mathfrak{X}}(F)\big) + \big(F - \varphi_L\big(\psi_L^{\mathfrak{X}}(F)\big)\big), \end{aligned} \tag{4.2}$$

we see, using (4.1), that the ev_a for $a \in \pi_L o_L$, resp., the ev_u for $u \in o_L^\times$, generate a dense subspace of $\varphi_L(\mathcal{O}_K(\mathfrak{X}))$, resp., of $\mathcal{O}_K(\mathfrak{X})^{\psi_L^{\mathfrak{X}}=0}$. In view of Lemma 4.1.2, the obvious formula $u_*(\mathrm{ev}_a) = \mathrm{ev}_{ua}$ together with the fact that the Dirac distributions δ_u, for $u \in o_L^\times$, generate a dense subspace of $D(o_L^\times, K)$ then implies that the decomposition (4.2) is $D(o_L^\times, K)$-invariant.

Lemma 4.1.6 (Mellin transform). *The natural inclusion $D(o_L^\times, K) \hookrightarrow D(o_L, K)$ combined with the Fourier isomorphism induces the map*

$$\begin{aligned} \mathfrak{M} : D(o_L^\times, K) \xrightarrow{\cong} D(o_L, K)^{\psi_L^D=0} &\cong \mathcal{O}_K(\mathfrak{X})^{\psi_L^{\mathfrak{X}}=0} \\ \lambda \mapsto \lambda(\delta_1) &\mathrel{\widehat{=}} \lambda(\mathrm{ev}_1) \end{aligned}$$

which is a topological isomorphism of $D(o_L^\times, K)$-modules.

Proof. The disjoint decomposition into open sets $o_L = \pi_L o_L \cup o_L^\times$ induces the linear topological decomposition $D(o_L, K) = \varphi_L(D(o_L, K)) \oplus D(o_L^\times, K)$. The assertion follows by comparing this with the decomposition (4.2).[1] ∎

The Robba ring

We recall a few facts from [8] about the analytic structure of the character variety $\mathfrak{X}$. As a general convention, all *radii* r which will occur throughout the paper are *assumed to lie in* $(0,1) \cap p^{\mathbb{Q}}$. Let $\mathbf{B}_1(r)$, resp., $\mathbf{B}(r)$, denote the $\mathbb{Q}_p$-affinoid disk of radius r around 1, resp., around 0, and let $\mathbf{B}_1^-(r)$ be the open disk of radius r around 1. We put

$$\mathfrak{X}_0(r) := \mathbf{B}_1(r) \otimes_{\mathbb{Z}_p} \mathrm{Hom}_{\mathbb{Z}_p}(o_L, \mathbb{Z}_p) \quad \text{and} \quad \mathfrak{X}(r) := \mathfrak{X} \cap \mathfrak{X}_0(r)_{/L}.$$

These are affinoid subgroups of $\mathfrak{X}_0$ and $\mathfrak{X}$, respectively, which are respected by the action of the monoid $o_L \setminus \{0\}$. Since $\mathfrak{X}(r) \hookrightarrow \mathfrak{X}_0(r)_{/L}$ is a closed immersion of affinoid varieties, the restriction map between the affinoid algebras $\mathcal{O}_K(\mathfrak{X}_0(r)) \twoheadrightarrow \mathcal{O}_K(\mathfrak{X}(r))$ is a strict surjection of Banach algebras. The families $\{\mathfrak{X}_0(r)\}_r$, resp., $\{\mathfrak{X}(r)\}_r$, form an increasing admissible covering of $\mathfrak{X}_0$, resp., $\mathfrak{X}$, which exhibits the latter as a quasi-Stein space. Hence, $\mathcal{O}_K(\mathfrak{X}_0(r))$, resp., $\mathcal{O}_K(\mathfrak{X}(r))$, is the completion of $\mathcal{O}_K(\mathfrak{X}_0)$, resp., $\mathcal{O}_K(\mathfrak{X})$, in the supremum norm $|\ |_{\mathfrak{X}_0(r)}$, resp., $|\ |_{\mathfrak{X}(r)}$.

[1] The map $D(o_L^\times, K) \to D(G, K)$ sending λ to $\lambda(\delta_1)$ is the inclusion map since $\delta_u(\delta_1) = \delta_u$.

The structure of the affinoid variety $\mathfrak{X}(r_0)$ is rather simple for any radius $r_0 < p^{-\frac{d}{p-1}}$. Then (cf. [8, Lem. 1.16]) the map

$$\begin{aligned} \mathbf{B}(r_0)_{/L} &\xrightarrow{\cong} \mathfrak{X}(r_0) \\ y &\mapsto \chi_y(a) := \exp(ay) \end{aligned} \tag{4.3}$$

is an isomorphism of L-affinoid groups. Taking, somewhat unconventionally, $\exp - 1$ as coordinate function on $\mathbf{B}(r_0)$, we may view $\mathcal{O}_K(\mathbf{B}(r_0))$ as the Banach algebra of all power series $f = \sum_{i \geq 0} c_i (\exp - 1)^i$ such that $c_i \in K$ and $\lim_{i \to \infty} |c_i| r_0^i = 0$; the norm is $|f|_{\mathbf{B}(r_0)} := \max_i |c_i| r_0^i$. Since $\exp - 1$ corresponds under the above isomorphism to the function $\mathrm{ev}_1 - 1$ on $\mathfrak{X}(r_0)$, we deduce that

$$\mathfrak{O}_K(\mathfrak{X}(r_0)) = \Big\{ f = \sum_{i \geq 0} c_i (\mathrm{ev}_1 - 1)^i : c_i \in K \text{ and } \lim_{i \to \infty} |c_i| r_0^i = 0 \Big\}$$

is a Banach algebra with the supremum norm $|f|_{\mathfrak{X}(r_0)} = \max_i |c_i| r_0^i$.

Next we need to explain the admissible open subdomains $\mathfrak{X}_I$ of $\mathfrak{X}$, where the $I \subseteq (0,1)$ are certain intervals (cf. [8, §2.1]). First of all, we have the admissible open subdomains

$$\mathfrak{X}_{(r,1)} := \mathfrak{X} \setminus \mathfrak{X}(r).$$

To introduce the relevant affinoid subdomains, we also need the open disk $\mathbf{B}_1^-(r)$ of radius r around 1. This allows us to first define the admissible open subdomains $\mathfrak{X}_0^-(r) := (\mathbf{B}_1^-(r) \otimes_{\mathbb{Z}_p} \mathrm{Hom}_{\mathbb{Z}_p}(o_L, \mathbb{Z}_p))_{/L}$ and $\mathfrak{X}^-(r) := \mathfrak{X} \cap \mathfrak{X}_0^-(r)$ of $\mathfrak{X}_0$ and $\mathfrak{X}$, respectively. For $r \leq s$ we then have the admissible open subdomains

$$\mathfrak{X}_0[r,s] := \mathfrak{X}_0(s) \setminus \mathfrak{X}_0^-(r) \subseteq \mathfrak{X}_0 \quad \text{and} \quad \mathfrak{X}_{[r,s]} := \mathfrak{X}(s) \setminus \mathfrak{X}^-(r) = \mathfrak{X} \cap \mathfrak{X}_0[r,s] \subseteq \mathfrak{X}.$$

We recall that the $\mathfrak{X}_{[r,s]}$ are actually affinoid varieties. There are the obvious inclusions $\mathfrak{X}_{[r,s]} \subseteq \mathfrak{X}(s)$ and $\mathfrak{X}_{[r,s]} \subseteq \mathfrak{X}_{(r',1)}$ provided $r' < r$. Moreover, $\mathfrak{X}_{(r',1)}$ is the increasing admissible union of the $\mathfrak{X}_{[r,s]}$ for $r' < r \leq s < 1$. Hence,

$$\mathcal{O}_K(\mathfrak{X}_{(r',1)}) = \varprojlim_{r' < r \leq s < 1} \mathcal{O}_K(\mathfrak{X}_{[r,s]}),$$

which exhibits the Fréchet algebra structure of the left side.

We point out that these subdomains $\mathfrak{X}_I$ all are invariant under $o_L^\times$. Their behavior with respect to π_L^* is more complicated. We recall from [8, Lem. 2.11] that, for any radius $p^{-\frac{dp}{p-1}} \leq r < 1$, we have

$$(\pi_L^*)^{-1}(\mathfrak{X}_{(r,1)}) \subseteq \mathfrak{X}_{(r^{1/p},1)}.$$

It is technically necessary in the following to sometimes only work with a smaller set of radii. We put

$$S_0 := [p^{-\frac{d}{e} - \frac{d}{p-1}}, p^{-\frac{d}{p-1}}) \cap p^{\mathbb{Q}} \subseteq [p^{-\frac{dp}{p-1}}, p^{-\frac{d}{p-1}}),$$

$S_n := S_0^{\frac{1}{p^n}}$ for $n \geq 1$, and $S_\infty := \bigcup_{n\geq 1} S_n$. Note that the sets S_n are pairwise disjoint. The point is (cf. [8, Prop. 1.20]) that for $s \in S_\infty$ we know that $\mathfrak{X}(s)$ becomes isomorphic to a closed disk over $\mathbb{C}_p$. Let s_n for $n \geq 0$ denote the left boundary point of the set S_n. Then we have the following result (cf. [8, Prop. 2.1]).

Proposition 4.1.7. *For any $n \geq 0$ the rigid variety $\mathfrak{X}_{(s_n,1)}$ is quasi-Stein with respect to the admissible covering $\{\mathfrak{X}_{[r,s]}\}$, where $s_n < r \leq s < 1$, $r \in S_n$, and $s \in \bigcup_{m\geq n} S_m$. In particular, the affinoid algebra $\mathcal{O}_K(\mathfrak{X}_{[r,s]})$ is the completion of $\mathcal{O}_K(\mathfrak{X}_{(s_n,1)})$ with respect to the supremum norm $|\ |_{\mathfrak{X}_{[r,s]}}$.*

Obviously, with $\mathfrak{X}$ each $\mathfrak{X}_{(s_n,1)}$ is one dimensional and smooth. But, in order to be able to apply later on Serre duality to the spaces $\mathfrak{X}_{(s_n,1)}$, we need to show that they are actually Stein spaces. This means that we have to check that the admissible covering in Proposition 4.1.7 has the property that $\mathfrak{X}_{[r',s']}$ is relatively compact in $\mathfrak{X}_{[r,s]}$ over L (cf. [14, §9.6.2]) for any $r < r' \leq s' < s$. We simply write $U \Subset X$ for an affinoid subdomain U being relatively compact over L in an L-affinoid variety X.

Lemma 4.1.8. *Let $U \subseteq X \subseteq X'$ be affinoid subdomains of the affinoid variety X'; we then have the following:*

(i) *if $U \Subset X$, then $U \Subset X'$;*

(ii) *suppose that $U = U_1 \cup \cdots \cup U_m$ is an affinoid covering; if $U_i \Subset X$ for any $1 \leq i \leq m$, then $U \Subset X$.*

Proof. Let $A \to B$ be the homomorphism of affinoid algebras which induces the inclusion $U = \mathrm{Sp}(B) \subseteq X = \mathrm{Sp}(A)$. It is not difficult to see that the property $U \Subset X$ is equivalent to the homomorphism $A \to B$ being inner with regard to L in the sense of [9, Def. 2.5.1]. Therefore, (i), resp., (ii), is a special case of Corollary 2.5.5, resp., Lemma 2.5.10, in [9]. ∎

Proposition 4.1.9. *$\mathfrak{X}_{(s_n,1)}$, for any $n \geq 0$, is a Stein space.*

Proof. By Proposition 4.1.7, we already know that the $\mathfrak{X}_{(s_n,1)}$ are quasi-Stein. Hence, it remains to show that $\mathfrak{X}_{[r',s']} \Subset \mathfrak{X}_{[r,s]}$ for any $r < r' \leq s' < s$. Looking first at $\mathfrak{X}_0$, let $\mathfrak{B}_1[r,s] \subseteq \mathfrak{B}_1$ denote the affinoid annulus of inner radius r and outer radius s. Fixing coordinate functions $z_1, \ldots, z_d$ on $\mathfrak{X}_0$, we have the admissible open covering

$$\mathfrak{X}_0[r,s] = \bigcup_{i=1}^{d} \mathfrak{X}_0^{(i)}[r,s] \quad \text{with } \mathfrak{X}_0^{(i)}[r,s] := \{x \in \mathfrak{X}_0(s) : |z_i(x)| \geq r\}.$$

The affinoid varieties of this covering have the direct product structure

$$\mathfrak{X}_0^{(i)}[r,s] = \mathfrak{B}_1(s) \times \cdots \mathfrak{B}_1(s) \times \mathfrak{B}_1[r,s] \times \mathfrak{B}_1(s) \times \cdots \mathfrak{B}_1(s)$$

with the annulus being the ith factor. It immediately follows that $\mathfrak{X}_0^{(i)}[r',s'] \Subset \mathfrak{X}_0^{(i)}[r,s]$ (cf. [14, Lem. 9.6.2.1]). Since relative compactness is preserved by passing to closed subvarieties, we deduce that

$$\mathfrak{X} \cap \mathfrak{X}_0^{(i)}[r', s'] \Subset \mathfrak{X} \cap \mathfrak{X}_0^{(i)}[r, s] \quad \text{for any } 1 \leq i \leq d.$$

Applying now Lemma 4.1.8, we conclude first that $\mathfrak{X} \cap \mathfrak{X}_0^{(i)}[r', s'] \Subset \mathfrak{X}_0[r, s]$ and then that $\mathfrak{X}_{[r',s']} \Subset \mathfrak{X}_{[r,s]}$. ∎

We finally recall that the Robba ring of $\mathfrak{X}$ over K is defined as the locally convex inductive limit $\varinjlim_{\mathfrak{Y}} \mathcal{O}_K(\mathfrak{X} \setminus \mathfrak{Y})$, where $\mathfrak{Y}$ runs over all affinoid subdomains of $\mathfrak{X}$. Since any such $\mathfrak{Y}$ is contained in some $\mathfrak{X}(r)$, we have

$$\mathcal{R}_K(\mathfrak{X}) = \varinjlim_{n \geq 0} \mathcal{O}_K(\mathfrak{X}_{(s_n,1)}),$$

and we view $\mathcal{R}_K(\mathfrak{X})$ as the locally convex inductive limit of the Fréchet algebras $\mathcal{O}_K(\mathfrak{X}_{(s_n,1)})$. By [8, Prop. 1.20], the system $\mathfrak{X}_{(s_n,1)/\mathbb{C}_p}$ is isomorphic to a decreasing system of one-dimensional annuli. This implies the following:

- $\mathcal{R}_K(\mathfrak{X})$ is the increasing union of the rings $\mathcal{O}_K(\mathfrak{X}_{(s_n,1)})$ and contains $\mathcal{O}_K(\mathfrak{X})$;
- each $\mathcal{O}_K(\mathfrak{X}_{(s_n,1)})$ and $\mathcal{R}_K(\mathfrak{X})$ are integral domains.

The action of the monoid $o_L \setminus \{0\}$ on $\mathcal{O}_K(\mathfrak{X})$ extends naturally to a continuous action on $\mathcal{R}_K(\mathfrak{X})$ (cf. [8, Lem. 2.12]). In fact, this action extends further uniquely to a separately continuous action of $D(o_L^\times, K)$-action on $\mathcal{R}_K(\mathfrak{X})$. This is a special case of the later Proposition 4.3.10 which implies that we will have such an action on any L-analytic (φ_L, Γ_L)-module over $\mathcal{R}_K(\mathfrak{X})$. Via the isomorphism $\chi_{\mathrm{LT}} : \Gamma_L \xrightarrow{\cong} o_L^\times$ we later on will view this as a $D(\Gamma_L, K)$-action.

In order to extend the ψ-operator to the Robba ring, we need the following fact.

Lemma 4.1.10. *The morphism $\pi_L^* : \mathfrak{X} \to \mathfrak{X}$ is finite, faithfully flat, and étale.*

Proof. The character variety $\mathfrak{X}'$ of the subgroup $\pi_L o_L \subseteq o_L$ is isomorphic to $\mathfrak{X}$ via

$$\begin{aligned} \mathfrak{X} &\xrightarrow{\cong} \mathfrak{X}' \\ \chi &\mapsto \chi'(\pi_L a) := \chi(a). \end{aligned}$$

We have the commutative diagram

$$\begin{array}{ccc} \mathfrak{X} & & \\ {\scriptstyle \pi_L^*}\downarrow & \searrow^{\chi \mapsto \chi|\pi_L o_L} & \\ \mathfrak{X} & \xrightarrow[\chi \mapsto \chi']{\cong} & \mathfrak{X}'. \end{array}$$

The oblique arrow is finite and faithfully flat by the proof of [29, Prop. 6.4.5]. For its étaleness it remains to check that all its fibers are unramified. This can be done after base change to $\mathbb{C}_p$. Then, since this arrow is a homomorphism of rigid groups, all fibers are isomorphic. But the fiber in the trivial character of $\pi_L o_L$ is isomorphic to $\mathrm{Sp}(\mathbb{C}_p[o_L/\pi_L o_L]) \cong \mathrm{Sp}(\mathbb{C}_p^q)$. It follows that π_L^* has these properties as well. ∎

Since the subsequent reasoning will be needed again in the next section in an analogous situation, we proceed in an axiomatic way. *Suppose* that the following hold:

- $\rho : \mathfrak{Y} \to \mathfrak{Z}$ is a finite and faithfully flat morphism of quasi-Stein spaces over K. In particular, the induced map $\rho^* : \mathcal{O}_K(\mathfrak{Z}) \to \mathcal{O}_K(\mathfrak{Y})$ is injective. Moreover, the finiteness of ρ implies that the preimage under ρ of any affinoid subdomain in $\mathfrak{Z}$ is an affinoid subdomain in $\mathfrak{Y}$ (cf. [14, Prop. 9.4.4.1 (i)]) and hence that ρ^* is continuous.
- $\mathcal{O}_K(\mathfrak{Y})$ is finitely generated free as a $\rho^*(\mathcal{O}_K(\mathfrak{Z}))$-module. Fix a corresponding basis $f_1, \ldots, f_h \in \mathcal{O}_K(\mathfrak{Y})$.

Proposition 4.1.11. *For any admissible open subset $\mathfrak{U} \subseteq \mathfrak{Z}$ we have*

$$\mathcal{O}_K\big(\rho^{-1}(\mathcal{U})\big) = \mathcal{O}_K(\mathfrak{Y}) \otimes_{\mathcal{O}_K(\mathfrak{Z})} \mathcal{O}_K(\mathfrak{U})$$

is free with basis $f_1, \ldots, f_h$ over $\mathcal{O}_K(\mathfrak{U})$.

Proof. Since ρ is finite, $\rho_*\mathcal{O}_{\mathfrak{Y}}$ is a coherent $\mathcal{O}_{\mathfrak{Z}}$-module by [14, Prop. 9.4.4.1 (ii)]. Gruson's theorem (cf. [8, Prop. 1.13]) then tells us that $\rho_*\mathcal{O}_{\mathfrak{Y}}$ is, in fact, a free $\mathcal{O}_{\mathfrak{Z}}$-module with basis $f_1, \ldots, f_h$. ∎

We observe that the definition of the Robba ring $\mathcal{R}_K(\mathfrak{X})$ above was completely formal and works precisely the same way for any quasi-Stein space. Hence, we have available the Robba rings $\mathcal{R}_K(\mathfrak{Y})$ and $\mathcal{R}_K(\mathfrak{Z})$. Since the morphism $\rho : \mathfrak{Y} \to \mathfrak{Z}$ is finite, the preimage under ρ of any affinoid subdomain in $\mathfrak{Z}$ is an affinoid subdomain in $\mathfrak{Y}$ (cf. [14, Prop. 9.4.4.1 (i)]). We note again that the preimage under ρ of any affinoid subdomain in $\mathfrak{Z}$ is an affinoid subdomain in $\mathfrak{Y}$. The injective map $\rho^* : \mathcal{O}_K(\mathfrak{Z}) \subseteq \mathcal{O}_K(\mathfrak{Y})$ therefore extends to a natural homomorphism of rings

$$\rho^* : \mathcal{R}_K(\mathfrak{Z}) \to \mathcal{R}_K(\mathfrak{Y}). \tag{4.4}$$

Remark 4.1.12. The homomorphism (4.4) is injective.

Proof. We fix an admissible covering $\mathfrak{Z} = \bigcup_{j \geq 1} \mathfrak{U}_j$ by an increasing sequence of affinoid subdomains $\mathfrak{U}_j \subseteq \mathfrak{Z}$. As ρ is a finite map, $\mathfrak{Y} = \bigcup_{j \geq 1} \rho^{-1}(\mathfrak{U}_j)$ again is an admissible covering by affinoid subdomains. It follows that

$$\mathfrak{R}_K(\mathfrak{Y}) = \varinjlim_j \mathfrak{O}_K\big(\mathfrak{Y} \setminus \rho^{-1}(\mathfrak{U}_j)\big),$$

and therefore it suffices to show the injectivity of the maps $\rho^* : \mathcal{O}_K(\mathfrak{Z} \setminus \mathfrak{U}_j) \to \mathcal{O}_K(\mathfrak{Y} \setminus \rho^{-1}(\mathfrak{U}_j))$. But this is clear since the map $\rho : \mathfrak{Y} \setminus \rho^{-1}(\mathfrak{U}_j) \to \mathfrak{Z} \setminus \mathfrak{U}_j$ is faithfully flat. ∎

Corollary 4.1.13. *$\mathcal{R}_K(\mathfrak{Y}) = \mathcal{O}_K(\mathfrak{Y}) \otimes_{\mathcal{O}_K(\mathfrak{Z})} \mathcal{R}_K(\mathfrak{Z})$ is free over $\rho^*(\mathcal{R}_K(\mathfrak{Z}))$ with the basis $f_1, \ldots, f_h$. In fact, the map*

$$\mathcal{R}_K(\mathfrak{Z})^h \xrightarrow{\cong} \mathcal{R}_K(\mathfrak{Y})$$
$$(z_1, \ldots, z_h) \mapsto \sum_{i=1}^{h} \rho^*(z_i) f_i$$

is a homeomorphism.

Proof. By passing to locally convex limits this follows from Proposition 4.1.11 which says that the map

$$\mathcal{O}_K(\mathfrak{U})^h \xrightarrow{\cong} \mathfrak{O}_K\big(\rho^{-1}(\mathfrak{U})\big)$$
$$(z_1, \ldots, z_h) \mapsto \sum_{i=1}^{h} \rho^*(z_i) f_i$$

is a continuous bijection between Fréchet spaces and hence a homeomorphism by the open mapping theorem. ■

By the Lemmata 4.1.1 and 4.1.10 the above applies to the morphism $\pi_L^* : \mathfrak{X} \to \mathfrak{X}$ and we obtain the following result.

Proposition 4.1.14. *Let $R \subseteq o_L$ be a set of representatives for the cosets in $o_L/\pi_L o_L$. Then the Robba ring $\mathcal{R}_K(\mathfrak{X})$ is a free module over $\varphi_L(\mathcal{R}_K(\mathfrak{X}))$ with basis $\{\mathrm{ev}_a\}_{a \in R}$.*

In particular, we have the trace map

$$\mathrm{trace}_{\mathcal{R}_K(\mathfrak{X})/\varphi_L(\mathcal{R}_K(\mathfrak{X}))} : \mathcal{R}_K(\mathfrak{X}) \to \varphi_L\big(\mathcal{R}_K(\mathfrak{X})\big)$$

and therefore may introduce the operator

$$\psi_L^{\mathfrak{X}} := \frac{1}{q} \varphi_L^{-1} \circ \mathrm{trace}_{\mathcal{R}_K(\mathfrak{X})/\varphi_L(\mathcal{R}_K(\mathfrak{X}))} : \mathcal{R}_K(\mathfrak{X}) \to \mathcal{R}_K(\mathfrak{X}).$$

Because of Remark 4.1.4, it extends the operator $\psi_L^{\mathfrak{X}}$ on $\mathcal{O}_K(\mathfrak{X})$, which justifies denoting it by the same symbol. By construction it is a left inverse of φ_L and satisfies the projection formula. Furthermore, as a consequence of Corollary 4.1.13, $\psi_L^{\mathfrak{X}}$ is continuous.

4.1.2 The multiplicative character variety and its Robba ring

In this section, we consider the multiplicative group $o_L^\times$ as a locally L-analytic group. We introduce the open subgroups $U_n := 1 + \pi_L^n o_L$ for $n \geq 1$. Correspondingly, we have the inclusion of distribution algebras $D(U_{n+1}, K) \subseteq D(U_n, K) \subseteq D(o_L^\times, K)$.

There is an integer $n_0 \geq 1$ such that, for any $n \geq n_0$, the logarithm series induces an isomorphism of locally L-analytic groups $\log : U_n \xrightarrow{\cong} \pi_L^n o_L$. We then introduce the isomorphisms $\ell_n := \pi_L^{-n} \log : U_n \xrightarrow{\cong} o_L$ together with the algebra isomorphisms

$$\ell_{n*} : D(U_n, K) \xrightarrow{\cong} D(o_L, K) \cong \mathcal{O}_K(\mathfrak{X})$$

which they induce.

As for o_L in the previous section, we have rigid analytic varieties (over L) of locally L-analytic characters $\mathfrak{X}^\times$ for $o_L^\times$ and $\mathfrak{X}_n^\times$ for U_n as well (cf. [74, Thm. 2.3, Lem. 2.4, Cor. 3.7] and [29, Props. 6.4.5 and 6.4.6]):

- $\ell_n^* : \mathfrak{X} \xrightarrow{\cong} \mathfrak{X}_n^\times$ is, for $n \geq n_0$, an isomorphism of group varieties.
- The restriction map $\rho_n : \mathfrak{X}^\times \to \mathfrak{X}_n^\times$ is a finite faithfully flat covering (cf. [29, proof of Prop. 6.4.5]).
- $\mathfrak{X}^\times$ and $\mathfrak{X}_n^\times$ are one-dimensional Stein spaces. (As group varieties, they are separated and equidimensional.)
- For $n \geq n_0$ the variety $\mathfrak{X}_n^\times$ is smooth and $\mathcal{O}_L(\mathfrak{X}_n^\times)$ is an integral domain.
- The Fourier transforms

$$D(o_L^\times, K) \xrightarrow{\cong} \mathcal{O}_K(\mathfrak{X}^\times) \quad \text{and} \quad D(U_n, K) \xrightarrow{\cong} \mathcal{O}_K(\mathfrak{X}_n^\times)$$

  sending a distribution μ to the function $F_\mu(\chi) := \mu(\chi)$ are isomorphisms of Fréchet algebras.

As a consequence of the properties of the morphism $\rho := \rho_n : \mathfrak{X}^\times \to \mathfrak{X}_n^\times$, the homomorphism $\rho^* : \mathcal{O}_K(\mathfrak{X}_n^\times) \to \mathcal{O}_K(\mathfrak{X}^\times)$ is injective and extends to an injective homomorphism $\rho^* : \mathcal{R}_K(\mathfrak{X}_n^\times) \to \mathcal{R}_K(\mathfrak{X}^\times)$ (cf. Remark 4.1.12).

Lemma 4.1.15. *We have the following equalities:*

(i) $\mathcal{O}_K(\mathfrak{X}^\times) = \mathbb{Z}[o_L^\times] \otimes_{\mathbb{Z}[U_n]} \mathcal{O}_K(\mathfrak{X}_n^\times)$.

(ii) $\mathcal{R}_K(\mathfrak{X}^\times) = \mathcal{O}_K(\mathfrak{X}^\times) \otimes_{\mathcal{O}_K(\mathfrak{X}_n^\times)} \mathcal{R}_K(\mathfrak{X}_n^\times) = \mathbb{Z}[o_L^\times] \otimes_{\mathbb{Z}[U_n]} \mathcal{R}_K(\mathfrak{X}_n^\times)$.

Proof. (i) Let $u_1, \ldots, u_h \in o_L^\times$ be a set of representatives for the cosets of U_n in $o_L^\times$. We then have the decomposition into open subsets $o_L^\times = u_1 U_n \cup \cdots \cup u_h U_n$. It follows that

$$D(o_L^\times, K) = \delta_{u_1} D(U_n, K) \oplus \cdots \oplus \delta_{u_h} D(U_n, K) = \mathbb{Z}[o_L^\times] \otimes_{\mathbb{Z}[U_n]} D(U_n, K)$$

is, in particular, a free $D(U_n, K)$-module of rank $h = [o_L^\times : U_n]$. Using the Fourier isomorphism, we obtain that $\mathcal{O}_K(\mathfrak{X}^\times)$ is a free $\mathcal{O}_K(\mathfrak{X}_n^\times)$-module over the basis $\mathrm{ev}_{u_1}, \ldots, \mathrm{ev}_{u_h}$.

(ii) Because of (i), the assumptions before Proposition 4.1.11 are satisfied and the present assertion is a special case of Corollary 4.1.13. ∎

Lemma 4.1.16. *The morphism ρ is étale.*

Proof. This is the same argument as in the proof of Lemma 4.1.10. ■

Corollary 4.1.17. *$\mathfrak{X}^\times$ is smooth.*

Proof. This follows from the lemma since $\mathfrak{X}_n^\times$ is smooth for $n \geq n_0$. ■

Remark 4.1.18. If $n \geq m$, then all the above assertions hold analogously for the finite morphism $\rho_{m,n} : \mathfrak{X}_m^\times \to \mathfrak{X}_n^\times$. In particular, all $\mathfrak{X}_n^\times$ are smooth.

Suppose that $n \geq n_0$. Then, due to the isomorphism $\ell_n^* : \mathfrak{X} \xrightarrow{\cong} \mathfrak{X}_n^\times$, everything which was defined for and recalled about $\mathfrak{X}$ in Section 4.1.1 holds correspondingly for $\mathfrak{X}_n^\times$. In particular, we have the admissible open subdomains $\mathfrak{X}_n^\times(r)$, $\mathfrak{X}_{n,(r,1)}^\times$, and $\mathfrak{X}_{n,[r,s]}^\times$. For $n \geq m \geq n_0$ we have the commutative diagram

$$\begin{array}{ccc} \mathfrak{X} & \xrightarrow[\cong]{\ell_m^*} & \mathfrak{X}_m^\times \\ {\scriptstyle (\pi_L^*)^{n-m}}\downarrow & & \downarrow{\scriptstyle \rho_{m,n}} \\ \mathfrak{X} & \xrightarrow[\cong]{\ell_n^*} & \mathfrak{X}_n^\times . \end{array} \tag{4.5}$$

Lemma 4.1.19. *Let $n \geq n_0$ and $m \geq 0$; for any $p^{-\frac{dp}{p-1}} \leq r < 1$ the map $\rho_{n,n+me}^* : \mathcal{O}_K(\mathfrak{X}_{n+me}^\times) \to \mathcal{O}_K(\mathfrak{X}_n^\times)$ extends to an isometric homomorphism of Banach algebras*

$$\big(\mathfrak{O}_K\big(\mathfrak{X}_{n+me}^\times(r)\big), |\,|_{\mathfrak{X}_{n+me}^\times(r)}\big) \to \big(\mathfrak{O}_K\big(\mathfrak{X}_n^\times(r^{\frac{1}{p^m}})\big), |\,|_{\mathfrak{X}_n^\times(r^{\frac{1}{p^m}})}\big).$$

Proof. By the above commutative diagram (4.5) our assertion amounts to the statement that the map $(\pi_L^{me})^* : \mathfrak{X} \to \mathfrak{X}$ restricts to a surjection $\mathfrak{X}(r^{\frac{1}{p^m}}) \to \mathfrak{X}(r)$. In [74, Lem. 3.3], this is shown to be the case for the map $(p^m)^*$. But p^m and π_L^{me} differ by a unit $u \in o_L^\times$, and u^* preserves $\mathfrak{X}(r)$. ■

4.1.3 Twisting

Consider any locally L-analytic group G and fix a locally L-analytic character $\chi : G \to L^\times$. Then multiplication by χ is a K-linear topological isomorphism

$$C^{\mathrm{an}}(G,K) \xrightarrow[\cong]{\chi\cdot} C^{\mathrm{an}}(G,K).$$

We denote the dual isomorphism by

$$\mathrm{Tw}_\chi^D : D(G,K) \xrightarrow{\cong} D(G,K),$$

i.e., $\mathrm{Tw}_\chi^D(\mu) = \mu(\chi -)$, and call it the twist by χ. For Dirac distributions we obtain $\mathrm{Tw}_\chi^D(\delta_g) = \chi(g)\delta_g$.

Suppose now that G is one of the groups o_L or $U_n \subseteq o_L^\times$ of the previous subsections, and let $\mathfrak{X}_G$ denote its character variety. Then χ is an L-valued point $z_\chi \in \mathfrak{X}_G(L)$. Using the product structure of the variety $\mathfrak{X}_G$, we similarly have the twist operator

$$\mathrm{Tw}_z^{\mathcal{X}_G} : \mathfrak{O}_K(\mathcal{X}_G) \xrightarrow{\cong} \mathcal{O}_K(\mathfrak{X}_G), \quad f \mapsto f(z-).$$

As any rigid automorphism multiplication by a rational point respects the system of affinoid subdomains and hence the system of their complements. Hence, $\mathrm{Tw}_z^{\mathfrak{X}_G}$ extends straightforwardly to an automorphism $\mathrm{Tw}_z^{\mathcal{X}_G} : \mathfrak{R}_K(\mathcal{X}_G) \xrightarrow{\cong} \mathcal{R}_K(\mathfrak{X}_G)$. The following properties are straightforward to check:

(i) Under the Fourier isomorphism, Tw_χ^D and $\mathrm{Tw}_{z_\chi}^{\mathfrak{X}_G}$ correspond to each other.

(ii) $\mathrm{Tw}_{z_1}^{\mathfrak{X}_G} \circ \mathrm{Tw}_{z_2}^{\mathfrak{X}_G} = \mathrm{Tw}_{z_1 \cdot z_2}^{\mathfrak{X}_G}$.

(iii) If $\alpha : G_1 \xrightarrow{\cong} G_2$ is an isomorphism between two of our groups, then, for any $z \in \mathfrak{X}_{G_2}(L)$, the twist operators $\mathrm{Tw}_{\alpha^*(z)}^{\mathfrak{X}_{G_1}}$ and $\mathrm{Tw}_z^{\mathfrak{X}_{G_2}}$ correspond to each other under the isomorphism

$$\alpha_* : \mathcal{R}_K(\mathfrak{X}_{G_1}) \xrightarrow{\cong} \mathcal{R}_K(\mathfrak{X}_{G_2}).$$

4.1.4 The LT-isomorphism, part 1

We write $\mathbf{B}$ for the open unit ball over L. The Lubin–Tate formal o_L-module gives $\mathbf{B}$ an o_L-action via $(a, z) \mapsto [a](z)$. If $\mathcal{O}_K(\mathbf{B})$ is the ring of power series in Z with coefficients in K which converge on $\mathbf{B}(\mathbb{C}_p)$, then the above o_L-action on $\mathbf{B}$ induces an action of the monoid $o_L \setminus \{0\}$ on $\mathcal{O}_K(\mathbf{B})$ by $(a, F) \mapsto F \circ [a]$. Similarly, as before, we let φ_L denote the action of π_L. Next we consider the continuous operator

$$\begin{aligned} \mathrm{tr} : \mathcal{O}_K(\mathbf{B}) &\to \mathcal{O}_K(\mathbf{B}) \\ f(z) &\mapsto \sum_{y \in \ker([\pi_L])} f(y +_{\mathrm{LT}} z). \end{aligned}$$

Coleman has shown (cf. [21, Lem. 3] or [80, §2]) that $\mathrm{tr}(Z^i) \in \mathrm{im}(\varphi_L)$ for any $i \geq 0$. Hence, since φ_L is a homeomorphism onto its image, we have $\mathrm{im}(\mathrm{tr}) \subseteq \mathrm{im}(\varphi_L)$ and hence, since φ_L is injective, we may introduce the K-linear operator

$$\psi_L : \mathcal{O}_K(\mathbf{B}) \to \mathcal{O}_K(\mathbf{B}) \quad \text{such that} \quad \pi_L^{-1}\mathrm{tr} = \varphi_L \circ \psi_L.$$

One easily checks that ψ_L is equivariant for the $o_L^\times$-action and satisfies the projection formula $\psi_L(f_1 \varphi_L(f_2)) = \psi_L(f_1) f_2$ as well as $\psi_L \circ \varphi_L = \frac{q}{\pi_L}$.

Furthermore, we fix a generator η' of T'_π as o_L-module and denote by $\Omega = \Omega_{\eta'}$ the corresponding period. In the following we *assume that* Ω *belongs to* K. From [74, Thm. 3.6] we recall the LT-isomorphism

$$\begin{aligned} \kappa^* : \mathcal{O}_K(\mathfrak{X}) &\xrightarrow{\cong} \mathcal{O}_K(\mathbf{B}) \\ F &\mapsto \big[z \mapsto F(\kappa_z)\big], \end{aligned} \tag{4.6}$$

where $\kappa_z(a) = 1 + F_{\eta'}([a](z))$ with $1 + F_{\eta'}(Z) := \exp(\Omega \log_{\mathrm{LT}}(Z))$. It is an isomorphism of topological rings which is equivariant with respect to the action by the monoid $o_L \setminus \{0\}$ (as a consequence of [74, Prop. 3.1]). Moreover, Lemma 4.1.2 implies that the $o_L^\times$-action on $\mathcal{O}_K(\mathbf{B})$ extends to a jointly continuous $D(o_L^\times, K)$-module structure (by descent even for general K) and that the LT-isomorphism is an isomorphism of $D(o_L^\times, K)$-modules.

By the construction of the LT-isomorphism we have

$$\kappa^*(\mathrm{ev}_a) = \exp\big(a\Omega \log_{\mathrm{LT}}(Z)\big) \in o_{\mathbb{C}_p}[\![Z]\!] \quad \text{for any } a \in o_L.$$

Hence, Lemma 4.1.3 implies that

$$\kappa^*\big(\ker(\psi_L^{\mathfrak{X}})\big) = \sum_{a \in R_0} \exp\big(a\Omega \log_{\mathrm{LT}}(Z)\big)\varphi_L\big(\mathcal{O}_K(\mathbf{B})\big),$$

where $R_0 \subseteq o_L$ denotes a set of representatives for the non-zero cosets in $o_L/\pi_L o_L$. Using that $\log_{\mathrm{LT}}(Z_1 +_{\mathrm{LT}} Z_2) = \log_{\mathrm{LT}}(Z_1) + \log_{\mathrm{LT}}(Z_2)$, we compute

$$\begin{aligned}
\mathrm{tr}\left(\exp\big(a\Omega \log_{\mathrm{LT}}(Z)\big)\right) &= \sum_{y \in \ker([\pi_L])} \exp\big(a\Omega \log_{\mathrm{LT}}(y +_{\mathrm{LT}} Z)\big) \\
&= \Big(\sum_{y \in \ker([\pi_L])} \exp\big(a\Omega \log_{\mathrm{LT}}(y)\big)\Big) \exp\big(a\Omega \log_{\mathrm{LT}}(Z)\big) \\
&= \Big(\sum_{y \in \ker([\pi_L])} \kappa_y(a)\Big) \exp\big(a\Omega \log_{\mathrm{LT}}(Z)\big).
\end{aligned}$$

But the κ_y for $y \in \ker([\pi_L])$ are precisely the characters of the finite abelian group $o_L/\pi_L oL$. Hence, $\sum_{y \in \ker([\pi_L])} \kappa_y(a) = 0$ for $a \in R_0$. It follows that $\kappa^*(\ker(\psi_L^{\mathfrak{X}})) = \ker(\psi_L)$. We conclude that under the LT-isomorphism ψ_L corresponds to $\frac{q}{\pi_L}\psi_L^{\mathfrak{X}}$ using the fact that we also have a decomposition

$$\mathcal{O}_K(\mathbf{B}) = \sum_{a \in o_L/\pi_L} \exp\big(a\Omega \log_{\mathrm{LT}}(Z)\big)\varphi_L\big(\mathcal{O}_K(\mathbf{B})\big). \tag{4.7}$$

In the following we denote by

$$\mathcal{M}_{\mathrm{LT}} : D(\Gamma_L, K) \xrightarrow{\cong} \mathcal{O}_K(\mathbf{B})^{\psi_L = 0}$$

the composite

$$D(\Gamma_L, K) \cong D(o_L^\times, K) \cong \mathcal{O}_K(\mathfrak{X})^{\psi_L^{\mathfrak{X}} = 0} \cong \mathcal{O}_K(\mathbf{B})^{\psi_L = 0},$$

where the first isomorphism is induced by the character $\chi_{\mathrm{LT}} : \Gamma_L \xrightarrow{\cong} o_L^\times$, the second one is the Mellin transform $\mathfrak{M}$ from Lemma 4.1.6, and the third one is the LT-isomorphism. By inserting the definitions we obtain the explicit formula

$$\mathfrak{M}_{\mathrm{LT}}(\lambda)(z) = \lambda(\kappa_z \circ \chi_{\mathrm{LT}}).$$

The following commutative diagram relates the pairing in [74, Lem. 4.6],

$$\begin{aligned}\{\ ,\ \} : \mathcal{O}_K(\mathbf{B}) \times C^{\mathrm{an}}(o_L, K) &\to K \\ (F, f) &\mapsto \mu(f),\end{aligned}$$

where $\mu \in D(o_L, K)$ is such that $\mu(\kappa_z) = F(z)$, to the construction of the above map $\mathfrak{M}_{\mathrm{LT}}$:

$$\begin{array}{ccccc}
D(\Gamma_L, K) & \times & C^{\mathrm{an}}(\Gamma_L, K) & \xrightarrow{(\lambda, f)\mapsto \lambda(f)} & K \\
\downarrow{\scriptstyle \mathfrak{M}_{\mathrm{LT}}} & & \downarrow{\scriptstyle (\chi_{\mathrm{LT}})_*} & & \| \\
\mathcal{O}_K(\mathbf{B})^{\psi_L=0} & \times & C^{\mathrm{an}}(o_L^\times, K) & & \| \\
\downarrow{\scriptstyle \subseteq} & & \downarrow{\scriptstyle \text{extension by } 0} & & \| \\
\mathcal{O}_K(\mathbf{B}) & \times & C^{\mathrm{an}}(o_L, K) & \xrightarrow{\{\ ,\ \}} & K.
\end{array}$$

Remark 4.1.20 ($\Omega \in K$). For any $F \in \mathcal{O}_K(\mathbf{B})^{\psi_L=0}$ and any $f \in C^{\mathrm{an}}(o_L, K)$ such that $f|o_L^\times = 0$ we have $\{F, f\} = 0$.

Proof. We have seen above that under the LT-isomorphism ψ_L corresponds, up to a non-zero constant, to $\psi_L^{\mathfrak{X}}$ and hence further under the Fourier isomorphism to ψ_L^D. It therefore suffices to show that for any $\mu \in D(o_L, K)^{\psi_L^D=0}$ we have $\mu(f) = 0$. For this we define $\tilde{f} := f(\pi_L -) \in C^{\mathrm{an}}(o_L, K)$ and note that $(\pi_L)_!(\tilde{f}) = f$. By the definition of ψ_L^D we therefore obtain, under our assumption on μ, that

$$\mu(f) = \mu(f) - \psi_L^D(\mu)(\tilde{f}) = \mu\big(f - (\pi_L)_!(\tilde{f})\big) = \mu(0) = 0. \qquad \blacksquare$$

Lemma 4.1.21 ($\Omega \in K$). *For any* $F \in \mathcal{O}_K(\mathbf{B})^{\psi_L=1}$ *and* $n \geq 0$ *we have*

$$\mathfrak{M}_{\mathrm{LT}}^{-1}\big(\big(1 - \tfrac{\pi_L}{q}\varphi_L\big)F\big)(\chi_{\mathrm{LT}}^n) = \Omega^{-n}\left(1 - \frac{\pi_L^{n+1}}{q}\right)(\partial_{\mathrm{inv}}^n F)_{|Z=0}.$$

Proof. Note that $(1 - \frac{\pi_L}{q}\varphi_L)F$ belongs to $\mathcal{O}_K(\mathbf{B})^{\psi_L=0}$. Let $\mathrm{inc}_! \in C^{\mathrm{an}}(o_L, K)$ denote the extension by zero of the inclusion $o_L^\times \subseteq o_L$, and let $\mathrm{id} : o_L \to K$ be the identity function. Using the above commutative diagram, the assertion reduces to the equality

$$\big\{\big(1 - \tfrac{\pi_L}{q}\varphi_L\big)F, \mathrm{inc}_!^n\big\} = \Omega^{-n}\left(1 - \frac{\pi_L^{n+1}}{q}\right)(\partial_{\mathrm{inv}}^n F)_{|Z=0}.$$

By Remark 4.1.20 we may replace on the left-hand side the function $\mathrm{inc}_!^n$ by the function id^n. Next we observe that $x\,\mathrm{id}^n(x) = \mathrm{id}^{n+1}(x)$. Hence, by [74, Lem. 4.6 (8)], i.e., $\{F, xf(x)\} = \{\Omega^{-1}\partial_{\mathrm{inv}} F, f\}$, and induction, the left-hand side is equal to

$$\begin{aligned}
&\{(1 - \tfrac{\pi_L}{q}\varphi_L)F, \mathrm{id}^n\} \\
&\quad = \{\Omega^{-n}\partial_{\mathrm{inv}}^n\big((1 - \tfrac{\pi_L}{q}\varphi_L)F\big), \mathrm{id}^0\} \\
&\quad = \{\Omega^{-n}\big(1 - \tfrac{\pi_L^{n+1}}{q}\varphi_L\big)(\partial_{\mathrm{inv}}^n F), \mathrm{id}^0\} \quad \text{since } \partial_{\mathrm{inv}}\varphi_L = \pi_L\varphi_L\partial_{\mathrm{inv}} \text{ by (2.3)} \\
&\quad = \Omega^{-n}\big(1 - \tfrac{\pi_L^{n+1}}{q}\varphi_L\big)(\partial_{\mathrm{inv}}^n F)_{|Z=0} \quad \text{since } \mathrm{id}^0 \text{ is the trivial character of } o_L \\
&\quad = \Omega^{-n}\big(1 - \tfrac{\pi_L^{n+1}}{q}\big)(\partial_{\mathrm{inv}}^n F)_{|Z=0} \quad \text{since } [\pi_L](0) = 0.
\end{aligned}$$

■

In the course of the previous proof we have seen that, for F in $\mathcal{O}_K(\mathbf{B})^{\psi_L=0}$ and $n \geq 0$,

$$\mathfrak{M}_{\mathrm{LT}}^{-1}(F)(\chi_{\mathrm{LT}}^n) = \Omega^{-n}(\partial_{\mathrm{inv}}^n F)_{|Z=0}. \tag{4.8}$$

Lemma 4.1.22 ($\Omega \in K$). *For any $F \in \mathcal{O}_K(\mathbf{B})^{\psi_L=0}$ and $n \geq 1$ we have*

$$\mathfrak{M}_{\mathrm{LT}}^{-1}(\log_{\mathrm{LT}} \cdot F)(\chi_{\mathrm{LT}}^n) = n\Omega^{-1}\mathfrak{M}_{\mathrm{LT}}^{-1}(F)(\chi_{\mathrm{LT}}^{n-1}).$$

Proof. First, using (2.3), observe that

$$\psi_L(\log_{\mathrm{LT}} \cdot F) = \psi_L(\pi_L^{-1}\varphi_L(\log_{\mathrm{LT}}) \cdot F) = \pi_L^{-1}\varphi_L(\log_{\mathrm{LT}})\psi_L(F) = 0.$$

Secondly, note that $\partial_{\mathrm{inv}} \log_{\mathrm{LT}} = 1$, i.e., $\partial_{\mathrm{inv}}^i \log_{\mathrm{LT}} = 0$ for $i \geq 2$; also $\log_{\mathrm{LT}}(0) = 0$. Using (4.8) twice, we have

$$\begin{aligned}
\mathfrak{M}_{\mathrm{LT}}^{-1}(\log_{\mathrm{LT}} F)(\chi_{\mathrm{LT}}^n) &= \Omega^{-n}\big(\partial_{\mathrm{inv}}^n(\log_{\mathrm{LT}} F)\big)_{|Z=0} \\
&= \Omega^{-n}\Big(\sum_{i+j=n}\binom{n}{i}(\partial_{\mathrm{inv}}^i \log_{\mathrm{LT}})(\partial_{\mathrm{inv}}^j F)\Big)_{|Z=0} \\
&= \Omega^{-n} n(\partial_{\mathrm{inv}}^{n-1} F)_{|Z=0} \\
&= n\Omega^{-1}\mathfrak{M}^{-1}(F)(\chi_{\mathrm{LT}}^{n-1}).
\end{aligned}$$

■

For the rest of this section we *assume* not only that K contains Ω but also that the action of G_L on $\mathbb{C}_p$ leaves K invariant.

The LT-isomorphism is a topological ring isomorphism

$$K \hat{\otimes}_L \mathcal{O}_L(\mathfrak{X}) = \mathcal{O}_K(\mathfrak{X}) \cong \mathcal{O}_K(\mathbf{B}) = K \hat{\otimes}_L \mathcal{O}_L(\mathbf{B})$$

(cf. [8, Prop. 2.1.5 (ii)] for the outer identities).

On both sides we have the obvious coefficientwise G_L-action induced by the Galois-action on the tensor-factor K. We use the following notation:

- $\sigma \in G_L$ acting *coefficientwise* on $\mathcal{O}_K(\mathbf{B})$ is denoted by: $F \mapsto {}^{\sigma}F$; the corresponding fixed ring is $\mathcal{O}_K(\mathbf{B})^{G_L} = \mathcal{O}_L(\mathbf{B})$.

- The coefficientwise action on $\mathcal{O}_K(\mathfrak{X})$ transfers to the *twisted action* on $\mathcal{O}_K(\mathbf{B})$ by [74, before Cor. 3.8] given as $F \mapsto^{\sigma *} F := {}^{\sigma}F \circ [\tau(\sigma^{-1})]$; the corresponding fixed ring is $\mathcal{O}_K(\mathbf{B})^{G_L,*} = \mathcal{O}_L(\mathfrak{X}) = D(o_L, L)$.

Remark 4.1.23. Note that the $o_L \setminus \{0\}$-action and hence the $D(o_L^\times, L)$-module structure commute with both G_L-actions. Moreover, ψ_L commutes with the G_L-actions as well.

Recall that using the notation from [74, Lem. 4.6 (1), (2)] the function

$$1 + F_{a\eta'}(Z) = \exp\left(a\Omega_{\eta'} \log_{\mathrm{LT}}(Z)\right)$$

corresponds to the Dirac distribution δ_a of $a \in o_L$ under the Fourier isomorphism.

Lemma 4.1.24. *Let σ be in G_L, $t' \in T'_\pi$ and $a \in o_L$. Then*

(i) $\sigma(\Omega_{t'}) = \Omega_{\tau(\sigma)t'} = \Omega_{t'}\tau(\sigma)$ *and*

(ii) ${}^{\sigma}F_{a\eta'} = F_{a\eta'} \circ [\tau(\sigma)] = F_{a\tau(\sigma)\eta'}$.

Proof. (i) The Galois equivariance of the pairing $(\ ,\ ) : T'_\pi \otimes_{o_L} \mathbb{C}_p \to \mathbb{C}_p$ from [74, before Prop. 3.1] with $(t', x) = \Omega_{t'}x$ implies that

$$\sigma(\Omega_{t'}) = \Omega_{\sigma(t')} = \Omega_{\tau(\sigma)t'},$$

while the o_L-invariance of that pairing implies that the latter expression equals $\Omega_{t'}\tau(\sigma)$.

(ii) This is immediate from (i) and the definition of F_a taking equation (2.3) into account. ∎

Proposition 4.1.25. *We equip the distribution rings $D(o_L, K)$ and $D(o_L^\times, K)$ with the G_L-action which is induced from the coefficientwise action on $\mathcal{O}_K(\mathbf{B})$ and $\mathcal{O}_K(\mathbf{B})^{\psi_L=0}$ via the LT-isomorphism and Mellin transform, respectively. Then the following holds true:*

(i) *The isomorphism $\mathfrak{K} : D(o_L, K) \cong \mathcal{O}_K(\mathfrak{X}) \cong \mathcal{O}_K(\mathbf{B})$ composed of LT together with Fourier restricts to an isomorphism*

$$D(o_L, K)^{G_L,\tilde{*}} = \mathcal{O}_K(\mathfrak{X})^{G_L} \cong \mathcal{O}_K(\mathbf{B})^{G_L} = \mathcal{O}_L(\mathbf{B})$$

of $D(o_L^\times, L)$-modules.

(ii) *The Mellin transform restricts to an isomorphism of $D(o_L^\times, L)$-modules*

$$D(o_L^\times, K)^{G_L,*} = \mathcal{O}_K(\mathfrak{X})^{G_L,\psi_L=0} \cong \mathcal{O}_L(\mathbf{B})^{\psi_L=0}.$$

Proof. (i) and (ii) follow from passing to the fixed vectors with respect to the coefficientwise G_L-action and Remark 4.1.23. ∎

In order to express the $D(o_L^\times, L)$-module $D(o_L^\times, K)^{G_L,*}$ in the above proposition more explicitly, we describe the previous two actions on $\mathcal{O}_K(\mathbf{B})$ now on $D(o_L, K)$:

- The coefficientwise G_L-action on $D(o_L, K) = K \hat{\otimes}_L D(o_L, L)$, which corresponds to the twisted action on $\mathcal{O}_K(\mathbf{B})$, will be written as $\lambda \mapsto {}^\sigma\lambda$.
- The G_L-action given by $\lambda \mapsto \tau(\sigma)_*({}^\sigma\lambda)$ corresponds to the coefficientwise action on $\mathcal{O}_K(\mathbf{B})$.

Note that for $\lambda \in D(o_L^\times, K)$ we have $\tau(\sigma)_*(\lambda) = \delta_{\tau(\sigma)}\lambda$, where the right-hand side refers to the product of λ and the Dirac distribution $\delta_{\tau(\sigma)}$ in the ring $D(o_L^\times, K)$. Then we conclude that

$$D(o_L^\times, K)^{G_L,*} = \left\{\lambda \in D(o_L^\times, K) \mid {}^\sigma\lambda = \delta_{\tau(\sigma)^{-1}}\lambda \text{ for all } \sigma \in G_L\right\}.$$

4.2 Consequences of Serre duality

Recall that in any quasi-separated rigid analytic variety the complement of any affinoid subdomain is admissible open (cf. [70, §3 Prop. 3 (ii)]). This applies in particular to quasi-Stein spaces since they are separated by definition. For a rigid analytic variety $\mathfrak{Y}$ we will denote by $\mathrm{Aff}(\mathfrak{Y})$ the set of all affinoid subdomains of $\mathfrak{Y}$.

We have seen that $\mathfrak{X}$, $\mathfrak{X}^\times$, and $\mathfrak{X}_n^\times$ for $n \geq 1$ all are one-(equi)dimensional smooth Stein spaces.

4.2.1 Cohomology with compact support

We slightly rephrase the definition of cohomology with compact support given in [90, §1] in the case of a Stein space $\mathfrak{Y}$ over L. For any abelian sheaf $\mathcal{F}$ on $\mathfrak{Y}$ and any $\mathfrak{U} \in \mathrm{Aff}(\mathfrak{Y})$ we put

$$H^0_{\mathfrak{U}}(\mathcal{Y}, \mathfrak{F}) := \ker\left(\mathcal{F}(\mathfrak{Y}) \to \mathcal{F}(\mathfrak{Y} \setminus \mathfrak{U})\right).$$

This is a left exact functor in $\mathcal{F}$, and we denote by $H^*_{\mathfrak{U}}(\mathfrak{Y}, \mathcal{F})$ its right derived functors. Since quasi-Stein spaces have no coherent cohomology, the relative cohomology sequence (cf. [90, Lem. 1.3]) gives rise, for a coherent sheaf $\mathcal{F}$, to the exact sequence

$$0 \to H^0_{\mathfrak{U}}(\mathfrak{Y}, \mathcal{F}) \to \mathcal{F}(\mathfrak{Y}) \to \mathcal{F}(\mathfrak{Y} \setminus \mathfrak{U}) \to H^1_{\mathfrak{U}}(\mathfrak{Y}, \mathcal{F}) \to 0. \tag{4.9}$$

We then define the cohomology with compact support as

$$H^*_c(\mathfrak{Y}, \mathcal{F}) := \varinjlim_{\mathfrak{U} \in \mathrm{Aff}(\mathfrak{Y})} H^*_{\mathfrak{U}}(\mathfrak{Y}, \mathcal{F}).$$

Again, if $\mathcal{F}$ is a coherent sheaf, we obtain the exact sequence

$$0 \to H^0_c(\mathfrak{Y}, \mathcal{F}) \to \mathcal{F}(\mathfrak{Y}) \to \varinjlim_{\mathfrak{U} \in \mathrm{Aff}(\mathfrak{Y})} \mathcal{F}(\mathfrak{Y} \setminus \mathfrak{U}) \to H^1_c(\mathfrak{Y}, \mathcal{F}) \to 0.$$

Suppose in the following that $j : \mathfrak{Y}_0 \to \mathfrak{Y}$ is an open immersion of Stein spaces (over L) which, for simplicity, we view as an inclusion.

Lemma 4.2.1. *For any* $\mathfrak{U} \in \mathrm{Aff}(\mathfrak{Y}_0)$ *the covering* $\mathfrak{Y} = (\mathfrak{Y} \setminus \mathfrak{U}) \cup \mathfrak{Y}_0$ *is admissible.*

Proof. This follows from [90, Lem. 1.1]. ∎

Lemma 4.2.2. *For any* $\mathfrak{U} \in \mathrm{Aff}(\mathfrak{Y}_0)$ *and any sheaf* $\mathcal{F}$ *on* $\mathfrak{Y}$ *the natural map*

$$H^*_{\mathfrak{U}}(\mathfrak{Y}, \mathcal{F}) \xrightarrow[\cong]{\text{res}} H^*_{\mathfrak{U}}(\mathfrak{Y}_0, \mathcal{F})$$

is bijective.

Proof. Recall that for an injective sheaf on $\mathfrak{Y}$ its restriction to $\mathfrak{Y}_0$ is injective as well. Hence, by using an injective resolution, it suffices to proof the assertion for $* = 0$. *Injectivity*: Let $f \in H^0_{\mathfrak{U}}(\mathfrak{Y}, \mathcal{F})$ such that $f|\mathfrak{Y}_0 = 0$. Since $f|\mathfrak{Y} \setminus \mathfrak{U} = 0$ as well, it follows from Lemma 4.2.1 that $f = 0$. *Surjectivity*: Let $g \in H^0_{\mathfrak{U}}(\mathfrak{Y}_0, \mathcal{F})$ so that $g|\mathfrak{Y}_0 \setminus \mathfrak{U} = 0$. Using Lemma 4.2.1 again, we may define a preimage f of g by $f|\mathfrak{Y} \setminus \mathfrak{U} := 0$ and $f|\mathfrak{Y}_0 := g$. ∎

By passing to inductive limits we obtain the composed map

$$\begin{aligned} j_! : H^*_c(\mathfrak{Y}_0, \mathcal{F}) \\ &= \varinjlim_{\mathfrak{U} \in \mathrm{Aff}(\mathfrak{Y}_0)} H^*_{\mathfrak{U}}(\mathfrak{Y}_0, \mathcal{F}) \xrightarrow[\cong]{\text{res}^{-1}} \varinjlim_{\mathfrak{U} \in \mathrm{Aff}(\mathfrak{Y}_0)} H^*_{\mathfrak{U}}(\mathfrak{Y}, \mathcal{F}) \to \varinjlim_{\mathfrak{U} \in \mathrm{Aff}(\mathfrak{Y})} H^*_{\mathfrak{U}}(\mathfrak{Y}, \mathcal{F}) \\ &= H^*_c(\mathfrak{Y}, \mathcal{F}). \end{aligned}$$

For later purposes we have to analyze the following situation. Let

$$\mathfrak{U}_1 \subseteq \mathfrak{U}_2 \subseteq \cdots \subseteq \mathfrak{U}_n \subseteq \cdots \subseteq \mathfrak{Y} = \bigcup_n \mathfrak{U}_n$$

be a Stein covering. We *assume* that each admissible open subset $\mathfrak{Y}_n := \mathfrak{Y} \setminus \mathfrak{U}_n$ also is a Stein space. Since $\cdots \subseteq \mathfrak{Y}_n \subseteq \cdots \subseteq \mathfrak{Y}_1 \subseteq \mathfrak{Y}$, we then have the projective system

$$\cdots \to H^*_c(\mathfrak{Y}_n, \mathcal{F}) \to \cdots \to H^*_c(\mathfrak{Y}_1, \mathcal{F}) \to H^*_c(\mathfrak{Y}, \mathcal{F}).$$

By Lemma 4.2.2 we may rewrite it as the projective system

$$\cdots \to \varinjlim_{\mathfrak{U} \in \mathrm{Aff}(\mathfrak{Y}_n)} H^*_{\mathfrak{U}}(\mathfrak{Y}, \mathcal{F}) \to \cdots \to \varinjlim_{\mathfrak{U} \in \mathrm{Aff}(\mathfrak{Y}_1)} H^*_{\mathfrak{U}}(\mathfrak{Y}, \mathcal{F}) \to \varinjlim_{\mathfrak{U} \in \mathrm{Aff}(\mathfrak{Y})} H^*_{\mathfrak{U}}(\mathfrak{Y}, \mathcal{F}). \tag{4.10}$$

Lemma 4.2.3. *In the above situation we assume in addition that $\mathcal{F}$ is a coherent sheaf and that the restriction maps $\mathcal{F}(\mathfrak{Y}) \to \mathcal{F}(\mathfrak{Y} \setminus \mathfrak{U})$ for any $\mathfrak{U} \in \mathrm{Aff}(\mathfrak{Y})$ are injective. We then have*

$$\varprojlim_n H_c^1(\mathcal{Y}_n, \mathfrak{F}) = \big(\varprojlim_n \varinjlim_{\mathcal{U} \in \mathrm{Aff}(\mathcal{Y}_n)} \mathfrak{F}(\mathcal{Y} \setminus \mathcal{U})\big)/\mathcal{F}(\mathfrak{Y}). \tag{4.11}$$

Proof. This is immediate from (4.10) and the relative cohomology sequence. ∎

For coherent sheaves $\mathcal{F}$ all the above cohomology vector spaces carry natural locally convex topologies, which we briefly recall. The global sections $\mathcal{F}(\mathfrak{Y})$ and $\mathcal{F}(\mathfrak{Y} \setminus \mathfrak{U})$, for $\mathfrak{U} \in \mathrm{Aff}(\mathfrak{Y})$, are Fréchet spaces. Using the relative cohomology sequence (4.9), we equip $H^1_{\mathfrak{U}}(\mathfrak{Y}, \mathcal{F})$ with the quotient topology from $\mathcal{F}(\mathfrak{Y} \setminus \mathfrak{U})$ (which might be non-Hausdorff) and then $H_c^1(\mathfrak{Y}, \mathcal{F})$ with the locally convex inductive limit topology (with regard to varying $\mathfrak{U}$).

Remark 4.2.4. Let

$$\begin{array}{ccc} M^0 & \xrightarrow{\alpha} & M^1 \\ \downarrow & & \downarrow \\ N^0 & \xrightarrow{\beta} & N^1 \end{array}$$

be a commutative diagram of Fréchet spaces such that the induced map $\mathrm{coker}(\alpha) \to \mathrm{coker}(\beta)$ is bijective; then the latter map is a topological isomorphism for the quotient topologies.

Proof. A more general statement can be found in [2, Chap. VII, Lem. 1.32]. ∎

Using this remark, we see that the bijection $H^1_{\mathfrak{U}}(\mathfrak{Y}, \mathcal{F}) \xrightarrow[\cong]{\mathrm{res}} H^1_{\mathfrak{U}}(\mathfrak{Y}_0, \mathcal{F})$ from Lemma 4.2.2 is a topological isomorphism. It follows that, in degree one at least, the above map $j_!$ is as well as the transition maps in the projective system (4.10) are continuous. Hence, both sides of (4.11) carry natural locally convex topologies by performing the operations $\varprojlim$ and $\varinjlim$ in the category of locally convex vector spaces.

Lemma 4.2.5. *The isomorphism* (4.11) *is topological.*

Proof. First of all, one checks that forming the projective limit on the right-hand side commutes with passing to the quotient space by $\mathcal{F}(\mathfrak{Y})$ (compare (ii) in the proof of [63, Thm. 4.3] for a more general statement). Secondly, as a special case of [15, II.28 Cor. 2], forming inductive limits commutes with passing to quotient spaces. This reduces us to $H^1_{\mathfrak{U}}(\mathfrak{Y} \setminus \mathfrak{U}, \mathcal{F}) = \mathcal{F}(\mathfrak{Y} \setminus \mathfrak{U})/\mathcal{F}(\mathfrak{Y})$ being a topological isomorphism, but which holds by definition. ∎

In the following, we compute (4.11) further in two concrete cases.

The open unit disk

Let $\mathbf{B} = \mathbf{B}_{[0,1)}$ denote the open unit disk over L. We recall our convention that all radii are assumed to lie in $(0,1) \cap p^{\mathbb{Q}}$. For any radii $r \leq s$ we introduce the affinoid disk $\mathbf{B}_{[0,r]}$ as well as the open disk $\mathbf{B}_{[0,r)}$ of radius r around 0 and the affinoid annulus $\mathbf{B}_{[r,s]} := \{r \leq |x| \leq s\}$. We put $\mathbf{B}_{(r,1)} := \mathbf{B} \setminus \mathbf{B}_{[0,r]}$, which are Stein spaces. By the identity theorem for Laurent series the assumptions of Lemma 4.2.3 are satisfied for the structure sheaf $\mathcal{O} = \mathcal{O}_{\mathbf{B}}$ of $\mathbf{B}$. We first fix a radius r and compute the cohomology $H_c^1(\mathbf{B}_{(r,1)}, \mathcal{O})$. By Lemma 4.2.2 and the relative cohomology sequence we have

$$H_c^1(\mathbf{B}_{(r,1)}, \mathcal{O}) = \varinjlim_{\mathcal{U} \in \mathrm{Aff}(\mathbf{B}_{(r,1)})} H^*_{\mathcal{U}}(\mathbf{B}, \mathcal{O}) = \Big(\varinjlim_{\mathcal{U} \in \mathrm{Aff}(\mathbf{B}_{(r,1)})} \mathcal{O}(\mathbf{B} \setminus \mathcal{U})\Big)/\mathcal{O}(\mathbf{B}).$$

Of course, it suffices to take the inductive limit over a cofinal sequence of larger and larger affinoid annuli in $\mathbf{B}_{(r,1)}$. For this we choose two sequences of radii $r < \cdots < r_m < \cdots < r_1, s_1 < \cdots < s_m < \cdots < 1$ with $(r_m)_m$ and $(s_m)_m$ converging to r and 1, respectively. Each space $\mathbf{B} \setminus \mathbf{B}_{[r_m,s_m]} = \mathbf{B}_{(s_m,1)} \,\dot{\cup}\, \mathbf{B}_{[0,r_m)}$ has two connected components. We see that

$$H_c^1(\mathbf{B}_{(r,1)}, \mathcal{O}) = \Big(\varinjlim_{m\to\infty} \mathcal{O}(\mathbf{B}_{(s_m,1)}) \oplus \varinjlim_{m\to\infty} \mathcal{O}(\mathbf{B}_{[0,r_m)})\Big)/\mathcal{O}(\mathbf{B}).$$

As explained in Lemma 4.2.5 this is a topological equality. We observe that

$$\mathcal{R}_L = \mathcal{R}_L(\mathbf{B}) = \varinjlim_{m\to\infty} \mathcal{O}(\mathbf{B}_{(s_m,1)})$$

is the usual Robba ring (over L), whereas $\mathcal{O}^\dagger(\mathbf{B}_{[0,r]}) = \varinjlim_{m\to\infty} \mathcal{O}(\mathbf{B}_{[0,r_m)})$ is the ring of overconvergent analytic functions on $\mathbf{B}_{[0,r]}$. Hence,

$$H_c^1(\mathbf{B}_{(r,1)}, \mathcal{O}) = \big(\mathcal{R}_L \oplus \mathcal{O}^\dagger(\mathbf{B}_{[0,r]})\big)/\mathcal{O}(\mathbf{B}).$$

Passing now to the projective limit with regard to $r \to 1$ of the continuous restriction maps $\mathcal{O}(\mathbf{B}) \to \mathcal{O}^\dagger(\mathbf{B}_{[0,r]}) \to \mathcal{O}(\mathbf{B}_{[0,r]})$, we observe that $\varprojlim_{r\to 1} \mathcal{O}^\dagger(\mathbf{B}_{[0,r]}) = \mathcal{O}(\mathbf{B})$ holds true topologically. We finally deduce that

$$\begin{aligned} \varprojlim_{r\to 1} H_c^1(\mathbf{B}_{(r,1)}, \mathcal{O}) &= \varprojlim_{r\to 1} \Big(\varinjlim_{\mathcal{U} \in \mathrm{Aff}(\mathbf{B}_{(r,1)})} \mathcal{O}(\mathbf{B} \setminus \mathcal{U})\Big)/\mathcal{O}(\mathbf{B}) \\ &= \big(\mathcal{R}_L \oplus \mathcal{O}(\mathbf{B})\big)/\mathcal{O}(\mathbf{B}) \cong \mathcal{R}_L \end{aligned} \tag{4.12}$$

as locally convex vector spaces.

The character variety $\mathfrak{X}$

Since $\mathfrak{X}_{/\mathbb{C}_p} \cong \mathbf{B}_{/\mathbb{C}_p}$ by [74], the injectivity of the restriction maps $\mathcal{O}(\mathfrak{X}) \to \mathcal{O}(\mathfrak{X} \setminus \mathfrak{U})$ for any $\mathfrak{U} \in \mathrm{Aff}(\mathfrak{X})$ follows from the corresponding fact for $\mathbf{B}$, which we saw already. According to Proposition 4.1.9, the admissible open subdomains $\mathfrak{X}_{(s_n,1)}$ of $\mathfrak{X}$ are Stein spaces. In order to compute their cohomology with compact support in the structure sheaf $\mathcal{O} = \mathcal{O}_{\mathfrak{X}}$, we fix an $n \geq 0$. We choose a sequence of radii $r_{n+1} > r_{n+2} > \cdots > s_n$ in S_n converging to s_n. Furthermore, we observe that the increasing sequence $(s_m)_{m>n}$ in S_∞ converges to 1. By Proposition 4.1.7 we then have the Stein covering

$$\mathfrak{X}_{(s_n,1)} = \bigcup_{m>n} \mathfrak{X}_{[r_m,s_m]}.$$

Hence, by Lemma 4.2.2, the relative cohomology sequence, and the explanation in Lemma 4.2.5, we have the topological equality

$$H_c^1(\mathcal{X}_{(s_n,1)}, \mathfrak{O}) = \big(\varinjlim_{m\to\infty} \mathcal{O}(\mathfrak{X} \setminus \mathfrak{X}_{[r_m,s_m]})\big)/\mathcal{O}(\mathfrak{X}).$$

The obvious set theoretic decomposition

$$\mathfrak{X} \setminus \mathfrak{X}_{[r_m,s_m]} = \mathfrak{X} \setminus \big(\mathfrak{X}(s_m) \setminus \mathfrak{X}^-(r_m)\big) = \big(\mathfrak{X} \setminus \mathfrak{X}(s_m)\big) \,\dot\cup\, \mathfrak{X}^-(r_m) = \mathfrak{X}_{(s_m,1)} \,\dot\cup\, \mathfrak{X}^-(r_m)$$

is in fact the decomposition of the space $\mathfrak{X} \setminus \mathfrak{X}_{[r_m,s_m]}$ into its connected components. This can be checked after base change to $\mathbb{C}_p$, where, by [8, Prop. 1.20 and proof of Prop. 2.1], the setting becomes isomorphic to the setting for the open unit disk which we discussed in the previous section. Entirely in the same way as in the previous subsection, it follows now that

$$H_c^1(\mathcal{X}_{(s_n,1)}, \mathfrak{O}) = \big(\mathcal{R}_L(\mathfrak{X}) \oplus \mathcal{O}^\dagger\big(\mathcal{X}(s_n)\big)\big)/\mathcal{O}(\mathfrak{X}),$$

where $\mathfrak{O}^\dagger(\mathfrak{X}(s_n)) := \varinjlim_{m\to\infty} \mathfrak{X}^-(r_m)$, and then

$$\begin{aligned} \varprojlim_{n\to\infty} H_c^1(\mathfrak{X}_{(s_n,1)}, \mathcal{O}) &= \varprojlim_{n\to\infty} \big(\varinjlim_{m\to\infty} \mathcal{O}(\mathfrak{X} \setminus \mathfrak{X}_{[r_m,s_m]})\big)/\mathcal{O}(\mathfrak{X}) \\ &= \big(\mathcal{R}_L(\mathfrak{X}) \oplus \mathcal{O}(\mathfrak{X})\big)/\mathcal{O}(\mathfrak{X}) \cong \mathcal{R}_L(\mathfrak{X}) \end{aligned} \tag{4.13}$$

as locally convex vector spaces.

4.2.2 Serre duality for Stein spaces

In the following, the continuous dual of a locally convex vector space is always equipped with the strong topology.

The Serre duality for smooth Stein spaces is established in [11, 20]. Let $\mathfrak{Y}$ be a one-(equi)dimensional smooth Stein space over L.

Theorem 4.2.6. *For any coherent sheaf $\mathcal{F}$ on $\mathfrak{Y}$ we have the following:*

(i) *$H_c^1(\mathfrak{Y}, \mathcal{F})$ is a complete reflexive Hausdorff space.*

(ii) $\mathrm{Hom}_{\mathfrak{Y}}(\mathcal{F}, \Omega^1_{\mathfrak{Y}}) = H^0(\mathfrak{Y}, \underline{\mathrm{Hom}}_{\mathfrak{Y}}(\mathcal{F}, \Omega^1_{\mathfrak{Y}}))$, *being the global sections of another coherent sheaf, is a reflexive Fréchet space strictly of countable type (cf.* [61, Def. 4.2.3]*).*

(iii) *There is a canonical trace map*

$$\mathrm{tr}_{\mathfrak{Y}} : H^1_c(\mathfrak{Y}, \Omega^1_{\mathfrak{Y}}) \to L$$

such that the Yoneda pairing

$$H^1_c(\mathfrak{Y}, \mathcal{F}) \times \mathrm{Hom}_{\mathfrak{Y}}(\mathcal{F}, \Omega^1_{\mathfrak{Y}}) \to H^1_c(\mathfrak{Y}, \Omega^1_{\mathfrak{Y}})$$

composed with the trace map induces isomorphisms of topological vector spaces

$$\mathrm{Hom}^{\mathrm{cont}}_L(H^1_c(\mathfrak{Y}, \mathcal{F}), L) \xrightarrow{\cong} \mathrm{Hom}_{\mathfrak{Y}}(\mathcal{F}, \Omega^1_{\mathfrak{Y}}),$$
$$\mathrm{Hom}^{\mathrm{cont}}_L(\mathrm{Hom}_{\mathfrak{Y}}(\mathcal{F}, \Omega^1_{\mathfrak{Y}}), L) \xrightarrow{\cong} H^1_c(\mathfrak{Y}, \mathcal{F})$$

which are natural in $\mathcal{F}$.

Proof. See [11, Thm. 7.1][2] and [20, Thm. 4.21] (as well as [90, Prop. 3.6]). ∎

In the special case of the structure sheaf $\mathcal{F} = \mathcal{O}_{\mathfrak{Y}}$, the above assertion gives us $\mathrm{Hom}_{\mathfrak{Y}}(\mathcal{O}_{\mathfrak{Y}}, \Omega^1_{\mathfrak{Y}}) = \Omega^1_{\mathfrak{Y}}(\mathfrak{Y})$ for trivial reasons. On the other hand, the relative cohomology sequence implies that

$$H^1_c(\mathfrak{Y}, \mathcal{O}_{\mathfrak{Y}}) = \mathcal{R}_L(\mathfrak{Y})/\mathcal{O}_{\mathfrak{Y}}(\mathfrak{Y}). \tag{4.14}$$

Hence, we have the following consequence of Serre duality.

Corollary 4.2.7. *Serre duality gives rise to an isomorphism of topological vector spaces*

$$\mathrm{Hom}^{\mathrm{cont}}_L\left(\mathfrak{R}_L(\mathfrak{Y})/\mathfrak{O}_{\mathfrak{Y}}(\mathfrak{Y}), L\right) \xrightarrow{\cong} \Omega^1_{\mathfrak{Y}}(\mathfrak{Y}).$$

Lemma 4.2.8. *Let $\alpha : \mathfrak{Y} \to \mathfrak{Y}'$ be a finite étale morphism of one-dimensional smooth Stein spaces over L. We then have, for any coherent sheaf $\mathcal{F}$ on $\mathfrak{Y}'$, the commutative diagram of Serre duality pairings*

$$\begin{array}{ccccccc}
H^1_c(\mathfrak{Y}, \alpha^*\mathcal{F}) & \times & \mathrm{Hom}_{\mathfrak{Y}}(\alpha^*\mathcal{F}, \Omega^1_{\mathfrak{Y}}) & \longrightarrow & H^1_c(\mathfrak{Y}, \Omega^1_{\mathfrak{Y}}) & \xrightarrow{\mathrm{tr}_{\mathfrak{Y}}} & L \\
\cong\uparrow & & \downarrow & & \downarrow & & \| \\
H^1_c(\mathfrak{Y}', \alpha_*\alpha^*\mathcal{F}) & \times & \mathrm{Hom}_{\mathfrak{Y}'}(\alpha_*\alpha^*\mathcal{F}, \Omega^1_{\mathfrak{Y}'}) & \longrightarrow & H^1_c(\mathfrak{Y}', \Omega^1_{\mathfrak{Y}'}) & \xrightarrow{\mathrm{tr}_{\mathfrak{Y}'}} & L \\
\uparrow & & \downarrow & & \| & & \| \\
H^1_c(\mathfrak{Y}', \mathcal{F}) & \times & \mathrm{Hom}_{\mathfrak{Y}'}(\mathcal{F}, \Omega^1_{\mathfrak{Y}'}) & \longrightarrow & H^1_c(\mathfrak{Y}', \Omega^1_{\mathfrak{Y}'}) & \xrightarrow{\mathrm{tr}_{\mathfrak{Y}'}} & L.
\end{array}$$

[2] This reference depends on the results in article [10], which unfortunately contains the following gaps. Firstly, in the proof of Lemma 4.2.2, Beyer quotes a result of Bosch concerning the connectedness of formal fibers without verifying the required assumptions. This is repaired by [54, Thm. 22, Cor. 23]. Secondly, Beyer claims implicitly and without proof that *special affinoid wide-open spaces* are *affinoid wide-open spaces* in the sense of [10, Def. 4.1.1, Rem. 4.1.2]. This crucial ingredient has now been shown explicitly in [54, §2.5].

Proof. The vertical arrows in the lower part of the diagram are induced by the adjunction homomorphism $\mathcal{F} \to \alpha_* \alpha^* \mathcal{F}$. It is commutative by the naturality of Serre duality in the coherent sheaf.

For the upper part we consider more generally a coherent sheaf $\mathcal{G}$ on $\mathfrak{Y}$ and check the commutativity of the diagram

$$\begin{array}{ccccccc}
H^1_c(\mathfrak{Y}, \mathcal{G}) & \times & \mathrm{Hom}_{\mathfrak{Y}}(\mathcal{G}, \Omega^1_{\mathfrak{Y}}) & \longrightarrow & H^1_c(\mathfrak{Y}, \Omega^1_{\mathfrak{Y}}) & \xrightarrow{\mathrm{tr}_{\mathfrak{Y}}} & L \\
\cong \uparrow & & \downarrow {\scriptstyle f \mapsto \alpha_*(f)} & & \cong \uparrow & & \| \\
H^1_c(\mathfrak{Y}', \alpha_* \mathcal{G}) & \times & \mathrm{Hom}_{\mathfrak{Y}'}(\alpha_* \mathcal{G}, \alpha_* \Omega^1_{\mathfrak{Y}}) & \longrightarrow & H^1_c(\mathfrak{Y}', \alpha_* \Omega^1_{\mathfrak{Y}}) & & \| \\
\| & & \downarrow & & \downarrow & & \| \\
H^1_c(\mathfrak{Y}', \alpha_* \mathcal{G}) & \times & \mathrm{Hom}_{\mathfrak{Y}'}(\alpha_* \mathcal{G}, \Omega^1_{\mathfrak{Y}'}) & \longrightarrow & H^1_c(\mathfrak{Y}', \Omega^1_{\mathfrak{Y}'}) & \xrightarrow{\mathrm{tr}_{\mathfrak{Y}'}} & L.
\end{array}$$

Here the second and third lower vertical arrows are induced by the relative trace map $\mathrm{tr}_\alpha : \alpha_* \Omega^1_{\mathfrak{Y}} \to \Omega^1_{\mathfrak{Y}'}$ (see below). The commutativity of the Yoneda pairings (before applying the horizontal trace maps) is a trivial consequence of functoriality properties. That the first and third upper vertical arrows are isomorphisms follows from the fact that for a finite morphism the functor α_* is exact on quasi-coherent sheaves. This reduces us to showing that the diagram

$$\begin{array}{ccccc}
 & & H^1_c(\mathfrak{Y}, \Omega^1_{\mathfrak{Y}}) & & \\
 & \overset{\cong}{\nearrow} & & \overset{\mathrm{tr}_{\mathfrak{Y}}}{\searrow} & \\
H^1_c(\mathfrak{Y}', \alpha_* \Omega^1_{\mathfrak{Y}}) & & & & L \\
 & \underset{H^1_c(\mathfrak{Y}', \mathrm{tr}_\alpha)}{\searrow} & & \underset{\mathrm{tr}_{\mathfrak{Y}'}}{\nearrow} & \\
 & & H^1_c(\mathfrak{Y}', \Omega^1_{\mathfrak{Y}'}) & &
\end{array} \tag{4.15}$$

is commutative.

For the convenience of the reader we briefly explain the definition of the relative trace map tr_α. But first we need to recall that any coherent $\alpha_* \mathcal{O}_{\mathfrak{Y}}$-module $\mathcal{M}$ can naturally be viewed (cf. [35, Prop. I.9.2.5]) as a coherent $\mathcal{O}_{\mathfrak{Y}}$-module $\widetilde{\mathcal{M}}$ such that $\alpha_* \widetilde{\mathcal{M}} = \mathcal{M}$ (for any open affinoid subdomain $\mathfrak{V} \subseteq \mathfrak{Y}'$ one has $\widetilde{\mathcal{M}}(\alpha^{-1}(\mathfrak{V})) = \mathcal{M}(\mathfrak{V})$). Since α is étale, we have (cf. [57, Thm. 25.1]) that

$$(\alpha_* \mathfrak{D}_{\mathfrak{Y}} \otimes_{\mathfrak{D}_{\mathfrak{Y}'}} \Omega^1_{\mathfrak{Y}'})^{\sim} \xrightarrow{\cong} \Omega^1_{\mathfrak{Y}}.$$

Since α is finite flat, the natural map

$$\underline{\mathrm{Hom}}_{\mathfrak{Y}'}(\alpha_* \mathcal{O}_{\mathfrak{Y}}, \mathcal{O}_{\mathfrak{Y}'}) \otimes_{\mathcal{O}_{\mathfrak{Y}'}} \Omega^1_{\mathfrak{Y}'} \xrightarrow{\cong} \underline{\mathrm{Hom}}_{\mathfrak{Y}'}(\alpha_* \mathcal{O}_{\mathfrak{Y}}, \Omega^1_{\mathfrak{Y}'})$$

is an isomorphism. Finally, since α is finite étale, the usual trace pairing is non-degenerate and induces an isomorphism[3]

$$\underline{\mathrm{Hom}}_{\mathcal{Y}'}(\alpha_* \mathfrak{O}_{\mathcal{Y}}, \mathfrak{O}_{\mathcal{Y}'}) \xrightarrow{\cong} \alpha_* \mathcal{O}_{\mathfrak{Y}}.$$

The relative trace map is now defined to be the composite map

$$\begin{aligned}\alpha_*\Omega^1_{\mathfrak{Y}} &\cong \alpha_*(\alpha_*\mathcal{O}_{\mathfrak{Y}} \otimes_{\mathcal{O}_{\mathfrak{Y}'}} \Omega^1_{\mathfrak{Y}'})^{\sim} \cong \alpha_*(\underline{\mathrm{Hom}}_{\mathfrak{Y}'}(\alpha_*\mathcal{O}_{\mathfrak{Y}}, \mathcal{O}_{\mathfrak{Y}'}) \otimes_{\mathcal{O}_{\mathfrak{Y}'}} \Omega^1_{\mathfrak{Y}'})^{\sim} \\ &\cong \alpha_*\underline{\mathrm{Hom}}_{\mathfrak{Y}'}(\alpha_*\mathcal{O}_{\mathfrak{Y}}, \Omega^1_{\mathfrak{Y}'})^{\sim} = \underline{\mathrm{Hom}}_{\mathfrak{Y}'}(\alpha_*\mathcal{O}_{\mathfrak{Y}}, \Omega^1_{\mathfrak{Y}'}) \xrightarrow{f \mapsto f(1)} \Omega^1_{\mathfrak{Y}'}.\end{aligned}$$

The commutativity of (4.15) is shown in detail in [54] and should also be a consequence of [1, Prop. 6.5.1 (2)] upon showing that their general construction boils down to the above description of the relative trace map. ■

We make the last lemma more explicit for the structure sheaf. Let $\rho : \mathfrak{Y} \to \mathfrak{Z}$ be a finite, faithfully flat, and étale morphism of one-dimensional smooth Stein spaces over L such that $\mathcal{O}_{\mathfrak{Y}}(\mathfrak{Y})$ is finitely generated free as an $\mathcal{O}_{\mathfrak{Z}}(\mathfrak{Z})$-module. Fix a basis $f_1, \ldots, f_h \in \mathcal{O}_{\mathfrak{Y}}(\mathfrak{Y})$. Going through the proof of Lemma 4.2.8, one checks that on global sections the relative trace map is given by

$$\begin{aligned}\mathrm{tr}_\rho : \Omega^1_{\mathfrak{Y}}(\mathfrak{Y}) = \mathcal{O}_{\mathfrak{Y}}(\mathfrak{Y}) \otimes_{\mathcal{O}_{\mathfrak{Z}}(\mathfrak{Z})} \Omega^1_{\mathfrak{Z}}(\mathfrak{Z}) &\to \Omega^1_{\mathfrak{Z}}(\mathfrak{Z}) \\ \omega = \sum_{i=1}^{h} f_i \otimes \omega_i &\mapsto \sum_{i=1}^{h} \mathrm{trace}_{\mathcal{O}_{\mathfrak{Y}}(\mathfrak{Y})/\mathcal{O}_{\mathfrak{Z}}(\mathfrak{Z})}(f_i)\omega_i.\end{aligned} \tag{4.16}$$

Hence, we have the commutative diagram of duality pairings

$$\begin{array}{ccc} \mathrm{Hom}_L^{\mathrm{cont}}(\mathcal{R}_L(\mathfrak{Y})/\mathcal{O}_{\mathfrak{Y}}(\mathfrak{Y}), L) & \xrightarrow{\cong} & \Omega^1_{\mathfrak{Y}}(\mathfrak{Y}) \\ \Big\downarrow {\scriptstyle \mathrm{Hom}(\rho^*, L)} & & \Big\downarrow {\scriptstyle \mathrm{tr}_\rho} \\ \mathrm{Hom}_L^{\mathrm{cont}}(\mathcal{R}_L(\mathfrak{Z})/\mathcal{O}_{\mathfrak{Z}}(\mathfrak{Z}), L) & \xrightarrow{\cong} & \Omega^1_{\mathfrak{Z}}(\mathfrak{Z}). \end{array} \tag{4.17}$$

It remains to explicitly compute the relative trace map in the cases of interest to us. But first we observe that, by the explanation at the end of Section 2.3 in [8], the sheaf of differentials $\Omega^1_{\mathfrak{Y}}$ on a smooth one-dimensional Stein group variety $\mathfrak{Y}$ is a free $\mathcal{O}_{\mathfrak{Y}}$-module. Furthermore, if $\mathfrak{Y}$ is one of our character varieties, say of the group G, then by the construction before Definition 1.27 in [8] we have the embedding

$$\begin{aligned} L = \mathrm{Lie}(G) &\to \mathcal{O}_{\mathfrak{Y}}(\mathfrak{Y}) \\ \mathfrak{x} &\mapsto \left[\chi \mapsto d\chi(\mathfrak{x})\right]\end{aligned}$$

and the function $\log_{\mathfrak{Y}}$ defined as the image of $1 \in L = \mathrm{Lie}(G)$.

[3] Cf. [82, §49.3].

Remark 4.2.9. The function $\log_{\mathfrak{X}}$ corresponds under the LT-isomorphism κ to the function $\Omega \log_{\mathrm{LT}}$ by the commutative diagram after [74, Lem. 3.4]. In particular, $d\log_{\mathfrak{X}}$ corresponds to $\Omega d\log_{\mathrm{LT}}$ and $\partial^{\mathfrak{X}}_{\mathrm{inv}}$ to $\frac{1}{\Omega}\partial_{\mathrm{inv}}$, where $df = \partial^{\mathfrak{X}}_{\mathrm{inv}} d\log_{\mathfrak{X}}$ defines the invariant derivation on $\mathcal{O}_K(\mathfrak{X})$ similarly as for ∂_{inv} in (2.2).

Proposition 4.2.10. *We have explicit formulae for the relative trace map in the following cases:*

(i) *For $\pi_L^* : \mathfrak{X} \to \mathfrak{X}$ we have $\Omega^1_{\mathfrak{X}}(\mathfrak{X}) = \mathcal{O}_{\mathfrak{X}}(\mathfrak{X}) d\log_{\mathfrak{X}}$ and*

$$\mathrm{tr}_{\pi_L^*}(f d\log_{\mathfrak{X}}) = \tfrac{q}{\pi_L}\psi_L^{\mathfrak{X}}(f) d\log_{\mathfrak{X}}\,.$$

(ii) *For $n \geq n_0$ and $\ell_n^* : \mathfrak{X} \xrightarrow{\cong} \mathfrak{X}_n^\times$ we have $\Omega^1_{\mathfrak{X}_n^\times}(\mathfrak{X}_n^\times) = \mathcal{O}_{\mathfrak{X}_n^\times}(\mathfrak{X}_n^\times) d\log_{\mathfrak{X}_n^\times}$ and*

$$\mathrm{tr}_{\ell_n^*}(f d\log_{\mathfrak{X}}) = \pi_L^n(\ell_n^*)_*(f) d\log_{\mathfrak{X}_n^\times}\,.$$

(iii) *For $n \geq m \geq 1$ and $\rho_{m,n} : \mathfrak{X}_m^\times \to \mathfrak{X}_n^\times$ we have $\Omega^1_{\mathfrak{X}_m^\times}(\mathfrak{X}_m^\times) = \mathcal{O}_{\mathfrak{X}_m^\times}(\mathfrak{X}_m^\times) d\log_{\mathfrak{X}_m^\times}$ and*

$$\mathrm{tr}_{\rho_{m,n}}(f d\log_{\mathfrak{X}_m^\times}) = q^{n-m} f_1 d\log_{\mathfrak{X}_n^\times} \quad \text{if } f = \sum_{i=1}^{h} \mathrm{ev}_{u_i}\, \rho_{m,n}^*(f_i),$$

where $u_1 = 1, u_2, \ldots, u_h \in U_m$ are representatives for the cosets of U_n in U_m (with $h := q^{n-m}$).

(iv) *For $n \geq 1$ and $\rho_n : \mathfrak{X}^\times \to \mathfrak{X}_n^\times$ we have $\Omega^1_{\mathfrak{X}^\times}(\mathfrak{X}^\times) = \mathcal{O}_{\mathfrak{X}^\times}(\mathfrak{X}^\times) d\log_{\mathfrak{X}^\times}$ and*

$$\mathrm{tr}_{\rho_n}(f d\log_{\mathfrak{X}^\times}) = (q-1)q^{n-1} f_1 d\log_{\mathfrak{X}_n^\times} \quad \text{if } f = \sum_{i=1}^{h} \mathrm{ev}_{u_i}\, \rho_n^*(f_i),$$

where $u_1 = 1, u_2, \ldots, u_h \in U_n$ are representatives for the cosets of U_n in $o_L^\times$ (with $h := (q-1)q^{n-1}$).

(v) *For the multiplication $\mu_\chi : \mathfrak{X}^\times \xrightarrow{\cong} \mathfrak{X}^\times$ by a fixed point $\chi \in \mathfrak{X}^\times(L)$ we have*

$$\mathrm{tr}_{\mu_\chi}(f d\log_{\mathfrak{X}^\times}) = \mu_{\chi *}(f) d\log_{\mathfrak{X}^\times} = \mu_{\chi^{-1}}^*(f) d\log_{\mathfrak{X}^\times}\,.$$

Proof. All subsequent computations start, of course, from the formula (4.16) for the relative trace map.

(i) The assumptions are satisfied by Lemmata 4.1.1 and 4.1.10. As explained at the end of Section 2.3 in [8], the sheaf of differentials $\Omega^1_{\mathfrak{X}}$ on $\mathfrak{X}$ is a free $\mathcal{O}_{\mathfrak{X}}$-module of rank one with basis the global differential $d\log_{\mathfrak{X}}$. By [8, Lem. 1.28 (ii)] we have $\varphi_L(\log_{\mathfrak{X}}) = \pi_L \log_{\mathfrak{X}}$. The formula for $\mathrm{tr}_{\pi_L^*}$ now follows from Remark 4.1.4.

(ii) The assumptions are trivially satisfied. The map $d\ell_n : L = \mathrm{Lie}(U_n) \to L = \mathrm{Lie}(o_L)$ is multiplication by π_L^{-n}. It follows that the isomorphism $(\ell_n^*)^* : \mathcal{O}_{\mathfrak{X}_n^\times}(\mathfrak{X}_n^\times) \to \mathcal{O}_{\mathfrak{X}}(\mathfrak{X})$ sends $\log_{\mathfrak{X}_n^\times}$ to $\pi_L^{-n} \log_{\mathfrak{X}}$. This implies the assertions.

(iii) The assumptions are satisfied by Remark 4.1.18. The inclusion $U_n \hookrightarrow U_m$ of an open subgroup induces the identity map on the Lie algebras. It follows that $\rho_{m,n}^*(\log_{\mathfrak{X}_n^\times}) = \log_{\mathfrak{X}_m^\times}$. Since $\rho_{m,n}$ is étale, we first may apply this with some $n \geq n_0$ and, using (ii), deduce that $\Omega^1_{\mathfrak{X}_m^\times}(\mathfrak{X}_m^\times) = \mathcal{O}_{\mathfrak{X}_m^\times}(\mathfrak{X}_m^\times) d \log_{\mathfrak{X}_m^\times}$ for any $m \geq 1$. The formula for $\mathrm{tr}_{\rho_{m,n}}$ follows by the same argument as in the proof of Remark 4.1.4.

(iv) The argument is the same as the one for (iii).

(v) The assumptions are trivially satisfied. Using that $d\chi(1) = \frac{d}{dt}\chi(\exp(t))_{|t=0}$, we check that

$$\log_{\mathfrak{X}^\times}(\chi_1\chi_2) = \log_{\mathfrak{X}^\times}(\chi_1) + \log_{\mathfrak{X}^\times}(\chi_2)$$

holds true. We deduce $\mu_\chi^*(d \log_{\mathfrak{X}^\times}) = d \log_{\mathfrak{X}^\times}$ and hence the formula for tr_{μ_χ}. ∎

We briefly remark on the case where our Stein space is the open unit disk $\mathbf{B}$ around zero. Then $\mathcal{R}_L(\mathbf{B})$ is the usual Robba ring of all Laurent series $f(Z) = \sum_{i \in \mathbb{Z}} c_i Z^i$ with coefficients $c_i \in L$ which converge in some annulus near 1. Analogously to (4.14), we have

$$H_c^1(\mathbf{B}, \Omega^1_{\mathbf{B}}) = \mathcal{R}_L(\mathbf{B}) dZ / \mathcal{O}_{\mathbf{B}}(\mathbf{B}) dZ$$

and the trace map sends $\sum_{i \in \mathbb{Z}} c_i Z^i dZ$ to its residue which is the coefficient c_{-1} (cf. [11, §3.1]).

4.2.3 Duality for boundary sections

First, we recall another functoriality property of Serre duality.

Proposition 4.2.11. *Let $j : \mathfrak{Y}_0 \to \mathfrak{Y}$ be an open immersion of one-(equi)dimensional smooth Stein spaces over L, and let $\mathcal{F}$ be a coherent sheaf on $\mathfrak{Y}$. Then the diagram*

$$\begin{array}{ccccccc}
H_c^1(\mathfrak{Y}, \mathcal{F}) & \times & \mathrm{Hom}_{\mathfrak{Y}}(\mathcal{F}, \Omega^1_{\mathfrak{Y}}) & \longrightarrow & H_c^1(\mathfrak{Y}, \Omega^1_{\mathfrak{Y}}) & \xrightarrow{\mathrm{tr}_{\mathfrak{Y}}} & L \\
{\scriptstyle j_!}\uparrow & & \downarrow{\scriptstyle \mathrm{res}} & & {\scriptstyle j_!}\uparrow & & \| \\
H_c^1(\mathfrak{Y}_0, \mathcal{F}) & \times & \mathrm{Hom}_{\mathfrak{Y}_0}(\mathcal{F}, \Omega^1_{\mathfrak{Y}_0}) & \longrightarrow & H_c^1(\mathfrak{Y}_0, \Omega^1_{\mathfrak{Y}_0}) & \xrightarrow{\mathrm{tr}_{\mathfrak{Y}_0}} & L
\end{array}$$

is commutative.

Proof. The commutativity of the Yoneda pairing (before applying trace maps) is immediate from the functoriality of the cohomology with compact support in the coefficient sheaf. The assertion that $\mathrm{tr}_{\mathfrak{Y}} \circ j_! = \mathrm{tr}_{\mathfrak{Y}_0}$ holds true is shown in [90, Thm. 3.7]. ∎

In order to combine the above functoriality property with Lemma 4.2.3 in the case of the structure sheaf $\mathcal{F} = \mathcal{O}_{\mathfrak{Y}}$, we first recall the setting of that lemma.

(1) $\mathfrak{Y} = \bigcup_n \mathfrak{U}_n$ is a Stein covering of the Stein space $\mathfrak{Y}$ such that the $\mathfrak{Y}_n = \mathfrak{Y} \setminus \mathfrak{U}_n$ are Stein spaces as well. In particular, $\mathcal{R}_L(\mathfrak{Y}) = \varinjlim_n \mathcal{O}_{\mathfrak{Y}}(\mathfrak{Y}_n)$ with the locally convex inductive limit topology.

(2) The restriction maps $\mathcal{O}_{\mathfrak{Y}}(\mathfrak{Y}) \to \mathcal{O}_{\mathfrak{Y}}(\mathfrak{Y} \setminus \mathfrak{U})$ are injective for any $\mathfrak{U} \in \mathrm{Aff}(\mathfrak{Y})$.

(3) The inductive system of Fréchet spaces

$$\Omega^1_{\mathfrak{Y}}(\mathfrak{Y}_1) \to \cdots \to \Omega^1_{\mathfrak{Y}}(\mathfrak{Y}_n) \to \Omega^1_{\mathfrak{Y}}(\mathfrak{Y}_{n+1}) \to \cdots$$

is regular (cf. [61, Def. 11.1.3 (ii)]). By [61, Thm. 11.2.4 (ii)] the locally convex inductive limit

$$\Omega^1_{\mathfrak{R}_L(\mathfrak{Y})} := \varinjlim_n \Omega^1_{\mathfrak{Y}}(\mathfrak{Y}_n) = \varinjlim_{\mathfrak{U} \in \mathrm{Aff}(\mathfrak{Y})} \Omega^1_{\mathfrak{Y}}(\mathfrak{Y} \setminus \mathfrak{U})$$

is a locally convex Hausdorff space.

Proposition 4.2.12. *In the above setting we have a natural topological isomorphism*

$$\mathrm{Hom}_L^{\mathrm{cont}}(\Omega^1_{\mathcal{R}_L(\mathfrak{Y})}, L) \cong \big(\varprojlim_n \varinjlim_{\mathcal{U} \in \mathrm{Aff}(\mathcal{Y}_n)} \mathfrak{O}_{\mathcal{Y}}(\mathcal{Y} \setminus \mathcal{U})\big)/\mathcal{O}_{\mathfrak{Y}}(\mathfrak{Y})$$

(where the left-hand side is equipped with the strong topology of bounded convergence).

Proof. The asserted isomorphism is the composite of the isomorphisms

$$\begin{aligned}\mathrm{Hom}_L^{\mathrm{cont}}(\Omega^1_{\mathcal{R}_L(\mathfrak{Y})}, L) &= \mathrm{Hom}_L^{\mathrm{cont}}(\varinjlim_n \Omega^1_{\mathfrak{Y}}(\mathfrak{Y}_n), L) \xrightarrow{\cong} \varprojlim_n \mathrm{Hom}_L^{\mathrm{cont}}(\Omega^1_{\mathfrak{Y}}(\mathfrak{Y}_n), L) \\ &= \varprojlim_n H^1_c(\mathfrak{Y}_n, \mathcal{O}_{\mathfrak{Y}}) \\ &= \big(\varprojlim_n \varinjlim_{\mathcal{U} \in \mathrm{Aff}(\mathcal{Y}_n)} \mathfrak{O}_{\mathcal{Y}}(\mathcal{Y} \setminus \mathcal{U})\big)/\mathcal{O}_{\mathfrak{Y}}(\mathfrak{Y}).\end{aligned}$$

The isomorphism in the first line comes from [61, Thm. 11.1.13]. The equality in the second, resp., third, line is a consequence of Theorem 4.2.6 and Proposition 4.2.11, resp., Lemmata 4.2.3 and 4.2.5. ∎

We now evaluate this latter result in the same concrete cases as in Section 4.2.1.

The open unit disk

First of all, the sheaf of differentials $\Omega^1_{\mathbf{B}}$ on the open unit disk $\mathbf{B}$ is a free $\mathcal{O}_{\mathbf{B}}$-module of rank one. Hence, by choosing, for example, the global differential dZ for a coordinate function Z, as a basis, we obtain a topological isomorphism $\mathcal{R}_L(\mathbf{B}) \cong \Omega^1_{\mathcal{R}_L(\mathbf{B})}$ as $\mathcal{R}_L(\mathbf{B})$-modules. The regularity assumption in (3) above therefore is reduced to the corresponding property for $\mathcal{R}_L(\mathbf{B})$, which is established in the proof of [8, Prop. 2.6 (i)]. Hence, Proposition 4.2.12 is available. By combining its assertion with (4.12) we obtain a natural topological isomorphism

$$\mathrm{Hom}_L^{\mathrm{cont}}\big(\mathcal{R}_L(\mathbf{B}), L\big) \cong \mathrm{Hom}_L^{\mathrm{cont}}(\Omega^1_{\mathcal{R}_L(\mathbf{B})}, L) \cong \mathcal{R}_L(\mathbf{B}). \tag{4.18}$$

This shows that $\mathcal{R}_L(\mathbf{B})$ is topologically self-dual. By going through the definitions and using the explicit description of the trace map in this case as the usual residue map (end of Section 4.2.2) one checks that this self-duality comes from the pairing

$$\begin{aligned}\mathcal{R}_L(\mathbf{B}) \times \mathcal{R}_L(\mathbf{B}) &\to L \\ \big(f_1(Z), f_2(Z)\big) &\mapsto \text{residue of } f_1(Z)f_2(Z)dZ.\end{aligned}$$

This latter form of the result was known (cf. [64, 67]) before Serre duality in rigid analysis was established. In this paper, it is more natural to use the self-duality given by the pairing

$$\begin{aligned}\langle\ ,\ \rangle_{\mathbf{B}} : \mathcal{R}_L(\mathbf{B}) \times \mathcal{R}_L(\mathbf{B}) &\to L \\ (f_1, f_2) &\mapsto \text{residue of } f_1 f_2 d\log_{\mathrm{LT}}.\end{aligned}$$

We will denote by $\mathrm{res}_{\mathbf{B}} : \Omega^1_{\mathcal{R}_L(\mathbf{B})} \to L$ the linear form which corresponds to $1 \in \mathcal{R}_L(\mathbf{B})$ under the second isomorphism in (4.18).

The character variety $\mathfrak{X}$

We recall that the sheaf of differentials $\Omega^1_{\mathfrak{X}}$ on $\mathfrak{X}$ is a free $\mathcal{O}_{\mathfrak{X}}$-module of rank one with basis the global differential $d\log_{\mathfrak{X}}$. Hence, again we have a topological isomorphism $\mathcal{R}_L(\mathfrak{X}) \cong \Omega^1_{\mathcal{R}_L(\mathfrak{X})}$ as $\mathcal{R}_L(\mathfrak{X})$-modules. The regularity assumption in (3) above therefore holds by [8, Prop. 2.6 (i)]. Hence, Proposition 4.2.12 is available. By combining its assertion with (4.13) we obtain a natural topological isomorphism

$$\mathrm{Hom}_L^{\mathrm{cont}}(\mathcal{R}_L(\mathfrak{X}), L) \cong \mathrm{Hom}_L^{\mathrm{cont}}(\Omega^1_{\mathcal{R}_L(\mathfrak{X})}, L) \cong \mathcal{R}_L(\mathfrak{X}). \tag{4.19}$$

This shows that $\mathcal{R}_L(\mathfrak{X})$ is topologically self-dual. Let $\mathrm{res}_{\mathfrak{X}} : \Omega^1_{\mathcal{R}_L(\mathfrak{X})} \to L$ be the linear form which corresponds to $1 \in \mathcal{R}_L(\mathfrak{X})$ under the above isomorphism. Then, as a consequence of the naturality of the Yoneda pairing, this self-duality comes from the pairing[4]

$$\begin{aligned}\langle\ ,\ \rangle_{\mathfrak{X}} : \mathcal{R}_L(\mathfrak{X}) \times \mathcal{R}_L(\mathfrak{X}) &\to L \\ (f_1, f_2) &\mapsto \mathrm{res}_{\mathfrak{X}}(f_1 f_2 d\log_{\mathfrak{X}}).\end{aligned} \tag{4.20}$$

Next we consider $\mathfrak{X}_n^\times$ for some $n \geq n_0$, where $n_0 \geq 1$ is the integer from Section 4.1.2. We then have the isomorphism of Stein group varieties $\ell_n^* : \mathfrak{X} \xrightarrow{\cong} \mathfrak{X}_n^\times$. Hence, all we have established for $\mathfrak{X}$ holds true correspondingly for $\mathfrak{X}_n^\times$. In particular, we have a natural topological isomorphism

$$\mathrm{Hom}_L^{\mathrm{cont}}(\mathcal{R}_L(\mathfrak{X}_n^\times), L) \cong \mathrm{Hom}_L^{\mathrm{cont}}(\Omega^1_{\mathcal{R}_L(\mathfrak{X}_n^\times)}, L) \cong \mathcal{R}_L(\mathfrak{X}_n^\times).$$

[4] Warning: If $L = \mathbb{Q}_p$, then $\mathfrak{X} = \mathbf{B}_1 \cong \mathbf{B}$ with the latter isomorphism given by $z \mapsto z - 1$; but the self-dualities (4.18) and (4.19) do not correspond to each other since in this case $d\log_{\mathfrak{X}}$ corresponds to $d\log(1+Z) = \frac{1}{1+Z}dZ$.

Let $\mathrm{res}_{\mathfrak{X}_n^\times} : \Omega^1_{\mathcal{R}_L(\mathfrak{X}_n^\times)} \to L$ be the linear form which corresponds to $1 \in \mathcal{R}_L(\mathfrak{X}_n^\times)$ under this isomorphism. We obtain that $\mathcal{R}_L(\mathfrak{X}_n^\times)$ is topologically self-dual with regard to the pairing

$$\begin{aligned} \langle\ ,\ \rangle_{\mathfrak{X}_n^\times} : \mathcal{R}_L(\mathfrak{X}_n^\times) \times \mathcal{R}_L(\mathfrak{X}_n^\times) &\to L \\ (f_1, f_2) &\mapsto \mathrm{res}_{\mathfrak{X}_n^\times}(f_1 f_2 d \log_{\mathfrak{X}_n^\times}). \end{aligned}$$

It follows from [90, Thm. 3.7] that the diagram

$$\begin{array}{ccc} \Omega^1_{\mathcal{R}_L(\mathfrak{X})} & \xrightarrow{\mathrm{res}_{\mathfrak{X}}} & \\ {\scriptstyle (\ell_n^*)_*}\downarrow \cong & & L \\ \Omega^1_{\mathcal{R}_L(\mathfrak{X}_n^\times)} & \xrightarrow{\mathrm{res}_{\mathfrak{X}_n^\times}} & \end{array}$$

is commutative. But in the proof of Proposition 4.2.10 (ii) we have already seen that $(\ell_n^*)_*(\log_{\mathfrak{X}}) = \pi_L^n \log_{\mathfrak{X}_n^\times}$. Therefore, the diagram of pairings

$$\begin{array}{ccccc} \mathcal{R}_L(\mathfrak{X}) & \times & \mathcal{R}_L(\mathfrak{X}) & \xrightarrow{\langle\ ,\ \rangle_{\mathfrak{X}}} & L \\ {\scriptstyle (\ell_n^*)^*}\uparrow \cong & & {\scriptstyle \pi_L^n(\ell_n^*)_*}\downarrow \cong & & \Vert \\ \mathcal{R}_L(\mathfrak{X}_n^\times) & \times & \mathcal{R}_L(\mathfrak{X}_n^\times) & \xrightarrow{\langle\ ,\ \rangle_{\mathfrak{X}_n^\times}} & L \end{array} \tag{4.21}$$

is commutative. Alternatively, we could have used the following observation.

Remark 4.2.13. Let $\rho : \mathfrak{Y} \to \mathfrak{Z}$ be one of the morphisms in Proposition 4.2.10. Then Proposition 4.1.11 applies, and it follows that, for any admissible open subset $\mathfrak{U} \subseteq \mathfrak{Z}$ which is Stein, the relative trace map $\mathrm{tr}_{\rho|\rho^{-1}(\mathfrak{U})}$ is given by the same formula as for tr_ρ.

In the case of the morphisms π_L^* and $\rho_{m,n}$ for $n \geq m \geq n_0$, this immediately leads to the following equalities of pairings.

Lemma 4.2.14. *We have the following:*

(i) $\langle \varphi_L(f_1), f_2\rangle_{\mathfrak{X}} = \frac{q}{\pi_L}\langle f_1, \psi_L^{\mathfrak{X}}(f_2)\rangle_{\mathfrak{X}}$ *for any* $f_1, f_2 \in \mathcal{R}_L(\mathfrak{X})$.

(ii) *Let* $n \geq m \geq n_0$ *and let* $u_1 = 1, u_2, \ldots, u_h \in U_m$ *be representatives for the cosets of* U_n *in* U_m *(with* $h := q^{n-m}$*); for any* $f' \in \mathcal{R}_L(\mathfrak{X}_n^\times)$ *and any* $f \in \mathcal{R}_L(\mathfrak{X})_m^\times$ *of the form* $f = \sum_{i=1}^h \mathrm{ev}_{u_i}\, \rho_{m,n}^*(f_i)$ *(cf. Remark 4.1.18) we have*

$$\langle \rho_{m,n}^*(f'), f\rangle_{\mathfrak{X}_m^\times} = q^{n-m}\langle f', f_1\rangle_{\mathfrak{X}_n^\times}.$$

The multiplicative character variety $\mathfrak{X}^\times$

We fix an $n \geq n_0$ for the moment as well as representatives $u_1 = 1, u_2, \ldots, u_h \in o_L^\times$ for the cosets of U_n in $o_L^\times$ (with $h := (q-1)q^{n-1}$). Recalling from Lemma 4.1.15 (ii) that

$$\mathcal{R}_L(\mathfrak{X}^\times) = \mathbb{Z}[o_L^\times] \otimes_{\mathbb{Z}[U_n]} \rho_n^*(\mathcal{R}_L(\mathfrak{X}_n^\times)),$$

we may write any $f \in \mathcal{R}_L(\mathfrak{X}^\times)$ as $f = \sum_{i=1}^h \mathrm{ev}_{u_i}\, \rho_n^*(f_i)$ with uniquely determined $f_i \in \mathcal{R}_L(\mathfrak{X}_n^\times)$. We now define $\mathrm{res}_{\mathfrak{X}^\times} : \Omega^1_{\mathcal{R}_L(\mathfrak{X}^\times)} \to L$ by

$$\mathrm{res}_{\mathfrak{X}^\times}(fd\log_{\mathfrak{X}^\times}) := (q-1)q^{n-1}\,\mathrm{res}_{\mathfrak{X}_n^\times}(f_1 d\log_{\mathfrak{X}_n^\times}) \tag{4.22}$$

and then the pairing

$$\begin{aligned} \langle\ ,\ \rangle_{\mathfrak{X}^\times} : \mathcal{R}_L(\mathfrak{X}^\times) \times \mathcal{R}_L(\mathfrak{X}^\times) &\to L \\ (f_1, f_2) &\mapsto \mathrm{res}_{\mathfrak{X}^\times}(f_1 f_2). \end{aligned} \tag{4.23}$$

These definitions are obviously independent of the choice of the representatives u_i. Moreover, due to Lemma 4.2.14 (ii), they are independent of the choice of n as well and we have

$$\big\langle \rho_n^*(f'), f\big\rangle_{\mathfrak{X}^\times} = (q-1)q^{n-1}\langle f', f_1\rangle_{\mathfrak{X}_n^\times} \tag{4.24}$$

for any $f' \in \mathcal{R}_L(\mathfrak{X}_n^\times)$ and any $f = \sum_{i=1}^{(q-1)q^{n-1}} \mathrm{ev}_{u_i}\, \rho_n^*(f_i) \in \mathcal{R}_L(\mathfrak{X})^\times$, where $u_i \in o_L^\times$ runs through representatives for the cosets of U_n in $o_L^\times$. The topological self-duality of $\mathcal{R}_L(\mathfrak{X}_n^\times)$ easily implies that this pairing makes $\mathcal{R}_L(\mathfrak{X}^\times)$ topological self-dual.

Lemma 4.2.15. *The twist morphism $\mu_\chi : \mathfrak{X}^\times \to \mathfrak{X}^\times$, for any $\chi \in \mathfrak{X}^\times(L)$, satisfies*

$$\big\langle \mu_\chi^*(f_1), \mu_\chi^*(f_2)\big\rangle_{\mathfrak{X}^\times} = \langle f_1, f_2\rangle_{\mathfrak{X}^\times} \quad \textit{for any } f_1, f_2 \in \mathcal{R}_L(\mathfrak{X}^\times).$$

Proof. The assertion immediately reduces to checking the equality $\mathrm{res}_{\mathfrak{X}^\times} \circ \mu_\chi^* = \mathrm{res}_{\mathfrak{X}^\times}$. Obviously, there are twist morphisms on $\mathfrak{X}_n^\times$ as well. One easily checks that

$$\mu_\chi^* \circ \rho_n^* = \rho_n^* \circ \mu_{\chi|U_n}^*$$

and that

$$\mu_\chi^*(\mathrm{ev}_u) = \chi(u)\,\mathrm{ev}_u \quad \text{for any } u \in o_L^\times.$$

Using Lemma 4.1.15 (ii), we write an $f \in \mathcal{R}_L(\mathfrak{X}^\times)$ as $f = \sum_{i=1}^h \mathrm{ev}_{u_i}\, \rho_n^*(f_i)$ and compute

$$\mu_\chi^*(f) = \sum_{i=1}^h \mu_\chi^*(\mathrm{ev}_{u_i})\mu_\chi^*\big(\rho_n^*(f)\big) = \sum_{i=1}^h \mathrm{ev}_{u_i}\, \rho_n^*\big(\chi(u_i)\mu_{\chi|U_n}^*(f_i)\big).$$

This shows that $\mu_\chi^*(f)_1 = \mu_{\chi|U_n}^*(f_1)$. This further reduces the claim to showing that $\mathrm{res}_{\mathfrak{X}_n^\times} \circ \mu_{\chi|U_n}^* = \mathrm{res}_{\mathfrak{X}_n^\times}$. But this follows from [90, Thm. 3.7] or, alternatively, from a version of Proposition 4.2.10 (v) for $\mathcal{R}_L(\mathfrak{X}_n^\times)$. ∎

Of course, everything in this entire Section 4.2 remains valid over any complete extension field K of L contained in $\mathbb{C}_p$. Moreover, our constructions above are compatible under (complete) base change: Let $\mathfrak{Y}_K$ denote the base change of $\mathfrak{Y}$ over L

to K (and similarly for affinoids). Then we obtain a commutative diagram

$$\begin{array}{ccccc} \mathcal{R}_L(\mathfrak{Y}) & \times & \Omega^1_{\mathcal{R}_L(\mathfrak{Y})} & \longrightarrow & L \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{R}_K(\mathfrak{Y}_K) & \times & \Omega^1_{\mathcal{R}_K(\mathfrak{Y}_K)} & \longrightarrow & K. \end{array} \tag{4.25}$$

Indeed, it is shown in [54] that Serre duality is compatible with base change in the sense that there is the following commutative diagram for any n, in which the horizontal lines are the Serre dualities over L and K, respectively:

$$\begin{array}{ccccc} H^1_c(\mathfrak{Y}_n, \mathcal{O}_{\mathfrak{Y}}) & \times & \Omega^1_{\mathfrak{Y}}(\mathfrak{Y}_n) & \longrightarrow & L \\ \downarrow & & \downarrow & & \downarrow \\ H^1_c(\mathfrak{Y}_{n,K}, \mathcal{O}_{\mathfrak{Y}_K}) & \times & \Omega^1_{\mathfrak{Y}_K}(\mathfrak{Y}_{n,K}) & \longrightarrow & K. \end{array}$$

Hence, taking limits as in the proof of Proposition 4.2.12, the claim follows upon observing that also the relative cohomology sequence (4.9) is compatible with base change. By inserting $1 \in \mathcal{R}_L(\mathfrak{Y}) \subseteq \mathcal{R}_K(\mathfrak{Y}_K)$ into the pairings of (4.25), we see that in any example discussed above the residue maps $\mathrm{res}_{\mathfrak{Y}}$ are compatible under base change as well as the pairings $\langle\ ,\ \rangle_{\mathfrak{Y}}$. Since $\mathcal{R}_K(\mathfrak{Y}_K) \cong K \hat{\otimes}_{L,\iota} \mathcal{R}_L(\mathfrak{Y})$ (and hence $\Omega^1_{\mathcal{R}_K(\mathfrak{Y}_K)} \cong K \hat{\otimes}_{L,\iota} \Omega^1_{\mathcal{R}_L(\mathfrak{Y})}$) by [8, Cor. 2.8] with respect to the (completed) inductive tensor product, we see that

$$\mathrm{res}_{\mathfrak{Y}_K} = K \hat{\otimes}_{L,\iota} \mathrm{res}_{\mathfrak{Y}} \tag{4.26}$$

and that we have the commutative diagram

$$\begin{array}{ccccc} K \hat{\otimes}_{L,\iota} \mathrm{Hom}^{\mathrm{cont}}_L(\mathcal{R}_L(\mathfrak{Y}), L) & \to & \mathrm{Hom}^{\mathrm{cont}}_K(K \hat{\otimes}_{L,\iota} \mathcal{R}_L(\mathfrak{Y}), K) & \xrightarrow{\cong} & \mathrm{Hom}^{\mathrm{cont}}_K(\mathcal{R}_K(\mathfrak{Y}_K), K) \\ \cong \downarrow \mathrm{id}_K \hat{\otimes}_{L,\iota} \mathrm{duality}_{\mathfrak{Y}_L} & & & & \cong \downarrow \mathrm{duality}_{\mathfrak{Y}_K} \\ K \hat{\otimes}_{L,\iota} \mathcal{R}_L(\mathfrak{Y}) & & \xrightarrow{\quad\cong\quad} & & \mathcal{R}_K(\mathfrak{Y}_K). \end{array}$$

4.3 (φ_L, Γ_L)-modules

As before, we let $L \subseteq K \subseteq \mathbb{C}_p$ be a complete intermediate field, and we denote by o_K its ring of integers.

4.3.1 The usual Robba ring

In Sections 4.2.2 and 4.2.3, we already had introduced the usual Robba ring $\mathcal{R} = \mathcal{R}_K = \mathcal{R}_K(\mathbf{B})$ of the Stein space $\mathbf{B}_{/K}$ in connection with Serre duality. We briefly review its construction in more detail. The ring of K-valued global holomorphic functions[5] $\mathcal{O}_K(\mathbf{B})$ on $\mathbf{B}$ is the Fréchet algebra of all power series in the variable Z with coefficients in K which converge on the open unit disk $\mathbf{B}(\mathbb{C}_p)$. The Fréchet topology on $\mathcal{O}_K(\mathbf{B})$ is given by the family of norms

$$\Big|\sum_{i\geq 0} c_i Z^i\Big|_r := \max_i |c_i| r^i \quad \text{for } 0 < r < 1.$$

In the commutative integral domain $\mathcal{O}_K(\mathbf{B})$, we have the multiplicative subset $Z^{\mathbb{N}} = \{Z^j : j \in \mathbb{N}\}$, so that we may form the corresponding localization $\mathcal{O}_K(\mathbf{B})_{Z^{\mathbb{N}}}$. Each norm $| \ |_r$ extends to this localization $\mathcal{O}_K(\mathbf{B})_{Z^{\mathbb{N}}}$ by setting

$$\Big|\sum_{i\gg -\infty} c_i Z^i\Big|_r := \max_i |c_i| r^i.$$

The Robba ring $\mathcal{R} \supseteq \mathcal{O}_K(\mathbf{B})$ is constructed as follows. For any $s > 0$, resp., any $0 < r \leq s$, in $p^{\mathbb{Q}}$ let $\mathbf{B}_{[0,s]}$, resp., $\mathbf{B}_{[r,s]}$, denote the affinoid disk around 0 of radius s, resp., the affinoid annulus of inner radius r and outer radius s, over K. For $I = [0, s]$ or $[r, s]$ we denote by

$$\mathcal{R}^I := \mathcal{R}^I_K(\mathbf{B}) := \mathcal{O}_K(\mathbf{B}_I)$$

the affinoid K-algebra of $\mathbf{B}_I$. The Fréchet algebra $\mathcal{R}^{[r,1)} := \varprojlim_{r<s<1} \mathcal{R}^{[r,s]}$ is the algebra of (infinite) Laurent series in the variable Z with coefficients in K which converge on the half-open annulus $\mathbf{B}_{[r,1)} := \bigcup_{r<s<1} \mathbf{B}_{[r,s]}$. The Banach algebra $\mathcal{R}^{[0,s]}$ is the completion of $\mathcal{O}_K(\mathbf{B})$ with respect to the norm $| \ |_s$. The Banach algebra $\mathcal{R}^{[r,s]}$ is the completion of $\mathcal{O}_K(\mathbf{B})_{Z^{\mathbb{N}}}$ with respect to the norm $| \ |_{r,s} := \max(| \ |_r, | \ |_s)$. It follows that the Fréchet algebra $\mathcal{R}^{[r,1)}$ is the completion of $\mathcal{O}_K(\mathbf{B})_{Z^{\mathbb{N}}}$ in the locally convex topology defined by the family of norms $(| \ |_{r,s})_{r<s<1}$. Finally, the Robba ring is $\mathcal{R} = \bigcup_{0<r<1} \mathcal{R}^{[r,1)}$.

Remark 4.3.1. $\mathcal{O}_K(\mathbf{B})_{Z^{\mathbb{N}}}$ is dense in $\mathcal{R}_K(\mathbf{B})$.

Let $p^{-\frac{d}{(q-1)e}} < r \leq s < 1$. Then we have a surjective map

$$\begin{aligned} \mathbf{B}_{[r,s]} &\to \mathbf{B}_{[r^q,s^q]} \\ z &\mapsto [\pi_L](z) \end{aligned} \tag{4.27}$$

according to [33, proof of Lem. 2.6].[6] It induces a ring homomorphism

$$\varphi_L^{[r^q,s^q]} : \mathcal{R}^{[r^q,s^q]} \to \mathcal{R}^{[r,s]} \tag{4.28}$$

[5] In the notation from [25, §1.2], this is the ring $\mathcal{R}^+$.

[6] The proof there is only written for the *special* Lubin–Tate group, i.e., such that $[\pi_L] = X^q + \pi_L X$, but generalizes easily by using the fact that $[\pi_L] = X^q + \pi_L X f(X)$ with $f(X) \in o_L[\![X]\!]^\times$.

which is isometric with respect to the supremum norms, i.e.,

$$\left|\varphi_L^{[r^q,s^q]}(f)\right|_{[r,s]} = |f|_{[r^q,s^q]} \quad \text{for any } f \in \mathcal{R}^{[r^q,s^q]}.$$

In particular, by taking first inverse and then direct limits we obtain a continuous ring homomorphism $\varphi_L : \mathcal{R} \to \mathcal{R}$. We shall often omit the interval in ${\varphi_L}^{[r,s]}$ and just write φ_L.

Similarly, we obtain a continuous Γ_L-action on $\mathcal{R}$: According to loc. cit., we have a bijective map

$$\begin{aligned} \mathbf{B}_{[r,s]} &\to \mathbf{B}_{[r,s]} \\ z &\mapsto [\chi_{\mathrm{LT}}(\gamma)](z) \end{aligned}$$

for any $\gamma \in \Gamma_L$, whence we obtain an isometric isomorphism

$$\gamma : \mathcal{R}^{[r,s]} \to \mathcal{R}^{[r,s]}$$

with respect to the supremum norms, i.e., $|\gamma(f)|_{[r,s]} = |f|_{[r,s]}$ for any $f \in \mathcal{R}^{[r,s]}$.

Finally, we extend the operator ψ_L to $\mathcal{R}$: For $y \in \ker([\pi_L])$ we have the isomorphism

$$\begin{aligned} \mathbf{B}_{[r,s]} &\to \mathbf{B}_{[r,s]} \\ z &\mapsto z +_{\mathrm{LT}} y \end{aligned}$$

of affinoid varieties because $|z +_{\mathrm{LT}} y| = |z + y| = |z|$. The latter equality comes from $|z| \geq r > p^{-\frac{d}{(q-1)e}} = q^{-\frac{1}{q-1}} = |y|$ for $y \neq 0$. Setting $\mathrm{tr}(f) := \sum_{y \in \ker([\pi_L])} f(z +_{\mathrm{LT}} y)$, we obtain a norm decreasing linear map $\mathrm{tr} : \mathcal{R}^{[r,s]} \to \mathcal{R}^{[r,s]}$. We claim that the image of tr is contained in the (closed) image of the isometry $\varphi_L^{[r^q,s^q]}$, whence there is a norm decreasing map

$$\psi_{\mathrm{Col}} : \mathcal{R}^{[r,s]} \to \mathcal{R}^{[r^q,s^q]},$$

such that $\varphi_L \circ \psi_{\mathrm{Col}} = \mathrm{tr}$. Indeed, by continuity it suffices to show that $\mathrm{tr}(Z^i)$ belongs to the image for any $i \in \mathbb{Z}$. For $i \geq 0$, Coleman has shown that $\mathrm{tr}(Z^i) = \varphi_L(\psi_{\mathrm{Col}}(Z^i))$ with $\psi_{\mathrm{Col}}(Z^i) \in o_L[\![Z]\!] \subseteq \mathcal{R}^{[r^q,s^q]}$, see [80, §2]. For $i < 0$, we calculate

$$\begin{aligned} \varphi_L\big(Z^i \psi_{\mathrm{Col}}\big([\pi_L](Z)^{-i} Z^i\big)\big) &= \varphi_L(Z^i)\Big(\sum_{y \in \ker([\pi_L])} [\pi_L](Z)^{-i} Z^i\Big)(Z +_{\mathrm{LT}} y) \\ &= \varphi_L(Z^i)\Big(\sum_{y \in \ker([\pi_L])} [\pi_L]\big((Z +_{\mathrm{LT}} y)\big)^{-i} (Z +_{\mathrm{LT}} y)^i\Big) \\ &= \varphi_L(Z^i)\Big(\sum_{y \in \ker([\pi_L])} \varphi_L(Z)^{-i} (Z +_{\mathrm{LT}} y)^i\Big) \\ &= \sum_{y \in \ker([\pi_L])} (Z +_{\mathrm{LT}} y)^i = \mathrm{tr}(Z^i), \end{aligned}$$

whence the claim follows. We put $\psi_L^{[r,s]} := \frac{1}{\pi_L}\psi_{\mathrm{Col}} : \mathcal{R}^{[r,s]} \to \mathcal{R}^{[r^q,s^q]}$ which induces the continuous operator $\psi_L : \mathcal{R} \to \mathcal{R}$ by taking first inverse limits and then direct limits. By definition of tr the operators $\psi_L^{[r,s]}$ and hence ψ_L satisfy the projection formula. We shall often omit the interval in $\psi_L^{[r,s]}$ and just write ψ_L.

As in Section 4.1.4, we fix a generator η' of the dual Tate module T'_π and denote by Ω the corresponding period. For the rest of this subsection we *assume* in addition that K contains Ω. Following Colmez in the notation, we introduce the power series $\eta(a, Z) := \exp(a\Omega \log_{\mathrm{LT}}(Z)) \in o_K[\![Z]\!]$ for $a \in o_L$. As noted in Section 4.1.4, the power series $\eta(a, Z)$ is nothing else than the image under the LT-isomorphism κ^* of the holomorphic function $\mathrm{ev}_a \in \mathcal{O}_K(\mathfrak{X})$. Generalizing the equality (4.7), we have the following decompositions of Banach spaces:

$$\mathcal{R}^{[r,s]} = \bigoplus_{a \in o_L/\pi_L^n} \varphi_L^n(\mathcal{R}^{[r^q,s^q]})\eta(a, Z) \tag{4.29}$$

and hence

$$\mathcal{R} = \bigoplus_{a \in o_L/\pi_L^n} \varphi_L^n(\mathcal{R})\eta(a, Z) \tag{4.30}$$

of LF-spaces using the formula

$$r = \left(\tfrac{\pi_L}{q}\right)^n \sum_a \varphi_L^n \psi_L^n\big(\eta(-a, Z)r\big)\eta(a, Z). \tag{4.31}$$

This can easily be reduced by induction on n to the case $n = 1$. Using the definition of tr and the orthogonality relations for the characters κ_y for $y \in \ker([\pi_L])$, the formula follows and, moreover, defines a continuous inverse to the continuous map

$$\begin{aligned} \mathbb{Z}[o_L] \otimes_{\mathbb{Z}[\pi_L o_L]} \mathcal{R}^{[r^q,s^q]} &\xrightarrow{\cong} \mathcal{R}^{[r,s]} \\ a \otimes f &\mapsto a\varphi_L(f). \end{aligned}$$

Inductively, we obtain canonical isomorphisms

$$\begin{aligned} \mathbb{Z}[o_L] \otimes_{\mathbb{Z}[\pi_L^n o_L]} \mathcal{R}^{[r^{q^n},s^{q^n}]} &\xrightarrow{\cong} \mathcal{R}^{[r,s]} \\ a \otimes f &\mapsto a\varphi_L^n(f). \end{aligned} \tag{4.32}$$

Moreover, immediately from the definitions, we have

$$\varphi_L\big(\eta(a, Z)\big) = \eta(\pi_L a, Z), \tag{4.33}$$

$$\sigma\big(\eta(a, Z)\big) = \eta\big(\chi_{\mathrm{LT}}(\sigma)a, Z\big) \quad \text{for } \sigma \in \Gamma_L, \tag{4.34}$$

$$\psi_L\big(\eta(a, Z)\big) = \begin{cases} \frac{q}{\pi_L}\eta\big(\frac{a}{\pi_L}, Z\big) & \text{for } a \in \pi_L o_L, \\ 0 & \text{otherwise.} \end{cases} \tag{4.35}$$

Remark 4.3.2. We have $\psi_L = \frac{1}{\pi_L}\varphi_L^{-1} \circ \mathrm{trace}_{\mathcal{R}/\varphi_L(\mathcal{R})}$.

Proof. Both maps tr and $\mathrm{trace}_{\mathcal{R}/\varphi_L(\mathcal{R})}$ are easily seen to be $\varphi_L(\mathcal{R})$-linear and to be multiplication by q on $\varphi_L(\mathcal{R})$. Hence, by (4.30), it suffices to compare their values on the elements $\eta(a, Z)$. By (4.35) we have

$$\mathrm{tr}\left(\eta(a,Z)\right) = \begin{cases} q\varphi_L\left(\eta(\pi_L^{-1}a, Z)\right) & \text{if } a \in \pi_L o_L, \\ 0 & \text{otherwise.} \end{cases}$$

On the other hand, a similar computation as in the proof of Remark 4.1.4 shows that $\mathrm{trace}_{\mathcal{R}/\varphi_L(\mathcal{R})}$ has exactly the same values. But $\psi_L = \frac{1}{\pi_L}\varphi_L^{-1} \circ \mathrm{tr}$. ∎

For uniformity of notation we put $\psi_L^{\mathbf{B}} := \frac{\pi_L}{q}\psi_L = \frac{1}{q}\varphi_L^{-1} \circ \mathrm{trace}_{\mathcal{R}/\varphi_L(\mathcal{R})}$.

4.3.2 The LT-isomorphism, part 2

We *assume* throughout this subsection that Ω is contained in K. Since the map $\kappa : \mathbf{B} \xrightarrow{\cong} \mathfrak{X}$ is an isomorphism of rigid varieties, it preserves the systems of affinoid subdomains on both sides. Hence, the LT-isomorphism (4.6) extends to a topological isomorphism

$$\kappa^* : \mathcal{R}_K(\mathfrak{X}) \xrightarrow{\cong} \mathcal{R}_K(\mathbf{B}).$$

In order to have a uniform notation, we usually write from now on

$$\mathcal{R}_K^I(\mathfrak{X}) := \mathcal{O}_K\left(\kappa(\mathbf{B}_I)\right)$$

for any closed interval $I \subseteq (0,1)$ so that we have the isomorphism of Banach algebras

$$\kappa^* : \mathcal{R}_K^I(\mathfrak{X}) \xrightarrow{\cong} \mathcal{R}_K^I(\mathbf{B}).$$

We warn the reader that only for specific closed intervals I there is another closed interval I' given by a complicated but explicit rule such that $\kappa(\mathbf{B}_I) = \mathfrak{X}_{I'}$. The precise statement can be worked out from [8, Prop. 1.20].

In the following, we list a few compatibilities under this extended LT-isomorphism.

First of all, under this isomorphism, the $\Gamma_L \cong o_L^\times$-action and the maps φ_L on both sides correspond to each other (cf. [8, §2.2]). Then it follows from Remarks 4.1.4 and 4.3.2 that the operators $\psi_L^{\mathfrak{X}}$ (defined at the end of Section 4.1.1) and $\psi_L^{\mathbf{B}}$ (defined in the previous section) also correspond under κ^*.

Secondly, as a consequence of [90, Thm. 3.7], we have the commutative diagram

$$\begin{array}{ccc} \Omega^1_{\mathcal{R}_K(\mathfrak{X})} & & \\ {\scriptstyle \kappa^*}\downarrow\cong & \searrow^{\mathrm{res}_{\mathfrak{X}}} & \\ & & K. \\ & \nearrow_{\mathrm{res}_{\mathbf{B}}} & \\ \Omega^1_{\mathcal{R}_K(\mathbf{B})} & & \end{array} \tag{4.36}$$

This combined with Remark 4.2.9 implies the explicit formula

$$\operatorname{res}_{\mathfrak{X}}(f d \log_{\mathfrak{X}}) = \Omega \operatorname{res}_{\mathbf{B}} \left(\kappa^*(f) g_{\mathrm{LT}} dZ\right). \tag{4.37}$$

4.3.3 φ_L-modules

Let $\mathfrak{Y}$ be either $\mathfrak{X}$ or $\mathbf{B}$ and $\mathcal{R} := \mathcal{R}_K(\mathfrak{Y})$. Henceforth, we will use the operator $\psi_L := \frac{q}{\pi_L}\psi_L^{\mathfrak{Y}}$ on $\mathcal{R}$. We also put

$$q_{\mathfrak{Y}} := \begin{cases} p & \text{if } \mathfrak{Y} = \mathfrak{X}, \\ q & \text{if } \mathfrak{Y} = \mathbf{B}. \end{cases}$$

Definition 4.3.3. A φ_L-module M over $\mathcal{R}$ is a finitely generated free $\mathcal{R}$-module M equipped with a semilinear endomorphism φ_M such that the $\mathcal{R}$-linear map

$$\varphi_M^{\mathrm{lin}} : \mathcal{R} \otimes_{\mathcal{R},\varphi_L} M \xrightarrow{\cong} M$$
$$f \otimes m \mapsto f\varphi_M(m)$$

is bijective.

Technically important is the following fact, which for $\mathfrak{X}$ is part of the proof of [8, Prop. 2.24]. The proof for $\mathbf{B}$ is entirely analogous. It allows us to extend the above maps and decompositions from the previous sections to φ_L-modules. For $r > 0$ we introduce the intervals

$$I(r, \mathfrak{Y}) := \begin{cases} (r, 1) & \text{if } \mathfrak{Y} = \mathfrak{X}, \\ [r, 1) & \text{if } \mathfrak{Y} = \mathbf{B}. \end{cases}$$

Proposition 4.3.4. *Let M be a φ_L-module M over $\mathcal{R}$. There exists a radius*

$$r_0 \geq \begin{cases} p^{-\frac{dp}{p-1}} & \text{if } \mathfrak{Y} = \mathfrak{X}, \\ p^{-\frac{dq}{(q-1)e}} & \text{if } \mathfrak{Y} = \mathbf{B} \end{cases}$$

and a finitely generated free $\mathcal{O}_K(\mathfrak{Y}_{I(r_0,\mathfrak{Y})})$-module M_0 equipped with a semilinear continuous homomorphism

$$\varphi_{M_0} : M_0 \to \mathcal{O}_K(\mathfrak{Y}_{I(r_0,\mathfrak{Y})^{1/q_{\mathfrak{Y}}}}) \otimes_{\mathcal{O}_K(\mathfrak{Y}_{I(r_0,\mathfrak{Y})})} M_0$$

such that the induced $\mathcal{O}_K(\mathfrak{Y}_{I(r_0,\mathfrak{Y})^{1/q_{\mathfrak{Y}}}})$-linear map

$$\varphi_{M_0}^{\mathrm{lin}} : \mathcal{O}_K(\mathfrak{Y}_{I(r_0,\mathfrak{Y})^{1/q_{\mathfrak{Y}}}}) \otimes_{\mathcal{O}_K(\mathfrak{Y}_{I(r_0,\mathfrak{Y})}),\varphi_L} M_0 \xrightarrow{\cong} \mathcal{O}_K(\mathfrak{Y}_{I(r_0,\mathfrak{Y})^{1/q_{\mathfrak{Y}}}}) \otimes_{\mathcal{O}_K(\mathfrak{Y}_{I(r_0,\mathfrak{Y})})} M_0$$

is an isomorphism and such that

$$\mathcal{R} \otimes_{\mathcal{O}_K(\mathfrak{Y}_{I(r_0,\mathfrak{Y})})} M_0 = M$$

with $\varphi_L \otimes \varphi_{M_0}$ and φ_M corresponding to each other.

The continuity condition for the φ_{M_0}, of course, refers to the product topology on $M_0 \cong (\mathcal{O}_K(\mathfrak{Y}_{I(r_0,\mathfrak{Y})}))^d$.

In the following, we fix a φ_L-module M over $\mathcal{R}$ and a pair (r_0, M_0) as in Proposition 4.3.4. For any $r_0 \leq r' < 1$ and any closed interval $I = [r,s] \subseteq I(r',\mathfrak{Y})$ we then have the finitely generated free modules

$$M^{I(r',\mathfrak{Y})} := \mathcal{O}_K(\mathfrak{Y}_{I(r',\mathfrak{Y})}) \otimes_{\mathcal{O}_K(\mathfrak{Y}_{I(r_0,\mathfrak{Y})})} M_0 \quad \text{over } \mathcal{R}^{[r,1)},$$
$$M^I := \mathcal{O}_K(\mathfrak{Y}_I) \otimes_{\mathcal{O}_K(\mathfrak{Y}_{I(r',\mathfrak{Y})})} M^{I(r',\mathfrak{Y})} \quad \text{over } \mathcal{O}_K(\mathfrak{Y}_I).$$

They satisfy

$$M^{I(r',\mathfrak{Y})} = \varprojlim_{s>r} M^I \quad \text{and} \quad M = \varinjlim_{r'} M^{I(r',\mathfrak{Y})}. \tag{4.38}$$

We equip M^I with the Banach norm $| \ |_{M^I}$ given by the maximum norm with respect to any fixed basis (the induced topology does not depend on the choice of basis) which is submultiplicative with respect to scalar multiplication and the norm $| \ |_I$ on $\mathcal{O}_K(\mathfrak{Y}_I)$.

Furthermore, base change with $\mathcal{O}_K(\mathfrak{Y}_{I^{1/q_\mathfrak{Y}}})$ over $\mathcal{O}_K(\mathfrak{Y}_{I(r_0,\mathfrak{Y})^{1/q_\mathfrak{Y}}})$ induces isomorphisms of Banach spaces

$$\varphi^I_{\mathrm{lin}} = \mathcal{O}_K(\mathfrak{Y}_{I^{1/q_\mathfrak{Y}}}) \otimes_{\mathcal{O}_K(\mathfrak{Y}_{I(r_0,\mathfrak{Y})^{1/q_\mathfrak{Y}}})} \varphi^{\mathrm{lin}}_{M_0} :$$
$$\mathfrak{O}_K(\mathfrak{Y}_{I^{1/q_\mathfrak{Y}}}) \otimes_{\mathfrak{O}_K(\mathfrak{Y}_I),\varphi_L} M^I \xrightarrow{\cong} M^{I^{1/q_\mathfrak{Y}}}$$

and hence injective, continuous maps

$$\varphi^I : M^I \to M^{I^{1/q_\mathfrak{Y}}}$$

by restriction.

Assuming that $I^{q_\mathfrak{Y}} \subseteq I(r',\mathfrak{Y})$, we define the additive, K-linear, continuous map $\psi^I : M^I \to M^{I^{q_\mathfrak{Y}}}$ as the composite

$$\psi^I : M^I \xrightarrow{(\varphi^{I^{q_\mathfrak{Y}}}_{\mathrm{lin}})^{-1}} \mathcal{O}_K(\mathfrak{Y}_I) \otimes_{\mathcal{O}_K(\mathfrak{Y}_{I^{q_\mathfrak{Y}}}),\varphi_L} M^{I^{q_\mathfrak{Y}}} \to M^{I^{q_\mathfrak{Y}}}, \tag{4.39}$$

where the last map sends $f \otimes m$ to $\psi^I(f)m$. By construction, it satisfies the projection formulae

$$\psi^I\big(\varphi^{I^{q_\mathfrak{Y}}}(f)m\big) = f\psi^I(m) \quad \text{and} \quad \psi^I\big(g\varphi^{I^{q_\mathfrak{Y}}}(m')\big) = \psi^I(g)m', \tag{4.40}$$

for any $f \in \mathcal{O}_K(\mathfrak{Y}_{I^{q_\mathfrak{Y}}})$, $g \in \mathcal{O}_K(\mathfrak{Y}_I)$, and $m \in M^I$, $m' \in M^{I^{q_\mathfrak{Y}}}$ as well as the formula

$$\psi^I \circ \varphi^{I^{q_\mathfrak{Y}}} = \frac{q_\mathfrak{Y}}{\pi_L} \cdot \mathrm{id}_{M^{I^{q_\mathfrak{Y}}}}.$$

Using Proposition 4.1.14 in case $\mathfrak{Y} = \mathfrak{X}$, resp., the decomposition (4.29) in case $\mathfrak{Y} = \mathbf{B}$ (under the assumption that Ω is contained in K), combined with (iterates of) φ_{lin}^I gives rise to decompositions

$$M^{I^{\frac{1}{q_{\mathfrak{Y}}^n}}} = \begin{cases} \bigoplus_{a \in (o_L/\pi_L^n)} \mathrm{ev}_a\, \varphi_L^n(M^I) & \text{if } \mathfrak{Y} = \mathfrak{X}, \\ \bigoplus_{a \in (o_L/\pi_L^n)} \eta(a, Z)\varphi_L^n(M^I) & \text{if } \mathfrak{Y} = \mathbf{B} \text{ and } \Omega \in K \end{cases} \tag{4.41}$$

of Banach spaces and

$$M = \begin{cases} \bigoplus_{a \in (o_L/\pi_L^n)} \mathrm{ev}_a\, \varphi_L^n(M) & \text{if } \mathfrak{Y} = \mathfrak{X}, \\ \bigoplus_{a \in (o_L/\pi_L^n)} \eta(a, Z)\varphi_L^n(M) & \text{if } \mathfrak{Y} = \mathbf{B} \text{ and } \Omega \in K \end{cases} \tag{4.42}$$

of LF-spaces, again given by the formula

$$m = \begin{cases} \left(\frac{\pi_L}{q}\right)^n \sum_a \varphi_M \psi_M(\mathrm{ev}_{-a}\, m)\, \mathrm{ev}_a & \text{if } \mathfrak{Y} = \mathfrak{X}, \\ \left(\frac{\pi_L}{q}\right)^n \sum_a \varphi_M \psi_M\big(\eta(-a, Z)m\big)\eta(a, Z) & \text{if } \mathfrak{Y} = \mathbf{B} \text{ and } \Omega \in K. \end{cases}$$

4.3.4 The Robba ring of a group

Recall that $L_n = L(\ker([\pi_L^n]))$. We set

$$\Gamma_n := \mathrm{Gal}(L_\infty/L_n) = \ker\big(\Gamma_L \xrightarrow{\chi_{\mathrm{LT}}} o_L^\times \to (o_L/\pi_L^n)^\times\big).$$

Also recall from Section 4.1.2 the notation $U_n := 1 + \pi_L^n o_L$ for $n \geq 1$ and the isomorphisms $\log : U_n \xrightarrow{\cong} \pi_L^n o_L$ and $\ell_n = \pi_L^{-n} \log : U_n \xrightarrow{\cong} o_L$ for $n \geq n_0$, where $n_0 \geq 1$ is minimal among n such that $\log : 1 + \pi_L^n o_L \to \pi_L^n o_L$ and $\exp : \pi_L^n o_L \to 1 + \pi_L^n o_L$ are mutually inverse isomorphisms.

Obviously, χ_{LT} restricts to isomorphisms $\Gamma_n \cong U_n$ for any $n \geq 1$. Consider the composed maps

$$\hat{\ell} := \log \circ \chi_{\mathrm{LT}} : \Gamma_L \to L \quad \text{and} \quad \hat{\ell}_n := \ell_n \circ \chi_{\mathrm{LT}} : \Gamma_n \xrightarrow{\cong} o_L \quad \text{for } n \geq n_0.$$

The latter isomorphisms induce isomorphisms of Fréchet algebras $D(\Gamma_n, K) \xrightarrow{\cong} D(o_L, K)$.

Because of the isomorphisms $\Gamma_L \cong o_L^\times$ and $\Gamma_n \cong U_n$, the formalism of character varieties and corresponding Robba rings applies to the groups Γ_L and Γ_n as well, giving us the corresponding character varieties $\mathfrak{X}_{\Gamma_L}$ and $\mathfrak{X}_{\Gamma_n}$, and the results of Section 4.1.2 transfer to this setting. To make a clear distinction, we put $\mathcal{R}_K(\Gamma_L) := \mathcal{R}_K(\mathfrak{X}_{\Gamma_L})$ and $\mathcal{R}_K(\Gamma_n) := \mathcal{R}_K(\mathfrak{X}_{\Gamma_n})$ and call them the Robba rings of the groups Γ_L and Γ_n. Clearly, the Lubin–Tate character χ_{LT} induces topological ring isomorphisms

$$\mathcal{R}_K(\Gamma_L) \xrightarrow{\cong} \mathcal{R}_K(\mathfrak{X}^\times) \quad \text{and} \quad \mathcal{R}_K(\Gamma_n) \xrightarrow{\cong} \mathcal{R}_K(\mathfrak{X}_n^\times) \text{ for } n \geq 1. \tag{4.43}$$

If Γ denotes any of these groups, then we will very often view, via the Fourier isomorphism, $K[\Gamma] \subseteq D(\Gamma, K)$ as subrings of $\mathcal{R}_K(\Gamma)$. In particular, we consider elements $\gamma \in \Gamma$ as elements of the Robba ring writing them in any of the forms $\gamma \mathrel{\hat{=}} \delta_\gamma \mathrel{\hat{=}} \mathrm{ev}_\gamma$.

Let $n \geq m \geq 1$. The inclusions $\iota_n : \Gamma_n \hookrightarrow \Gamma_L$ and $\iota_{n,m} : \Gamma_n \hookrightarrow \Gamma_m$ induce, by the transfer of the results in Section 4.1.2, ring monomorphisms $\iota_{n*} : \mathcal{R}_K(\Gamma_n) \hookrightarrow \mathcal{R}_K(\Gamma_L)$ and $\iota_{n,m*} : \mathcal{R}_K(\Gamma_n) \hookrightarrow \mathcal{R}_K(\Gamma_m)$. More precisely, we have (by Lemma 4.1.15 and Remark 4.1.18) topological ring isomorphisms

$$\mathbb{Z}[\Gamma_L] \otimes_{\mathbb{Z}[\Gamma_n]} \mathcal{R}_K(\Gamma_n) \xrightarrow{\cong} \mathcal{R}_K(\Gamma_L), \tag{4.44}$$

$$\mathbb{Z}[\Gamma_m] \otimes_{\mathbb{Z}[\Gamma_n]} \mathcal{R}_K(\Gamma_n) \xrightarrow{\cong} \mathcal{R}_K(\Gamma_m). \tag{4.45}$$

Here the left-hand sides are viewed as free $\mathcal{R}_K(\Gamma_n)$-modules endowed with the product topology.

We also note that, for $n \geq m \geq n_0$, the commutative diagram

$$\begin{array}{ccc} \Gamma_n & \xrightarrow{\hat{\ell}_n} & o_L \\ {\scriptstyle \iota_{n,m}}\downarrow & & \downarrow{\scriptstyle \pi_L^{n-m}} \\ \Gamma_m & \xrightarrow{\hat{\ell}_m} & o_L \end{array}$$

induces the commutative diagrams

$$\begin{array}{ccccc} D(\Gamma_n, K) & \xrightarrow{\hat{\ell}_{n*}} & D(o_L, K) & \xrightarrow[\text{Fourier}]{\cong} & \mathcal{O}_K(\mathfrak{X})_{\varphi_L^{n-m}} \\ {\scriptstyle \iota_{n,m*}}\big\downarrow & & {\scriptstyle (\pi_L^{n-m})_*}\big\downarrow & & \big\downarrow \\ D(\Gamma_m, K) & \xrightarrow{\hat{\ell}_{m*}} & D(o_L, K) & \xrightarrow[\text{Fourier}]{\cong} & \mathcal{O}_K(\mathfrak{X}) \end{array}$$

and

$$\begin{array}{ccc} \mathcal{R}_K(\Gamma_n) & \xrightarrow[\cong]{\hat{\ell}_{n*}} & \mathcal{R}_K(\mathfrak{X})^{\varphi_L^{n-m}} \\ {\scriptstyle \iota_{n,m*}}\big\downarrow & & \big\downarrow \\ \mathcal{R}_K(\Gamma_m) & \xrightarrow[\cong]{\hat{\ell}_{m*}} & \mathcal{R}_K(\mathfrak{X}). \end{array} \tag{4.46}$$

For the rest of this subsection we *assume* that Ω is contained in K. Let $n \geq n_0$. We then have the isomorphisms of rigid varieties

$$\mathbf{B} \xrightarrow[\kappa]{\cong} \mathfrak{X} \xrightarrow[\hat{\ell}_n^*]{\cong} \mathfrak{X}_{\Gamma_n}.$$

For any closed interval $I \subseteq (0, 1)$ we therefore have the affinoid subdomain $\hat{\ell}_n^* \circ \kappa(\mathbf{B}_I)$ in $\mathfrak{X}_{\Gamma_n}$ and we may introduce the Banach algebra $\mathcal{R}_K^I(\Gamma_n) := \mathcal{O}_K(\hat{\ell}_n^* \circ \kappa(\mathbf{B}_I))$.

By its very construction the diagram

$$\begin{array}{ccccc}
\mathcal{R}_K^{I^{q^{n-m}}}(\Gamma_n) & \xrightarrow[\cong]{\hat{\ell}_{n*}} & \mathcal{R}_K^{I^{q^{n-m}}}(\mathfrak{X}) & \xrightarrow[\cong]{\kappa^*} & \mathcal{R}_K^{I^{q^{n-m}}}(\mathbf{B}) \\
\iota_{n,m*}\big\downarrow & & \big\downarrow \varphi_L^{n-m} & & \big\downarrow \varphi_L^{n-m} \\
\mathcal{R}_K^{I}(\Gamma_m) & \xrightarrow[\cong]{\hat{\ell}_{m*}} & \mathcal{R}_K^{I}(\mathfrak{X}) & \xrightarrow[\cong]{\kappa^*} & \mathcal{R}_K^{I}(\mathbf{B}),
\end{array}$$

for $n \geq m \geq n_0$, is commutative. Together with (4.32) it implies the canonical isomorphism

$$\mathbb{Z}[\Gamma_m] \otimes_{\mathbb{Z}[\Gamma_n]} \mathcal{R}_K^{I^{q^{n-m}}}(\Gamma_n) \xrightarrow{\cong} \mathcal{R}_K^{I}(\Gamma_m). \tag{4.47}$$

We will denote the composite of Fourier and LT-isomorphism by

$$\mathfrak{L} : D(o_L, K) \xrightarrow{\cong} \mathcal{O}_K(\mathfrak{X}) \xrightarrow{\cong} \mathcal{O}_K(\mathbf{B}).$$

Recall that $\mathcal{O}_K(\mathbf{B})$ is a space of certain power series in the variable Z. We put

$$X := \mathfrak{L}^{-1}(Z) \in D(o_L, K) \quad \text{and} \quad Y_n := \hat{\ell}_{n*}^{-1}(X) \in D(\Gamma_n, K) \text{ for } n \geq n_0.$$

In this way, we can express elements in these distribution algebras as power series in these variables. This will later on be an important technical tool for our proofs.

As an immediate consequence of Remark 4.3.1, we find the following.

Remark 4.3.5. The subsequent inclusions are dense:

(i) $D(o_L, K)_{Z^{\mathbb{N}}} \subseteq \mathcal{R}_K(\mathfrak{X})$ and

(ii) $D(\Gamma_n, K)_{Y_n^{\mathbb{N}}} \subseteq \mathcal{R}_K(\Gamma_n)$ for $n \geq n_0$.

4.3.5 Locally $\mathbb{Q}_p$-analytic versus locally L-analytic distribution algebras

We fix a $\mathbb{Z}_p$-basis $h_1 = 1, \dots, h_d$ of o_L and set $b_i := h_i - 1$ and, for any multi-index $\mathbf{k} = (k_1, \dots, k_d) \in \mathbb{N}_0^d$, $\mathbf{b}^{\mathbf{k}} := \prod_{i=1}^d b_i^{k_i} \in \mathbb{Z}_p[o_L]$. We write $D_{\mathbb{Q}_p}(G, K)$ for the algebra of K-valued locally $\mathbb{Q}_p$-analytic distributions on a $\mathbb{Q}_p$-Lie group G. Any $\lambda \in D_{\mathbb{Q}_p}(o_L, K)$ has a unique convergent expansion $\lambda = \sum_{\mathbf{k} \in \mathbb{N}_0^d} \alpha_{\mathbf{k}} \mathbf{b}^{\mathbf{k}}$ with $\alpha_{\mathbf{k}} \in K$ such that, for any $0 < r < 1$, the set $\{\alpha_{\mathbf{k}} r^{\diamond|\mathbf{k}|}\}_{\mathbf{k} \in \mathbb{N}_0^d}$ is bounded, where $\diamond := 2$ if $p = 2$ and $\diamond := 1$ otherwise. The completion with respect to the norm

$$\|\lambda\|_{\mathbb{Q}_p, r} := \sup_{\mathbf{k} \in \mathbb{N}_0^d} |\alpha_{\mathbf{k}}| r^{\diamond|\mathbf{k}|}$$

for $0 < r < 1$ is denoted by

$$D_{\mathbb{Q}_p, r}(o_L, K) = \Big\{ \sum_{\mathbf{k} \in \mathbb{N}_0^d} \alpha_{\mathbf{k}} \mathbf{b}^{\mathbf{k}} | \alpha_{\mathbf{k}} \in K \text{ and } |\alpha_{\mathbf{k}}| r^{\diamond|\mathbf{k}|} \to 0 \text{ as } |\mathbf{k}| \to \infty \Big\}.$$

By [68, Prop. 2.1], the group o_L satisfies the hypothesis (HYP) of [76] with p-valuation ω satisfying $\omega(h_i) = \diamond$. Thus, by [76, Thm. 4.5], restricting to the subfamily $q^{-e} < r < 1$, $r \in p^{\mathbb{Q}}$, the norms $\| \|_{\mathbb{Q}_p, r}$ are multiplicative. If not otherwise specified, we denote by $V \otimes_K W$ the projective tensor product of locally convex K-vector spaces V, W.

Lemma 4.3.6. *Let*

$$0 \to V \to W \to X \to 0$$

be a strict exact sequence of locally convex topological K-vector spaces with W metrizable and X Hausdorff, and then

(i) *the sequence of the associated Hausdorff completed spaces*

$$0 \to \widehat{V} \to \widehat{W} \to \widehat{X} \to 0$$

is again strict exact,

(ii) *for a complete valued field extension F of K the associated sequence of completed base extension*

$$0 \to F \mathbin{\widehat{\otimes}}_K V \to F \mathbin{\widehat{\otimes}}_K W \to F \mathbin{\widehat{\otimes}}_K X \to 0$$

is again strict exact,

(iii) *if W is a K-Banach space, V a closed subspace with induced norm and $X = W/V$ endowed with the quotient norm, then in* (ii) *the quotient norm coincides with the tensor product norm on $F \mathbin{\widehat{\otimes}}_K X$.*

Proof. By [15, I.17 §2] with W also V, X, and all their completions are metrizable. Hence, the first statement follows from [17, IX.26 Prop. 5]. For the second statement we first obtain the exact sequence

$$0 \to F \otimes_K V \to F \otimes_K W \to F \otimes_K X \to 0$$

of metrizable locally convex spaces (cf. [61, Thm. 10.3.13]). The first non-trivial map is strict by [61, Thm. 10.3.8]. Regarding the strictness of the second map, one easily checks that $F \otimes_K W / F \otimes_K V$ endowed with the quotient topology satisfies the universal property of the projective tensor product $F \otimes_K X$. Now apply (i). The third item is contained in [36, §3, n^o 2, Thm. 1], see also [91, Thm. 4.28]. ∎

The kernel of the surjection of Fréchet spaces $D_{\mathbb{Q}_p}(o_L, K) \twoheadrightarrow D(o_L, K)$ is generated as a closed ideal by $\mathfrak{a} := \ker(L \otimes_{\mathbb{Q}_p} \mathrm{Lie}_{\mathbb{Q}_p}(o_L) \xrightarrow{a \otimes \mathfrak{x} \mapsto a\mathfrak{x}} \mathrm{Lie}_L(o_L))$. For $K = L$ this is [68, Lem. 5.1]. As seen in the proof of Lemma 4.1.2, we have

$$K \mathbin{\widehat{\otimes}}_L D_{\mathbb{Q}_p}(o_L, L) = D_{\mathbb{Q}_p}(o_L, K) \quad \text{and} \quad K \mathbin{\widehat{\otimes}}_L D(o_L, L) = D(o_L, K).$$

Hence, the assertion for general K follows from Lemma 4.3.6 (ii). We write $D_r(o_L, K)$ for the completion of $D(o_L, K)$ with respect to the quotient norm $\| \, \|_r$ of $\| \, \|_{\mathbb{Q}_p,r}$. By the proof of [76, Prop. 3.7] we have the exact sequence of K-Banach algebras

$$0 \to \hat{\mathfrak{a}}_r \to D_{\mathbb{Q}_p,r}(o_L, K) \to D_r(o_L, K) \to 0, \tag{4.48}$$

where $\hat{\mathfrak{a}}_r$ denotes the closed ideal generated by $\mathfrak{a}$. Moreover, the K-Banach algebras $D_r(o_L, K)$ realize a Fréchet–Stein structure on $D(o_L, K)$. For convenience we set $\mathfrak{r}_m := q^{-\frac{\diamond e}{p^m}}$ for $m \geq 0$. We, of course, have

$$D(o_L, K) = \varprojlim_m D_{\mathfrak{r}_m}(o_L, K).$$

Moreover, according to [69, Cor. 5.13], one has

$$D_{\mathfrak{r}_m}(o_L, K) = \mathbb{Z}[o_L] \otimes_{\mathbb{Z}[p^m o_L]} D_{\mathfrak{r}_0}(p^m o_L, K).$$

We have corresponding results and will be using analogous notation for groups isomorphic to o_L. This applies, in particular, to Γ_n for any $n \geq n_0$. Note that $\Gamma_n^{p^m} = \Gamma_{n+me}$.

4.3.6 (φ, Γ)-modules

We recall the definition of as well as a few known facts about (φ_L, Γ_L)-modules (cf. [8]). Let $\mathfrak{Y}$ be either $\mathfrak{X}$ or $\mathbf{B}$ and $\mathcal{R} := \mathcal{R}_K(\mathfrak{Y})$. Any (φ_L, Γ_L)-module M over $\mathcal{R}$ is, by definition, in particular an $\mathcal{R}$-module with a semilinear action of the group Γ_L. Our aim in this section is to show that these two structures on M give rise to a module structure on M under the "group" Robba ring $\mathcal{R}_K(\Gamma_L)$.

Definition 4.3.7. A (φ_L, Γ_L)-module M over $\mathcal{R}$ is a φ_L-module M (see Definition 4.3.3) equipped with a semilinear continuous action of Γ_L which commutes with the endomorphism φ_M. We shall write $\mathcal{M}(\mathcal{R})$ for the category of (φ_L, Γ_L)-modules over $\mathcal{R}$.

The continuity condition for the Γ_L-action on M, of course, refers to the product topology on $M \cong \mathcal{R}^d$.

According to [8, Prop. 2.25], the Γ_L-action on a (φ_L, Γ_L)-module M is differentiable so that the derived action of the Lie algebra $\mathrm{Lie}(o_L^\times)$ on M is available.

Definition 4.3.8. The (φ_L, Γ_L)-module M over $\mathcal{R}$ is called L-analytic if the derived action $\mathrm{Lie}(\Gamma_L) \times M \to M$ is L-bilinear, i.e., if the induced action $\mathrm{Lie}(\Gamma_L) \to \mathrm{End}(M)$ of the Lie algebra $\mathrm{Lie}(\Gamma_L)$ of Γ_L is L-linear (and not just $\mathbb{Q}_p$-linear). We shall write $\mathcal{M}^{\mathrm{an}}(\mathcal{R})$ for the category of L-analytic (φ_L, Γ_L)-modules over $\mathcal{R}$.

In [8], a (φ_L, Γ_L)-module M over $\mathscr{R}$ is only required to be projective instead of free as in our definition. Since throughout this paper we are exclusively interested in L-analytic modules, that makes no difference as by [8, Thm. 3.17] any L-analytic (φ_L, Γ_L)-module M is actually a free $\mathscr{R}$-module.

We have the following variant of Proposition 4.3.4 (cf. [8, Prop. 2.24]).

Proposition 4.3.9. *Let M be a (φ_L, Γ_L)-module over $\mathscr{R}$. Then there exists a model (M_0, r_0) as in Proposition 4.3.4 equipped with a semilinear continuous action of Γ_L such that*

$$\mathscr{R} \otimes_{\mathscr{R}^{[r_0,1)}} M_0 = M$$

respects the Γ_L-actions (acting diagonally on the left-hand side).

From now on in this subsection *we fix a (φ, Γ)-module M over $\mathscr{R}$ and a pair (r_0, M_0) as in Proposition 4.3.9*. We then have available the objects introduced after Proposition 4.3.4. But now the finitely generated free modules $M^{I(r',\mathfrak{Y})}$ and M^I are each in addition equipped with a semilinear continuous Γ_L-action, compatible with the identities (4.38). Moreover, the Γ_L-actions commute with the ψ^I-operators (4.39), and the decompositions (4.42) and (4.41) are Γ_L-equivariant.

Assume henceforth in this subsection that M is an *L-analytic* (φ_L, Γ_L)-module over $\mathscr{R}$.

Proposition 4.3.10. *The Γ_L-action on M extends uniquely to a separately continuous action of the locally L-analytic distribution algebra $D(\Gamma_L, K)$ of Γ_L with coefficients in K. If $M \xrightarrow{f} N$ is a homomorphism of L-analytic (φ_L, Γ_L)-modules, then f is $D(\Gamma_L, K)$-equivariant with regard to this action.*

Proof. First of all, we observe that the Dirac distributions generate a dense L-subspace in $D(\Gamma_L, L)$ by [75, Lem. 3.1]. Since $\Gamma_L \cong o_L^\times$, we have seen in the proof of Lemma 4.1.2 that $D(\Gamma_L, K) = K \hat{\otimes}_L D(\Gamma_L, L)$. Hence, the Dirac distributions also generate a dense K-linear subspace of $D(\Gamma_L, K)$. Therefore, the extended action is unique provided it exists.

Our assertion is easily reduced to the analogous statement concerning the Banach spaces M^I for a closed interval $I = [r, s]$. From [8, Props. 2.16 and 2.17] we know that the Γ_L-action on M^I is locally $\mathbb{Q}_p$-analytic. But since we assume M to be L-analytic, it is actually locally L-analytic (cf. the addendum to [8, Prop. 2.25] and the argument at the end of the proof of [8, Prop. 2.17]).

For our purpose we show more generally the existence, for any K-Banach space W, of a continuous K-linear map

$$I : \mathscr{C}^{an}(\Gamma_L, W) \to \mathscr{L}_b\big(D(\Gamma_L, K), W\big)$$

satisfying $I(f)(\delta_g) = f(g)$. Note that this map, if it exists, is unique by our initial observation. Recall (cf. [72, §12]) that the locally convex vector space $\mathscr{C}^{an}(\Gamma_L, W)$

is the locally convex inductive limit of finite products of Banach spaces of the form $B \hat{\otimes}_K W$ with a Banach space B, and that its strong dual $D(\Gamma_L, K)$ is the corresponding projective limit of the finite sums of dual Banach spaces B'. We therefore may construct the map I as the inductive limit of finite products of maps of the form

$$\begin{aligned} B \hat{\otimes}_K W &\to \mathcal{L}_b(B', W) \\ x \otimes y &\mapsto \big[\ell \mapsto \ell(x)y\big]. \end{aligned}$$

Since B as a Banach space is barreled, this map is easily seen to be continuous (cf. the argument in the proof of [71, Lem. 9.9]).

Now suppose that W carries a locally L-analytic Γ_L-action (e.g., $W = M^I$). For $y \in W$ let $\rho_y(g) := gy$ denote the orbit map in $\mathcal{C}^{\mathrm{an}}(\Gamma_L, W)$. We then define

$$\begin{aligned} D(\Gamma_L, K) \times W &\to W \\ (\mu, y) &\mapsto I(\rho_y)(\mu). \end{aligned}$$

Due to our initial observation, the proof of [75, Prop. 3.2], that the above is a separately continuous module structure, remains valid even though K is not assumed to be spherically complete.

By [8, Rem. 2.20], the homomorphism f is continuous and hence the $D(\Gamma_L, K)$-equivariance of f follows from the Γ_L-invariance by the argument in the first paragraph of this proof. ■

Recall that M^I, for each $I = [r, s]$ with $r \geq r_0$, bears a natural Γ_L-action. Now, for each $n \geq 1$, we will define a different action of Γ_n on $M^{[r,s]}$, which is motivated by Lemma 4.3.11 below and which is crucial for analyzing the structure of $M^{\psi_M = 0}$ in the next subsection. To this end, consider for each $\gamma \in \Gamma_n$ the operator $H_n(\gamma)$ on $M^{[r,s]}$ defined by

$$H_n(\gamma)(m) := \begin{cases} \mathrm{ev}_{\pi_L^{-n}(\chi_{\mathrm{LT}}(\gamma)-1)}\, \gamma m & \text{if } \mathfrak{Y} = \mathfrak{X}, \\ \eta\big(\pi_L^{-n}(\chi_{\mathrm{LT}}(\gamma) - 1), Z\big)\gamma m & \text{if } \mathfrak{Y} = \mathbf{B} \text{ and } \Omega \in K. \end{cases}$$

Note that, since Γ_n acts on $\mathcal{O}_K(\mathfrak{Y})$ via χ_{LT} and the $o_L^\times$-action, we may form the skew group ring $\mathcal{O}_K(\mathfrak{Y})[\Gamma_n]$, which due to the semilinear action of Γ_L on M maps into the K-Banach algebra $\mathcal{E}nd_K(M^I)$ of continuous K-linear endomorphisms of M^I, endowed with the operator norm $\| \, \|_{M^I}$. Hence, we obtain the ring homomorphism

$$\begin{aligned} H_n : K[\Gamma_n] &\to \mathcal{O}_K(\mathfrak{Y})[\Gamma_n] \to \mathcal{E}nd_K(M^I) \\ \gamma &\mapsto \begin{cases} \mathrm{ev}_{\pi_L^{-n}(\chi_{\mathrm{LT}}(\gamma)-1)}\, \gamma & \text{if } \mathfrak{Y} = \mathfrak{X}, \\ \eta\big(\pi_L^{-n}\big(\chi_{\mathrm{LT}}(\gamma) - 1\big), Z\big)\gamma & \text{if } \mathfrak{Y} = \mathbf{B} \text{ and } \Omega \in K. \end{cases} \end{aligned}$$

The next lemma holds true in both cases. We spell it out only in the **B**-case since we technically need it only there.

Lemma 4.3.11. *Suppose that Ω is contained in K, and let $n \geq m \geq 1$.*

(i) *We have for all $\sigma \in \Gamma_n$*

$$\sigma\big(\eta(1,Z)\varphi_L^n(y)\big) = \eta(1,Z)\varphi_L^n\big(H_n(\sigma)(y)\big),$$

i.e., the isomorphisms

$$M \xrightarrow{\cong} \eta(1,Z)\varphi_L^n(M),$$
$$M^{[r,s]} \xrightarrow{\cong} \eta(1,Z)\varphi_L^n(M^{[r,s]})$$
$$y \mapsto \eta(1,Z)\varphi_L^n(y)$$

are Γ_n-equivariant with respect to the natural action on the right-hand side and the action via H_n on the left-hand side.

(ii) *The map*

$$\mathbb{Z}[\Gamma_m] \otimes_{\mathbb{Z}[\Gamma_n],H_n} M^{[r,s]} \to M^{[r^{1/q^{n-m}}, s^{1/q^{n-m}}]}$$
$$\gamma \otimes y \mapsto \eta\Big(\frac{\chi_{\mathrm{LT}}(\gamma)-1}{\pi_L^m}, Z\Big)\varphi_M^{n-m}(\gamma y)$$

is a homeomorphism of Banach spaces, where the left-hand side is viewed as the direct sum of Banach spaces $\bigoplus_{\gamma\in\Gamma_m/\Gamma_n} \gamma \otimes M^{[r,s]}$. Moreover, the map is Γ_m-equivariant with respect to the H_m-action on the right-hand side.

(iii) *If $H_n : K[\Gamma_n] \to \mathcal{E}nd_K(M^I)$ extends to a continuous ring homomorphism $R_K^I(\Gamma_n) \to \mathcal{E}nd_K(M^I)$, then $H_m : K[\Gamma_m] \to \mathcal{E}nd_K(M^{I^{1/q^{n-m}}})$ similarly extends to a continuous homomorphism*

$$R_K^{I^{1/q^{n-m}}}(\Gamma_m) \to \mathcal{E}nd_K(M^{I^{1/q^{n-m}}}).$$

If the first extension is unique, so is the second one.

Proof. (i) Setting $b := \frac{\chi_{\mathrm{LT}}(\sigma)-1}{\pi_L^n}$, we calculate

$$\begin{aligned}\sigma\big(\eta(1,Z)\varphi_L^n(m)\big) &= \sigma(\eta(1,Z))\varphi_L^n(\sigma m)\\ &= \eta(1+\pi_L^n b, Z)\varphi_L^n(\sigma m)\\ &= \eta(1,Z)\eta(\pi^n b, Z)\varphi_L^n(\sigma m)\\ &= \eta(1,Z)\varphi_L^n\big(\eta(b,Z)\sigma m\big),\end{aligned}$$

where we used the multiplicativity of η in the first variable in the third and (4.33) in the last equality.

(ii) By a straightforward computation one first checks that the map is well defined. The bijectivity follows from (4.41) using the isomorphism $1+\pi_L^m o_L/1+\pi_L^n o_L \xrightarrow{\cong} o_L/\pi_L^{n-m}o_L$, $\gamma \mapsto \frac{\chi_{\mathrm{LT}}(\gamma)-1}{\pi_L^m}$ and that $M^{[r,s]} = \gamma M^{[r,s]}$.

(iii) Base change induces the $R_K^{I^{1/q^{n-m}}}(\Gamma_m)$-action on

$$R_K^{I^{1/q^{n-m}}}(\Gamma_m) \otimes_{R_K^I(\Gamma_n),H_n} M^I \cong \mathbb{Z}[\Gamma_m] \otimes_{\mathbb{Z}[\Gamma_n]} R_K^I(\Gamma_n) \otimes_{R_K^I(\Gamma_n),H_n} M^I$$
$$\cong \mathbb{Z}[\Gamma_m] \otimes_{\mathbb{Z}[\Gamma_n],H_n} \otimes M^I$$
$$\cong M^{I^{1/q^{n-m}}}, \tag{4.49}$$

where we used (4.47) and (ii). The continuity is easily checked by considering "matrix entries" which are built by composites of the original continuous map by other continuous transformations. Here we use that the identifications (4.47) and (4.49) are homeomorphisms when we endow the left-hand side with the maximum norm. Finally, the claim regarding uniqueness follows from (4.49) as the action of Γ_m is already determined by the original H_m. ∎

For the rest of this subsection we *assume that* Ω *is contained in* K and we will work exclusively in the **B**-case, i.e., $\mathcal{R} = \mathcal{R}_K(\mathbf{B})$ and $\mathcal{R}^I = \mathcal{R}_K^I(\mathbf{B})$. The consequences for the $\mathfrak{X}$-case will be given in Section 4.3.8.

There is a natural ring homomorphism $\mathcal{R}^I \to \mathcal{E}nd_K(M^I)$ by assigning to $f \in \mathcal{R}^I$ the operator which multiplies by f, and which we denote by the same symbol f. Part (iii) of the following remark means that this ring homomorphism has operator norm 1.

Remark 4.3.12. We have the following:

(i) $\sup_{x\in o_L} |\eta(x,Z)-1|_I < 1$ and $|\eta(x,Z)|_I = 1$ for all $x \in o_L$,[7]

(ii) $|\eta(px,Z)-1|_I \le \max\{|\eta(x,Z)-1|_I^p, \frac{1}{q^e}|\eta(x,Z)-1|_I\}$, where the right-hand side is equal to $|\eta(x,Z)-1|_I^p$ if $|\eta(x,Z)-1|_I \ge q^{-\frac{e}{p-1}}$, and

(iii) $|f|_I = \|f\|_{M^I}$ for all $f \in \mathcal{R}^I$.

Proof. It is known (cf. [74]) that $\eta(x,Z) = \eta(1,[x](Z))$ belongs to $1 + Zo_{\mathbb{C}_p}[[Z]]$, whence we have, for any $x \in o_L$, that $|\eta(x,Z)-1|_I < 1$ from the definition of $|\;|_I$, and (i) follows from the fact that the map $o_L \to \mathbb{R}$, $x \mapsto |\eta(x,Z)-1|_I$ is continuous with compact source. Affirmation (ii) is a consequence of the expansion

$$\eta(px,Z)-1 = \big(\eta(x,Z)-1+1\big)^p - 1$$
$$= \big(\eta(x,Z)-1\big)^p + \sum_{k=1}^{p-1}\binom{p}{k}\big(\eta(x,Z)-1\big)^k$$

and $\left|\binom{p}{k}\right| = q^{-e}$ for $k = 1,\ldots,p-1$. (iii) follows from the submultiplicativity of $|\;|_I$ plus the fact that $1 \in \mathcal{R}^I$, which implies the statement on $M \cong (\mathcal{R}^I)^m$. ∎

[7] $|\eta(x,Z)-1|_I = |\eta(1,Z)-1|_I < 1$ for all $x \in o_L^\times$ because any $x \in o_L^\times$ induces an isomorphism $[x](-)$ of B_I.

Remark 4.3.12 allows us to fix a natural number $m_0 = m_0(r_0)$ such that for all $m \geq m_0$ we have that

$$\begin{aligned} &\left|\eta(x,Z)-1\right|_I < \mathfrak{r}_m \text{ for all } x \in o_L \text{ and } \left|\eta(x,Z)-1\right|_I \leq \mathfrak{r}_0 \text{ for all } x \in p^m o_L, \\ &r_0^{1/q} < \mathfrak{r}_m, \end{aligned} \tag{4.50}$$

for any of the intervals $I = [r_0, r_0]$, $[r_0, r_0^{1/q}]$ and $[r_0^{1/q}, r_0^{1/q}]$. In the following, let I always denote one of those intervals.

Lemma 4.3.13. *Let $\varepsilon > 0$ be arbitrary. Then there exists $n_1 \gg 0$ such that, for any $n \geq n_1$, the operator norm $\| \ \|_{M^I}$ on M^I satisfies*

$$\|\gamma - 1\|_{M^I} \leq \varepsilon \quad \textit{for all } \gamma \in \Gamma_n. \tag{4.51}$$

Proof. We first prove the statement for the module $M = \mathcal{R}$. For the convenience of the reader we adopt the proof of [41, Lem. 5.2]. First note that for any fixed $f \in R^I$ by the continuity of the action of Γ_L there exists an open normal subgroup H of Γ_L such that

$$\left|(\gamma - 1)f\right|_I < \varepsilon |f|_I \tag{4.52}$$

holds for all $\gamma \in H$. So we may assume that the latter inequality holds for Z and Z^{-1} simultaneously. Using the twisted Leibniz rule

$$(\gamma - 1)(gf) = (\gamma - 1)(g)f + \gamma(g)(\gamma - 1)(f)$$

and induction, we get (4.52) for all powers $Z^{\mathbb{Z}}$. Since the latter form an orthogonal basis, the claim follows using that $|\gamma(g)|_I = |g|_I$ for any $\gamma \in H, g \in \mathcal{R}^I$. If $M \cong \bigoplus_{i=1}^d \mathcal{R}e_i$ and $m = \sum f_i e_i$, we may assume that

$$\left|(\gamma - 1)e_i\right|_{M^I} < \varepsilon |e_i|_{M^I} \tag{4.53}$$

holds for $1 \leq i \leq d$, and apply the same Leibniz rule to $f_i e_i$ instead of gf, whence the result follows, noting that $|e_i|_{M^I} = 1$ by the definition of the maximum norm and that $|\gamma(e_i)|_{M^I} = |e_i|_{M^I} = 1$ for any $\gamma \in H$ and $1 \leq i \leq d$ as a consequence of (4.53). ∎

We fix $n_1 = n_1(r_0) \geq n_0$ such that the Lemma holds for $\varepsilon = \mathfrak{r}_0$. Then, for any $n \geq n_1$, $m \geq m_0$, the above H_n extends to continuous ring homomorphisms

$$\begin{aligned} \widetilde{H}_n : D_{\mathbb{Q}_p,\mathfrak{r}_m}(\Gamma_n, K) &\to \mathcal{E}nd_K(M^I) \\ \sum_{\mathbf{k} \in \mathbb{N}_0^d} \alpha_{\mathbf{k}} \hat{\ell}_{n,*}^{-1}(\mathbf{b})^{\mathbf{k}} &\mapsto \sum_{k \geq 0} \alpha_{\mathbf{k}} \prod_{i=1}^d H_n\big(\hat{\ell}_{n,*}^{-1}(b_i)\big)^{k_i} \end{aligned}$$

and

$$\widetilde{\mathbb{H}}_n := \widetilde{H}_n \circ \hat{\ell}_{n,*}^{-1} : D_{\mathbb{Q}_p,\mathfrak{r}_m}(o_L, K) \xrightarrow{\hat{\ell}_{n,*}^{-1}} D_{\mathbb{Q}_p,\mathfrak{r}_m}(\Gamma_n, K) \to \mathcal{E}nd_K(M^I)$$

$$\sum_{\mathbf{k}\in\mathbb{N}_0^d} \alpha_{\mathbf{k}} \mathbf{b}^{\mathbf{k}} \mapsto \sum_{k\geq 0} \alpha_{\mathbf{k}} \prod_{i=1}^{d} H_n\big(\hat{\ell}_{n,*}^{-1}(b_i)\big)^{k_i}.$$

Indeed, we have

$$H_n\big(\hat{\ell}_{n,*}^{-1}(b_i)\big) = \eta\Big(\frac{\hat{\ell}_n^{-1}(h_i)-1}{\pi_L^n}, Z\Big) - 1 + \eta\Big(\frac{\hat{\ell}_n^{-1}(h_i)-1}{\pi_L^n}, Z\Big)\big(\hat{\ell}_n^{-1}(h_i)-1\big)$$

and since

$$\begin{aligned}
&\Big\|\eta\Big(\frac{\hat{\ell}_n^{-1}(h_i)-1}{\pi_L^n}, Z\Big) - 1 + \eta\Big(\frac{\hat{\ell}_n^{-1}(h_i)-1}{\pi_L^n}, Z\Big)\big(\hat{\ell}_n^{-1}(h_i)-1\big)\Big\|_{M^I} \\
&\leq \max\Big\{\Big\|\eta\Big(\frac{\hat{\ell}_n^{-1}(h_i)-1}{\pi_L^n}, Z\Big) - 1\Big\|_{M^I}, \Big\|\eta\Big(\frac{\hat{\ell}_n^{-1}(h_i)-1}{\pi_L^n}, Z\Big)\big(\hat{\ell}_n^{-1}(h_i)-1\big)\Big\|_{M^I}\Big\} \\
&\leq \mathfrak{r}_m
\end{aligned}$$

by (4.51), (4.50) and Remark 4.3.12 (i), the above defining sum converges with respect to the operator norm. Moreover, we have

$$\big\|\widetilde{\mathbb{H}}_n(\lambda)\big\|_{M^I} \leq \sup_{\mathbf{k}} |\alpha_{\mathbf{k}}| \mathfrak{r}_m^{|\mathbf{k}|} = \|\lambda\|_{\mathbb{Q}_p,\mathfrak{r}_m} \tag{4.54}$$

for all $\lambda \in D_{\mathbb{Q}_p,\mathfrak{r}_m}(o_L, K)$; i.e., the operator norm of $\widetilde{\mathbb{H}}_n$ is less than or equal to 1.

Since M is assumed to be L-analytic, $\widetilde{H}_n$ factorizes over the desired ring homomorphism

$$H_n : \big(D(\Gamma_n, K) \subseteq\big) D_{\mathfrak{r}_m}(\Gamma_n, K) \to \mathcal{E}nd_K(M^I)$$

and $\widetilde{\mathbb{H}}_n$ over

$$\mathbb{H}_n : \big(D(o_L, K) \subseteq\big) D_{\mathfrak{r}_m}(o_L, K) \to \mathcal{E}nd_K(M^I)$$

by (4.48). As $D_{\mathfrak{r}_m}(o_L, K)$ carries the quotient norm of $D_{\mathbb{Q}_p,\mathfrak{r}_m}(o_L, K)$, we obtain from (4.54)

$$\big\|\mathbb{H}_n(\lambda)\big\|_{M^I} \leq \inf_{\widetilde{\lambda}, \mathrm{pr}(\widetilde{\lambda})=\lambda} \|\lambda\|_{\mathbb{Q}_p,\mathfrak{r}_m} = \|\lambda\|_{\mathfrak{r}_m} \tag{4.55}$$

for all $\lambda \in D_{\mathfrak{r}_m}(o_L, K)$; i.e., the operator norm of $\mathbb{H}_n$ is again less than or equal to 1. By a similar, but simpler, reasoning one shows the following lemma.

Lemma 4.3.14. *The isomorphism $\mathfrak{L} : D(o_L, K) \cong \mathcal{O}_K(\mathbf{B}), \delta_a \mapsto \eta(a, Z)$ (composed of LT and Fourier) induces, for all $m \geq m_0$, a commutative diagram of continuous maps*

$$\begin{array}{ccc} D_{\mathbb{Q}_p,\mathfrak{r}_m}(o_L, K) & & \\ \mathrm{pr}\downarrow & \searrow & \\ D_{\mathfrak{r}_m}(o_L, K) & \xrightarrow{\mathfrak{L}} & \mathcal{R}^I \end{array}$$

with operator norms less than or equal to 1*. Moreover, the operator norm of the* scalar *action via* $\mathfrak{L}$

$$\mathrm{scal} : \big(D(o_L, K) \subseteq \big) D_{\mathfrak{r}_m}(o_L, K) \xrightarrow{\mathfrak{L}} \mathcal{R}^I \to \mathcal{E}nd_K(M^I) \tag{4.56}$$

is also bounded by 1*, see Remark 4.3.12* (iii).

Remark 4.3.15. The maps $\widetilde{H}_n$ and $\widetilde{\mathbb{H}}_n$, as well as H_n and $\mathbb{H}_n$, are uniquely determined by their restriction to $K[\Gamma_n]$ and $K[o_L]$, respectively, because these group algebras are dense in the sources $D_{\mathbb{Q}_p,\mathfrak{r}_m}(\Gamma_n, K)$, $D_{\mathfrak{r}_m}(\Gamma_n, K)$ and $D_{\mathbb{Q}_p,\mathfrak{r}_m}(o_L, K)$, $D_{\mathfrak{r}_m}(o_L, K)$, respectively.

Applying our convention before Remark 4.3.12, we usually abbreviate scal(μ) by $\mathfrak{L}(\mu)$ for $\mu \in D(o_L, K)$ below when we refer to this scalar action on M^I. For the proof of Theorem 4.3.20 below it will be crucial to compare the two actions scal and $\mathbb{H}_n$ of $D(o_L, K)$ on M^I.

Finally, for $n \geq n_1$, we obtain similar maps for the original (multiplicative) action of $\Gamma_n \subseteq \Gamma$ on M^I:

$$D_{\mathbb{Q}_p,\mathfrak{r}_m}(\Gamma_n, K) \to \mathcal{E}nd_K(M^I)$$
$$\sum_{\mathbf{k}\in\mathbb{N}_0^d} \alpha_{\mathbf{k}} \hat{\ell}_{n,*}^{-1}(\mathbf{b})^{\mathbf{k}} \mapsto \sum_{k\geq 0} \alpha_{\mathbf{k}} \prod_{i=1}^{d} \hat{\ell}_{n,*}^{-1}(b_i)^{k_i}$$

with operator norm bounded by 1.

A special case of the following lemma was pointed out to us by Rustam Steingart.

Lemma 4.3.16. *Let $m \in \mathbb{N}$ be arbitrary. Setting $u_n(a) := \frac{\exp(\pi_L^n a)-1}{\pi_L^n a}$ for $a \in o_L \setminus \{0\}$ and $u_n(0) = 1$, there exist $n_2 = n_2(m)$ such that $u_n(a) \in 1 + \pi_L^m o_L$ for all $a \in o_L$ and $n \geq n_2$.*

Proof. This is easily checked using $v_p(n!) \leq \frac{n}{p-1}$. ∎

In order to distinguish Dirac distributions for elements γ in the *multiplicative* group Γ_n from those for elements a in the *additive* group o_L, we often shall write $\delta_\gamma^\times$ in contrast to δ_a.

Lemma 4.3.17. *Let $0 < \varepsilon < 1$ be arbitrary and $\Delta = \sum_k c_k(\delta_{a_k} - 1) \in D(o_L, K)$ a finite sum with $a_k \in o_L$. Then there exists $n_3 = n_3(\varepsilon, \Delta, r_0)$ such that for all $n \geq n_3$ it holds that*

$$\left\|\mathfrak{L}(\Delta) - \mathbb{H}_n(\Delta)\right\|_{M^I} < \varepsilon.$$

Proof. Put $\xi := \sup_k |c_k|$ and choose $\varepsilon' \leq \varepsilon$ such that $\varepsilon'\xi < \varepsilon$. Then choose $m_1 \geq m_0$ such that

$$\|\delta_\gamma^\times - 1\|_{M^I} < \varepsilon' \quad \text{and} \quad \|\delta_\gamma^\times - 1\|_{\mathcal{R}^I} < \varepsilon'$$

for all $\gamma \in \Gamma_{m_1}$ (see Lemma 4.3.13). Now according to Lemma 4.3.16, we choose $n_3 := n_2(m_1) \geq m_1$. Observing that for $a \in o_L$

$$\mathbb{H}_n(\delta_a) = \mathfrak{L}(\delta_{u_n(a)a}) \circ \delta^\times_{\hat{\ell}_n^{-1}(a)},$$

we estimate, for $n \geq n_3$,

$$\begin{aligned}
&\left\|\mathfrak{L}(\Delta) - \mathbb{H}_n(\Delta)\right\|_{M^I} \\
&= \Big\| \sum_k c_k \big\{\mathfrak{L}(\delta_{a_k}) - 1 - \big(\mathfrak{L}(\delta_{u_n(a_k)a_k}) \circ \delta^\times_{\hat{\ell}_n^{-1}(a_k)} - 1\big)\big\} \Big\|_{M^I} \\
&= \Big\| \sum_k c_k \big\{\mathfrak{L}(\delta_{a_k}) - 1 - \big(\mathfrak{L}(\delta_{u_n(a_k)a_k}) \circ \big(\delta^\times_{\hat{\ell}_n^{-1}(a_k)} - 1\big) + \mathfrak{L}(\delta_{u_n(a_k)a_k}) - 1\big)\big\} \Big\|_{M^I} \\
&= \Big\| \sum_k c_k \big\{\mathfrak{L}(\delta_{a_k}) - 1 - \big(\mathfrak{L}(\delta_{u_n(a_k)a_k}) \circ \big(\delta^\times_{\hat{\ell}_n^{-1}(a_k)} - 1\big) + \delta^\times_{u_n(a_k)}\big(\mathfrak{L}(\delta_{a_k}) - 1\big)\big)\big\} \Big\|_{M^I} \\
&= \Big\| \sum_k c_k \big\{(1 - \delta^\times_{u_n(a_k)})\big(\mathfrak{L}(\delta_{a_k}) - 1\big) - \big(\mathfrak{L}(\delta_{u_n(a_k)a_k}) \circ \big(\delta^\times_{\hat{\ell}_n^{-1}(a_k)} - 1\big)\big)\big\} \Big\|_{M^I} \\
&\leq \sup_k \Big(|c_k| \max\big\{\big\|(1 - \delta^\times_{u_n(a_k)})\big(\mathfrak{L}(\delta_{a_k}) - 1\big)\big\|_{M^I}, \big\|\mathfrak{L}(\delta_{u_n(a_k)a_k}) \circ \big(\delta^\times_{\hat{\ell}_n^{-1}(a_k)} - 1\big)\big\|_{M^I}\big\}\Big) \\
&\leq \varepsilon' \sup_k |c_k| = \varepsilon'\xi < \varepsilon,
\end{aligned}$$

where we used for the last line the estimate

$$\begin{aligned}
\big\|(1 - \delta^\times_{u_n(a_k)})\big(\mathfrak{L}(\delta_{a_k}) - 1\big)\big\|_{M^I} &= \big|(1 - \delta^\times_{u_n(a_k)})\big(\mathfrak{L}(\delta_{a_k}) - 1\big)\big|_I \\
&\leq \big\|(1 - \delta^\times_{u_n(a_k)})\big\|_{\mathcal{R}^I} \big|\mathfrak{L}(\delta_{a_k}) - 1\big|_I \\
&\leq \varepsilon' \big|\eta(a_k, Z) - 1\big|_I \leq \varepsilon'
\end{aligned}$$

(by Remark 4.3.12 (i), (iii) and due to the choice of m_1 and n_3) for the first term as well as the estimate

$$\begin{aligned}
\big\|\mathfrak{L}(\delta_{u_n(a_k)a_k}) \circ \big(\delta^\times_{\hat{\ell}_n^{-1}(a_k)} - 1\big)\big\|_{M^I} &\leq \big|\eta\big(u_n(a_k)a_k, Z\big)\big|_I \big\|\delta^\times_{\hat{\ell}_n^{-1}(a_k)} - 1\big\|_{M^I} \\
&= \big\|\delta^\times_{\hat{\ell}_n^{-1}(a_k)} - 1\big\|_{M^I} < \varepsilon'
\end{aligned}$$

for the second term (again by Remark 4.3.12 (i), (iii) and due to the choice of m_1 and $n_3 \geq m_1$). ∎

Lemma 4.3.18. *Let $0 < \varepsilon < 1$ be arbitrary and $\mu \in D(o_L, K)$ any element. Then there exists $\Delta = \sum_k c_k(\delta_{a_k} - 1) \in D(o_L, K)$ a finite sum with $a_k \in o_L$ such that $\|\mu - \Delta\|_{\mathfrak{r}_{m_0}} < \varepsilon$. Moreover, for $n_3 = n_3(\varepsilon, \Delta, r_0)$ from the previous lemma and all $n \geq n_3$, we have*

$$\left\|\mathfrak{L}(\mu) - \mathbb{H}_n(\mu)\right\|_{M^I} < \varepsilon.$$

In particular, if $\mathfrak{L}(\mu)$ is invertible as an operator in $\mathcal{E}nd_K(M^I)$ or equivalently invertible as an element of $\mathcal{R}^I$,[8] *then firstly there exists $n_4 = n_4(\mu, r_0)$ such that*

$$\|\mathfrak{L}(\mu) - \mathbb{H}_n(\mu)\|_{M^I} < |\mathfrak{L}(\mu)^{-1}|_I^{-1} \quad \text{and} \quad \|\mathbb{H}_n(\mu)^{-1} - \mathfrak{L}(\mu)^{-1}\|_{M^I} < |\mathfrak{L}(\mu)^{-1}|_I$$

for any $n \geq n_4$ and secondly the operator $\mathbb{H}_n(\mu)$ is invertible, too.

Proof. The existence of such Δ is clear because such elements form a dense subset of $D(o_L, K)$ in the Fréchet topology (as noted at the beginning of the proof of Proposition 4.3.10). Consider the estimation for $n \geq n_3$

$$\begin{aligned}&\left\|\mathfrak{L}(\mu) - \mathbb{H}_n(\mu)\right\|_{M^I} \\ &\qquad \leq \max\left(\left\|\mathfrak{L}(\mu - \Delta)\right\|_{M^I}, \left\|\mathfrak{L}(\Delta) - \mathbb{H}_n(\Delta)\right\|_{M^I}, \left\|\mathbb{H}_n(\mu - \Delta)\right\|_{M^I}\right) < \varepsilon,\end{aligned}$$

where we use the estimate

$$\left\|\mathfrak{L}(\mu - \Delta)\right\|_{M^I} = \left|\mathfrak{L}(\mu - \Delta)\right|_I \leq \|\mu - \Delta\|_{\mathfrak{r}_{m_0}} < \varepsilon$$

by (4.56) for the first, Lemma 4.3.17 for the second, and (4.55) for the last term.

Now suppose that $\mathfrak{L}(\mu)$ as an operator on M^I is invertible. We choose suitable $\varepsilon \leq \|\mathfrak{L}(\mu)^{-1}\|_{M^I}^{-1}$, Δ accordingly and put $n_4 = n_3(\varepsilon, \Delta, r_0)$. Then, for $n \geq n_4$, we have

$$\left\|1 - \mathfrak{L}(\mu)^{-1}\mathbb{H}_n(\mu)\right\|_{M^I} = \left\|\mathfrak{L}(\mu)^{-1}\big(\mathfrak{L}(\mu) - \mathbb{H}_n(\mu)\big)\right\|_{M^I} < 1,$$

whence $\sum_{k\geq 0}(1 - \mathfrak{L}(\mu)^{-1}\mathbb{H}_n(\mu))^k$ converges in $\mathcal{E}nd_K(M^I)$ and

$$\mathbb{H}_n(\mu)^{-1} := \Big(\sum_{k\geq 0}\big(1 - \mathfrak{L}(\mu)^{-1}\mathbb{H}_n(\mu)\big)^k\Big)\mathfrak{L}(\mu)^{-1}$$

is the inverse of $\mathbb{H}_n(\mu) = \mu(1 - (1 - \mathfrak{L}(\mu)^{-1}\mathbb{H}_n(\mu)))$.

Furthermore,

$$\begin{aligned}\left\|\mathbb{H}_n(\mu)^{-1} - \mathfrak{L}(\mu)^{-1}\right\|_{M^I} &= \left\|\Big(\sum_{k\geq 1}\big(1 - \mathfrak{L}(\mu)^{-1}\mathbb{H}_n(\mu)\big)^k\Big)\mathfrak{L}(\mu)^{-1}\right\|_{M^I} \\ &\leq \sup_{k\geq 1}\left\|1 - \mathfrak{L}(\mu)^{-1}\mathbb{H}_n(\mu)\right\|_{M^I}^k\left|\mathfrak{L}(\mu)^{-1}\right|_I < \left|\mathfrak{L}(\mu)^{-1}\right|_I. \ \blacksquare\end{aligned}$$

[8] M^I is a free $\mathcal{R}^I$-module on which $\mathfrak{L}(\mu)$ acts via the diagonal matrix with all diagonal entries equal to $\mathfrak{L}(\mu)$.

Note that the above lemma applies to the variable X and from now on we set $n_4 := n_4(X, r_0)$. In view of Lemma 4.3.11 (iii), the following lemma is crucial for the main result Theorem 4.3.20 of this section.

Lemma 4.3.19. *For $n \geq n_4$,*

(i) *the map $\Theta_n : D(\Gamma_n, K) \xrightarrow{H_n} \mathcal{E}nd_K(M^I)$ extends uniquely to a continuous ring homomorphism*

$$\mathcal{R}_K^I(\Gamma_n) \to \mathcal{E}nd_K(M^I).$$

If $M \xrightarrow{f} N$ is a homomorphism of L-analytic (φ_L, Γ_L)-modules, then $f^I : M^I \to N^I$ is $\mathcal{R}_K^I(\Gamma_n)$-equivariant with regard to this action.

(ii) *M^I is a free $\mathcal{R}_K^I(\Gamma_n)$-module of rank $\operatorname{rk}_{\mathcal{R}} M$. Any basis as $\mathcal{R}^I$-module also is a basis as $\mathcal{R}_K^I(\Gamma_n)$-module.*

(iii) *The natural maps*

$$\mathcal{R}_K^{[r_0,r_0]}(\Gamma_n) \otimes_{\mathcal{R}_K^{[r_0,r_0^{\frac{1}{q}}]}(\Gamma_n)} M^{[r_0,r_0^{\frac{1}{q}}]} \xrightarrow{\cong} M^{[r_0,r_0]},$$

$$\mathcal{R}_K^{[r_0^{\frac{1}{q}},r_0^{\frac{1}{q}}]}(\Gamma_n) \otimes_{\mathcal{R}_K^{[r_0,r_0^{\frac{1}{q}}]}(\Gamma_n)} M^{[r_0,r_0^{\frac{1}{q}}]} \xrightarrow{\cong} M^{[r_0^{\frac{1}{q}},r_0^{\frac{1}{q}}]}$$

are isomorphisms.

Proof. (i) Inductively, for $n \geq n_4$, we obtain from Lemma 4.3.18 – by expressing $(\mathbb{H}_n(\mu)^{\pm})^k - (\mathfrak{L}(\mu)^{\pm})^k$ as $\sum_{l=1}^{k} \binom{k}{l} (\mathbb{H}_n(\mu)^{\pm} - \mathfrak{L}(\mu)^{\pm})^l (\mathfrak{L}(\mu)^{\pm})^{k-l}$ – that

$$\left\| \mathbb{H}_n(\mu)^k - \mathfrak{L}(\mu)^k \right\|_{M^I} \leq \begin{cases} \left|\mathfrak{L}(\mu)\right|_I^k & \text{for } k \geq 0, \\ \left|\mathfrak{L}(\mu)^{-1}\right|_I^{-k} \leq \left|\mathfrak{L}(\mu)\right|_I^k \leq \left|\mathfrak{L}(\mu)^k\right|_I & \text{for } k < 0 \end{cases} \tag{4.57}$$

for all $k \in \mathbb{Z}$. Evaluating for $\mu = X$, it follows that if $\sum_{k \in \mathbb{Z}} a_k Z^k \in \mathcal{R}^I$ with $a_i \in K$, then $\sum_{k \in \mathbb{Z}} a_k \mathbb{H}_n(X)^k$ converges in $\mathcal{E}nd_K(M^I)$ because

$$\begin{aligned} \left\| a_k \mathbb{H}_n(X)^k \right\|_{M^I} &\leq \max \left\{ \left\| a_k \left(\mathbb{H}_n(X)^k - \mathfrak{L}(\mu)^k \right) \right\|_{M^I}, \left\| a_k \mathfrak{L}(\mu)^k \right\|_{M^I} \right\} \\ &\leq \begin{cases} |a_k||Z|_I^k & \text{for } k \geq 0, \\ |a_k||Z^{-1}|_I^{-k} & \text{for } k < 0 \end{cases} \end{aligned}$$

goes to zero for k going to $\pm\infty$. In other words, we have extended the continuous ring homomorphism Θ_n to a continuous ring homomorphism

$$\mathcal{R}^I \to \mathcal{E}nd_K(M^I), \quad Z \mapsto \mathbb{H}_n(X).$$

As by definition $\kappa^* \circ \hat{\ell}_{n,*}$ extends to a continuous ring isomorphism $\mathcal{R}_K^I(\Gamma_n) \xrightarrow{\cong} \mathcal{R}_K^I(\mathbf{B}) = \mathcal{R}^I$, we have constructed a continuous ring homomorphism

$$\mathcal{R}_K^I(\Gamma_n) \to \mathcal{E}nd_K(M^I)$$

as claimed.

The uniqueness is a consequence of the fact that $\mathcal{R}_K^I(\Gamma_n)$ is the completion of the localization $D(\Gamma_n, K)_{Y_n^{\mathbb{N}}}$ by a certain norm, for which the extended map is continuous.

Concerning functoriality, observe that the maps f and f^I are automatically continuous by [8, Rem. 2.20] (with respect to the canonical topologies). Without loss of generality, we may assume that the estimates of Lemma 4.3.18 hold for M and N simultaneously. By the invariance under the distribution algebra and $\mathcal{R}$-linearity of f, the map f^I is compatible with respect to the operators $\mathbb{H}_n(X)^{\pm}$ of M^I and N^I. By continuity this extends to arbitrary elements of $\mathcal{R}_K^I(\Gamma_n)$.

(ii) follows in the same way as in [42]: Recall that (e_k) denotes an $\mathcal{R}^I$-basis of M^I and consider the maps

$$\Phi : \bigoplus_{k=1}^{m} \mathcal{R}_K^I(\mathbf{B}) \cong M^I, \quad (f_k) \mapsto \sum_{k=1}^{m} f_k e_k,$$
$$\Phi' : \bigoplus_{k=1}^{m} \mathcal{R}_K^I(\Gamma_n) \to M^I, \quad (f_k) \mapsto \sum_{k=1}^{m} f_k(e_k),$$

and

$$\Upsilon : \bigoplus_{k=1}^{m} \mathcal{R}_K^I(\mathbf{B}) \xrightarrow{\cong} \bigoplus_{k=1}^{m} \mathcal{R}_K^I(\Gamma_n),$$

which in each component is given by $(\kappa^* \circ \hat{\ell}_{n,*})^{-1}$. Then we have from (4.57) that

$$\left|\Phi' \circ \Upsilon \circ \Phi^{-1}(m) - m\right|_I < |m|_I,$$

i.e.,

$$\|\Phi' \circ \Upsilon \circ \Phi^{-1} - \mathrm{id}\|_I < 1,$$

whence with Φ and Υ also Φ' is an isomorphism because $\Phi' \circ \Upsilon \circ \Phi^{-1}$ is invertible by the usual argument using the geometric series.

(iii) The base change property follows from the fact that Φ' is compatible with changing the interval. ∎

Theorem 4.3.20. *Suppose that Ω is contained in K.*

(i) *Let J be any of the intervals*

$$[r_0, r_0]^{1/q^n} \quad \text{or} \quad [r_0, r_0^{1/q}]^{1/q^n} \quad \text{for } n \geq 0.$$

Then the Γ_{n_4}-action on M^J via H_{n_4} can be uniquely extended to a continuous $\mathcal{R}_K^J(\Gamma_{n_4})$-module structure. Moreover, M^J is a finitely generated free $\mathcal{R}_K^J(\Gamma_{n_4})$-module; any $\mathcal{R}^{[r_0,1)}$-basis of M_0 is also an $\mathcal{R}_K^J(\Gamma_{n_4})$-basis of M^J. If $M \xrightarrow{f} N$ is a homomorphism of L-analytic (φ_L, Γ_L)-modules, then $f^J : M^J \to N^J$ is $\mathcal{R}_K^J(\Gamma_{n_4})$-equivariant with regard to this action.

(ii) *The Γ_1-action on M via H_1 extends uniquely to a separately continuous $\mathcal{R}_K(\Gamma_1)$-module structure. Moreover, M is a finitely generated free $\mathcal{R}_K(\Gamma_1)$-module; any $\mathcal{R}^{[r_0,1)}$-basis of M_0 is also an $\mathcal{R}_K(\Gamma_1)$-basis of M. If $M \xrightarrow{f} N$ is a homomorphism of L-analytic (φ_L, Γ_L)-modules, then f is $\mathcal{R}_K(\Gamma_1)$-equivariant with regard to this action.*

Proof. (i) From Lemma 4.3.19 we obtain, for any $n \geq n_4$, the H_n-action of $\mathcal{R}_K(\Gamma_n)$ on M^I for the original three intervals I. Using Lemma 4.3.11 (iii), we deduce the H_{n_4}-action of $\mathcal{R}_K^{I^{1/q^{n-n_4}}}(\Gamma_{n_4})$-action on $M^{I^{1/q^{n-n_4}}}$. The asserted properties of these actions follow from the same lemmata.

(ii) By the uniqueness part in (i) we may glue the $\mathcal{R}_K^J(\Gamma_{n_4})$-actions on the M^J to a continuous $\mathcal{R}_K^{[r_0,1)}(\Gamma_{n_4})$-action on $M^{[r_0,1)}$. By Remark 4.3.5 (ii) it is uniquely determined by the Γ_{n_4}-action. Therefore, we may vary r_0 now and obtain in the inductive limit a separately continuous H_{n_4}-action of $\mathcal{R}_K(\Gamma_{n_4})$ on M. Using (4.45) and Lemma 4.3.11 (ii), we deduce the separately continuous H_1-action of $\mathcal{R}_K(\Gamma_1)$ on M. Again by Remark 4.3.5 this action is uniquely determined by the Γ_1-action. The remaining assertions follow from the corresponding ones in (i). ■

4.3.7 The structure of $M^{\psi_M=0}$

We still *assume that Ω is contained in K and let M be an L-analytic (φ_L, Γ)-module over $\mathcal{R} = \mathcal{R}_K(\mathbf{B})$*. We want to show that $M^{\psi_L=0}$ carries a natural $\mathcal{R}_K(\Gamma_L)$-action extending the action of $D(\Gamma_L, K)$.

From (4.42) and using formula (4.35) and (4.34) we have

$$M^{\psi_L=0} = \bigoplus_{a \in (o_L/\pi_L)^\times} \eta(a, Z)\varphi_M(M) = \mathbb{Z}[\Gamma_L] \otimes_{\mathbb{Z}[\Gamma_1]} \big(\eta(1, Z)\varphi_M(M)\big). \quad (4.58)$$

Theorem 4.3.21. *The Γ_L action on M extends to a unique separately continuous $\mathcal{R}_K(\Gamma_L)$-action on $M^{\psi_L=0}$ (with respect to the LF-topology on $\mathcal{R}_K(\Gamma_L)$ and the subspace topology on $M^{\psi_L=0}$); moreover, the latter is a free $\mathcal{R}_K(\Gamma_L)$-module of rank $\mathrm{rk}_{\mathcal{R}} M$. If $e_1, \ldots, e_r$ is a basis of M over $\mathcal{R}$, an $\mathcal{R}_K(\Gamma_L)$-basis of $M^{\psi_L=0}$ is given by $\eta(1, Z)\varphi_M(e_1), \ldots, \eta(1, Z)\varphi_M(e_r)$. If $M \xrightarrow{f} N$ is a homomorphism of L-analytic (φ_L, Γ_L)-modules, then $M \xrightarrow{f^{\psi_L=0}} N$ is $\mathcal{R}_K(\Gamma_L)$-equivariant with regard to this action.*

Proof. By Lemma 4.3.11 (i) we can transfer the $\mathcal{R}_K(\Gamma_1)$-action on M from Theorem 4.3.20 (ii) to the space $\eta(1, Z)\varphi_M(M)$. Note that the resulting action is separately continuous for the subspace topology of $\eta(1, Z)\varphi_M(M)$ because the map $\varphi_L : M \to M$ is a homeomorphism onto its image. The latter is a consequence of the existence of the continuous operator ψ_L and the relation $\psi_L \circ \varphi_L = \frac{q}{\pi_L}\mathrm{id}_M$. Finally, because of (4.44) and (4.58), the $\mathcal{R}_K(\Gamma_1)$-action extends to the asserted $\mathcal{R}_K(\Gamma_L)$-action. Similarly as before, since Γ_L spans a dense subspace of $D(\Gamma_L, K)$, the uniqueness of the action follows from Remark 4.3.5. ■

4.3.8 Descent

For the proof of Theorem 4.3.21 we had to work over a field K containing the period Ω since only then we were able to write elements in $\mathcal{R}_K(\Gamma_L)$ or rather $\mathcal{R}_K(\Gamma_n)$ as certain Laurent series in one variable Y_n by means of the Lubin–Tate isomorphism $\mathcal{R}_K(\mathfrak{X}) \cong \mathcal{R}_K(\mathbf{B})$, which in general does not exist over L. In this section, we are going to explore to which extent the structure theorem over K descends to L. We shall consider two situations; i.e., we now start with an L-analytic (φ_L, Γ)-module M over $\mathcal{R}_L(\mathfrak{X})$ or $\mathcal{R}_L(\mathbf{B})$, respectively. Thus, in what follows let $\mathfrak{Y}$ be either $\mathfrak{X}$ or $\mathbf{B}$ and $\mathcal{R} = \mathcal{R}_L(\mathfrak{Y})$. Then we consider the functor

$$\mathcal{M}^{\mathrm{an}}\big(\mathcal{R}_L(\mathfrak{Y})\big) \to \mathcal{M}^{\mathrm{an}}\big(\mathcal{R}_K(\mathfrak{Y})\big)$$
$$M \mapsto M_K := \mathcal{R}_K(\mathfrak{Y}) \otimes_{\mathcal{R}_L(\mathfrak{Y})} M \cong K \,\widehat{\otimes}_{\iota,L}\, M,$$

where the last isomorphism and the well-definedness of the functor are established in [8, Lem. 2.23]. Moreover, there is a natural action of G_L on both $\mathcal{R}_K(\mathfrak{Y}) \cong K \,\widehat{\otimes}_{\iota,L}\, \mathcal{R}_L(\mathfrak{Y})$ and M_K via the first tensor factor (and the identity on the second). We have

$$\mathcal{R}_K(\mathfrak{Y})^{G_L} \cong \mathcal{R}_L(\mathfrak{Y}) \tag{4.59}$$

by [8, Prop. 2.7 (iii)], whence also

$$(M_K)^{G_L} = M$$

because M is finitely generated free over $\mathcal{R}_L(\mathfrak{Y})$ (by definition or [8, Thm. 3.17]) and hence M_K has a G_L-invariant basis over $\mathcal{R}_K(\mathfrak{Y})$.

Since the φ_L-operator on $\mathcal{R}_K(\mathfrak{Y})$ is induced from that on $\mathcal{R}_L(\mathfrak{Y})$, it commutes with the action of G_L. Similarly, one checks that this action commutes with the operator ψ_L of $\mathcal{R}_K(\mathfrak{Y})$. Indeed, by Proposition 4.1.14 there exists a G_L-invariant basis of $\mathcal{R}_K(\mathfrak{Y})$ over $\varphi_L(\mathcal{R}_K(\mathfrak{Y}))$, whence the trace commutes with the G_L-action. From this and the construction of the operator ψ_M, one derives easily that also ψ_M commutes with the G_L-action. As a consequence, we obtain natural isomorphisms

$$M^{\psi_M=0} \cong \big((M_K)^{G_L}\big)^{\psi_M=0} \cong \big((M_K)^{\psi_M=0}\big)^{G_L}. \tag{4.60}$$

Since the Lubin–Tate isomorphism $\mathcal{R}_K(\mathfrak{X}) \cong \mathcal{R}_K(\mathbf{B})$ respects the (φ_L, Γ_L)-module structure, Theorem 4.3.21 applies for both choices of $\mathfrak{Y}$; i.e., we obtain a separately continuous action

$$\mathcal{R}_K(\Gamma_L) \times (M_K)^{\psi=0} \to (M_K)^{\psi=0}. \tag{4.61}$$

Moreover, if $M = \bigoplus_{i=1}^r \mathcal{R}_L(\mathfrak{Y})e_i$, then the families $(\eta(1, Z)\varphi_M(e_i))$ and $(\mathrm{ev}_1\varphi_M(e_i))$ form bases of $(M_K)^{\psi=0}$ as $\mathcal{R}_K(\Gamma_L)$-modules in case $\mathbf{B}$ and $\mathfrak{X}$, respectively. Therefore, we consider next a natural G_L-action on $\mathcal{R}_K(\Gamma_L)$ and show that (4.61) is G_L-equivariant. For the first aim we use the canonical isomorphisms (4.44) and (4.46)

$$\mathcal{R}_K(\Gamma_L) \xleftarrow{\cong} \mathbb{Z}[\Gamma_L] \otimes_{\mathbb{Z}[\Gamma_n]} \mathcal{R}_K(\Gamma_n) \xrightarrow[\hat{\ell}_{n*}]{\cong} \mathbb{Z}[\Gamma_L] \otimes_{\mathbb{Z}[\Gamma_n]} \mathcal{R}_K(\mathfrak{X})$$

to extend the G_L-action from $\mathcal{R}_K(\mathfrak{X})$ to $\mathcal{R}_K(\Gamma_L)$; clearly, we obtain from (4.59) and the fact that the isomorphism

$$\mathcal{R}_K(\Gamma_n) \xrightarrow[\cong]{\ell_{n*}} \mathcal{R}_K(\mathfrak{X})$$

is defined over L that

$$\mathcal{R}_K(\Gamma_L)^{G_L} \cong \mathcal{R}_L(\Gamma_L). \tag{4.62}$$

For the second aim we prove the following lemma.

Lemma 4.3.22. *The action (4.61) is G_L-equivariant.*

Proof. We fix $\sigma \in G_L$ and define a second separately continuous action

$$\mathcal{R}_K(\Gamma_L) \times (M_K)^{\psi=0} \to (M_K)^{\psi=0}$$

by sending (λ, x) to $\sigma^{-1}(\sigma(\lambda)(\sigma(x)))$ (using that σ and ψ_L commute and that σ is a homeomorphism). By the uniqueness statement of Theorem 4.3.21, it suffices to show that the new and original actions coincide on $\Gamma \times (M_K)^{\psi=0}$. We shall show that these actions coincide even as actions $\Gamma_L \times M_K \to M_K$: For $\gamma \in \Gamma$, $f \in \mathcal{R}_K(\mathfrak{Y})$, and $m \in M$ we calculate

$$\begin{aligned}
\sigma^{-1}\big(\sigma(\gamma)\big(\sigma(f \otimes m)\big)\big) &= \sigma^{-1}\big(\gamma\big(\sigma(f) \otimes m\big)\big) \\
&= \sigma^{-1}\big(\gamma\big(\sigma(f)\big) \otimes \gamma(m)\big) \\
&= \sigma^{-1}\big(\sigma\big(\gamma(f)\big)\big) \otimes \gamma(m) \\
&= \gamma(f) \otimes \gamma(m) \\
&= \gamma(f \otimes m).
\end{aligned}$$

Here we used firstly that σ acts trivially on γ (or rather ev_γ) as they are already defined over L (via the Fourier transformation) and secondly that the G_L- and Γ_L-actions commute. Since this equality holds for all $\sigma \in G_L$, the claim follows. ∎

Taking G_L-invariants of (4.61) therefore induces – upon using (4.60) and (4.62) – the following separately continuous action:

$$\mathcal{R}_L(\Gamma_L) \times M^{\psi=0} \to M^{\psi=0},$$

which extends the Γ_L-action.

Theorem 4.3.23. *We have the following:*

(i) *The Γ_L-action on M (in $\mathfrak{M}^{an}(\mathcal{R}_L(\mathfrak{X}))$ or $\mathfrak{M}^{an}(\mathcal{R}_L(\mathbf{B}))$) extends to a separately continuous $\mathcal{R}_L(\Gamma_L)$-action on $M^{\psi_L=0}$ (with respect to the LF-topology on $\mathcal{R}_L(\Gamma_L)$ and the subspace topology on $M^{\psi_L=0}$). If $M \xrightarrow{f} N$ is a homomorphism of L-analytic (φ_L, Γ_L)-modules, then $M \xrightarrow{f^{\psi_L=0}} N$ is $\mathcal{R}_L(\Gamma_L)$-equivariant with regard to this action.*

(ii) *If $\mathfrak{Y} = \mathfrak{X}$, then $M^{\psi_L=0}$ is a free $\mathcal{R}_L(\Gamma_L)$-module of rank $\mathrm{rk}_{\mathcal{R}} M$. More precisely, if $e_1, \ldots, e_r$ is a basis of M over $\mathcal{R}_L(\mathfrak{X})$, then an $\mathcal{R}_L(\Gamma_L)$-basis of $M^{\psi_L=0}$ is given by $\mathrm{ev}_1\varphi_M(e_1), \ldots, \mathrm{ev}_1\varphi_M(e_r)$.*

Proof. It is easy to check that also the $\mathcal{R}_L(\Gamma_L)$-equivariance of $f^{\psi_L=0}$ follows by descent. Therefore, only (ii) remains to be shown. But this is an immediate consequence of the fact noted above, that the family $(\mathrm{ev}_1\varphi_M(e_i))$ forms a G_L-invariant basis of $(M_K)^{\psi=0}$ as $\mathcal{R}_K(\Gamma_L)$-module, by just taking G_L-invariants again. ■

Remark 4.3.24. As for possible generalizations, we note the following:

(i) For each complete intermediate field $L \subseteq K' \subseteq \mathbb{C}_p$ we obtain an analogous structure theorem for $(M_{K'})^{\psi=0}$ over $\mathcal{R}_{K'}(\Gamma_L)$ by replacing L by K' everywhere in the above reasoning.

(ii) Since for $\mathfrak{Y} = \mathbf{B}$ the basis $(\eta(1, Z)\varphi_M(e_i))$ of $(M_K)^{\psi=0}$ as $\mathcal{R}_K(\Gamma_L)$-module is visibly not G_L-invariant, the analogue of Theorem 4.3.23 (ii) cannot hold true in this case.

4.3.9 The Mellin transform and twists

Extending the Mellin transform from Lemma 4.1.6, we introduce the map

$$\mathcal{M} : \mathfrak{R}_K(\Gamma_L) \xrightarrow{\cong} \mathcal{R}_K(\mathfrak{X})^{\psi_L=0}, \quad \lambda \mapsto \lambda(\mathrm{ev}_1),$$

which is an isomorphism by Theorem 4.3.23. If $\Omega \in K$, then its composite with the LT-isomorphism is the isomorphism

$$\mathcal{M}_{\mathrm{LT}} : \mathcal{R}_K(\Gamma_L) \xrightarrow{\cong} \mathcal{R}_K(\mathbf{B})^{\psi_L=0}, \quad \lambda \mapsto \lambda\big(\eta(1, Z)\big).$$

Recall the twist operators Tw_χ from Section 4.1.3.

Lemma 4.3.25. *The diagram*

$$\begin{array}{ccc} \mathcal{R}_K(\Gamma_L) & \xrightarrow{\mathfrak{M}} & \mathcal{R}_K(\mathfrak{X})^{\psi_L=0} \\ {\scriptstyle \mathrm{Tw}_{\chi_{LT}}}\downarrow & & \cong \downarrow {\scriptstyle \partial^{\mathfrak{X}}_{\mathrm{inv}}} \\ \mathcal{R}_K(\Gamma_L) & \xrightarrow{\mathfrak{M}} & \mathcal{R}_K(\mathfrak{X})^{\psi_L=0} \end{array} \tag{4.63}$$

is commutative; in particular, the right-hand vertical map is an isomorphism.

Proof. The commutativity can be checked after base change. Assuming $\Omega \in K$, the diagram corresponds by Remark 4.2.9 to the diagram

$$\begin{array}{ccc} \mathcal{R}_K(\Gamma_L) & \xrightarrow{\mathfrak{M}_{\mathrm{LT}}} & \mathcal{R}_K(\mathbf{B})^{\psi_L=0} \\ {\scriptstyle \mathrm{Tw}_{\chi_{LT}}}\downarrow & & \cong \downarrow {\scriptstyle \frac{1}{\Omega}\partial_{\mathrm{inv}}} \\ \mathcal{R}_K(\Gamma_L) & \xrightarrow{\mathfrak{M}_{\mathrm{LT}}} & \mathcal{R}_K(\mathbf{B})^{\psi_L=0}. \end{array} \tag{4.64}$$

Now, the corresponding result for $\mathcal{R}_K(\Gamma_L)$ replaced by $D(\Gamma_L, K)$ is implicitly given in Sections 4.1.3 and 4.1.4. Section 1.2.4 of [25] establishes, for $\gamma \in \Gamma_L$, the relation $\partial_{\mathrm{inv}} \circ \gamma = \chi_{\mathrm{LT}}(\gamma)\gamma \circ \partial_{\mathrm{inv}}$ as operators on $\mathcal{R}_K(\mathbf{B})$. It follows by K-linearity and continuity that the relation of operators $\partial_{\mathrm{inv}} \circ \lambda = \mathrm{Tw}_{\chi_{\mathrm{LT}}}(\lambda) \circ \partial_{\mathrm{inv}}$ holds for all $\lambda \in D(\Gamma_L, K)$. By continuity of the action of $\mathcal{R}_K(\Gamma_L) = \mathbb{Z}[\Gamma_L] \otimes_{\mathbb{Z}[\Gamma_n]} \mathcal{R}_K(\Gamma_n)$ on $\mathcal{R}_K(\mathbf{B})^{\psi_L=0}$ it suffices to check the compatibility for the element Y_n^{-1}, where $Y_n \in D(\Gamma_n, K)$, for $n \gg 0$, has been defined at the end of Section 4.3.4. Using that $\mathrm{Tw}_{\chi_{\mathrm{LT}}}$ is multiplicative and that $\partial_{\mathrm{inv}}(\eta(1, Z)) = \Omega\eta(1, Z)$, the claim follows from the relation

$$\begin{aligned} \mathrm{Tw}_{\chi_{\mathrm{LT}}}(Y_n^{-1})\eta(1, Z) &= \mathrm{Tw}_{\chi_{\mathrm{LT}}}(Y_n)^{-1}\tfrac{1}{\Omega}\partial_{\mathrm{inv}}\big(Y_n Y_n^{-1}\eta(1, Z)\big) \\ &= \mathrm{Tw}_{\chi_{\mathrm{LT}}}(Y_n)^{-1}\mathrm{Tw}_{\chi_{\mathrm{LT}}}(Y_n)\tfrac{1}{\Omega}\partial_{\mathrm{inv}}\big(Y_n^{-1}\eta(1, Z)\big) \\ &= \tfrac{1}{\Omega}\partial_{\mathrm{inv}}\big(Y_n^{-1}\eta(1, Z)\big). \end{aligned}$$

∎

Lemma 4.3.26. *Assume $\Omega \in K$ and let n_1 be as in Lemma 4.3.13. Then, for $n \geq n_1$, the map $\mathfrak{M}_{\mathrm{LT}}$ induces isomorphisms*

$$\mathcal{R}_K(\Gamma_n) \cong \varphi_L^n\big(\mathcal{R}_K(\mathbf{B})\big)\eta(1, Z)\big(\subseteq \mathcal{R}_K(\mathbf{B})^{\psi_L=0}\big)$$

of $\mathcal{R}_K(\Gamma_n)$-modules and

$$D(\Gamma_n, K) \cong \varphi_L^n\big(\mathcal{O}_K(\mathbf{B})\big)\eta(1, Z)\big(\subseteq \mathcal{O}_K(\mathbf{B})^{\psi_L=0}\big)$$

of $D(\Gamma_n, K)$-modules.

Proof. By taking limits the first isomorphism follows from Lemma 4.3.19 (ii) in combination with Lemma 4.3.11 (i), both applied to $M^I = \mathscr{R}_K(\mathbf{B}_I)$. The isomorphism of the latter restricts visibly to the isomorphism $\mathscr{O}_K(\mathbf{B}_I) \cong \varphi_L^n(\mathscr{O}_K(\mathbf{B}_I))\eta(1, Z)$, while $\mathscr{O}_K(\mathbf{B}_I)$ is a free $\mathscr{O}_K(\hat{\ell}_n^* \circ \kappa(\mathbf{B}_I))$-module with basis 1 by an obvious analogue of the former reference. Hence, we obtain the second isomorphism by the same reasoning. ■

4.4 Explicit elements

There are two sources for explicit elements in the distribution algebras $D(o_L^\times, L)$ and $D(U_n, L)$, where in this section we fix an $n \geq n_1$; i.e., $\log : U_n \xrightarrow{\cong} \pi_L^n o_L$ is an isomorphism. First of all, we have, for any group element $u \in o_L^\times$, resp., $u \in U_n$, the Dirac distribution δ_u in $D(o_L^\times, L)$, resp., in $D(U_n, L)$. As in Section 4.1.1, the corresponding holomorphic function $F_{\delta_u} = \mathrm{ev}_u$ is the function of evaluation in u.

Lemma 4.4.1. *We have the following:*

(i) *Let $u \in o_L^\times$ be any element not of finite order; then the zeros of the function $\mathrm{ev}_u - 1$ on $\mathfrak{X}^\times$ are exactly the characters χ of finite order such that $\chi(u) = 1$.*

(ii) *For any $1 \neq u \in U_n$ the zeros of the function $\mathrm{ev}_u - 1$ on $\mathfrak{X}_n^\times$ all have multiplicity one.*

Proof. (i) Obviously, the zeros of $\mathrm{ev}_u - 1$ are the characters χ such that $\chi(u) = 1$. On the other hand, consider any locally L-analytic character $\chi : o_L^\times \to \mathbb{C}_p^\times$. Its kernel $H := \ker(\chi)$ is a closed locally L-analytic subgroup of $o_L^\times$. Hence, its Lie algebra $\mathrm{Lie}(H)$ is an L-subspace of $\mathrm{Lie}(o_L^\times) = L$. We see that either $\mathrm{Lie}(H) = L$, in which case H is open in $o_L^\times$ and hence χ is a character of finite order, or $\mathrm{Lie}(H) = 0$, in which case H is zero dimensional and hence is a finite subgroup of $o_L^\times$. If $\chi(u) = 1$, then, by our assumption on u, the second case cannot happen.

(ii) (We will recall the concept of multiplicity further below.) Because of the isomorphism $\mathfrak{X}_n^\times \cong \mathfrak{X}$, it suffices to prove the corresponding assertion in the additive case. Let $0 \neq a \in o_L$ and let $\chi \in \mathfrak{X}(\mathbb{C}_p)$ be a character of finite order such that $\chi(a) = 1$. By [74] we have an isomorphism between $\mathfrak{X}_{/\mathbb{C}_p}$ and the open unit disk $\mathbf{B}_{/\mathbb{C}_p}$. Let $z \in \mathbf{B}(\mathbb{C}_p)$ denote the image of χ under this isomorphism. By [74, Prop. 3.1 and Eq. $(\diamond\diamond)$], the function $\mathrm{ev}_a - 1$ corresponds under this isomorphism to the holomorphic function on $\mathbf{B}(\mathbb{C}_p)$ given by the formal power series

$$F_{at_0'}(Z) = \exp\big(g\Omega \log_{\mathrm{LT}}(Z)\big) - 1,$$

where $\Omega \neq 0$ is a certain period. By assumption we have $F_{at_0'}(z) = 0$. On the other hand, the formal derivative of this power series is

$$\frac{d}{dZ} F_{gt_0'}(Z) = g\Omega g_{\mathrm{LT}}(Z)\big(F_{gt_0'}(Z) + 1\big).$$

Since $g_{\mathrm{LT}}(Z)$ is a unit in $o_L[\![Z]\!]$, we see that z is not a zero of this derivative. It follows that z has multiplicity one as a zero of $F_{at_0'}(Z)$. ∎

The other source comes from the Lie algebra $\mathrm{Lie}(U_n) = \mathrm{Lie}(o_L^\times) = L$. We have the element

$$\nabla := 1 \in \mathrm{Lie}(o_L^\times) = L.$$

On the other hand, there is the L-linear embedding (cf. [75, §2])

$$\begin{aligned} \mathrm{Lie}(U_n) &\to D(U_n, L) \\ \mathfrak{x} &\mapsto \left[f \mapsto \tfrac{d}{dt} f\left(\exp_{U_n}(t\mathfrak{x})\right)_{|t=0}\right], \end{aligned}$$

which composed with the Fourier isomorphism becomes the map

$$\begin{aligned} \mathrm{Lie}(U_n) &\to \mathcal{O}_L(\mathfrak{X}_n^\times) \\ \mathfrak{x} &\mapsto \left[\chi \mapsto d\chi(\mathfrak{x})\right]. \end{aligned}$$

On the one hand, we therefore may and will view ∇ always as a distribution on U_n or $o_L^\times$. On the other hand, using the formula before [8, Lem. 1.28], one checks that the function F_∇ (corresponding to ∇ via the Fourier isomorphism) on $\mathfrak{X}_n^\times$ is explicitly given by

$$F_\nabla(\chi) = \pi_L^{-n} \log\left(\chi\left(\exp(\pi_L^n)\right)\right). \tag{4.65}$$

Lemma 4.4.2. *The zeros of the function F_∇ on $\mathfrak{X}_n^\times$ are precisely the characters of finite order each with multiplicity one.*

Proof. Once again because of the isomorphism $\mathfrak{X}_n^\times \cong \mathfrak{X}$, it suffices to prove the corresponding assertion in the additive case. This is done in [8, Lem. 1.28]. ∎

To recall from [8, §1.1] the concept of multiplicity used above and to explain a divisibility criterion in these rings of holomorphic functions, we let $\mathfrak{Y}$ be any one-dimensional smooth rigid analytic quasi-Stein space over L such that $\mathcal{O}_L(\mathfrak{Y})$ is an integral domain. Under these assumptions, the local ring in a point y of the structure sheaf $\mathcal{O}_{\mathfrak{Y}}$ is a discrete valuation ring. Let $\mathfrak{m}_y$ denote its maximal ideal. The divisor $\mathrm{div}(f)$ of any non-zero function $f \in \mathcal{O}_L(\mathfrak{Y})$ is defined to be the function $\mathrm{div}(f) : \mathfrak{Y} \to \mathbb{Z}_{\geq 0}$ given by $\mathrm{div}(f)(y) = n$ if and only if the germ of f in y lies in $\mathfrak{m}_y^n \setminus \mathfrak{m}_y^{n+1}$. By Lemma 1.1 in loc. cit. for any affinoid subdomain $\mathfrak{Z} \subseteq \mathfrak{Y}$ the set

$$\left\{x \in \mathfrak{Z} \mid \mathrm{div}(f)(x) > 0\right\} \text{ is finite.} \tag{4.66}$$

Lemma 4.4.3. *For any two non-zero functions $f_1, f_2 \in \mathcal{O}_L(\mathfrak{Y})$, we have $f_2 \in f_1 \mathcal{O}_L(\mathfrak{Y})$ if and only if* $\mathrm{div}(f_2) \geq \mathrm{div}(f_1)$.

Proof. We consider the principal ideal $f_1\mathcal{O}_L(\mathfrak{Y})$. As a consequence of [8, Props. 1.6 and 1.4], we have

$$f_1\mathcal{O}_L(\mathfrak{Y}) = \left\{f \in \mathcal{O}_L(\mathfrak{Y}) \setminus \{0\} : \operatorname{div}(f) \geq \operatorname{div}(f_1)\right\} \cup \{0\}. \qquad \blacksquare$$

We now apply these results to exhibit a few more explicit elements in the distribution algebra $D(U_n, L)$, which will be used later on.

Lemma 4.4.4. *For any $1 \neq u \in U_n$ the fraction $\frac{\nabla}{\delta_u - 1}$ is a well-defined element in the integral domain $D(U_n, L)$.*

Proof. By the Fourier isomorphism we may equivalently establish that the fraction $\frac{F_\nabla}{\mathrm{ev}_u - 1}$ exists in $\mathcal{O}_L(\mathfrak{X}_u^\times)$. But for this we only need to combine the Lemmata 4.4.1 to 4.4.3. $\blacksquare$

The next elements will only lie in the Robba ring of U_n. Since $\mathfrak{X}_n^\times \cong \mathfrak{X}$, we deduce from Proposition 4.1.7 and the subsequent discussion that there is an admissible covering $\mathfrak{X}_n^\times = \bigcup_{j \geq 1} \mathfrak{V}_{n,j}$ by an increasing sequence $\mathfrak{V}_{n,1} \subseteq \cdots \subseteq \mathfrak{V}_{n,j} \subseteq \cdots$ of affinoid subdomains $\mathfrak{V}_{n,j}$ with the following properties:

- The system $(\mathfrak{X}_n^\times \setminus \mathfrak{V}_{n,j})_{/\mathbb{C}_p}$ is isomorphic to an increasing system of one-dimensional annuli. This implies that
  - $\mathcal{R}_L(\mathfrak{X}_n^\times)$ is the increasing union of the rings $\mathcal{O}_L(\mathfrak{X}_n^\times \setminus \mathfrak{V}_{n,j})$ and contains $\mathcal{O}_L(\mathfrak{X}_n^\times)$;
  - each $\mathcal{O}_L(\mathfrak{X}_n^\times \setminus \mathfrak{V}_{n,j})$ and $\mathcal{R}_L(\mathfrak{X}_n^\times)$ are integral domains.
- Each $\mathfrak{X}_n^\times \setminus \mathfrak{V}_{n,j}$ is a one-dimensional smooth quasi-Stein space.

In particular, the $\mathcal{O}_L(\mathfrak{X}_n^\times \setminus \mathfrak{V}_{n,j})$ are naturally Fréchet algebras, and we may view $\mathcal{R}_L(\mathfrak{X}_n^\times)$ as their locally convex inductive limit. We also conclude that Lemma 4.4.3 applies to each $\mathfrak{X}_n^\times \setminus \mathfrak{V}_{n,j}$.

We now fix a basis $b = (b_1, \ldots, b_d)$ of U_n as a $\mathbb{Z}_p$-module such that $b_i \neq 1$ for any $1 \leq i \leq d$.

Proposition 4.4.5. *The fraction*

$$\Xi_b := \frac{F_\nabla^{d-1}}{\prod_{i=1}^d (\mathrm{ev}_{b_i} - 1)}$$

is well defined in the Robba ring $\mathcal{R}_L(\mathfrak{X}_n^\times)$.

Proof. The zeros of the fraction $\frac{F_\nabla}{\mathrm{ev}_{b_i} - 1} \in \mathcal{O}_L(\mathfrak{X}_n^\times)$ are precisely those finite-order characters which are non-trivial on b_i. Hence, if we fix $1 \leq j \leq d$, then the product $\prod_{i \neq j} \frac{F_\nabla}{\mathrm{ev}_{b_i} - 1}$ still has a zero in any finite-order character which is non-trivial on b_i for at least one $i \neq j$. On the other hand, the zeros of $\mathrm{ev}_{b_j} - 1$ are those finite-order characters which are trivial on b_j (and they have multiplicity one). Since only the trivial character is trivial on all $b_1, \ldots, b_d$, we see that all zeros of $\mathrm{ev}_{b_j} - 1$ with the exception of the trivial character occur also as zeros of the product $\frac{F_\nabla^{d-1}}{\prod_{i \neq j}(\mathrm{ev}_{b_i} - 1)}$. It follows that the asserted fraction Ξ_b exists in $\mathcal{O}_L(\mathfrak{X}_n^\times \setminus \mathfrak{V}_{n,j})$ provided j is large enough so that the trivial character is a point in $\mathfrak{V}_{n,j}$. Since $(\prod_{i=1}^d(\mathrm{ev}_{b_i} - 1))\Xi_b = F_\nabla^{d-1}$ and $\mathcal{R}_L(\mathfrak{X}_\Gamma^\times)$ is an integral domain, we see the independence of j. ∎

In fact, the proof of Proposition 4.4.5 shows that Ξ_b is a meromorphic function on $\mathfrak{X}_n^\times$ with a single pole at the trivial character, which moreover is a simple pole. We abbreviate $\ell(b) := \prod_{i=1}^d \log(b_i)$.

Proposition 4.4.6. *For any other basis $b' = (b'_1, \ldots, b'_d)$ of U_n as a $\mathbb{Z}_p$-module with $b'_i \neq 1$ we have*

$$\ell(b')\Xi_{b'} - \ell(b)\Xi_b \in \mathcal{O}_L(\mathfrak{X}_n^\times).$$

Proof. We only have to check that the asserted difference does not any longer have a pole at the trivial character. Both $\Xi_{b'}^{-1}$ and Ξ_b^{-1} are uniformizers in the local ring $\mathcal{O}_1$ of $\mathfrak{X}_n^\times$ in the trivial character. Hence, we have in $\mathcal{O}_1$ an equality of the form

$$\frac{\Xi_{b'}}{\Xi_b} = x + \Xi_b^{-1} \cdot G$$

with some $x \in L$ and $G \in \mathcal{O}_1$. Our assertion amounts to the claim that $x = \prod_i \frac{\log(b_i)}{\log(b'_i)}$. To compute x, we use (4.3) which leads to the open embedding

$$\mathbf{B}(r_0)_{/L} \xrightarrow{\cong} \mathfrak{X}(r_0) \xrightarrow{\subseteq} \mathfrak{X} \xrightarrow{\ell_n^*} \mathfrak{X}_n^\times$$

which maps y to the character $\chi_y(u) := \exp(\pi_L^{-n}\log(u)y)$ (and, in particular, 0 to the trivial character). Using (4.65), we see that F_∇ pulls back to the function $y \mapsto \pi_L^{-n} y$ on $\mathbf{B}(r_0)$. On the other hand, $\mathrm{ev}_{b_i} - 1$ pulls back to $y \mapsto \exp(\pi_L^{-n}\log(b_i)y) - 1$. Hence, Ξ_b pulls back to the meromorphic function

$$y \mapsto \frac{\pi_L^{-n(d-1)} y^{d-1}}{\prod_i \left(\exp\left(\pi_L^{-n}\log(b_i)y\right) - 1\right)}.$$

Its germ at zero lies in

$$\frac{1}{\pi_L^{-n}\left(\prod_i \log(b_i)\right)y}(1 + y\mathcal{O}_0),$$

where $\mathcal{O}_0$ denotes the local ring of $\mathbf{B}(r_0)_{/L}$ in zero. It follows that the germ of the pull back of $\frac{\Xi_{b'}}{\Xi_b}$ lies in $\left(\prod_i \frac{\log(b_i)}{\log(b'_i)}\right)(1 + y\mathcal{O}_0)$. ∎

By Lemma 4.4.2 the function $F_\nabla \Xi_b$ is holomorphic on $\mathfrak{X}_n^\times$ and has no zero in the trivial character.

Lemma 4.4.7. *The value of $F_\nabla \Xi_b$ at the trivial character is $\ell(b)^{-1}$.*

Proof. We use the same strategy as in the previous proof. The function Θ_b pulls back to the function

$$y \mapsto \frac{\pi_L^{-nd} y^d}{\prod_i \left(\exp\left(\pi_L^{-n} \log(b_i) y\right) - 1\right)}$$

on $\mathbf{B}(r_0)_{/L}$, and we have to compute its value at 0. But visibly the above right-hand side is a power series in y with constant term $\frac{1}{\prod_{i=1}^d \log(b_i)}$. ■

These last two facts suggest renormalizing our functions by setting

$$\bar{\Xi}_b := \ell(b)\Xi_b \quad \text{and} \quad \Theta_b := F_\nabla \bar{\Xi}_b.$$

Choosing a field K containing Ω, we also let $\widetilde{\Xi_b}$ denote the image of $\bar{\Xi}_b$ under the composite map

$$\mathcal{R}_L(\mathfrak{X}_n^\times) \xrightarrow{\ell_{n*}} \mathcal{R}_L(\mathfrak{X}) \subseteq \mathcal{R}_K(\mathfrak{X}) \xrightarrow{\kappa^*} \mathcal{R}_K(\mathbf{B}). \tag{4.67}$$

Remark 4.4.8. Suppose that K contains Ω. We have

$$\widetilde{\Xi_b} = \frac{\ell(b)\left(\frac{\Omega}{\pi_L^n} \log_{\mathrm{LT}}(Z)\right)^{d-1}}{\prod_j \left(\exp\left(\log(b_j)\frac{\Omega}{\pi_L^n} \log_{\mathrm{LT}}(Z)\right) - 1\right)},$$

and it follows from the proof of Proposition 4.4.5 that $Z\widetilde{\Xi_b}$ belongs to $\mathcal{O}_K(\mathbf{B})$ with constant term $(\frac{\Omega}{\pi_L^n})^{-1}$, whence

$$\widetilde{\Xi_b} \equiv \frac{\pi_L^n}{\Omega Z} \mod \mathcal{O}_K(\mathbf{B}).$$

Proof. One checks that the map (4.67) sends a distribution μ to the map

$$g_\mu(z) = \mu\Big(\exp\Big(\Omega\frac{\log(-)}{\pi_L^n} \log_{\mathrm{LT}}(z)\Big)\Big).$$

In particular, a Dirac distribution δ_a is sent to $\exp(\Omega\frac{\log(a)}{\pi_L^n} \log_{\mathrm{LT}}(z))$. Recall that the action of ∇ as distribution is given as sending a locally L-analytic function f to $-(\frac{d}{dt} f(\exp(-t)))_{|t=0}$, whence ∇ is sent to

$$\begin{aligned}\nabla\Big(\exp\Big(\Omega\frac{\log(-)}{\pi_L^n} \log_{\mathrm{LT}}(z)\Big)\Big) &= -\Big(\tfrac{d}{dt} \exp\Big(\Omega\frac{\log(\exp(-t))}{\pi_L^n} \log_{\mathrm{LT}}(z)\Big)\Big)_{|t=0} \\ &= \frac{\Omega}{\pi_L^n} \log_{\mathrm{LT}}(z).\end{aligned}$$

■

Remark 4.4.9. Recall that Θ_b lies in $\mathcal{O}_L(\mathfrak{X}_n^\times)$ and therefore can be viewed, via the Fourier transform, as a distribution in $D(U_n, L) \subseteq D(o_L^\times, L)$. If K contains Ω and for sufficiently large n, the Mellin transform $\mathfrak{M}$ in Lemma 4.1.6 then satisfies

$$\kappa^* \circ \mathfrak{M}(\Theta_b) = \varphi_L^n(\xi_b)\eta(1, Z)$$

with

$$\xi_b \equiv \frac{\log_{\mathrm{LT}}(Z)}{Z} \mod \log_{\mathrm{LT}}(Z)\mathcal{O}_K(\mathbf{B}).$$

Proof. Consider the element

$$F(X) = \frac{X}{\exp(X) - 1} = 1 + XQ(X)$$

with $Q(X) \in \mathbb{Q}_p[\![X]\!]$ and let $r > 0$ be such that $Q(X)$ converges on $|X| \leq r$. We shall proof the claim within the Banach algebra $\mathcal{R}_K^I(\mathbf{B})$ for $I = [0, r]$ (which contains $\mathcal{O}_K(\mathbf{B})$), using that the actions on both rings are compatible. We assume for the operator norm that $\|\delta_{b_i} - 1\|_I < \min(p^{-\frac{1}{p-1}}, r)$ for all i (otherwise, we enlarge n according to Lemma 4.3.13). From [8, Cor. 2.3.2, proof of Lem. 2.3.1] it follows that $\nabla = \frac{\log(\delta_{b_i})}{\log(b_i)}$ as operators in the Banach algebra A of continuous linear endomorphisms of $\mathcal{R}_K^I(\mathbf{B})$ and

$$\exp\big(\log(b_i)\nabla\big) = \exp\big(\log(\delta_{b_i})\big) = \delta_{b_i}$$

in A. Moreover,

$$\big\|\log(\delta_{b_i})\big\|_I < \min(p^{-\frac{1}{p-1}}, r) \tag{4.68}$$

for all i, whence $\|\nabla\|_I < \min(p^{-\frac{1}{p-1}}, r)|\log(b_i)|$. Then, as operators in A, we have

$$\begin{aligned}\log(b_i)^{-1} + \nabla Q\big(\log(b_i)\nabla\big) &= \log(b_i)^{-1} F\big(\log(b_i)\nabla\big)\\ &= \frac{\nabla}{\exp\big(\log(b_i)\nabla\big) - 1}\\ &= \frac{\nabla}{\delta_{b_i} - 1}.\end{aligned}$$

Hence,

$$\Theta_b = \frac{\ell(b)\nabla^d}{\prod\big(\exp\big(\log(b_j)\nabla\big) - 1\big)} = 1 + \ell(b)\nabla g\big(\log(b_j)\nabla\big)$$

for some power series $g \in \mathcal{R}_K^I(\mathbf{B})$. It follows that

$$\kappa^* \circ \mathfrak{M}(\Theta_b) = \big(1 + \ell(b)\Omega \log_{\mathrm{LT}}(Z) f(Z)\big)\eta(1, Z) \tag{4.69}$$

for some $f(Z) \in \mathcal{R}_K^I(\mathbf{B})$. Indeed, concerning the derived action, we have

$$\nabla\big(\eta(1, Z)\big) = \big(\tfrac{d}{dt} \exp\big(\Omega \exp(t) \log_{\mathrm{LT}}(Z)\big)\big)_{|t=0} = \Omega \log_{\mathrm{LT}}(Z)\eta(1, Z)$$

using from [8, end of §2.3] the fact that

$$\nabla \text{ acts as } \log_{\mathrm{LT}}(Z)\partial_{\mathrm{inv}} \text{ on } \mathcal{O}_K(\mathbf{B}), \tag{4.70}$$

and

$$\nabla\big(\Omega \log_{\mathrm{LT}}(Z)\big) = \Omega \log_{\mathrm{LT}}(Z).$$

Furthermore, we obtain inductively that

$$\nabla^i \eta(1, Z) = \left(\prod_{k=0}^{i-1} \big(\Omega \log_{\mathrm{LT}}(Z) + k\big)\right)\eta(1, Z)$$

for all $i \geq 0$. The convergence of $f(Z)$ can be deduced using the operator norm (4.68).

On the other hand, according to [7, Lem. 2.4.2], we have

$$\Theta_b \eta(1, Z) = \frac{\ell(b) \log_{\mathrm{LT}}(Z)}{\varphi_L^n(Z)} g(Z)$$

for some $g(Z) \in \mathcal{O}_K(\mathbf{B})$. Since the element $\Theta_b\eta(1, Z)$ lies in $(\mathcal{O}_K(\mathbf{B}))^{\psi_L=0}$, we conclude from

$$0 = \psi_L\left(\frac{\pi_L^{-1}\varphi_L\big(\log_{\mathrm{LT}}(Z)\big)}{\varphi_L^n(Z)} g(Z)\right) = \frac{\pi_L^{-1} \log_{\mathrm{LT}}(Z)}{\varphi_L^{n-1}(Z)} \psi_L\big(g(Z)\big)$$

that $g(Z)$ belongs to $(\mathcal{O}_K(\mathbf{B}))^{\psi_L=0}$, whence it is of the form

$$g(Z) = \sum_{a \in (o_L/\pi_L)^\times} \varphi_L\big(g_a(Z)\big)\eta(a, Z)$$

for some $g_a \in \mathcal{O}_K(\mathbf{B})$ by the analogue of (4.58) for $\mathcal{O}_K(\mathbf{B})$. From Lemma 4.3.26, we derive that, for some $a(Z) \in \mathcal{O}_K(\mathbf{B})$, we have

$$\Theta_b \eta(1, Z) = \ell(b)\varphi_L^n\big(a(Z)\big)\eta(1, Z).$$

Since the decomposition in (4.58) is direct, we conclude that

$$g(Z) = \varphi_L\big(g_1(Z)\big)\eta(1, Z) \quad \text{and} \quad \frac{\log_{\mathrm{LT}}(Z)}{\varphi_L^n(Z)}\varphi_L\big(g_1(Z)\big) = \varphi_L^n\big(a(Z)\big),$$

whence $\log_{\mathrm{LT}}(Z)$ divides $\varphi_L^n(a(Z)Z)$. Since φ_L^n sends the zeros of $\log_{\mathrm{LT}}(Z)$, i.e., the points in $LT(\pi_L) = \bigcup_k \mathrm{LT}[\pi_L^k]$, surjectively onto itself, we conclude by Lemma 4.4.3 that $\log_{\mathrm{LT}}(Z)$ divides also $a(Z)Z$ in $\mathcal{O}_K(\mathbf{B})$ and that there exists $c(Z) \in \mathcal{O}_K(\mathbf{B})$ such that

$$\kappa^* \circ \mathfrak{M}(\Theta_b) = \ell(b)\varphi_L^n\Big(\frac{\log_{\mathrm{LT}}(Z)}{Z} c(Z)\Big)\eta(1, Z). \tag{4.71}$$

Comparing (4.71) with the first description (4.69) gives the claim as $c(0) = \ell(b)^{-1}$ because evaluation at 0 is compatible with the embedding $\mathcal{O}_K(\mathbf{B}) \subseteq \mathcal{R}_K^I(\mathbf{B})$ and $\frac{\log_{\mathrm{LT}}(Z)}{Z}(0) = 1$ by (2.1). ∎

4.5 Pairings

In this section, we discuss various kinds of pairings. The starting point is Serre duality on $\mathfrak{X}$ which induces a (residue) pairing

$$\langle\ ,\ \rangle_{\mathfrak{X}} : \mathcal{R}_L(\mathfrak{X}) \times \mathcal{R}_L(\mathfrak{X}) \to L,$$

as we have seen in (4.20). Similarly, Serre duality on $\mathfrak{X}^\times$ induces a pairing

$$\langle\ ,\ \rangle_{\Gamma_L} : \mathcal{R}_L(\Gamma_L) \times \mathcal{R}_L(\Gamma_L) \to L \tag{4.72}$$

for the Robba ring of Γ_L, which by definition is the Robba ring of its character variety $\mathfrak{X}_{\Gamma_L} \cong \mathfrak{X}^\times$ (induced by the isomorphism $\chi_{\mathrm{LT}} : \Gamma_L \to o_L^\times$) as constructed in (4.23). This pairing, as defined in Section 4.5.2, is actually already characterized by its restriction to $\mathcal{R}_L(\Gamma_n)$ for any $n \geq n_0$ and thus is by construction and the functoriality properties of Section 4.2 closely related to the pairing $\langle\ ,\ \rangle_{\mathfrak{X}}$ using the "logarithm" $\mathcal{R}_L(\Gamma_n) \xrightarrow{(\ell_n)_*} \mathcal{R}_L(\mathfrak{X})$, see (4.21).

In contrast, the commutative diagram

$$\begin{array}{ccc}
\mathcal{R}_L(\mathfrak{X}^\times) & \xrightarrow{\cdot d\log_{\mathfrak{X}^\times}} & \Omega^1_{\mathcal{R}_L(\mathfrak{X}^\times)} \\
 & & \big\downarrow {\scriptstyle \mathrm{res}_{\mathfrak{X}^\times}} \\
{\scriptstyle (-)(\mathrm{ev}_1\, d\log_{\mathfrak{X}})} \big\downarrow & & L \\
 & & \big\uparrow {\scriptstyle \mathrm{res}_{\mathfrak{X}}} \\
(\Omega^1_{\mathcal{R}_L(\mathfrak{X})})^{\psi=0} & \xrightarrow{\mathrm{ev}_{-1}\cdot} & \Omega^1_{\mathcal{R}_L(\mathfrak{X})}
\end{array}$$

from Theorem 4.5.12 in Section 4.5.3 relates the pairing $\langle\ ,\ \rangle_{\Gamma_L}$ to the pairing $\langle\ ,\ \rangle_{\mathfrak{X}}$ *in a highly non-trivial, non-obvious way*. The resulting description of $\langle\ ,\ \rangle_{\Gamma_L}$ in (4.84) (resp., (4.86)) forms one main ingredient in the proof of the abstract reciprocity law in Theorem 4.5.32 in Section 4.5.5 below.

Based on the (generalized) residue pairings (4.73) in Section 4.5.1,

$$\{\ ,\ \} : \check{M} \times M \to L,$$

with $\check{M} := \mathrm{Hom}_{\mathcal{R}}(M, \Omega^1_{\mathcal{R}})$, the pairing (4.72) induces for any (analytic) (φ_L, Γ_L)-module M over $\mathcal{R}_L(\mathfrak{X})$ an Iwasawa pairing (4.88)

$$\{\ ,\ \}_{\mathrm{Iw}} : \check{M}^{\psi_L = \frac{q}{\pi_L}} \times M^{\psi_L = 1} \to D(\Gamma_L, L)$$

in Section 4.5.4, which behaves well with twisting (cf. Lemma 4.5.22).

By construction and the comparison isomorphism (4.91) for Kisin–Ren modules – the second main ingredient – the pairing $\{\ ,\ \}_{\mathrm{Iw}}$ is closely related to a pairing

$$[\ ,\ ] : \mathcal{R}_L(\mathfrak{X})^{\psi_L=0} \otimes_L D_{\mathrm{cris},L}\big(V^*(1)\big) \times \mathcal{R}_L(\mathfrak{X})^{\psi_L=0} \otimes_L D_{\mathrm{cris},L}\big(V(\tau^{-1})\big) \to \mathcal{R}_L(\Gamma_L)$$

induced from the natural pairing for $D_{\mathrm{cris},L}$. The precise relationship is the content of an abstract form of a reciprocity formula, see Theorem 4.5.32. As a consequence, we shall later derive a concrete reciprocity formula, i.e., the adjointness of Berger's and Fourquaux's big exponential map with our regulator map, see Theorem 5.2.1.

4.5.1 The residuum pairing for modules

Throughout our coefficient field K is a complete intermediate extension $L \subseteq K \subseteq \mathbb{C}_p$. Let $\mathfrak{Y}$ be either $\mathfrak{X}$ or $\mathbf{B}$ and $\mathcal{R} = \mathcal{R}_K(\mathfrak{Y})$. Consider the residuum map $\mathrm{res}_{\mathfrak{X}}$ defined after (4.19) and the residuum map

$$\mathrm{res}_{\mathbf{B}} : \Omega^1_{\mathcal{R}} \to K, \quad \sum_i a_i Z^i dZ \mapsto a_{-1}.$$

Recall that we are using the operator $\psi_L := \frac{q}{\pi}\psi_L^{\mathfrak{Y}}$ on $\mathcal{R}$.

Moreover, we define $\iota_* : \mathcal{R}_K(\Gamma_L) \to \mathcal{R}_K(\Gamma_L)$ to be the map which is induced by sending $\gamma \in \Gamma_L$ to its inverse γ^{-1}; i.e., the involution of the group induces an isomorphism on the multiplicative character variety, which in turn gives rise to ι_*. The corresponding involution on $\mathcal{R}_K(\Gamma_{n_0})$, also denoted by ι_*, satisfies the commutative diagram

$$\begin{array}{ccc} \mathcal{R}_K(\Gamma_{n_0}) & \xrightarrow[\cong]{\hat{\ell}_{n_0 *}} & \mathcal{R}_K(\mathfrak{X}) \\ {\scriptstyle \iota_*}\downarrow{\scriptstyle \cong} & & {\scriptstyle \cong}\downarrow{\scriptstyle \iota} \\ \mathcal{R}_K(\Gamma_{n_0}) & \xrightarrow[\cong]{\hat{\ell}_{n_0 *}} & \mathcal{R}_K(\mathfrak{X}), \end{array}$$

where the involution ι on $\mathcal{R}_K(\mathfrak{X})$ sends ev_x to ev_{-x}.

Setting $\check{M} := \mathrm{Hom}_{\mathcal{R}}(M, \Omega^1_{\mathcal{R}}) \cong \mathrm{Hom}_{\mathcal{R}}(M, \mathcal{R})(\chi_{\mathrm{LT}})$, for any finitely generated projective $\mathcal{R}$-module M, we obtain more generally the pairing

$$\{\ ,\ \} := \{\ ,\ \}_M : \check{M} \times M \to K, \quad (g, f) \mapsto \mathrm{res}_{\mathfrak{Y}}\big(g(f)\big), \tag{4.73}$$

which satisfies the following properties.

Lemma 4.5.1. *For M in $\mathcal{M}(\mathcal{R})$ we have the following:*

(i) $\{\ ,\ \}$ *identifies M and $\check{M}$ with the (strong) topological duals of $\check{M}$ and M, respectively,*

(ii) $\{\varphi_L(g), \varphi_L(f)\} = \frac{q}{\pi_L}\{g, f\}$ *for all $g \in \check{M}$ and $f \in M$,*

(iii) $\{\sigma(g), \sigma(f)\} = \{g, f\}$ *for all $g \in \check{M}$, $f \in M$, and $\sigma \in \Gamma_L$,*

(iv) $\{\psi_L(g), f\} = \{g, \varphi_L(f)\}$ *and* $\{\varphi_L(g), f\} = \{g, \psi_L(f)\}$ *for all $g \in \check{M}$ and $f \in M$.*

Proof. (i) follows from the discussion in Section 4.2.3. (ii) is a purely formal consequence of (iv). (iii) follows as in (4.36) with σ_* instead of κ_*. For (iv) we refer to Lemma 4.2.14. For $\mathfrak{Y} = \mathbf{B}$ see also [80, Prop. 3.17, Cor. 3.18, Prop. 3.19]. ∎

Convention. For coherence of our notation we set $\log_{\mathbf{B}} := \log_{\mathrm{LT}}$ although in general this is *not* the standard logarithm!

Proposition 4.5.2. *The pairing* $\langle\ ,\ \rangle_{\mathfrak{Y}} : \mathcal{R} \times \mathcal{R} \to K$, $(f, g) \mapsto \mathrm{res}_{\mathfrak{Y}}(fgd\log_{\mathfrak{Y}})$, *induces topological isomorphisms*

$$\mathrm{Hom}_{K,\mathrm{cont}}(\mathcal{R}, K) \cong \mathcal{R} \quad \text{and} \quad \mathrm{Hom}_{K,cts}(\mathcal{R}/\mathcal{O}_K(\mathfrak{Y}), K) \cong \mathcal{O}_K(\mathfrak{Y}).$$

Proof. See Section 4.2.3. ∎

Remark 4.5.3. If we assume $\Omega \in K$, then these pairings can be compared via the LT-isomorphism κ. By (4.37) we have

$$\Omega\big\langle\kappa^*(f), \kappa^*(g)\big\rangle_{\mathbf{B}} = \langle f, g\rangle_{\mathfrak{X}}$$

for $f, g \in \mathcal{R}_K(\mathfrak{X})$.

Assume henceforth that M is an analytic (φ_L, Γ_L)-module over $\mathcal{R}$ and recall from Proposition 4.3.10 that the Γ_L-action on M extends continuously to a $D(\Gamma_L, K)$-module structure.

Corollary 4.5.4. *The isomorphism* $\check{M} \cong \mathrm{Hom}_{K,\mathrm{cont}}(M, K)$ *(induced by* $\{\ ,\ \}$*) is* $D(\Gamma_L, K)$*-linear.*

Proof. This follows from Lemma 4.5.1 (iii) since Γ_L generates a dense subspace of $D(\Gamma_L, K)$. ∎

Since $\frac{\pi_L}{q}\psi_L \circ \varphi_L = \mathrm{id}_M$, we have a canonical decomposition $M = \varphi_L(M) \oplus M^{\psi_L=0}$. By Lemma 4.5.1 we see that $M^{\psi_L=0}$ is the exact orthogonal complement of $\varphi_L(\check{M})$; i.e., we obtain a canonical isomorphism

$$\check{M}^{\psi_L=0} \cong \mathrm{Hom}_{K,\mathrm{cont}}(M^{\psi_L=0}, K). \tag{4.74}$$

Lemma 4.5.5. *The isomorphism* (4.74) *is* $\mathcal{R}_K(\Gamma_L)$*-equivariant; i.e., we have for all* $\check{m} \in \check{M}^{\psi_L=0}$, $m \in M^{\psi_L=0}$, *and* $\lambda \in \mathcal{R}_K(\Gamma_L)$ *that*

$$\{\lambda\check{m}, m\} = \big\{\check{m}, \iota_*(\lambda)m\big\}.$$

Proof. This is clear for $D(\Gamma_L, K)$ by Corollary 4.5.4. Without loss of generality, we may and do assume that Ω belongs to K. It then follows for the localization $D(\Gamma_L, K)_{Y_{n_1}^{\mathbb{N}}}$, where we use the notation and considerations from Section 4.3.6, especially Lemma 4.3.19 and its proof. Since $D(\Gamma_L, K)_{Y_{n_1}^{\mathbb{N}}}$ is dense in $\mathcal{R}_K(\Gamma_L)$, Remark 4.3.5, the assertion now is a consequence of the continuity property in Theorem 4.3.21. ∎

4.5.2 The duality pairing $\langle\ ,\ \rangle_{\Gamma_L}$ for the group Robba ring

Using the isomorphisms (4.43) induced by the Lubin–Tate character χ_{LT}, we now carry over structures concerning the (multiplicative) character varieties $\mathfrak{X}^\times$, $\mathfrak{X}_n^\times$ to those of the groups Γ_L, Γ_n. In particular, we use analogous notation res_{Γ_L}, res_{Γ_n}, $\log_{\Gamma_L}$, $\log_{\Gamma_n}$ for corresponding objects. In this sense we introduce and recall from (4.23) the pairing

$$\langle\ ,\ \rangle_{\Gamma_L} : \mathcal{R}_K(\Gamma_L) \times \mathcal{R}_K(\Gamma_L) \to K \tag{4.75}$$

and similarly $\langle\ ,\ \rangle_{\Gamma_n}$ from (4.21). This pairing is of the form

$$\langle\ ,\ \rangle_{\Gamma_L} : \mathcal{R}_K(\Gamma_L) \times \mathcal{R}_K(\Gamma_L) \xrightarrow{\text{mult}} \mathcal{R}_K(\Gamma_L) \xrightarrow{\varrho} K,$$

where

$$\begin{aligned} \varrho = \langle 1, - \rangle_{\Gamma_L} : \mathcal{R}_K(\Gamma_L) &\to \Omega^1_{\mathcal{R}_K(\Gamma_L)} \xrightarrow{\mathrm{res}_{\Gamma_L}} K \\ f &\mapsto f d\log_{\Gamma_L} \mapsto \mathrm{res}_{\Gamma_L}(f d\log_{\Gamma_L}) \end{aligned}$$

has also the following description – writing $\mathrm{pr}_{n,m}$ and similarly $\mathrm{pr}_{L,m}$ for the projection maps induced by (4.44), (4.45) –

$$\begin{aligned} \varrho : \mathcal{R}_K(\Gamma_L) &= \mathbb{Z}[\Gamma_L] \otimes_{\mathbb{Z}[\Gamma_{n_0}]} \mathcal{R}(\Gamma_{n_0}) \to K \\ f &\mapsto \tfrac{q-1}{q}\left(\tfrac{q}{\pi_L}\right)^{n_0} \mathrm{res}_{\mathfrak{X}}\left(\hat{\ell}_{n_0 *} \circ \mathrm{pr}_{L,n_0}(f) d\log_{\mathfrak{X}}\right) \end{aligned} \tag{4.76}$$

with n_0 as defined at the beginning of Section 4.3.4. Indeed, using (4.22), (4.21) we obtain

$$\begin{aligned} \langle 1, f \rangle_{\Gamma_L} &= \mathrm{res}_{\Gamma_L}(f d\log_{\Gamma_L}) \\ &= \tfrac{q-1}{q} q^{n_0} \mathrm{res}_{\Gamma_{n_0}}\left(\mathrm{pr}_{L,n_0}(f) d\log_{\Gamma_{n_0}}\right) \\ &= \tfrac{q-1}{q}\left(\tfrac{q}{\pi_L}\right)^{n_0} \mathrm{res}_{\mathfrak{X}}\left(\hat{\ell}_{n_0 *} \circ \mathrm{pr}_{L,n_0}(f) d\log_{\mathfrak{X}}\right) \end{aligned}$$

because $(\ell_n^*)^*(1) = 1$.

The following properties follow immediately from the definition.

Lemma 4.5.6. *We have for all $f, \lambda, \mu \in \mathcal{R}_K(\Gamma_L)$ that*

(i) $\langle \lambda, f\mu \rangle_{\Gamma_L} = \langle f\lambda, \mu \rangle_{\Gamma_L}$,

(ii) $\langle \lambda, \mu \rangle_{\Gamma_L} = \langle \mu, \lambda \rangle_{\Gamma_L}$.

Remark 4.5.7. For $n \geq n_0$ we have the projection formula

$$\mathrm{pr}_{L,n}\left(\iota_{n,*}(x) y\right) = x\, \mathrm{pr}_{L,n}(y)$$

and (4.24) translates into

$$\left\langle \iota_{n,*}(x), y \right\rangle_{\Gamma_L} = (q-1) q^{n-1} \left\langle x, \mathrm{pr}_{L,n}(y) \right\rangle_{\Gamma_n} \tag{4.77}$$

for $x \in \mathcal{R}(\Gamma_n)$, $y \in \mathcal{R}(\Gamma_L)$ and the canonical inclusion $\mathcal{R}(\Gamma_n) \xrightarrow{\iota_{n,*}} \mathcal{R}(\Gamma_L)$. Analogous formulae hold for Γ_m with $n \geq m \geq n_0$ instead of Γ_L Lemma 4.2.14 (ii).

Remark 4.5.8 (Frobenius reciprocity). The projection map $\mathrm{pr}_{\Gamma_L,\Gamma_n} : \mathcal{R}_K(\Gamma_L) \to \mathcal{R}_K(\Gamma_n)$ induces an isomorphism

$$\mathrm{Hom}_{\mathcal{R}_K(\Gamma_L)}\left(N, \mathcal{R}_K(\Gamma_L)\right) \cong \mathrm{Hom}_{\mathcal{R}_K(\Gamma_n)}\left(N, \mathcal{R}_K(\Gamma_n)\right)$$

for any $\mathcal{R}_K(\Gamma_L)$-module N; the inverse sends f to the homomorphism given by $x \mapsto \sum_{g \in \Gamma_L/U} g\iota_{n,*} \circ f(g^{-1}x)$.

The following proposition translates the results at the end of Section 4.2.3 into the present setting.

Proposition 4.5.9. *The pairing* $\langle\ ,\ \rangle_{\Gamma_L} : \mathcal{R}_K(\Gamma_L) \times \mathcal{R}_K(\Gamma_L) \to K$ *induces topological isomorphisms*

$$\mathrm{Hom}_{K,\mathrm{cont}}\left(\mathcal{R}_K(\Gamma_L), K\right) \cong \mathcal{R}_K(\Gamma_L),$$
$$\mathrm{Hom}_{K,\mathrm{cont}}\left(\mathcal{R}_K(\Gamma_L)/D(\Gamma_L, K), K\right) \cong D(\Gamma_L, K).$$

Proposition 4.5.10. *Assume* $\Omega \in K$ *and* M *in* $\mathcal{M}(\mathcal{R})$. *Then the map*

$$\begin{aligned} \mathrm{Hom}_{\mathcal{R}_K(\Gamma_L)}\left(M^{\psi_L=0}, \mathcal{R}_K(\Gamma_L)\right)^{\iota} \xrightarrow{\cong} \mathrm{Hom}_{K,\mathrm{cont}}(M^{\psi_L=0}, K) \xrightarrow[(4.74)]{\cong} \check{M}^{\psi_L=0} \\ F \mapsto \rho \circ F \end{aligned} \tag{4.78}$$

is an isomorphism of $\mathcal{R}_K(\Gamma_L)$*-modules, where the superscript "*ι*" on the left-hand side indicates that* $\mathcal{R}_K(\Gamma_L)$ *acts through the involution* ι_*.

Proof. According to Theorem 4.3.21, the $\mathcal{R}_K(\Gamma_L)$-module $M^{\psi_L=0}$ is finitely generated free. Hence, it suffices to show that the map

$$\begin{aligned} \mathrm{Hom}_{\mathcal{R}_K(\Gamma_L)}\left(\mathcal{R}_K(\Gamma_L), \mathcal{R}_K(\Gamma_L)\right) \to \mathrm{Hom}_{K,\mathrm{cont}}\left(\mathcal{R}_K(\Gamma_L), K\right) \\ F \mapsto \rho \circ F \end{aligned}$$

is bijective. But this map is nothing else than the duality isomorphism in Proposition 4.5.9. ∎

The following twist invariance is just Lemma 4.2.15.

Proposition 4.5.11. *Let* U *be* Γ_L *or* Γ_n *for* $n \geq n_0$. *Then, for all* $\lambda, \mu \in \mathcal{R}(U)$ *we have*

$$\left\langle \mathrm{Tw}_{\chi_{LT}}(\mu), \mathrm{Tw}_{\chi_{LT}}(\lambda)\right\rangle_U = \langle \mu, \lambda \rangle_U.$$

4.5.3 A residuum identity and an alternative description of $\langle\ ,\ \rangle_{\Gamma_L}$

Let $\sigma_{-1} \in \Gamma_L$ be the element with $\chi_{\mathrm{LT}}(\sigma_{-1}) = -1$. Consider the continuous K-linear map

$$\begin{aligned} \varsigma : \mathcal{R}_K(\Gamma_L) &\to K \\ \lambda &\mapsto \operatorname{res}_{\mathfrak{X}}\big(\mathfrak{M}(\sigma_{-1})\mathfrak{M}^{\Omega^1}(\lambda)\big), \end{aligned}$$

where $\mathcal{M}^{\Omega^1} : \mathfrak{R}_K(\Gamma_L) \xrightarrow{\cong} {\Omega^1_{\mathcal{R}(\mathfrak{X})}}^{\psi=0} \subseteq \Omega^1_{\mathcal{R}(\mathfrak{X})}$ sends λ to

$$\lambda(\mathrm{ev}_1\, d\log_{\mathfrak{X}}) = \big(\mathrm{Tw}_{\chi_{\mathrm{LT}}}(\lambda)(\mathrm{ev}_1)\big) d\log_{\mathfrak{X}}, \tag{4.79}$$

whence we also have

$$\operatorname{res}_{\mathfrak{X}}\big(\mathfrak{M}(\sigma_{-1})\mathfrak{M}^{\Omega^1}(\lambda)\big) = \operatorname{res}_{\mathfrak{X}}\big(\mathfrak{M}(\sigma_{-1})\mathfrak{M}\big(\mathrm{Tw}_{\chi_{\mathrm{LT}}}(\lambda)\big) d\log_{\mathfrak{X}}\big). \tag{4.80}$$

Recall the definition of ϱ from (4.75).

Theorem 4.5.12. *We have*

$$\varsigma = \tfrac{q}{q-1}\varrho;$$

i.e., the following identity for the residue map holds:

$$\left(\tfrac{q}{\pi_L}\right)^{n_0} \operatorname{res}_{\mathfrak{X}}\big(\hat{\ell}_{n_0 *} \circ \mathrm{pr}_{L,n_0}(\lambda) d\log_{\mathfrak{X}}\big) = \operatorname{res}_{\mathfrak{X}}\big(\mathrm{ev}_{-1}\, \lambda(\mathrm{ev}_1\, d\log_{\mathfrak{X}})\big)$$

for all $\lambda \in \mathcal{R}_K(\Gamma_L)$; i.e., the following diagram commutes:

$$\begin{array}{ccc} \mathcal{R}_K(\mathfrak{X}^\times) & \xrightarrow{\cdot d\log_{\mathfrak{X}^\times}} & \Omega^1_{\mathcal{R}_K(\mathfrak{X}^\times)} \\ & & \big\downarrow \operatorname{res}_{\mathfrak{X}^\times} \\ {\scriptstyle (-)(\mathrm{ev}_1\, d\log_{\mathfrak{X}})} \big\downarrow & & K \\ & & \big\uparrow \operatorname{res}_{\mathfrak{X}} \\ (\Omega^1_{\mathcal{R}_K(\mathfrak{X})})^{\psi=0} & \xrightarrow{\mathrm{ev}_{-1}\cdot} & \Omega^1_{\mathcal{R}_K(\mathfrak{X})}. \end{array}$$

Remark 4.5.13. Compare with [3, Props. 2.2.1, 3.2.1] where residue identities also play a crucial role in the proof of his reciprocity formula.

The proof of this theorem requires some preparation.

Lemma 4.5.14. *For all $\lambda \in \mathcal{R}_K(\Gamma_L)$ and $j \in \mathbb{Z}$ we have*

$$\varsigma\big(\mathrm{Tw}_{\chi_{\mathrm{LT}}^j}(\lambda)\big) = \varsigma(\lambda).$$

Proof. For the proof we may and do assume that Ω belongs to K. Since then

$$\mathrm{res}_{\mathfrak{X}}\big(\partial^{\mathfrak{X}}_{\mathrm{inv}}(f)d\log_{\mathfrak{X}}\big) = \Omega\mathrm{res}_{\mathbf{B}}\big(\tfrac{1}{\Omega}\partial_{\mathrm{inv}}\big(\kappa^*(f)\big)d\log_{\mathrm{LT}}\big) = \mathrm{res}_{\mathbf{B}}\big(d\kappa^*(f)\big) = 0$$

for any f by Remark 4.2.9, (4.37), and [33, Prop. 2.12], the case $j = 1$ follows directly from (4.80) using with $g := \mathrm{Tw}_{\chi_{\mathrm{LT}}}(\lambda)$ that

$$\begin{aligned}\partial^{\mathfrak{X}}_{\mathrm{inv}}\big(\mathfrak{M}(\sigma_{-1})\mathfrak{M}(g)\big) &= \partial^{\mathfrak{X}}_{\mathrm{inv}}\big(\mathfrak{M}(\sigma_{-1})\big)\mathfrak{M}(g) + \mathfrak{M}(\sigma_{-1})\partial^{\mathfrak{X}}_{\mathrm{inv}}\big(\mathfrak{M}(g)\big)\\ &\overset{(4.63)}{=} \mathfrak{M}\big(\mathrm{Tw}_{\chi_{\mathrm{LT}}}(\sigma_{-1})\big)\mathfrak{M}(g) + \mathfrak{M}(\sigma_{-1})\mathfrak{M}\big(\mathrm{Tw}_{\chi_{\mathrm{LT}}}(g)\big)\\ &= -\mathfrak{M}(\sigma_{-1})\mathfrak{M}(g) + \mathfrak{M}(\sigma_{-1})\mathfrak{M}\big(\mathrm{Tw}_{\chi_{\mathrm{LT}}}(g)\big).\end{aligned}$$

From this the general case is immediate. ■

Lemma 4.5.15. *Let $\lambda \in D(\Gamma_L, \mathbb{C}_p)$ with $\mathrm{ev}_{\chi^j_{\mathrm{LT}}}(\lambda) = 0$ for infinitely many j; then $\lambda = 0$.*

Proof. On the character variety the characters χ^j_{LT} correspond to points which converge to the trivial character. It follows that λ corresponds to the trivial function since otherwise its divisor of zeros would have only finitely many zeros in any disk with fixed radius strictly smaller than 1 by (4.66), which would contradict the assumptions. ■

Now fix a $\mathbb{Z}_p$-basis $b = (b_1, \ldots, b_d)$ of U_{n_0} with all $b_i \neq 1$ and set $\ell^*(b) := \ell^*_\Gamma(b) := q^{-n_0}\ell(b) \in o_L^\times$ with $\Gamma := \Gamma_{n_0}$. According to Section 4.4, we may define the operator

$$\widehat{\Xi_b} := q^{-n_0}\chi^*_{\mathrm{LT}}(\overline{\Xi}_b) = \ell^*(b)\chi^*_{\mathrm{LT}}(\Xi_b)$$

in $\mathcal{R}_K(\Gamma)$. Let $\mathrm{aug} : D(\Gamma, K) \to K$ denote the augmentation map, induced by the trivial map $\Gamma \to \{1\}$.

Lemma 4.5.16. *The element $\widehat{\Xi_b}$ induces – up to the constant q^{-n_0} – the augmentation map*

$$\langle \widehat{\Xi_b}, -\rangle_{\Gamma_{n_0}} = q^{-n_0}\mathrm{aug} : D(\Gamma_{n_0}, K) \to K. \tag{4.81}$$

Moreover, we have

$$\varsigma(\widehat{\Xi_b}) = 1 = \frac{q}{q-1}\varrho(\widehat{\Xi_b}). \tag{4.82}$$

Proof. We may and do assume $\Omega \in K$ by compatibility of res with respect to (complete) base change (4.26). Since

$$\kappa^*\big(\hat{\ell}_{n_0 *}(\widehat{\Xi_b})\big) = q^{-n_0}\widetilde{\Xi}_b \equiv \frac{\pi_L^{n_0}}{q^{n_0}\Omega}\frac{1}{Z} \mod \mathcal{O}_K(\mathbf{B})$$

by Remark 4.4.8, one has for every $\lambda \in D(\Gamma, K)$

$$\begin{aligned}
\langle \widehat{\Xi_b}, \lambda\rangle_\Gamma &\overset{(4.77)}{=} \tfrac{1}{(q-1)q^{n_0-1}} \langle \widehat{\Xi_b}, \lambda\rangle_{\Gamma_L} \\
&\overset{(4.76)}{=} q^{-n_0} \mathrm{res}_{\mathfrak{X}}\big((\tfrac{q}{\pi_L})^{n_0} \kappa^*\big(\hat{\ell}_{n_0 *}(\widehat{\Xi_b}\lambda)\big) d\log_{\mathfrak{X}}\big) \\
&\overset{(4.37)}{=} q^{-n_0} \mathrm{res}_{\mathbf{B}}\big(\Omega (\tfrac{q}{\pi_L})^{n_0} \kappa^*\big(\hat{\ell}_{n_0 *}(\widehat{\Xi_b}\lambda)\big) g_{\mathrm{LT}} dZ\big) \\
&= q^{-n_0} \mathrm{res}_{\mathbf{B}}\big(\tfrac{1}{Z} \kappa^*\big(\hat{\ell}_{n_0 *}(\lambda)\big) g_{\mathrm{LT}} dZ\big) \\
&= q^{-n_0} \mathrm{aug}(\lambda),
\end{aligned}$$

where we use for the last equation that $g_{\mathrm{LT}}(Z)$ has constant term 1 and the fact that the augmentation map corresponds via Fourier theory and the LT-isomorphism to the "evaluation at $Z = 0$" map. Taking $\lambda = 1$ we see that $\varrho(\widehat{\Xi_b}) = \langle \widehat{\Xi_b}, 1\rangle_{\Gamma_L} = \frac{q-1}{q}$.

For the other equation of the second claim one has by definition of ς

$$\begin{aligned}
\varsigma(\widehat{\Xi_b}) &\overset{(4.37)}{=} \Omega \ell^*(b) \mathrm{res}_{\mathbf{B}}\Big(\kappa^*\big(\mathfrak{M}(\sigma_{-1}) \mathfrak{M}\big(\mathrm{Tw}_{\chi_{\mathrm{LT}}}\big(\chi_{\mathrm{LT}}^*(\Xi_b)\big)\big)\big) d\log_{\mathrm{LT}}\Big) \\
&= \Omega \ell^*(b) \mathrm{res}_{\mathbf{B}}\Big(\mathfrak{M}_{\mathrm{LT}}(\sigma_{-1}) \mathfrak{M}_{\mathrm{LT}}\big(\mathrm{Tw}_{\chi_{\mathrm{LT}}}\big(\chi_{\mathrm{LT}}^*(\Xi_b)\big)\big) d\log_{\mathrm{LT}}\Big) \\
&= \ell^*(b) \mathrm{res}_{\mathbf{B}}\left(\mathfrak{M}_{\mathrm{LT}}(\sigma_{-1}) \log_{\mathrm{LT}}(Z) \partial_{\mathrm{inv}} \mathfrak{M}_{\mathrm{LT}}\big(\chi_{\mathrm{LT}}^*(\Xi_b)\big) \frac{d\log_{\mathrm{LT}}}{\log_{\mathrm{LT}}(Z)} \right) \\
&= \ell^*(b) \mathrm{res}_{\mathbf{B}}\left(\mathfrak{M}_{\mathrm{LT}}(\sigma_{-1}) \mathfrak{M}_{\mathrm{LT}}\big(\nabla \chi_{\mathrm{LT}}^*(\Xi_b)\big) \frac{d\log_{\mathrm{LT}}}{\log_{\mathrm{LT}}(Z)} \right) \\
&= \ell^*(b) \mathrm{res}_{\mathbf{B}}\left(\mathfrak{M}_{\mathrm{LT}}(\sigma_{-1}) \frac{\pi_L^{n_0} \log_{\mathrm{LT}}(Z)}{\varphi_L^{n_0}\big(Z\ell(b)\big)} \eta(1, Z) \frac{d\log_{\mathrm{LT}}}{\log_{\mathrm{LT}}(Z)} \right) \\
&= \ell^*(b) \pi_L^{n_0} \mathrm{res}_{\mathbf{B}}\left(\eta(-1, Z) \frac{1}{\varphi_L^{n_0}\big(Z\ell(b)\big)} \eta(1, Z) d\log_{\mathrm{LT}} \right) \\
&= \ell^*(b) \pi_L^{n_0} \mathrm{res}_{\mathbf{B}}\left(\eta(1-1, Z) \frac{1}{\varphi_L^{n_0}\big(Z\ell(b)\big)} d\log_{\mathrm{LT}} \right) \\
&= \ell^*(b) \pi_L^{n_0} \mathrm{res}_{\mathbf{B}}\left(\varphi_L^{n_0}\left(\eta(0, Z) \frac{1}{Z\ell(b)} d\log_{\mathrm{LT}} \right)\right) \\
&= \frac{\ell^*(b)}{\ell(b)} \pi_L^{n_0} (\tfrac{q}{\pi_L})^{n_0} \mathrm{res}_{\mathbf{B}}\big(\tfrac{1}{Z} g_{\mathrm{LT}} dZ\big) \\
&= 1,
\end{aligned}$$

where we use (4.64) in the third equation, the fact that ∇ acts on $\mathcal{R}$ as $\log_{\mathrm{LT}}(Z)\partial_{\mathrm{inv}}$ (cf. (4.70)) in the fourth equation, Remark 4.4.9 for the fifth equation, Lemma 4.5.1 (iv) with $\psi_L(1) = \frac{q}{\pi_L}$ for the penultimate equation, and finally for the last equation that $g_{\mathrm{LT}}(Z)$ has constant term 1. ■

Proof of Theorem 4.5.12. Since the equality can also be checked after base change by (4.26), we may and do assume that Ω belongs to K. Due to Proposition 4.5.9, there exists $g \in D(\Gamma_L, K)$ such that $\varsigma(\lambda) = \langle g, \lambda\rangle_{\Gamma_L}$ for all $\lambda \in \mathcal{R}_K(\Gamma_L)$ because ς sends $D(\Gamma_L, K)$ to zero. We claim that

$$\mathrm{Tw}_{\chi_{LT}^j}(g) = g \tag{4.83}$$

for all $j \in \mathbb{Z}$: By Proposition 4.5.11 and Lemma 4.5.14 we have

$$\begin{aligned}\langle Tw_{\chi_{LT}^j}(g), f\rangle_{\Gamma_L} &= \langle g, \mathrm{Tw}_{\chi_{LT}^{-j}}(f)\rangle_{\Gamma_L}\\ &= \varsigma\big(\mathrm{Tw}_{\chi_{LT}^{-j}}(f)\big)\\ &= \varsigma(f)\\ &= \langle g, f\rangle_{\Gamma_L}\end{aligned}$$

for all $f \in \mathcal{R}_K(\Gamma_L)$.

Now it follows from (4.83) combined with Lemma 4.5.15 that g is constant (and equal to $\mathrm{ev}_{\chi_{LT}^0}(g)$), i.e., $\varsigma(-) = g\langle 1, -\rangle_{\Gamma_L} = g\varrho(-)$. Finally, it follows from (4.82) that $g = \frac{q}{q-1}$. ∎

Corollary 4.5.17. *The pairing* $\frac{q}{q-1}\langle\ ,\ \rangle_{\Gamma_L}$ *makes the following diagram commutative:*

$$\begin{array}{ccccccc}\mathcal{R}_K(\mathfrak{X})^{\psi_L=0} & \times & (\Omega^1_{\mathcal{R}(\mathfrak{X})})^{\psi_L=0} & \xrightarrow{\text{mult}} & \Omega^1_{\mathcal{R}(\mathfrak{X})} & \xrightarrow{\mathrm{res}_{\mathfrak{X}}} & K\\ \uparrow{\scriptstyle \sigma_{-1}\mathfrak{M}\circ\iota_*} & & \uparrow{\scriptstyle \mathfrak{M}^{\Omega^1}} & & & & \|\\ \mathcal{R}_K(\Gamma_L) & \times & \mathcal{R}_K(\Gamma_L) & & \xrightarrow{\frac{q}{q-1}\langle\ ,\ \rangle_{\Gamma_L}} & & K;\end{array} \tag{4.84}$$

i.e., we have

$$\begin{aligned}\tfrac{q}{q-1}\langle\mu,\lambda\rangle_{\Gamma_L} &= \big\{\mathfrak{M}\big(\sigma_{-1}\iota_*(\mu)\big), \mathfrak{M}^{\Omega^1}(\lambda)\big\}_{\Omega^1}\\ &= \mathrm{res}_{\mathfrak{X}}\big(\sigma_{-1}\mathfrak{M}\big(\iota_*(\mu)\big)\mathfrak{M}^{\Omega^1}(\lambda)\big)\\ &= \mathrm{res}_{\mathfrak{X}}\big(\mathfrak{M}\big(\sigma_{-1}\iota_*(\mu)\big)\mathfrak{M}\big(\mathrm{Tw}_{\chi_{LT}}(\lambda)\big)d\log_{\mathfrak{X}}\big)\\ &= \mathrm{res}_{\mathfrak{X}}\big(\mathfrak{M}\big(\iota_*(\mu)\big)\mathfrak{M}(\mathrm{Tw}_{\chi_{LT}}\big(\sigma_{-1}\lambda)\big)d\log_{\mathfrak{X}}\big).\end{aligned} \tag{4.85}$$

Proof. By Theorem 4.5.12 and the definition of ς and of $\langle\ ,\ \rangle_{\Gamma_L}$ we have

$$\begin{aligned}\tfrac{q}{q-1}\langle\mu,\lambda\rangle_{\Gamma_L} &= \frac{q}{q-1}\langle 1, \mu\lambda\rangle_{\Gamma_L}\\ &= \big\{\mathfrak{M}(\sigma_{-1}), \mathfrak{M}^{\Omega^1}(\mu\lambda)\big\}_{\Omega^1}\\ &= \big\{\mathfrak{M}\big(\sigma_{-1}\iota_*(\mu)\big), \mathfrak{M}^{\Omega^1}(\lambda)\big\}_{\Omega^1},\end{aligned} \tag{4.86}$$

where we use Lemma 4.5.5 for the last equation. ∎

Lemma 4.5.18. *We have for all* $\lambda, \mu \in \mathcal{R}_K(\Gamma_L)$ *that* $\langle\lambda,\mu\rangle_{\Gamma_L} = -\langle\iota_*(\lambda), \iota_*(\mu)\rangle_{\Gamma_L}$.

Proof. Using (4.85) for the first and third equation, property (3) in Section 4.1.3 applied to ι and the fact that $\mathrm{Tw}_{\chi_{LT}}(\sigma_{-1}) = -\sigma_{-1}$ for the second equation, and Proposition 4.5.11 for the last one, we see that

$$\begin{aligned}
\tfrac{q}{q-1}\langle \mu, \lambda\rangle_{\Gamma_L} &= \mathrm{res}_{\mathfrak{X}}\big(\mathfrak{M}\big(\mathrm{Tw}_{\chi_{LT}}(\sigma_{-1}\lambda)\big)\mathfrak{M}\big(\iota_*(\mu)\big)d\log_{\mathfrak{X}}\big)\\
&= -\mathrm{res}_{\mathfrak{X}}\big(\mathfrak{M}\big(\sigma_{-1}\iota_*\big(\mathrm{Tw}_{\chi_{LT}^{-1}}\big(\iota_*(\lambda)\big)\big)\big)\mathfrak{M}\big(\iota_*(\mu)\big)d\log_{\mathfrak{X}}\big)\\
&= -\tfrac{q}{q-1}\big\langle \mathrm{Tw}_{\chi_{LT}^{-1}}\big(\iota_*(\lambda)\big), \mathrm{Tw}_{\chi_{LT}^{-1}}\big(\iota_*(\mu)\big)\big\rangle_{\Gamma_L}\\
&= -\tfrac{q}{q-1}\big\langle \iota_*(\lambda), \iota_*(\mu)\big\rangle_{\Gamma_L}.
\end{aligned}$$

∎

4.5.4 The Iwasawa pairing for (φ_L, Γ_L)-modules over the Robba ring

As before, let $\mathfrak{Y}$ be either $\mathfrak{X}$ or $\mathbf{B}$ and $\mathcal{R} = \mathcal{R}_K(\mathfrak{Y})$ and M in $\mathcal{M}^{\mathrm{an}}(\mathcal{R})$, where K is any complete intermediate extension $L \subseteq K \subseteq \mathbb{C}_p$. Using Proposition 4.5.9, we define the pairing

$$\{\ ,\ \}^0_{\mathrm{Iw}} := \{\ ,\ \}^0_{M,\mathrm{Iw}} : \check{M}^{\psi_L=0} \times M^{\psi_L=0} \to \mathcal{R}_K(\Gamma_L),$$

which is $\mathcal{R}_K(\Gamma_L)$-ι_*-sesquilinear in the sense that

$$\lambda\{\check{m}, m\}^0_{\mathrm{Iw}} = \{\lambda\check{m}, m\}^0_{\mathrm{Iw}} = \big\{\check{m}, \iota_*(\lambda)m\big\}^0_{\mathrm{Iw}} \tag{4.87}$$

for all $\lambda \in \mathcal{R}_K(\Gamma_L)$ and $\check{m} \in \check{M}^{\psi_L=0}$, $m \in M^{\psi_L=0}$. This requires the commutativity of the diagram

$$\begin{array}{ccccc}
\mathcal{R}_K(\Gamma_L) & \times & \check{M}^{\psi_L=0} \times M^{\psi_L=0} & \longrightarrow & K\\
\Big\| & & \Big\downarrow{\scriptstyle \{\ ,\ \}^0_{\mathrm{Iw}}} & & \Big\|\\
\mathcal{R}_K(\Gamma_L) & \times & \mathcal{R}_K(\Gamma_L) & \xrightarrow{\ \langle\ ,\ \rangle_{\Gamma_L}\ } & K,
\end{array}$$

in which the upper line sends (f, x, y) to $\{f(x), y\}_M$, where the latter pairing is (4.73). Indeed, the property

$$\{\lambda\check{m}, m\}^0_{\mathrm{Iw}} = \big\{\check{m}, \iota_*(\lambda)m\big\}^0_{\mathrm{Iw}}$$

follows from the corresponding property for $\{\ ,\ \}_M$ by Lemma 4.5.5, while with regard to the second one

$$\lambda\{\check{m}, m\}^0_{\mathrm{Iw}} = \{\lambda\check{m}, m\}^0_{\mathrm{Iw}}$$

we have for all $f \in \mathcal{R}_K(\Gamma_L)$

$$\big\langle f, \{\lambda\check{m}, m\}^0_{\mathrm{Iw}}\big\rangle_{\Gamma_L} = \{f\cdot\lambda\check{m}, m\} = \big\langle \lambda f, \{\check{m}, m\}^0_{\mathrm{Iw}}\big\rangle_{\Gamma_L} = \big\langle f, \lambda\{\check{m}, m\}^0_{\mathrm{Iw}}\big\rangle_{\Gamma_L}$$

by Lemma 4.5.6. Note that the pairing $\{\ ,\ \}^0_{\mathrm{Iw}}$ induces the isomorphism in (4.78).

We set

$$\mathcal{C} := \big(\tfrac{\pi_L}{q}\varphi_L - 1\big)M^{\psi_L=1} \quad \text{and} \quad \check{\mathcal{C}} := (\varphi_L - 1)\check{M}^{\psi_L=\frac{q}{\pi_L}}$$

and we shall need the following lemma.

Lemma 4.5.19. *For $f \in D(\Gamma_L, K)$ we have $\{f \cdot (\varphi_L - 1)x, (\frac{\pi_L}{q}\varphi_L - 1)y\} = 0$ for all $x \in \check{M}^{\psi_L=\frac{q}{\pi_L}}$ and $y \in M^{\psi_L=1}$.*

Proof. The result follows by straightforward calculation using Lemma 4.5.1 (cf. [42, Lem. 4.2.7]). ■

This lemma combined with the second statement of Proposition 4.5.9 implies that the restriction of $\{\ ,\ \}^0_{\mathrm{Iw}}$ to $\check{\mathcal{C}} \times \mathcal{C}$, which by abuse of notation we denote by the same symbol, is characterized by the commutativity of the diagram

$$\begin{array}{ccccc} \check{\mathcal{C}} \times \mathcal{C} & \times & \mathcal{R}_K(\Gamma_L)/D(\Gamma_L, K) & \longrightarrow & K \\ {\scriptstyle \{\,,\}^0_{\mathrm{Iw}}}\downarrow & & \| & & \| \\ D(\Gamma_L, K) & \times & \mathcal{R}_K(\Gamma_L)/D(\Gamma_L, K) & \xrightarrow{\langle\,,\rangle_{\Gamma_L}} & K, \end{array}$$

in which the upper line sends (x, y, f) to $\{f(x), y\}_M$. In particular, it takes values in $D(\Gamma_L, K)$.

Finally, we obtain a $D(\Gamma_L, K)$-ι_*-sesquilinear pairing $\{\ ,\ \}_{\mathrm{Iw}} := \{\ ,\ \}_{M,\mathrm{Iw}}$ which by definition fits into the following commutative diagram:

$$\begin{array}{ccccc} \check{M}^{\psi_L=\frac{q}{\pi_L}} & \times & M^{\psi_L=1} & \overset{\{\,,\}_{M,Iw}}{\dashrightarrow} & D(\Gamma_L, K) \\ {\scriptstyle \varphi_L-1}\downarrow & & {\scriptstyle \frac{\pi_L}{q}\varphi_L-1}\downarrow & & \| \\ \check{\mathcal{C}} & \times & \mathcal{C} & \xrightarrow{\{\,,\}^0_{M,Iw}} & D(\Gamma_L, K). \end{array}$$

Altogether we obtain the following theorem.

Theorem 4.5.20. *There is a $D(\Gamma_L, K)$-ι_*-sesquilinear pairing*

$$\{\ ,\ \}_{\mathrm{Iw}} : \check{M}^{\psi_L=\frac{q}{\pi_L}} \times M^{\psi_L=1} \to D(\Gamma_L, K). \tag{4.88}$$

It is characterized by the following equality:

$$\big\langle f, \{\check{m}, m\}_{\mathrm{Iw}}\big\rangle_{\Gamma_L} = \big\{f \cdot (\varphi_L - 1)\check{m}, \big(\tfrac{\pi_L}{q}\varphi_L - 1\big)m\big\} \tag{4.89}$$

for all $f \in \mathcal{R}_K(\Gamma_L)$, $\check{m} \in \check{M}$, $m \in M$.

Remark 4.5.21. For any $n \geq n_0$, we obtain in the same way as in (4.88) a $D(\Gamma_n, K)$-ι_*-sesquilinear pairing

$$\{\ ,\ \}_{\mathrm{Iw},\Gamma_n} : \check{M}^{\psi_L=\frac{q}{\pi_L}} \times M^{\psi_L=1} \to D(\Gamma_n, K).$$

It follows immediately from the definitions, the projection formulae (4.77), and the Frobenius reciprocity (Remark 4.5.8) that

$$\{\ ,\ \}_{\mathrm{Iw},\Gamma_n} := (q-1)q^{n-1}\,\mathrm{pr}_{L,n} \circ \{\ ,\ \}_{\mathrm{Iw}}.$$

If $\chi : \Gamma_L \to o_L^\times$ is any continuous character with representation module $W_\chi = o_L \eta_\chi$, then, for any (φ_L, Γ_L)-module M over $\mathcal{R}$, we have the twisted (φ_L, Γ_L)-module $M(\chi)$, where $M(\chi) := M \otimes_{o_L} W_\chi$ as $\mathcal{R}$-module, $\varphi_{M(\chi)}(m \otimes w) := \varphi_M(m) \otimes w$, and $\gamma(m \otimes w) := \gamma(m) \otimes \gamma(w) = \chi(\gamma) \cdot \gamma(m) \otimes w$ for $\gamma \in \Gamma_L$. It follows that

$$\psi_{M(\chi)}(m \otimes w) = \psi_M(m) \otimes w.$$

For the character χ_{LT} we take $W_{\chi_{\mathrm{LT}}} = T = o_L\eta$ and $W_{\chi_{\mathrm{LT}}^{-1}} = T^* = o_L\eta^*$ as representation module, where T^* denotes the o_L-dual with dual basis η^* of η.

Consider the $\mathcal{R}_K$-linear (but of course not $\mathcal{R}_K(\Gamma_L)$-linear) map

$$\mathrm{tw}_\chi : M \to M(\chi), \quad m \mapsto m \otimes \eta_\chi.$$

Lemma 4.5.22. *There is a commutative diagram*

$$\begin{array}{ccccc}
\check{M}(\chi_{\mathrm{LT}}^{-j})^{\psi_L = \frac{q}{\pi_L}} & \times & M(\chi_{\mathrm{LT}}^{j})^{\psi_L = 1} & \xrightarrow{\{\,,\}_{\mathrm{Iw}}} & D(\Gamma_L, \mathbb{C}_p) \\
\uparrow{\scriptstyle \mathrm{tw}_{\chi_{\mathrm{LT}}^{-j}}} & & \uparrow{\scriptstyle \mathrm{tw}_{\chi_{\mathrm{LT}}^{j}}} & & \uparrow{\scriptstyle \mathrm{Tw}_{\chi_{\mathrm{LT}}^{j}}} \\
\check{M}^{\psi_L = \frac{q}{\pi_L}} & \times & M^{\psi_L = 1} & \xrightarrow{\{\,,\}_{\mathrm{Iw}}} & D(\Gamma_L, \mathbb{C}_p).
\end{array}$$

Proof. We have for all $f \in \mathcal{R}_K(\Gamma_L)$,

$$\begin{aligned}
&\left\langle f, \{\mathrm{tw}_{\chi_{\mathrm{LT}}^{-j}}(\check{m}), \mathrm{tw}_{\chi_{\mathrm{LT}}^{j}}(m)\}_{\mathrm{Iw}} \right\rangle_{\Gamma_L} \\
&\qquad = \left\{ f \cdot \left((\varphi_L - 1)\check{m} \otimes \eta^{\otimes -j} \right), \left(\tfrac{\pi_L}{q}\varphi_L - 1 \right) m \otimes \eta^{\otimes j} \right\} \\
&\qquad = \left\{ \left(\mathrm{Tw}_{\chi_{\mathrm{LT}}^{-j}}(f) \cdot (\varphi_L - 1)\check{m} \right) \otimes \eta^{\otimes -j}, \left(\tfrac{\pi_L}{q}\varphi_L - 1 \right) m \otimes \eta^{\otimes j} \right\} \\
&\qquad = \left\{ \left(\mathrm{Tw}_{\chi_{\mathrm{LT}}^{-j}}(f) \cdot (\varphi_L - 1)\check{m} \right), \left(\tfrac{\pi_L}{q}\varphi_L - 1 \right) m \right\} \\
&\qquad = \left\langle \mathrm{Tw}_{\chi_{\mathrm{LT}}^{-j}}(f), \{\check{m}, m\}_{\mathrm{Iw}} \right\rangle_{\Gamma_L} \\
&\qquad = \left\langle f, \mathrm{Tw}_{\chi_{\mathrm{LT}}^{j}}\left(\{\check{m}, m\}_{\mathrm{Iw}} \right) \right\rangle_{\Gamma_L},
\end{aligned}$$

where we used Proposition 4.5.11 for the last equation. The second equation is clear for δ-distributions and hence extends by the uniqueness result of Theorem 4.3.21, cf. the proof of Theorem 4.3.20. ∎

4.5.5 The abstract reciprocity formula

We keep the notation from the preceding subsection and set $t_{\mathfrak{Y}} := \log_{\mathfrak{Y}}$.

Compatibility of the Iwasawa pairing under comparison isomorphisms

Let M, N be (not necessarily étale) L-analytic (φ_L, Γ_L)-modules over $\mathcal{R}$. We extend the action of Γ_L, φ_L, and ψ_L to the $\mathcal{R}[\frac{1}{t_{\mathfrak{Y}}}]$-module $M[\frac{1}{t_{\mathfrak{Y}}}]$ (and in the same way to $N[\frac{1}{t_{\mathfrak{Y}}}]$) as follows:[9]

$$\gamma\frac{m}{t_{\mathfrak{Y}}^k} := \frac{\gamma m}{\gamma t_{\mathfrak{Y}}^k} = \frac{\frac{\gamma m}{\chi_{\mathrm{LT}}^k(\gamma)}}{t_{\mathfrak{Y}}^k},$$

$$\varphi_L\left(\tfrac{m}{t_{\mathfrak{Y}}^k}\right) := \frac{\varphi_L(m)}{\varphi_L(t_{\mathfrak{Y}}^k)} = \frac{\frac{\varphi_L(m)}{\pi_L^k}}{t_{\mathfrak{Y}}^k},$$

$$\psi_L\left(\tfrac{m}{t_{\mathfrak{Y}}^k}\right) := \frac{\pi_L^k\psi_L(m)}{t_{\mathfrak{Y}}^k}.$$

Lemma 4.5.23. *We have the following:*

(i) $(M[\frac{1}{t_{\mathfrak{Y}}}])^{\psi_L=0} = (M^{\psi_L=0})[\frac{1}{t_{\mathfrak{Y}}}] := \{\frac{m}{t_{\mathfrak{Y}}^k} | m \in M^{\psi_L=0},\ k \geq 0\}.$

(ii) *The (separately continuous) $\mathcal{R}_K(\Gamma_L)$-action on $M^{\psi_L=0}$ extends to a (separately continuous with respect to direct limit topology) action of $\mathcal{R}_K(\Gamma_L)$ on $(M[\frac{1}{t_{\mathfrak{Y}}}])^{\psi_L=0}$.*

Proof. For (i) note that $0 = \psi_L(\frac{m}{t_{\mathfrak{Y}}^k}) = \frac{\pi_L^k\psi_L(m)}{t_{\mathfrak{Y}}^k}$ if and only if $\psi_L(m) = 0$. For (ii) take for any $f \in \mathcal{R}_K(\Gamma_L)$ the direct limit of the following commutative diagram:

$$\begin{array}{ccccccccc}
M^{\psi_L=0} & \xrightarrow{t_{\mathfrak{Y}}} & M^{\psi_L=0} & \xrightarrow{t_{\mathfrak{Y}}} & \cdots & \xrightarrow{t_{\mathfrak{Y}}} & M^{\psi_L=0} & \xrightarrow{t_{\mathfrak{Y}}} & \cdots \\
\downarrow f & & \downarrow \mathrm{Tw}_{\chi_{\mathrm{LT}}^{-1}}(f) & & & & \downarrow \mathrm{Tw}_{\chi_{\mathrm{LT}}^{-i}}(f) & & \\
M^{\psi_L=0} & \xrightarrow{t_{\mathfrak{Y}}} & M^{\psi_L=0} & \xrightarrow{t_{\mathfrak{Y}}} & \cdots & \xrightarrow{t_{\mathfrak{Y}}} & M^{\psi_L=0} & \xrightarrow{t_{\mathfrak{Y}}} & \cdots\ .
\end{array}$$

This defines a (separately continuous) action. ∎

Now we assume that there is an isomorphism

$$c : \mathcal{R}\left[\tfrac{1}{t_{\mathcal{Y}}}\right] \otimes_{\mathcal{R}} M \xrightarrow{\cong} \mathcal{R}\left[\tfrac{1}{t_{\mathcal{Y}}}\right] \otimes_{\mathcal{R}} N$$

of (φ_L, Γ_L)-modules over $\mathcal{R}[\frac{1}{t_{\mathfrak{Y}}}]$.

---

[9] Since $t_{\mathfrak{Y}}^k = \varphi_L\left(\frac{t_{\mathfrak{Y}}^k}{\pi_L^k}\right)$, one checks that $\psi_L(t_{\mathfrak{Y}}^k m) = \frac{t_{\mathfrak{Y}}^k}{\pi_L^k}\psi_L(m)$ by the projection formula. In particular, the definition is independent of the chosen denominator.

Consider the composite map

$$\begin{aligned}\check{c} : \mathcal{R}[\tfrac{1}{t_{\mathfrak{Y}}}] \otimes_{\mathcal{R}} \check{M} &\cong \operatorname{Hom}_{\mathcal{R}[\frac{1}{t_{\mathfrak{Y}}}]}\big(\mathcal{R}[\tfrac{1}{t_{\mathfrak{Y}}}] \otimes_{\mathcal{R}} M, \mathcal{R}[\tfrac{1}{t_{\mathfrak{Y}}}] \otimes_{\mathcal{R}} \Omega^1_{\mathcal{R}}\big)\\ &\cong \operatorname{Hom}_{\mathcal{R}[\frac{1}{t_{\mathfrak{Y}}}]}\big(\mathcal{R}[\tfrac{1}{t_{\mathfrak{Y}}}] \otimes_{\mathcal{R}} N, \mathcal{R}[\tfrac{1}{t_{\mathfrak{Y}}}] \otimes_{\mathcal{R}} \Omega^1_{\mathcal{R}}\big)\\ &\cong \mathcal{R}[\tfrac{1}{t_{\mathfrak{Y}}}] \otimes_{\mathcal{R}} \check{N},\end{aligned}$$

where the second isomorphism is $(c^{-1})^*$.

Lemma 4.5.24. *$c^{\psi_L=0}$ and $\check{c}^{\psi_L=0}$ are $\mathcal{R}_K(\Gamma_L)$-equivariant.*

Proof. Consider, for $n \in \mathbb{Z}$, the (φ_L, Γ_L)-modules (!) $M_n := t_{\mathfrak{Y}}^{-n} M$ over $\mathcal{R}$ and note that the inclusion $(M_n)^{\psi_L=0} \subseteq (M[\frac{1}{t_{\mathfrak{Y}}}])^{\psi_L=0}$ is $\mathcal{R}_K(\Gamma_L)$-equivariant by construction of the action. Now, since M, N are finitely generated over $\mathcal{R}$, there exists $n_0 \geq 0$ such that c restricts to a homomorphism $c_0 : M \to N_{n_0}$ of (φ_L, Γ_L)-modules over $\mathcal{R}$, whence $c_0^{\psi_L=0} : M^{\psi_L=0} \to N_{n_0}^{\psi_L=0} \subseteq (N[\frac{1}{t_{\mathfrak{Y}}}])^{\psi_L=0}$ is $\mathcal{R}_K(\Gamma_L)$-equivariant by the functoriality of Theorem 4.3.23 and similarly for the induced maps $c_n : M_n \to N_{n_0+n}$ for all $n \geq 0$. The equivariance for $c^{\psi_L=0}$ follows by taking direct limits.

Similarly, for some $n_0 \geq 0$, the inverse b of c induces homomorphisms $b_n : N_{-n_0-n} \to M_{-n}$ of (φ_L, Γ_L)-modules over $\mathcal{R}$ all $n \in \mathbb{Z}$. We obtain homomorphisms of (φ_L, Γ_L)-modules over $\mathcal{R}$

$$\begin{aligned}\check{c}_n : (\check{M})_n &= \operatorname{Hom}_{\mathcal{R}}(M, t_{\mathfrak{Y}}^{-n}\Omega^1_{\mathcal{R}})\\ &\cong \operatorname{Hom}_{\mathcal{R}}(M_{-n}, \Omega^1_{\mathcal{R}})\\ &\cong \operatorname{Hom}_{\mathcal{R}}(N_{-n_0-n}, \Omega^1_{\mathcal{R}})\\ &\cong (\check{N})_{n_0+n},\end{aligned}$$

where the third isomorphism is $(b_n)^*$. As above $(\check{c}_n)^{\psi_L=0}$ is $\mathcal{R}_K(\Gamma_L)$-equivariant and the claim follows by taking direct limits. ■

Lemma 4.5.25. *The following diagram commutes on the vertical intersections:*

$$\begin{array}{ccccc}
\check{M}^{\psi_L=0} & \times & M^{\psi_L=0} & \xrightarrow{\{\,,\}^0_{M,\mathrm{Iw}}} & \mathcal{R}_K(\Gamma_L)\\
\big\downarrow & & \big\downarrow & & \Big\| \\
\big(\mathcal{R}[\frac{1}{t_{\mathfrak{Y}}}] \otimes_{\mathcal{R}} \check{M}\big)^{\psi_L=0} & \times & \big(\mathcal{R}[\frac{1}{t_{\mathfrak{Y}}}] \otimes_{\mathcal{R}} M\big)^{\psi_L=0} & & \Big\| \\
{\scriptstyle \check{c}}\big\downarrow{\scriptstyle \cong} & & {\scriptstyle c}\big\downarrow{\scriptstyle \cong} & & \Big\| \\
\big(\mathcal{R}[\frac{1}{t_{\mathfrak{Y}}}] \otimes_{\mathcal{R}} \check{N}\big)^{\psi_L=0} & \times & \big(\mathcal{R}[\frac{1}{t_{\mathfrak{Y}}}] \otimes_{\mathcal{R}} N\big)^{\psi_L=0} & & \Big\| \\
\big\uparrow & & \big\uparrow & & \Big\| \\
\check{N}^{\psi_L=0} & \times & N^{\psi_L=0} & \xrightarrow{\{\,,\}^0_{N,\mathrm{Iw}}} & \mathcal{R}_K(\Gamma_L);
\end{array}$$

i.e., if $\check{m} \in \check{M}$, $m \in M$, $\check{n} \in \check{N}$, $n \in N$ *with* $\check{c}(\check{m}) = \check{n}$ *and* $c(m) = n$, *then*

$$\{\check{m}, m\}^0_{M,\mathrm{Iw}} = \{\check{n}, n\}^0_{N,\mathrm{Iw}}.$$

Proof. By definition of the Iwasawa pairings, we have for all $f \in \mathcal{R}_K(\Gamma_L)$

$$\begin{aligned}\left\langle f, \{\check{n}, n\}^0_{N,\mathrm{Iw}}\right\rangle_{\Gamma_L} &= \{f \cdot \check{n}, n\}_N \\ &= \left\{f \cdot \check{c}(\check{m}), c(m)\right\}_N \\ &= \left\{\check{c}(f \cdot \check{m}), c(m)\right\}_N \\ &= \mathrm{res}_{\mathfrak{Y}}\big(\check{c}(f \cdot \check{m})\big(c(m)\big)\big) \\ &= \mathrm{res}_{\mathfrak{Y}}\big(\big((f \cdot \check{m}) \circ c^{-1}\big)\big(c(m)\big)\big) \\ &= \mathrm{res}_{\mathfrak{Y}}\big((f \cdot \check{m})(m)\big) \\ &= \{f \cdot \check{m}, m\}_M \\ &= \left\langle f, \{\check{m}, m\}^0_{M,\mathrm{Iw}}\right\rangle_{\Gamma_L},\end{aligned}$$

whence the claim. Here, we use the $\mathcal{R}_K(\Gamma_L)$-equivariance of $\check{c}$ in the third equality. ■

Now let D be any φ_L-module over L of finite dimension, say d (with trivial Γ_L-action), and consider the (φ_L, Γ_L)-module $N := \mathcal{R} \otimes_L D$ over $\mathcal{R}$ (with diagonal actions). Since $N \cong \mathcal{R}^d$ as Γ_L-module, it is L-analytic. Moreover, we have $\check{N} \cong \Omega^1_{\mathcal{R}} \otimes D^*$ with $D^* = \mathrm{Hom}_L(D, L)$ being the dual φ_L-module. We set

$$\widetilde{\Omega} := \begin{cases} 1 & \text{if } \mathfrak{Y} = \mathfrak{X}, \\ \Omega & \text{if } \mathfrak{Y} = \mathbf{B} \text{ (and } \Omega \in K). \end{cases}$$

Lemma 4.5.26. *If* $\mathfrak{Y} = \mathbf{B}$, *we assume* $\Omega \in K$. *There is a commutative diagram*

$$\begin{array}{ccccc} (\Omega^1_{\mathcal{R}} \otimes D^*)^{\psi_L = 0} & \times & (\mathcal{R} \otimes_L D)^{\psi_L = 0} & \xrightarrow{\widetilde{\Omega}\frac{q-1}{q}\{\ ,\ \}^0_{N,\mathrm{Iw}}} & \mathcal{R}_K(\Gamma_L) \\ {\scriptstyle \mathfrak{M}^{\Omega^1} \otimes \mathrm{id}} \uparrow \cong & & {\scriptstyle \sigma_{-1}\mathfrak{M} \circ \iota_* \otimes \mathrm{id}} \uparrow \cong & & \| \\ \mathcal{R}_K(\Gamma_L) \otimes_L D^* & \times & \mathcal{R}_K(\Gamma_L) \otimes_L D & \longrightarrow & \mathcal{R}_K(\Gamma_L), \end{array}$$

where the bottom line is the $\mathcal{R}_K(\Gamma_L)$*-linear extension of the canonical pairing between* D^* *and* D*; i.e., it maps* $(\lambda \otimes l, \mu \otimes d)$ *to* $\lambda \mu l(d)$.

Proof. Let $\check{d}_j$ and d_i be a basis of D^* and D, respectively, and $x = \sum_j \lambda_j \cdot \check{d}_j$ and $y = \sum_i \mu_i \cdot d_i$. Then, by definition of $\{\ ,\ \}^0_{\mathrm{Iw}}$ we have for all $\lambda \in \mathcal{R}_K(\Gamma_L)$

$$\begin{aligned}&\left\langle \lambda, \left\{(\mathfrak{M}^{\Omega^1} \otimes \mathrm{id})(x), (\sigma_{-1}\mathfrak{M} \circ \iota_* \otimes \mathrm{id})(y)\right\}^0_{\mathrm{Iw}}\right\rangle_{\Gamma_L} \\ &\quad = \left\{(\lambda\mathfrak{M}^{\Omega^1} \otimes \mathrm{id})(x), (\sigma_{-1}\mathfrak{M} \circ \iota_* \otimes \mathrm{id})(y)\right\}\end{aligned}$$

$$
\begin{aligned}
&= \Big\{ \sum_j (\lambda\lambda_j) \cdot (\mathrm{ev}_1\, d\log_{\mathfrak{Y}} \otimes \check{d}_j), \sum_i \iota_*(\mu_i) \cdot \mathrm{ev}_{-1} \otimes d_i \Big\} \\
&= \sum_{i,j} \{(\lambda\lambda_j \mu_i) \cdot (\mathrm{ev}_1\, d\log_{\mathfrak{Y}}) \otimes \check{d}_j, \mathrm{ev}_{-1} \otimes d_i\} \\
&= \sum_{i,j} \mathrm{res}_{\mathfrak{Y}}\big(\check{d}_j(d_i)\, \mathrm{ev}_{-1}(\lambda\lambda_j \mu_i) \cdot (\mathrm{ev}_1\, d\log_{\mathfrak{Y}})\big) \\
&= \sum_{i,j} \check{d}_j(d_i) \mathrm{res}_{\mathfrak{Y}}\big(\mathrm{ev}_{-1}(\lambda\lambda_j \mu_i) \cdot (\mathrm{ev}_1\, d\log_{\mathfrak{Y}})\big).
\end{aligned}
$$

Here, for the third equation we used property (iii) in Lemma 4.5.1. On the other hand, we can pair the image $\sum_{i,j} \lambda_j \mu_i \check{d}_j(d_i)$ of (x, y) under the bottom pairing with λ using the description (4.86)

$$
\begin{aligned}
\tfrac{q}{q-1}\Big\langle \lambda, \sum_{i,j} \lambda_j \mu_i \check{d}_j(d_i) \Big\rangle_{\Gamma_L} &= \sum_{i,j} \check{d}_j(d_i) \{\mathfrak{M}(\sigma_{-1}), \mathfrak{M}^{\Omega^1}(\lambda\lambda_j \mu_i)\} \\
&= \sum_{i,j} \check{d}_j(d_i) \mathrm{res}_{\mathfrak{X}}\big(\mathrm{ev}_{-1}(\lambda\lambda_j \mu_i) \cdot (\mathrm{ev}_1\, d\log_{\mathfrak{X}})\big),
\end{aligned}
$$

whence comparing with the above gives the result for $\mathfrak{Y} = \mathfrak{X}$, using Proposition 4.5.9. If $\mathfrak{Y} = \mathbf{B}$, we obtain the factor Ω due to Remark 4.5.3. ■

Definition 4.5.27. We write:

(i) $\mathrm{Rep}_L^{\mathrm{an}}(G_L)$ for the full subcategory of the category $\mathrm{Rep}_L(G_L)$ consisting of *L-analytic* representations V, i.e., $\mathbb{C}_p \otimes_{\sigma,L} V$ is the trivial semilinear $\mathbb{C}_p$-representation $\mathbb{C}_p^{\dim_L V}$ for all embeddings $\sigma : L \to \mathbb{C}_p$ different from the fixed inclusion $\iota : L \to \mathbb{C}_p$;

(ii) $\mathcal{M}^{\mathrm{an,ét}}(\mathcal{R})$ for the category of étale, analytic (φ_L, Γ_L)-modules over $\mathcal{R}$ (remember that an L-analytic (φ_L, Γ_L)-module M over $\mathcal{R}$ is called étale if it is semistable and of slope 0).

The following theorem is crucial.

Theorem 4.5.28. *There are equivalences of categories*

$$
\begin{aligned}
\mathrm{Rep}_L^{\mathrm{an}}(G_L) &\leftrightarrow \mathcal{M}^{\mathrm{an,ét}}\big(\mathcal{R}_L(\mathbf{B})\big) \\
V &\mapsto D^{\dagger}_{\mathrm{rig}}(V)
\end{aligned}
$$

and

$$
\begin{aligned}
\mathrm{Rep}_L^{\mathrm{an}}(G_L) &\leftrightarrow \mathcal{M}^{\mathrm{an,ét}}\big(\mathcal{R}_L(\mathfrak{X})\big) \\
V &\mapsto D^{\dagger}_{\mathrm{rig}}(V)_{\mathfrak{X}}.
\end{aligned}
$$

Proof. The first equivalence is Theorem D in [6]. By [8, Thm. 3.27] one has an equivalence of categories

$$\begin{aligned} \mathcal{M}^{\mathrm{an,ét}}\big(\mathcal{R}_L(\mathbf{B})\big) &\leftrightarrow \mathcal{M}^{\mathrm{an,ét}}\big(\mathcal{R}_L(\mathfrak{X})\big) \\ M &\mapsto M_{\mathfrak{X}}. \end{aligned} \tag{4.90}$$

■

We recall the definition of the subring $\mathbf{B}_L^\dagger$ of $\mathcal{R}_L(\mathbf{B})$ by defining first $\widetilde{\mathbf{A}} := W(\mathbb{C}_p^\flat)_L$ and

$$\widetilde{\mathbf{A}}^\dagger := \Big\{x = \sum_{n\geq 0} \pi_L^n[x_n] \in \widetilde{\mathbf{A}} : |\pi_L^n||x_n|_\flat^r \xrightarrow{n\to\infty} 0 \text{ for some } r > 0\Big\}.$$

Then we set $\boldsymbol{A}^\dagger := \widetilde{\boldsymbol{A}}^\dagger \cap \boldsymbol{A}$, $\boldsymbol{B}^\dagger := \boldsymbol{A}^\dagger[\frac{1}{\pi_L}]$ as well as $\boldsymbol{A}_L^\dagger := (\boldsymbol{A}^\dagger)^{H_L}$ and $\boldsymbol{B}_L^\dagger := (\boldsymbol{B}^\dagger)^{H_L}$.

It follows from the proof of [6, Thm. 10.1] that for $V \in \mathrm{Rep}_L^{\mathrm{an}}(G_L)$ we have $D_{\mathrm{rig}}^\dagger(V) = \mathcal{R}_L(\mathbf{B}) \otimes_{\mathbf{B}_L^\dagger} D^\dagger(V)$, where $D^\dagger(V)$ belongs to $\mathfrak{M}^{\mathrm{ét}}(\mathbf{B}_L^\dagger)$. From the theory of Wach modules we actually know that $D_{\mathrm{LT}}(V)$ is even of finite height if V is crystalline in addition:

$$D^\dagger(V) = \mathbf{B}_L^\dagger \otimes_{\mathbf{A}_L^+} N(T) = \mathbf{B}_L^\dagger \otimes_{\mathbf{B}_L^+} N(V)$$

for any Galois stable o_L-lattice $T \subseteq V$. From the diagram in (3.2), we thus obtain the following diagram, in which the horizontal maps are equivalences of categories:

$$\begin{array}{ccc}
\mathrm{Mod}_{\mathbf{B}_L^+}^{\varphi_L,\Gamma_L,\mathrm{an}} & \xrightarrow[\simeq]{\mathbf{B}_L\otimes_{\mathbf{B}_L^+}-} & \mathfrak{M}^{\mathrm{et,cris}}(\mathbf{B}_L) \\
{\scriptstyle \mathcal{O}\otimes_{\mathbf{B}_L^+}-}\big\downarrow & \nwarrow{\scriptstyle N(-)} & \big\uparrow{\scriptstyle \simeq\ D_{\mathrm{LT}}(-)} \\
\mathrm{Mod}_{\mathcal{O}}^{\varphi_L,\Gamma_L,0} & \underset{\mathcal{M}\circ D_{\mathrm{cris},L}}{\overset{V_L\circ D}{\rightleftarrows}}\ \simeq & \mathrm{Rep}_L^{\mathrm{cris,an}}(G_L) \\
{\scriptstyle \mathcal{R}_L(\mathbf{B})\otimes_{\mathcal{O}}-}\big\downarrow & & \big\downarrow{\scriptstyle \subseteq} \\
\mathcal{M}\big(\mathcal{R}_L(\mathbf{B})\big)^{\mathrm{an,ét}} & \xleftarrow[\simeq]{D^\dagger(V)} & \mathrm{Rep}_L^{\mathrm{an}}(G_L).
\end{array}$$

Here $\mathfrak{M}^{\mathrm{et,cris}}(\mathbf{B}_L)$ denotes the essential image of $\mathrm{Rep}_L^{\mathrm{cris,an}}(G_L)$ under $D_{\mathrm{LT}}(-)$ in $\mathfrak{M}^{\mathrm{et}}(\mathbf{B}_L)$ with $\mathbf{B}_L := \mathbf{A}_L[\frac{1}{\pi_L}]$.

Now let T be an o_L-lattice in an L-linear continuous representation of G_L such that $V^*(1)$ (and hence $V(\tau^{-1})$) is L-analytic and crystalline. Then it follows from [44] and the discussion above that

$$\begin{aligned} M := D_{\mathrm{rig}}^\dagger\big(V(\tau^{-1})\big) &= \mathcal{R}_L(\mathbf{B}) \otimes_{\mathcal{O}_K(\mathbf{B})} \mathcal{M}\Big(D_{\mathrm{cris},L}\big(V(\tau^{-1})\big)\Big) \\ &= \mathcal{R}_L(\mathbf{B}) \otimes_{\mathbb{A}_L^+} N\big(T(\tau^{-1})\big) \end{aligned}$$

as well as

$$\check{M} = D^{\dagger}_{\mathrm{rig}}\big(V^*(1)\big) = \mathcal{R}_L(\mathbf{B}) \otimes_{\mathcal{O}_K(\mathbf{B})} \mathcal{M}\Big(D_{\mathrm{cris},L}\big(V^*(1)\big)\Big)$$
$$= \mathcal{R}_L(\mathbf{B}) \otimes_{\mathbb{A}_L^+} N\big(T^*(1)\big)$$

and the comparison isomorphism in (3.16) induces isomorphisms

$$\mathrm{comp}_M : M\big[\tfrac{1}{t_{\mathfrak{Y}}}\big] \cong \mathfrak{R}_L(\mathbf{B})\big[\tfrac{1}{t_{\mathfrak{Y}}}\big] \otimes_L D_{\mathrm{cris},L}\big(V(\tau^{-1})\big)$$

and

$$\mathrm{comp}_{\check{M}} : \check{M}\big[\tfrac{1}{t_{\mathfrak{Y}}}\big] \cong \mathfrak{R}_L(\mathbf{B})\big[\tfrac{1}{t_{\mathfrak{Y}}}\big] \otimes_L D_{\mathrm{cris},L}\big(V^*(1)\big).$$

By [8, §3.4 and §3.5] an analogue of Kisin–Ren modules exists for $\mathfrak{Y} = \mathfrak{X}$; i.e., if we take $M := D^{\dagger}_{\mathrm{rig}}(V(\tau^{-1}))_{\mathfrak{X}}$ and $\check{M} = D^{\dagger}_{\mathrm{rig}}(V^*(1))_{\mathfrak{X}}$, we obtain analogous comparison isomorphisms

$$\mathrm{comp}_M : M\big[\tfrac{1}{t_{\mathfrak{Y}}}\big] \cong \mathfrak{R}_L(\mathfrak{X})\big[\tfrac{1}{t_{\mathfrak{Y}}}\big] \otimes_L D_{\mathrm{cris},L}\big(V(\tau^{-1})\big) \tag{4.91}$$

and

$$\mathrm{comp}_{\check{M}} : \check{M}\big[\tfrac{1}{t_{\mathfrak{Y}}}\big] \cong \mathfrak{R}_L(\mathfrak{X})\big[\tfrac{1}{t_{\mathfrak{Y}}}\big] \otimes_L D_{\mathrm{cris},L}\big(V^*(1)\big),$$

which this time stem from [8, Prop. 3.42] by base change $\mathcal{R}_L(\mathfrak{X}) \otimes_{\mathcal{O}_K(\mathfrak{X})} -$ using the inclusion $\mathcal{O}_L(\mathfrak{X})[Z^{-1}] \subseteq \mathcal{R}_L(\mathfrak{X})[\frac{1}{t_{\mathfrak{Y}}}]$. Moreover, these comparison isomorphisms for $\mathbf{B}$ and $\mathfrak{X}$ are compatible with regard to the equivalence of categories (4.90) by [8, Thm. 3.48]. Note that for $c = \mathrm{comp}_M$ and $D = D_{\mathrm{cris},L}(V(\tau^{-1}))$ we have

$$\mathrm{comp}_{\check{M}} = (\mathrm{comp}_{\Omega^1_{\mathcal{R}}} \otimes_L \mathrm{id}_{D^*}) \circ \check{c} \tag{4.92}$$

using the identifications $\Omega^1_{\mathcal{R}} \cong \mathcal{R}(\chi_{\mathrm{LT}})$ and

$$D_{\mathrm{cris},L}\big(V^*(1)\big) \cong D^* \otimes D_{\mathrm{cris},L}\big(L(\chi_{\mathrm{LT}})\big).$$

We set $b := \mathrm{comp}_{\Omega^1_{\mathcal{R}}}(t_{\mathfrak{Y}}^{-1} d\log_{\mathrm{LT}}) = \frac{1}{t_{\mathfrak{Y}}} \otimes \eta \in D_0 := D_{\mathrm{cris},L}(L(\chi_{\mathrm{LT}}))$ and

$$\tilde{\nabla} := \begin{cases} \nabla & \text{if } \mathfrak{Y} = \mathfrak{X}, \\ \frac{\nabla}{\Omega}, & \text{if } \mathfrak{Y} = \mathbf{B} \text{ (and } \Omega \in K). \end{cases}$$

Remark 4.5.29. As operators on $\mathcal{R}$, we have the equalities

$$\nabla = t_{\mathfrak{Y}} \partial^{\mathfrak{Y}}_{\mathrm{inv}} \quad \text{and} \quad \tilde{\nabla} = t_{\mathfrak{Y}} \tilde{\partial}^{\mathfrak{Y}}_{\mathrm{inv}},$$

where we define $\partial^{\mathbf{B}}_{\mathrm{inv}} := \partial_{\mathrm{inv}}$ and

$$\tilde{\partial}^{\mathfrak{Y}}_{\mathrm{inv}} := \begin{cases} \partial^{\mathfrak{X}}_{\mathrm{inv}} & \text{if } \mathfrak{Y} = \mathfrak{X}, \\ \frac{\partial_{\mathrm{inv}}}{\Omega} & \text{if } \mathfrak{Y} = \mathbf{B} \text{ (and } \Omega \in K). \end{cases}$$

Indeed, for $\mathfrak{Y} = \mathbf{B}$ (4.70) grants these equalities of operators on the subring $\mathcal{O}_K(\mathbf{B})$. Concerning the ring $\mathcal{R}_K(\mathbf{B})$, we note that ∇ is acting as a continuous derivation as can be shown in the same way as in [44, Lem. 2.1.2], while for the operator $t_{\mathbf{B}}\partial_{\mathrm{inv}}$ this is clear anyway. Thus, the same equalities hold for $\mathcal{R}$, Remark 4.3.1. Indeed, on the localization $\mathcal{O}_K(\mathbf{B})_{Z^{\mathbb{N}}}$ it extends uniquely by the derivation property and then it extends uniquely by continuity to $\mathcal{R}_K(\mathbf{B})$. Regarding $\mathfrak{Y} = \mathfrak{X}$, note that all operators are defined over K. Since the equality can be checked over $\mathbb{C}_p$, the claim follows from Remark 4.2.9 and the previous case $\mathfrak{Y} = \mathbf{B}$.

Lemma 4.5.30. *Assuming $\Omega \in K$ if $\mathfrak{Y} = \mathbf{B}$, the following diagram commutes:*

$$
\begin{array}{ccc}
\Omega^1_{\mathcal{R}}[\frac{1}{t_{\mathfrak{Y}}}] \otimes D^* & \xrightarrow{\mathrm{comp}_{\Omega^1_{\mathcal{R}}} \otimes_L \mathrm{id}_{D^*}} & \mathcal{R}[\frac{1}{t_{\mathfrak{Y}}}] \otimes D^* \otimes D_0 \\
\uparrow & & \uparrow \tilde{\nabla} \\
(\Omega^1_{\mathcal{R}} \otimes_L D^*)^{\psi_L=0} & & \mathcal{R}^{\psi_L=0} \otimes D^* \otimes D_0 \\
\mathfrak{M}^{\Omega^1} \otimes \mathrm{id}_{D^*} \uparrow \cong & & \mathfrak{M} \otimes \mathrm{id}_{D^* \otimes D_0} \uparrow \cong \\
\mathcal{R}_K(\Gamma_L) \otimes_L D^* & \xrightarrow{\mathrm{id}_{\mathcal{R}_K(\Gamma_L) \otimes_L D^*} \otimes b} & \mathcal{R}_K(\Gamma_L) \otimes_L D^* \otimes D_0 .
\end{array}
$$

Proof. We first give the proof for $\mathfrak{Y} = \mathbf{B}$. Observe, since on D^* we have the identity throughout, that the commutativity of the above diagram follows from the commutativity of

$$
\begin{array}{ccccc}
\mathcal{R}_K(\Gamma_L) & \xrightarrow{\mathfrak{M}^{\Omega^1}} & (\Omega^1_{\mathcal{R}})^{\psi_L=0} & \hookrightarrow & \Omega^1_{\mathfrak{R}}[\frac{1}{t_{\mathfrak{Y}}}]^{\psi_L=0} \\
\| & & \cong \uparrow & & \\
\| & & \mathcal{R}(\chi_{\mathrm{LT}})^{\psi_L=0} & & \cong \downarrow \mathrm{comp}_{\Omega^1_{\mathcal{R}}} \\
\| & & \tilde{\partial}^{\mathfrak{Y}}_{\mathrm{inv}} \otimes t_{\mathfrak{Y}} \uparrow & & \\
\mathcal{R}_K(\Gamma_L) & \xrightarrow{\mathfrak{M} \otimes b} & \mathcal{R}^{\psi_L=0} \otimes_L D_{\mathrm{cris},L}\big(L(\chi_{\mathrm{LT}})\big) & & \\
\tilde{\nabla} \downarrow & & t_{\mathfrak{Y}} \tilde{\partial}^{\mathfrak{Y}}_{\mathrm{inv}} \otimes \mathrm{id} \downarrow & & \\
\mathcal{R}_K(\Gamma_L) & \xrightarrow{\mathfrak{M} \otimes b} & \mathcal{R}^{\psi_L=0} \otimes_L D_{\mathrm{cris},L}\big(L(\chi_{\mathrm{LT}})\big) & \hookrightarrow & \big(\mathfrak{R}[\frac{1}{t_{\mathfrak{Y}}}] \otimes_L D_{\mathrm{cris},L}\big(L(\chi_{\mathrm{LT}})\big)\big)^{\psi_L=0},
\end{array}
\tag{4.93}
$$

where the map $\tilde{\partial}^{\mathfrak{Y}}_{\mathrm{inv}} \otimes t_{\mathfrak{Y}} : \mathcal{R} \otimes_L D_{\mathrm{cris},L}(L(\chi_{\mathrm{LT}})) \to \mathcal{R}(\chi_{\mathrm{LT}})$ sends $f \otimes \frac{1}{t_{\mathfrak{Y}}} \otimes \eta$ to $\tilde{\partial}^{\mathfrak{Y}}_{\mathrm{inv}} f \otimes \eta$ and the composite with the natural identification $\mathcal{R}(\chi_{\mathrm{LT}}) \cong \Omega^1$, which sends η to $d\log_{\mathrm{LT}}$, is the map $\frac{d}{\Omega} : \mathcal{R} \to \Omega^1_{\mathcal{R}}$ upon identifying $\mathcal{R} \otimes_L D_{\mathrm{cris},L}(L(\chi_{\mathrm{LT}}))$ with $\mathcal{R}$ by sending $f \otimes \frac{1}{t_{\mathfrak{Y}}} \otimes \eta$ to f. Remark 4.5.29 implies the commutativity of the left lower corner. For the upper left corner the commutativity follows from (4.64), the easily checked identity $\widetilde{\partial}^{\mathfrak{Y}}_{\mathrm{inv}}\eta(1,Z) = \eta(1,Z)$, and (4.79)

$$\begin{aligned}
\mathfrak{M}^{\Omega^1}(\lambda) &= \big(\mathrm{Tw}_{\chi_{\mathrm{LT}}}(\lambda)\cdot\eta(1,Z)\big)d\log_{\mathrm{LT}}\\
&= \big(\mathrm{Tw}_{\chi_{\mathrm{LT}}}(\lambda)\cdot\widetilde{\partial}^{\mathfrak{Y}}_{\mathrm{inv}}\eta(1,Z)\big)d\log_{\mathrm{LT}}\\
&= \widetilde{\partial}^{\mathfrak{Y}}_{\mathrm{inv}}\big(\lambda\cdot\eta(1,Z)\big)d\log_{\mathrm{LT}}.
\end{aligned}$$

Finally, since $\eta(1,Z)\otimes b \in \mathcal{R}^{\psi_L=0}\otimes_L D_{\mathrm{cris},L}(L(\chi_{\mathrm{LT}}))$ is sent up to $\eta(1,Z)d\log_{\mathrm{LT}}$ and down to $t_{\mathfrak{Y}}\eta(1,Z)\otimes b$, the compatibility with $\mathrm{comp}_{\Omega^1_{\mathcal{R}}}$ is easily checked. The same proof works for $\mathfrak{Y}=\mathfrak{X}$ by using (4.63) instead of (4.64) and replacing $\eta(1,Z)$ and $\frac{d}{\Omega}$ by ev_1 and d, respectively. ■

Now we introduce a pairing – if $\mathfrak{Y}=\mathbf{B}$ assuming $\Omega\in K$ as usual –

$$[\ ,\ ]_{D_{\mathrm{cris},L}(V(\tau^{-1}))}:\quad \mathcal{R}^{\psi_L=0}\otimes_L D_{\mathrm{cris},L}\big(V^*(1)\big)\times\mathcal{R}^{\psi_L=0}\otimes_L D_{\mathrm{cris},L}\big(V(\tau^{-1})\big)\to\mathcal{R}_K(\Gamma_L)$$

which we will abbreviate as $[\ ,\ ] := [\ ,\ ]_{D_{\mathrm{cris},L}(V(\tau^{-1}))}$, by requiring that the following diagram becomes commutative:

$$\begin{array}{ccccc}
\mathcal{R}^{\psi_L=0}\otimes D^*\otimes D_0 & \times & \mathcal{R}^{\psi_L=0}\otimes_L D & \xrightarrow{[\ ,\ ]} & \mathcal{R}_K(\Gamma_L)\\
\mathfrak{M}\otimes\mathrm{id}_{D^*\otimes D_0}\uparrow\cong & & \sigma_{-1}\mathfrak{M}\circ\iota_*\otimes\mathrm{id}\uparrow\cong & & \|\\
\mathcal{R}_K(\Gamma_L)\otimes_L D^*\otimes D_0 & \times & \mathcal{R}_K(\Gamma_L)\otimes_L D & \longrightarrow & \mathcal{R}_K(\Gamma_L),
\end{array} \tag{4.94}$$

where the bottom line sends $(\lambda\otimes l\otimes\beta b, \mu\otimes d)$ to $\lambda\mu\beta l(d)$.

Combining the Lemmata 4.5.26 and 4.5.30, we obtain the following lemma for $N=\mathcal{R}\otimes_L D_{\mathrm{cris},L}(V(\tau^{-1}))$.

Lemma 4.5.31. $[\ ,\ ]_{D_{\mathrm{cris},L}(V(\tau^{-1}))} = \frac{q-1}{q}\{\nabla(\mathrm{comp}_{\Omega^1_{\mathcal{R}}}\otimes_L \mathrm{id}_{D^*})^{-1}(-),-\}^0_{N,\mathrm{Iw}}$.

Setting

$$\begin{aligned}
M' &:= \mathrm{comp}^{-1}\big(\mathcal{R}^{\psi_L=0}\otimes_L D_{\mathrm{cris},L}\big(V(\tau^{-1})\big)\big),\\
\check{M}' &:= \mathrm{comp}^{-1}\big(\mathcal{R}^{\psi_L=0}\otimes_L D_{\mathrm{cris},L}\big(V^*(1)\big)\big)
\end{aligned}$$

we obtain the subsequent result.

Theorem 4.5.32. *Assume $\Omega\in K$ if $\mathfrak{Y}=\mathbf{B}$. For all $x\in\check{M}'\cap(\check{M}^{\psi_L=0})$ and $y\in M'\cap(M^{\psi_L=0})$ it holds that*

$$\tfrac{q-1}{q}\{\nabla x, y\}^0_{\mathrm{Iw}} = [x,y];$$

i.e., the following diagram commutes on the vertical intersections:

$$\begin{array}{ccccc} \check{M}^{\psi_L=0} & \times & M^{\psi_L=0} & \xrightarrow{\frac{q-1}{q}\nabla\{\,,\,\}^0_{M,Iw}} & \mathcal{R}_K(\Gamma_L) \\ \big\downarrow & & \big\downarrow & & \Big\| \\ \big(\mathcal{R}[\frac{1}{t_{\mathcal{Y}}}]\otimes_{\mathcal{R}}\check{M}\big)^{\psi_L=0} & \times & \big(\mathcal{R}[\frac{1}{t_{\mathcal{Y}}}]\otimes_{\mathcal{R}} M\big)^{\psi_L=0} & & \\ \mathrm{comp}_{\check{M}}\big\downarrow\cong & & \mathrm{comp}_M\big\downarrow\cong & & \\ \big(\mathfrak{R}[\frac{1}{t_{\mathfrak{Y}}}]\otimes_L D_{\mathrm{cris},L}(V^*(1))\big)^{\psi_L=0} & \times & \big(\mathfrak{R}[\frac{1}{t_{\mathfrak{Y}}}]\otimes_L D_{\mathrm{cris},L}(V(\tau^{-1}))\big)^{\psi_L=0} & & \\ \big\uparrow & & \big\uparrow & & \\ \mathcal{R}^{\psi_L=0}\otimes_L D_{\mathrm{cris},L}(V^*(1)) & \times & \mathcal{R}^{\psi_L=0}\otimes_L D_{\mathrm{cris},L}(V(\tau^{-1})) & \xrightarrow{[\,,\,]} & \mathcal{R}_K(\Gamma_L). \end{array}$$

Proof. Combine Lemmata 4.5.25 and 4.5.31 using (4.92). ■

Interpretation of the abstract reciprocity formula in terms of the $D_{\mathrm{cris},L}$-pairing

Assuming as usual that $\Omega \in K$ if $\mathfrak{Y} = \mathbf{B}$, the canonical pairing

$$D_{\mathrm{cris},L}\big(V^*(1)\big) \times D_{\mathrm{cris},L}\big(V(\tau^{-1})\big) \to D_{\mathrm{cris},L}\big(L(\chi_{\mathrm{LT}})\big)$$

extends to a pairing

$$\mathcal{R}^{\psi_L=0} \otimes_L D_{\mathrm{cris},L}\big(V^*(1)\big) \times \mathcal{R}^{\psi_L=0} \otimes_L D_{\mathrm{cris},L}\big(V(\tau^{-1})\big) \\ \to \mathcal{R}^{\psi_L=0} \otimes_L D_{\mathrm{cris},L}\big(L(\chi_{\mathrm{LT}})\big)$$

denoted by $[\,,\,]_{\mathrm{cris}}$, by requiring that the following diagram is commutative (in which the lower one is induced by multiplication within $\mathcal{R}_K(\Gamma_L)$ and the natural duality paring on $D_{\mathrm{cris},L}$)

$$\begin{array}{ccccc} \mathcal{R}^{\psi_L=0}\otimes_L D_{\mathrm{cris},L}\big(V^*(1)\big) \times & \mathcal{R}^{\psi_L=0}\otimes_L D_{\mathrm{cris},L}\big(V(\tau^{-1})\big) & \xrightarrow{[\,,\,]_{\mathrm{cris}}} & \mathcal{R}^{\psi_L=0}\otimes_L D_{\mathrm{cris},L}\big(L(\chi_{\mathrm{LT}})\big) \\ \big\uparrow \mathfrak{M}\otimes\mathrm{id} & \big\uparrow \sigma_{-1}\mathfrak{M}\circ\iota_*\otimes\mathrm{id} & & \big\uparrow \mathfrak{M}\otimes\mathrm{id} \\ \mathcal{R}_K(\Gamma_L)\otimes_L D_{\mathrm{cris},L}\big(V^*(1)\big) \times & \mathcal{R}_K(\Gamma_L)\otimes_L D_{\mathrm{cris},L}\big(V(\tau^{-1})\big) & \to & \mathcal{R}_K(\Gamma_L)\otimes_L D_{\mathrm{cris},L}\big(L(\chi_{\mathrm{LT}})\big). \end{array}$$

Note that

$$\mathrm{comp}\big([x,y]\cdot \mathrm{ev}_1 \otimes (t_{\mathfrak{Y}}^{-1}\otimes \eta)\big) = [x,y]_{\mathrm{cris}}.$$

Hence, using the diagram in (4.93), Theorem 4.5.32 is also equivalent to

$$\mathrm{comp}\circ \mathfrak{M}^{\Omega^1}\circ \widetilde{\Omega}\frac{q-1}{q}\{x,y\}^0_{Iw} = \big[\mathrm{comp}(x),\mathrm{comp}(y)\big]_{\mathrm{cris}},$$

i.e., the "commutativity" (whenever it makes sense) of the following diagram:

$$\begin{array}{ccccc}
D^\dagger_{\mathrm{rig}}\big(V^*(1)\big)^{\psi_L=\frac{q}{\pi_L}}\big[\frac{1}{t_{\mathfrak{Y}}}\big] & \times & D^\dagger_{\mathrm{rig}}\big(V(\tau^{-1})\big)^{\psi_L=1}\big[\frac{1}{t_{\mathfrak{Y}}}\big] & \overset{\tilde{\Omega}\frac{q-1}{q}\{\,,\}_{\mathrm{Iw}}}{\dashrightarrow} & cR_K(\Gamma_L) \\
\downarrow{\scriptstyle 1-\varphi_L} & & \downarrow{\scriptstyle 1-\frac{\pi_L}{q}\varphi_L} & & \| \\
D^\dagger_{\mathrm{rig}}\big(V^*(1)\big)^{\psi_L=0}\big[\frac{1}{t_{\mathfrak{Y}}}\big] & \times & D^\dagger_{\mathrm{rig}}\big(V(\tau^{-1})\big)^{\psi_L=0}\big[\frac{1}{t_{\mathfrak{Y}}}\big] & \overset{\tilde{\Omega}\frac{q-1}{q}\{\,,\}^0_{\mathrm{Iw}}}{\dashrightarrow} & \mathcal{R}_K(\Gamma_L) \\
\| & & \| & & \downarrow{\scriptstyle \mathfrak{M}^{\Omega^1}} \\
D^\dagger_{\mathrm{rig}}\big(V^*(1)\big)^{\psi_L=0}\big[\frac{1}{t_{\mathfrak{Y}}}\big] & \times & D^\dagger_{\mathrm{rig}}\big(V(\tau^{-1})\big)^{\psi_L=0}\big[\frac{1}{t_{\mathfrak{Y}}}\big] & \overset{\tilde{\Omega}\frac{q-1}{q}\{\,,\}^0_{\mathrm{Iw}}}{\dashrightarrow} & \mathfrak{R}(\chi_{\mathrm{LT}})\big[\frac{1}{t_{\mathfrak{Y}}}\big]^{\psi_L=0} \\
\downarrow{\scriptstyle \mathrm{comp}\ \cong} & & \downarrow{\scriptstyle \mathrm{comp}\ \cong} & & \downarrow{\scriptstyle \cong\ \mathrm{comp}} \\
\mathfrak{R}^{\psi_L=0}\big[\frac{1}{t_{\mathfrak{Y}}}\big]\otimes_L D_{\mathrm{cris},L}\big(V^*(1)\big) & \times & \mathfrak{R}^{\psi_L=0}\big[\frac{1}{t_{\mathfrak{Y}}}\big]\otimes_L D_{\mathrm{cris},L}\big(V(\tau^{-1})\big) & \overset{[\,,]_{\mathrm{cris}}}{\dashrightarrow} & \mathfrak{R}^{\psi_L=0}\big[\frac{1}{t_{\mathfrak{Y}}}\big]\otimes_L D_{\mathrm{cris},L}\big(L(\chi_{\mathrm{LT}})\big)
\end{array}$$

for $\mathfrak{Y} = \mathbf{B}$, while for $\mathfrak{Y} = \mathfrak{X}$ one has to decorate the $D^\dagger_{\mathrm{rig}}$s with index $\mathfrak{X}$.

Question. Is it possible to extend the definition of $[\,,\,]$ and $\{\,,\,\}$ to $(\check{M}[\frac{1}{t_{\mathfrak{Y}}}])^{\psi_L=0} \times (M[\frac{1}{t_{\mathfrak{Y}}}])^{\psi_L=0}$ by perhaps enlarging the target $\mathcal{R}_K(\Gamma_L)$ by an appropriate localization, which reflects the inversion of $t_{\mathfrak{Y}}$ somehow?