

Chapter 5

Application

For the convenience of the reader we recall some definitions. Firstly, for $V \in \text{Rep}_L(G_L)$ we put

$$D_{\text{cris},L}(V) := (B_{\text{cris}} \otimes_{L_0} V)^{G_L} \quad \text{and} \quad D_{\text{dR},L}(V) := (B_{\text{dR}} \otimes_{\mathbb{Q}_p} V)^{G_L}.$$

The second definition will be used also in the more general form

$$D_{\text{dR},L'}(V) := D_{\text{dR},L'}(V|_{G_{L'}}) = (B_{\text{dR}} \otimes_{\mathbb{Q}_p} V)^{G_{L'}}$$

for L' any finite extension of L . Moreover, we write $D_{\text{dR},L'}^0(V) := \text{Fil}^0(D_{\text{dR},L'}(V))$. Recall that $T_\pi = {}_{o_L}\eta$ denotes the Tate module of the Lubin–Tate formal group LT . We set $V_\pi := L \otimes_{o_L} T_\pi \in \text{Rep}_L(G_L)$. We write $\tau = \chi_{\text{LT}}^{-1} \chi_{\text{cyc}}$, $t_{\text{LT}} = \log_{\text{LT}}(\omega_{\text{LT}}) \in B_{\text{cris},L}$, and analogously $t_{\mathbb{G}_m} := \log_{\widehat{\mathbb{G}}_m}(\omega_{\widehat{\mathbb{G}}_m})$ for the Lubin–Tate formal group $\widehat{\mathbb{G}}_m$. Then $\mathbf{d}_r := t_{\text{LT}}^r t_{\mathbb{Q}_p}^{-1} \otimes (\eta^{\otimes -r} \otimes \eta_{\text{cyc}})$, where η_{cyc} is a generator of the cyclotomic Tate module $\mathbb{Z}_p(1)$, is an L -basis of $D_{\text{dR},L}^0(V_\pi^{\otimes -r}(1))$.

The generalized Iwasawa cohomology of $T \in \text{Rep}_{o_L}(G_L)$ is defined by

$$H_{\text{Iw}}^*(L_\infty/L, T) := \varprojlim_{L'} H^*(L', T),$$

where L' runs through the finite Galois extensions of L contained in L_∞ and the transition maps in the projective system are the cohomological corestriction maps. For $V := T \otimes_{o_L} L \in \text{Rep}_L(G_L)$ we define

$$H_{\text{Iw}}^*(L_\infty/L, V) := H_{\text{Iw}}^*(L_\infty/L, T) \otimes_{o_L} L,$$

which is independent of the choice of T . As usual, we denote by

$$\text{cor} : H_{\text{Iw}}^*(L_\infty/L, T) \rightarrow H^*(L', T)$$

the projection map and analogously for rational coefficients.

5.1 The regulator map

Let T be in $\text{Rep}_{o_L, f}^{\text{cris}}(G_L)$ such that $T(\tau^{-1})$ belongs to $\text{Rep}_{o_L, f}^{\text{cris, an}}(G_L)$ with all Hodge–Tate weights in $[0, r]$, and such that $V := L \otimes_{o_L} T$ does not have any quotient isomorphic to $L(\tau)$. Then the assumptions of Lemma 3.3.6 are satisfied and we may define the regulator maps

$$L_V : H_{\text{Iw}}^1(L_\infty/L, T) \rightarrow D(\Gamma_L, \mathbb{C}_p) \otimes_L D_{\text{cris},L}(V(\tau^{-1})),$$

$$\begin{aligned} \mathcal{L}_V^0 : H_{\text{Iw}}^1(L_\infty/L, T) &\rightarrow \mathcal{O}_L(\mathbf{B})^{\psi_L=0} \otimes_L D_{\text{cris},L}(V(\tau^{-1})), \\ \mathcal{L}_V : H_{\text{Iw}}^1(L_\infty/L, T) &\rightarrow D(\Gamma_L, \mathbb{C}_p) \otimes_L D_{\text{cris},L}(V) \end{aligned}$$

as (part of) the composite

$$\begin{aligned} H_{\text{Iw}}^1(L_\infty/L, T) &\cong D_{\text{LT}}(T(\tau^{-1}))^{\psi_L=1} \\ &= N(T(\tau^{-1}))^{\psi_{D_{\text{LT}}(T(\tau^{-1}))}=1} \xrightarrow{(1-\frac{\pi_L}{q}\varphi_L)} \varphi_L^*(N(V(\tau^{-1})))^{\psi_L=0} \\ &\hookrightarrow \mathcal{O}^{\psi_L=0} \otimes_L D_{\text{cris},L}(V(\tau^{-1})) \subseteq \mathcal{O}_{\mathbb{C}_p}(\mathbf{B})^{\psi_L=0} \otimes_L D_{\text{cris},L}(V(\tau^{-1})) \\ &\xrightarrow{\mathfrak{M}^{-1} \otimes \text{id}} D(\Gamma_L, \mathbb{C}_p) \otimes_L D_{\text{cris},L}(V(\tau^{-1})) \rightarrow D(\Gamma_L, \mathbb{C}_p) \otimes_L D_{\text{cris},L}(V) \end{aligned} \tag{5.1}$$

using [80, Thm. 5.13], Lemma 3.3.6, the inclusion (3.17), and where the last map sends $\mu \otimes d \in D(\Gamma_L, \mathbb{C}_p) \otimes_L D_{\text{cris},L}(V(\tau^{-1}))$ to $\mu \otimes d \otimes \mathbf{d}_1 \in D(\Gamma_L, \mathbb{C}_p) \otimes_L D_{\text{cris},L}(V(\tau^{-1})) \otimes_L D_{\text{cris},L}(L(\tau)) \cong D(\Gamma_L, \mathbb{C}_p) \otimes_L D_{\text{cris},L}(V)$. Note that we have $D := D_{\text{cris},L}(L(\tau)) = D_{\text{dR},L}^0(L(\tau)) = L\mathbf{d}_1$ with $\mathbf{d}_1 = t_{\text{LT}} t_{\mathbb{Q}_p}^{-1} \otimes (\eta^{\otimes -1} \otimes \eta_{\text{cyc}})$, where $L(\chi_{\text{LT}}) = L\eta$ and $L(\chi_{\text{cyc}}) = L\eta_{\text{cyc}}$.

Alternatively, in order to stress that the regulator is essentially the map $1 - \varphi_L$, one can rewrite this as

$$\begin{aligned} H_{\text{Iw}}^1(L_\infty/L, T) &\cong D_{\text{LT}}(V(\tau^{-1}))^{\psi_L=1} = N(T(\tau^{-1}))^{\psi_{D_{\text{LT}}(T(\tau^{-1}))}=1} \\ &\hookrightarrow N(V(\tau^{-1}))^{\psi_{D_{\text{LT}}(V(\tau^{-1}))}=1} \otimes_L D \xrightarrow{1-\varphi_L} \varphi_L^*(N(V(\tau^{-1})))^{\psi_L=0} \otimes_L D \\ &\hookrightarrow \mathcal{O}^{\psi_L=0} \otimes_L D_{\text{cris},L}(V(\tau^{-1})) \otimes_L D \subseteq \mathcal{O}_{\mathbb{C}_p}(\mathbf{B})^{\psi_L=0} \otimes_L D_{\text{cris},L}(V) \\ &\xrightarrow{\mathfrak{M}^{-1} \otimes \text{id}} D(\Gamma_L, \mathbb{C}_p) \otimes_L D_{\text{cris},L}(V), \end{aligned}$$

where the \hookrightarrow in the second line sends n to $n \otimes \mathbf{d}_1$ and the φ_L now acts diagonally. By construction, this regulator map \mathcal{L}_V takes values in $D(\Gamma_L, K)^{G_L,*} \otimes_L D_{\text{cris},L}(V)$, where the twisted action of G_L on the distribution algebra is induced by the Mellin transform as in (ii) of Proposition 4.1.25.

We write $\nabla_{\text{Lie}} \in \text{Lie}(\Gamma_L)$ for the element in the Lie algebra of Γ_L corresponding to 1 under the identification $\text{Lie}(\Gamma_L) = L$.

Proposition 5.1.1. *The regulator maps for V and $V(\chi_{\text{LT}})$ – assuming that both representations satisfy the conditions above – are related by*

$$\mathcal{L}_{V(\chi_{\text{LT}})}(x \otimes \eta) = \nabla_{\text{Lie}} \cdot \left(\frac{1}{\Omega} \text{Tw}_{\chi_{\text{LT}}}^{-1}(\mathcal{L}_V(x)) \otimes t_{\text{LT}}^{-1} \eta \right);$$

i.e., the following Γ_L -equivariant diagram commutes:

$$\begin{array}{ccc} H_{\text{Iw}}^1(L_\infty/L, V(\chi_{\text{LT}})) & \xrightarrow{\mathcal{L}_{V(\chi_{\text{LT}})}} & D(\Gamma_L, \mathbb{C}_p) \otimes_L D_{\text{cris},L}(V(\chi_{\text{LT}})) \\ \cong \downarrow & & \uparrow \frac{\nabla_{\text{Lie}} \text{Tw}_{\chi^{-1}}}{\Omega} \otimes t_{\text{LT}}^{-1} \\ H_{\text{Iw}}^1(L_\infty/L, V) \otimes_{\mathcal{O}_L} L(\chi_{\text{LT}}) & \xrightarrow{\mathcal{L}_V \otimes L(\chi_{\text{LT}})} & D(\Gamma_L, \mathbb{C}_p) \otimes_L D_{\text{cris},L}(V) \otimes_L L(\chi_{\text{LT}}). \end{array}$$

Proof. The proof is analogous to [51, Prop. 3.1.4]. Note that the period Ω enters due to (4.64). ■

This twisting property can be used to drop the condition concerning the Hodge–Tate weights in the definition of the regulator map; i.e., upon replacing $D(\Gamma_L, \mathbb{C}_p)$ in the target by its total ring of quotients, one can extend the regulator map as usual to all T in $\text{Rep}_{o_L, f}^{\text{cris}}(G_L)$ such that $T(\tau^{-1})$ belongs to $\text{Rep}_{o_L, f}^{\text{cris, an}}(G_L)$.

In order to better understand the effect of twisting, we have the following lemma.

Lemma 5.1.2. *For $\mu \in D(\Gamma_L, K)$ we have*

$$\frac{1}{\Omega} (\nabla_{\text{Lie}} \text{Tw}_{\chi^{-1}})(\mu) = \mathfrak{M}^{-1}(t_{\text{LT}} \mathfrak{M}(\mu))$$

and for all $n \geq 1$

$$(\nabla_{\text{Lie}} \text{Tw}_{\chi^{-1}}(\mu))(\chi_{\text{LT}}^n) = n\mu(\chi_{\text{LT}}^{n-1}).$$

Proof. The first claim follows by combining (5.5) with (4.64), while the second claim is just Lemma 4.1.22 applied to the first. ■

One significance of regulator maps is that it should interpolate (dual) Bloch–Kato exponential maps. We shall prove such interpolation formulae in Section 5.2.4 by means of a reciprocity formula.

5.1.1 The basic example

Setting $U := \varprojlim_{\leftarrow n} o_{L_n}^\times$ with transition maps given by the norm, we have the maps

$$\begin{aligned} CW_r : U \otimes_{\mathbb{Z}} T_\pi^{\otimes -r} &\rightarrow D_{\text{dR}, L}^0(V_\pi^{\otimes -r}(1)) = L\mathbf{d}_r \\ u \otimes a\eta^{\otimes -r} &\mapsto a \frac{1 - \pi_L^{-r}}{(r-1)!} \partial_{\text{inv}}^r \log g_{u, \eta}(Z)|_{Z=0} \mathbf{d}_r \end{aligned}$$

for all $r \geq 1$. Furthermore, $g_{u, \eta}$ denotes the Coleman power series (cf. [21, Thm. A, Cor. 17]) attached to $u = (u_n)_n$ and $\eta = (\eta_n)_n$ satisfying $g_{u, \eta}(\eta_n) = u_n$ for all $n \geq 1$.

We are looking for a map

$$\mathcal{L} : U \otimes_{\mathbb{Z}} T_\pi^* \rightarrow D(\Gamma_L, \mathbb{C}_p) \otimes_L D_{\text{cris}, L}(L(\tau))$$

such that

$$\frac{\Omega^r}{r!} \frac{1 - \pi_L^{-r}}{1 - \frac{\pi_L^r}{q}} \mathcal{L}(u \otimes a\eta^*) (\chi_{\text{LT}}^r) \otimes (t_{\text{LT}}^{r-1} \otimes \eta^{\otimes -r+1}) = CW_r(u \otimes a\eta^{\otimes -r}) \quad (5.2)$$

for all $r \geq 1, u \in U, a \in o_L$. Recall that $D_{\text{cris}, L}(L(\tau)) = L\mathbf{d}_1$ and note the equalities $D_{\text{dR}, L}^0(V_\pi^{\otimes -r}(1)) = L\mathbf{d}_r = L\mathbf{d}_1 \otimes (t_{\text{LT}}^{r-1} \otimes \eta^{\otimes -r+1})$.

Theorem 5.1.3 (A special case of Kato’s explicit reciprocity law [80, Cor. 8.7]). *For $r \geq 1$ the diagram*

$$\begin{array}{ccc}
 U \otimes_{\mathbb{Z}} T_{\pi}^{\otimes -r} & & \\
 \downarrow -\kappa \otimes \text{id} & \searrow & \\
 H_{\text{Iw}}^1(L_{\infty}/L, T_{\pi}^{\otimes -r}(1)) & \xrightarrow{CW_r} & \\
 \downarrow \text{cor} & & \\
 H^1(L, T_{\pi}^{\otimes -r}(1)) & \xrightarrow{\text{exp}^*} & D_{\text{dR},L}^0(V_{\pi}^{\otimes -r}(1)) = \mathbf{Ld}_r
 \end{array}$$

commutes.

For the definition of the Kummer map κ see [80, §6], while for the dual Bloch–Kato exponential map exp^* we refer the reader to [12].

We set $\mathcal{L} = \mathfrak{L} \otimes \mathbf{d}_1$ with \mathfrak{L} given as the composition

$$\begin{aligned}
 \mathcal{L} : U \otimes T_{\pi}^* &\xrightarrow{\nabla} o_L[\omega_{\text{LT}}]^{\psi_L=1} \xrightarrow{(1-\frac{\pi_L}{q}\varphi)} \mathcal{O}_{\mathbb{C}_p}(\mathbf{B})^{\psi_L=0} \\
 &\xrightarrow{\log_{\text{LT}}} \mathfrak{D}_{\mathbb{C}_p}(\mathbf{B})^{\psi_L=0} \xrightarrow{\mathfrak{M}^{-1}} D(\Gamma_L, \mathbb{C}_p),
 \end{aligned}$$

where the map ∇ has been defined in [80, §6] as the homomorphism

$$\begin{aligned}
 \nabla : U \otimes_{\mathbb{Z}} T^* &\rightarrow o_L[\omega_{\text{LT}}]^{\psi=1} \\
 u \otimes a\eta^* &\mapsto a \frac{\partial_{\text{inv}}(g_{u,\eta})}{g_{u,\eta}}(\omega_{\text{LT}}).
 \end{aligned}$$

Note that due to the multiplication by \log_{LT} , the maps $\mathcal{L}, \mathfrak{L}$ are not Γ_L -equivariant. Using Lemmata 4.1.21 and 4.1.22, we obtain

$$\begin{aligned}
 \mathfrak{L}(u \otimes a\eta^*)(\chi_{\text{LT}}^r) &= a\mathfrak{M}^{-1}(\log_{\text{LT}}(1 - \frac{\pi_L}{q}\varphi)\partial_{\text{inv}} \log g_{u,\eta})(\chi_{\text{LT}}^r) \\
 &= a\Omega^{-1}r\mathfrak{M}^{-1}(1 - \frac{\pi_L}{q}\varphi)(\partial_{\text{inv}} \log g_{u,\eta})(\chi_{\text{LT}}^{r-1}) \\
 &= ar\Omega^{-r}(1 - \frac{\pi_L}{q}\pi_L^{r-1})(\partial_{\text{inv}}^{r-1} \partial_{\text{inv}} \log g_{u,\eta})|_{Z=0} \\
 &= ar\Omega^{-r}(1 - \frac{\pi_L^r}{q})(\partial_{\text{inv}}^{r-1} \partial_{\text{inv}} \log g_{u,\eta})|_{Z=0};
 \end{aligned} \tag{5.3}$$

i.e., \mathcal{L} satisfies (5.2), indeed. By construction and Proposition 4.1.25 the image of \mathcal{L} actually lies in the G_L -invariants:

$$\mathcal{L} : U \otimes_{\mathbb{Z}} T_{\pi}^* \rightarrow D(\Gamma_L, K)^{G_L} \otimes_L D_{\text{cris},L}(L(\tau)).$$

We claim that the composition

$$\begin{array}{ccc}
 U \otimes_{\mathbb{Z}} T_{\pi}^* & \dashrightarrow & \\
 \downarrow \kappa \otimes T_{\pi}^* & & \\
 H_{\text{Iw}}^1(L_{\infty}/L, o_L(\tau)) & \xrightarrow{\mathfrak{L}_{L(\tau\chi_{\text{LT}})} \otimes o_L(\chi_{\text{LT}}^{-1}) \otimes \iota_{\text{LT}}} & D(\Gamma_L, \mathbb{C}_p) \otimes_L D_{\text{cris},L}(L(\tau))
 \end{array} \tag{5.4}$$

coincides with

$$\mathcal{L} : U \otimes_{\mathbb{Z}} T_{\pi}^* \rightarrow D(\Gamma_L, \mathbb{C}_p) \otimes_L D_{\text{cris},L}(L(\tau)).$$

Indeed, from the commutativity of the diagram in Figure 5.1 (cf. with [51, Appx. C] for $L = \mathbb{Q}_p$) the above claim immediately follows by tensoring the diagram for $r = 1$ with $o_L(\chi_{\text{LT}}^{-1})$ and then composing with the multiplication by t_{LT} . In this diagram, we use the following abbreviations:

$$\begin{aligned} \diamond_r &:= \mathcal{L}_{L(\tau\chi_{\text{LT}}^r)} \otimes \mathbf{d}_1^{\otimes -1} \text{ for any } r \geq 1, \\ e_r &:= t_{\text{LT}}^{-r} \otimes \eta^{\otimes r} \in D_{\text{cris},L}(L(\chi_{\text{LT}}^r)), \\ \mathfrak{I}_i &:= t_{\text{LT}} \partial_{\text{inv}} - i, \end{aligned}$$

and we recall that $\partial_{\text{inv}} = \frac{d}{dt_{\text{LT}}}$.

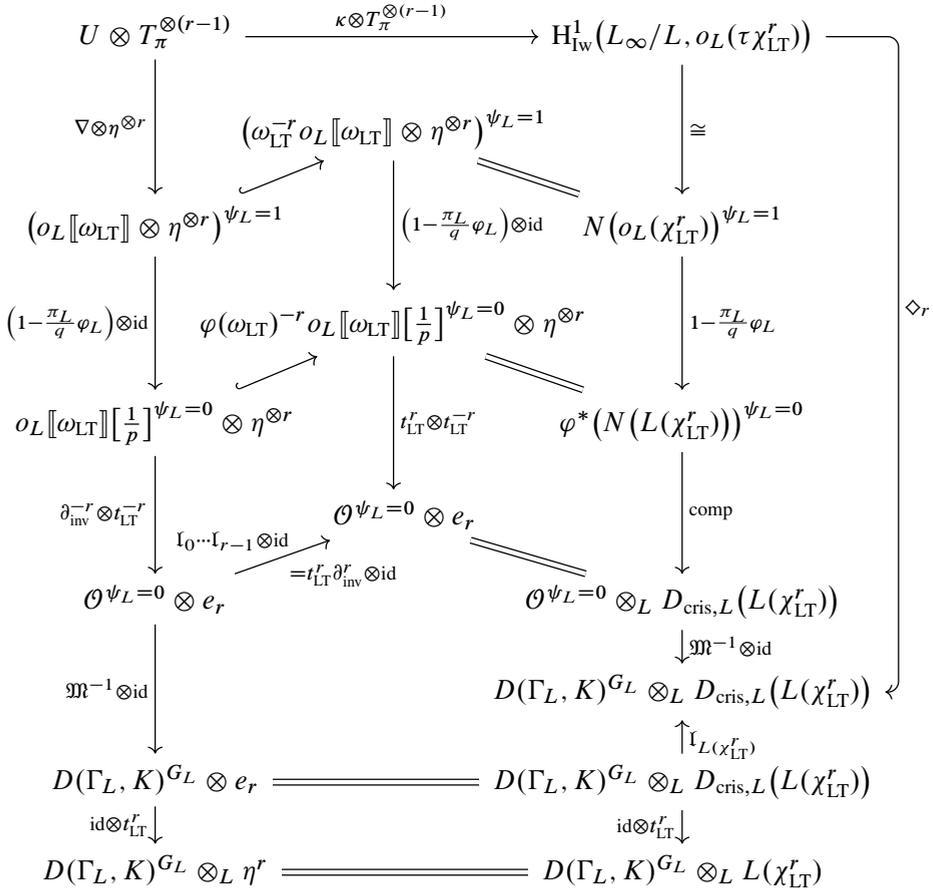


Figure 5.1. Comparison of \mathcal{L} and $\mathcal{L}_{L(\chi_{\text{LT}}^r)}$.

Note that we have

$$\mathfrak{M}^{-1}(i_0 f) = \lim_{\gamma \rightarrow 1} \frac{\delta_\gamma(\mathfrak{M}^{-1}(f)) - \mathfrak{M}^{-1}(f)}{\ell(\gamma)} = \nabla_{\text{Lie}} \mathfrak{M}^{-1}(f), \quad (5.5)$$

see [44, Lem. 2.1.4] for the fact that

$$\nabla_{\text{Lie}} = t_{\text{LT}} \partial_{\text{inv}}$$

as operators on \mathcal{O} . By abuse of notation we thus also write

$$\mathfrak{L}_i := \nabla_{\text{Lie}} - i$$

for the corresponding element in $D(\Gamma_L, K)$; compare [75, §2.3] for the action of $\text{Lie}(\Gamma_L)$ on and its embedding into $D(\Gamma_L, K)$. Moreover, we set

$$\mathfrak{L}_{L(\chi_{\text{LT}}^r)} := \prod_{i=0}^{r-1} \mathfrak{L}_i.$$

Note that ∂_{inv} is invertible on $\mathcal{O}^{\psi_L=0}$ by [33, Prop. 3.12]. Finally, the map

$$\text{comp} : \varphi^*(N(o_L(\chi_{\text{LT}}^r)))^{\psi_L=0} \rightarrow \mathcal{O}^{\psi_L=0} \otimes_L D_{\text{cris},L}(L(\chi_{\text{LT}}^r))$$

is given by Corollary 3.1.14. Note that the commutativity of the upper part of the diagram in Figure 5.1 is an immediate consequence of [80, Thm. 6.2].

Inspired by Proposition 5.1.1, we define $\mathcal{L}_{L(\tau)}$ – since $L(\tau)$ does not satisfy the conditions from the beginning of this chapter while $L(\tau\chi_{\text{LT}})$ does – as a twist of $\mathcal{L}_{L(\tau\chi_{\text{LT}})}$ by requiring the commutativity of the diagram in Figure 5.2, which is possible due to the commutativity of the diagram in Figure 5.1.

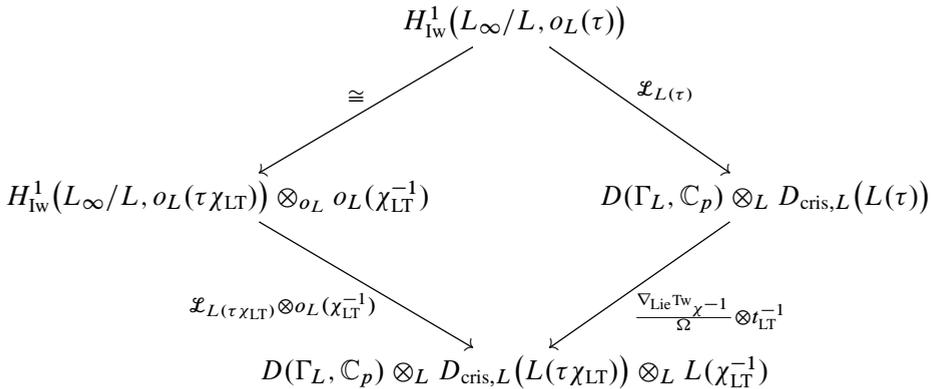


Figure 5.2. Comparison of $\mathcal{L}_{L(\tau)}$ and $\mathcal{L}_{L(\tau\chi_{\text{LT}})}$.

Then

$$\mathcal{L} : U \otimes_{\mathbb{Z}} T_{\pi}^* \rightarrow D(\Gamma_L, \mathbb{C}_p) \otimes_L D_{\text{cris},L}(L(\tau))$$

also coincides with

$$U \otimes_{\mathbb{Z}} T_{\pi}^* \xrightarrow{\kappa \otimes T_{\pi}^*} H_{\text{Iw}}^1(L_{\infty}/L, o_L(\tau)) \xrightarrow{(\frac{1}{\Omega} \nabla_{\text{Lie Tw}_{\chi^{-1}} \otimes \text{id}}) \circ \mathcal{L}_L(\tau)} D(\Gamma_L, \mathbb{C}_p) \otimes_L D_{\text{cris},L}(L(\tau)) \quad (5.6)$$

by Proposition 5.1.1.

We refer the interested reader to [74, §5] for an example of a CM-elliptic curve E with supersingular reduction at p in which they attach to a norm-compatible sequence of elliptic units $e(\alpha)$ (in the notation of [27, §II 4.9]) a distribution $\mu(\alpha) \in D(\Gamma_L, K)$ in [74, Prop. 5.2] satisfying a certain interpolation property with respect to the values of the attached (partial) Hecke- L -function. Without going into any detail concerning their setting and instead referring the reader to the notation in loc. cit., we just want to point out that up to twisting this distribution is the image of $\kappa(e(\alpha)) \otimes \eta^{-1}$ under the regulator map $\mathcal{L}_L(\tau)$:

$$\mathcal{L}_L(\tau)(\kappa(e(\alpha)) \otimes \eta^{-1}) = \Omega \text{Tw}_{\chi_{\text{LT}}}(\mu(\alpha)) \otimes \mathbf{d}_1.$$

Here, $L = \mathbf{K}_p = \mathbf{F}_{\wp}$ (in their notation) is the unique unramified extension of \mathbb{Q}_p of degree 2, $\pi_L = p$, $q = p^2$, and the Lubin–Tate formal group is \widehat{E}_{\wp} , while $K = \widehat{L}_{\infty}$.

Indeed, we have a commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{\text{Col}} & D(\Gamma_L, K) \\ \kappa(-) \otimes \eta^{-1} \downarrow & & \downarrow \Omega \text{Tw}_{\chi_{\text{LT}}} \otimes \mathbf{d}_1 \\ H_{\text{Iw}}^1(L_{\infty}/L, o_L(\tau)) & \xrightarrow{\mathcal{L}_L(\tau)} & D(\Gamma_L, K) \otimes_L D_{\text{cris},L}(L(\tau)), \end{array} \quad (5.7)$$

where the Coleman map Col is given as the composite in the upper line of the following commutative diagram:

$$\begin{array}{ccccccc} U & \xrightarrow{\log g^{-1}} & \mathcal{O}^{\psi_L = \frac{1}{\pi_L}} & \xrightarrow{1 - \frac{\varphi_L}{p^2}} & \mathcal{O}^{\psi_L = 0} & \xrightarrow{\mathfrak{M}^{-1}} & D(\Gamma_L, K) \\ \downarrow & & \downarrow \partial_{\text{inv}} & & \downarrow \partial_{\text{inv}} & & \downarrow \nabla_{\text{Lie}} \\ U \otimes T_p^* & \xrightarrow{\nabla} & \mathcal{O}^{\psi_L = 1} & \xrightarrow{1 - \frac{\pi_L}{q} \varphi_L} & \mathcal{O}^{\psi_L = 0} & \xrightarrow{\log_{\text{LT}}} & \mathcal{O}^{\psi_L = 0} & \xrightarrow{\mathfrak{M}^{-1}} & D(\Gamma_L, K), \end{array} \quad (5.8)$$

in which the second line is just \mathcal{L} . Then the commutativity of (5.7) follows by comparing (5.8) with (5.6). Finally, $\text{Col}(e(\alpha)) = \mu(\alpha) (= \mathfrak{M}^{-1}(g_{\alpha}(Z)))$ in their notation) holds by construction in loc. cit. upon noting that on $\mathcal{O}^{\psi_L = \frac{1}{\pi_L}}$ the operator $1 - \frac{\pi}{p^2} \varphi_L \circ \psi_L$, which is used implicitly to define $g_{\alpha}(Z) (= (1 - \frac{\pi}{p^2} \varphi_L \circ \psi_L) \log Q_{\alpha}(Z))$, equals $1 - \frac{\varphi_L}{p^2}$.

5.2 Relation to Berger’s and Fourquaux’s big exponential map

Let V denote an L -analytic representation of G_L and take an integer $h \geq 1$ such that $\text{Fil}^{-h} D_{\text{cris},L}(V) = D_{\text{cris},L}(V)$ and such that $D_{\text{cris},L}(V)^{\varphi_L = \pi_L^{-h}} = 0$ holds. Under these conditions in [7], a big exponential map à la Perrin-Riou

$$\Omega_{V,h} : (\mathcal{O}^{\psi_L=0} \otimes_L D_{\text{cris},L}(V))^{\Delta=0} \rightarrow D_{\text{rig}}^\dagger(V)^{\psi_L = \frac{q}{\pi_L}}$$

is constructed as follows: According to [7, Lem. 3.5.1], there is an exact sequence

$$\begin{aligned} 0 \rightarrow \bigoplus_{k=0}^h t_{\text{LT}}^k D_{\text{cris},L}(V)^{\varphi_L = \pi_L^{-k}} &\rightarrow (\mathcal{O} \otimes_{o_L} D_{\text{cris},L}(V))^{\psi_L = \frac{q}{\pi_L}} \\ \xrightarrow{1-\varphi_L} \mathcal{O}^{\psi_L=0} \otimes_L D_{\text{cris},L}(V) &\xrightarrow{\Delta} \bigoplus_{k=0}^h D_{\text{cris},L}(V)/(1 - \pi_L^k \varphi_L) D_{\text{cris},L}(V) \rightarrow 0, \end{aligned}$$

where, for $f \in \mathcal{O} \otimes_L D_{\text{cris},L}(V)$, we denote by $\Delta(f)$ the image of the map

$$\bigoplus_{k=0}^h (\partial_{\text{inv}}^k \otimes \text{id}_{D_{\text{cris},L}(V)})(f)(0) \rightarrow \bigoplus_{k=0}^h D_{\text{cris},L}(V)/(1 - \pi_L^k \varphi_L) D_{\text{cris},L}(V).$$

Hence, if already $f \in (\mathcal{O}^{\psi_L=0} \otimes_L D_{\text{cris},L}(V))^{\Delta=0}$, then there exists an element $y \in (\mathcal{O} \otimes_{o_L} D_{\text{cris},L}(V))^{\psi_L = \frac{q}{\pi_L}}$ such that $f = (1 - \varphi_L)y$. Setting $\nabla_i := \nabla - i$ for any integer i , one observes that $\nabla_{h-1} \circ \dots \circ \nabla_0$ annihilates $\bigoplus_{k=0}^{h-1} t_{\text{LT}}^k D_{\text{cris},L}(V)^{\varphi_L = \pi_L^{-k}}$, whence $\Omega_{V,h}(f) := \nabla_{h-1} \circ \dots \circ \nabla_0(y)$ is well defined and belongs under the comparison isomorphism in (3.16) to $D_{\text{rig}}^\dagger(V)^{\psi_L = \frac{q}{\pi_L}}$ by Proposition 3.1.13.

Note that

$$(\mathcal{O}^{\psi_L=0} \otimes_L D_{\text{cris},L}(V))^{\Delta=0} = \mathcal{O}^{\psi_L=0} \otimes_L D_{\text{cris},L}(V)$$

if $D_{\text{cris},L}(V)^{\varphi_L = \pi_L^{-k}} = 0$ for all $0 \leq k \leq h$. If this does not hold for V itself, it does hold for $V(\chi_{\text{LT}}^{-r})$ for r sufficiently large (with respect to the same h).

In the special case $L = \mathbb{Q}_p$, the above map becomes the exponential map due to Perrin-Riou and satisfies the following adjointness property with Loeffler’s and Zerbes’ regulator map, see [51, Prop. A.2.2], where the upper pairing and notation are introduced:

$$\begin{array}{ccc} D(\Gamma, \mathbb{Q}_p) \otimes_{\Lambda_L} H_{\text{Iw}}(\mathbb{Q}_p, V^*(1)) & \times & D(\Gamma, \mathbb{Q}_p) \otimes_{\Lambda_{\mathbb{Q}_p}} H_{\text{Iw}}(\mathbb{Q}_p, V) \rightarrow D(\Gamma, \mathbb{Q}_p) \\ \uparrow \Omega_{V^*(1),1} & & \gamma_{-1} \mathcal{L}_V \downarrow \parallel \\ D(\Gamma, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} D_{\text{cris},\mathbb{Q}_p}(V^*(1)) & \times & D(\Gamma, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} D_{\text{cris},\mathbb{Q}_p}(V) \rightarrow D(\Gamma, \mathbb{Q}_p). \end{array}$$

In fact, this is a variant of Perrin-Riou’s reciprocity law comparing $\Omega_{V,h}$ with $\Omega_{V^*(1),1-h}$.

For $L \neq \mathbb{Q}_p$ the issue of L -analyticity requires that $V^*(1)$ is L -analytic for the construction of $\Omega_{V^*(1),1-h}$, which then implies that V is not L -analytic. Instead, our regulator map is available and the purpose of this subsection is to prove an analogue of the above adjointness for arbitrary L .

Theorem 5.2.1 (Reciprocity formula/Adjointness of big exponential and regulator map). *Assume that $V^*(1)$ is L -analytic with $\text{Fil}^{-1} D_{\text{cris},L}(V^*(1)) = D_{\text{cris},L}(V^*(1))$ and $D_{\text{cris},L}(V^*(1))^{\varphi_L = \pi_L^{-1}} = D_{\text{cris},L}(V^*(1))^{\varphi_L = 1} = 0$. Then the following diagram consisting of $D(\Gamma_L, K)$ - ι_* -sesquilinear pairings (in the sense of (4.87)) commutes:*

$$\begin{array}{ccc}
 D_{\text{rig}}^\dagger(V^*(1))^{\psi_L = \frac{q}{\pi_L}} & \times & D(V(\tau^{-1}))^{\psi_L = 1} \xrightarrow{\frac{q-1}{q}\{\cdot, \cdot\}_{\text{Iw}}} D(\Gamma_L, \mathbb{C}_p) \\
 \uparrow \Omega_{V^*(1),1} & & \mathfrak{L}_V^0 \downarrow \\
 \mathcal{O}^{\psi_L = 0} \otimes_L D_{\text{cris},L}(V^*(1)) & \times & \mathcal{O}^{\psi_L = 0} \otimes_L D_{\text{cris},L}(V(\tau^{-1})) \xrightarrow{[\cdot, \cdot]} D(\Gamma_L, \mathbb{C}_p).
 \end{array}$$

Note that the terms on the right-hand side of the pairings are all defined over L !

Proof. This follows from the abstract reciprocity law in Theorem 4.5.32 (where we set $M := D_{\text{rig}}^\dagger(V(\tau^{-1}))$ as before). Indeed, assuming that $z \in \mathcal{O}^{\psi_L = 0} \otimes_L D_{\text{cris},L}(V^*(1))$ and $y \in D(V(\tau^{-1}))^{\psi_L = 1}$, we have that $(1 - \frac{\pi_L}{q}\varphi_L)y \in M' \cap (M^{\psi_L = 0})$ (see (5.1)) and

$$\text{comp}^{-1}((1 - \varphi_L)x) \in \check{M}' \quad \text{for } x \in (\mathcal{O} \otimes_L D_{\text{cris},L}(V^*(1)))^{\psi_L = \frac{q}{\pi_L}}$$

such that $z = (1 - \varphi_L)x$. Moreover, $\text{comp}^{-1}((1 - \varphi_L)x) \in \check{M}^{\psi_L = 0}$ by Proposition 3.1.13 as $V^*(1)$ is positive by assumption. Recall that $\text{comp}^{-1}(\nabla x)$ is an element in $D_{\text{rig}}^\dagger(V^*(1))^{\psi_L = \frac{q}{\pi_L}}$ again by Proposition 3.1.13. We thus obtain

$$\begin{aligned}
 \frac{q-1}{q} \{ \text{comp}^{-1}(\nabla x), y \}_{\text{Iw}} &= \frac{q-1}{q} \{ \nabla \text{comp}^{-1}((1 - \varphi_L)x), (1 - \frac{\pi_L}{q}\varphi_L)y \}_{\text{Iw}}^0 \\
 &= [(1 - \varphi_L)x, \text{comp}((1 - \frac{\pi_L}{q}\varphi_L)y)].
 \end{aligned}$$

By definition of the big exponential and regulator map the latter is equivalent to

$$\{ \Omega_{V^*(1),1}(z), y \}_{\text{Iw}} = [z, \mathfrak{L}_V^0(y)]. \quad \blacksquare$$

We also could consider the following variant of the big exponential map (under the assumptions of the theorem):

$$\Omega_{V,h} : D(\Gamma_L, \mathbb{C}_p) \otimes_L D_{\text{cris},L}(V^*(1)) \rightarrow D_{\text{rig}}^\dagger(V)^{\psi_L = \frac{q}{\pi_L}}$$

by extending scalars from L to \mathbb{C}_p and composing the original one with Ω^{-h} times¹

$$D(\Gamma_L, \mathbb{C}_p) \otimes_L D_{\text{cris},L}(V^*(1)) \xrightarrow{\mathcal{M} \otimes \text{id}} (\mathcal{O}_K(\mathbf{B}))^{\psi_L = 0} \otimes_L D_{\text{cris},L}(V^*(1)).$$

¹This means to replace ∇ by $\frac{\nabla}{\Omega}$ in order to achieve twist invariance of the big exponential map, see the remark below.

Corollary 5.2.2 (Reciprocity formula/Adjointness of big exponential and regulator map). *Under the assumptions of the theorem, the following diagram of $D(\Gamma_L, K)$ - ι_* -sesquilinear pairings commutes:*

$$\begin{array}{ccc}
 D_{\text{rig}}^\dagger(V^*(1))^{\psi_L = \frac{q}{\pi L}} & \times & D(V(\tau^{-1}))^{\psi_L = 1} \xrightarrow{\frac{q-1}{q}\{\iota, \text{tw}\}} D(\Gamma_L, \mathbb{C}_p) \\
 \uparrow \Omega_{V^*(1), 1} & & \downarrow \frac{\sigma_{-1} L_V}{\Omega} \\
 D(\Gamma_L, \mathbb{C}_p) \otimes_L D_{\text{cris}, L}(V^*(1)) & \times & D(\Gamma_L, \mathbb{C}_p) \otimes_L D_{\text{cris}, L}(V(\tau^{-1})) \xrightarrow{[\iota, \text{tw}]^0} D(\Gamma_L, \mathbb{C}_p)
 \end{array}$$

where $[-, -]^0 = [\mathfrak{M} \otimes \text{id}(-), \sigma_{-1} \mathfrak{M} \otimes \text{id}(-)]$, i.e.,

$$[\lambda \otimes \check{d}, \mu \otimes d]^0 \cdot \eta(1, Z) \otimes (t_{\text{LT}}^{-1} \otimes \eta) = \lambda \iota_*(\mu) \cdot \eta(1, Z) \otimes [\check{d}, d]_{\text{cris}}, \tag{5.9}$$

where $D_{\text{cris}, L}(V^*(1)) \times D_{\text{cris}, L}(V(\tau^{-1})) \xrightarrow{[\iota, \text{tw}]_{\text{cris}}} D_{\text{cris}, L}(L(\chi_{\text{LT}}))$ is the canonical pairing.

Remark 5.2.3. By [7, Cor. 3.5.4], we have

$$\Omega_{V, h}(x) \otimes \eta^{\otimes j} = \Omega_{V(\chi_{\text{LT}}^j), h+j}(\partial_{\text{inv}, x}^{-j} \otimes t_{\text{LT}}^{-j} \eta^{\otimes j})$$

and $\mathfrak{I}_h \circ \Omega_{V, h} = \Omega_{V, h+1}$, whence we obtain the equality

$$\Omega_{V, h}(x) \otimes \eta^{\otimes j} = \Omega_{V(\chi_{\text{LT}}^j), h+j}(\text{Tw}_{\chi_{\text{LT}}}^{-j}(x) \otimes t_{\text{LT}}^{-j} \eta^{\otimes j})$$

and $\mathfrak{I}_h \circ \Omega_{V, h} = \Omega_{V, h+1}$.

5.2.1 Some homological algebra

Let $X \xrightarrow{f} Y$ be a morphism of cochain complexes. Its mapping cone $\text{cone}(f)$ is defined as $X[1] \oplus Y$ with differential $d_{\text{cone}(f)}^i := \begin{pmatrix} d_X^i & 0 \\ f[1]^i & d_Y^i \end{pmatrix}$ (using column notation) and we define the mapping fiber of f as $\text{Fib}(f) := \text{cone}(f)[-1]$. Here the translation $X[n]$ of a complex X is given by $X[n]^i := X^{i+n}$ and $d_{X[n]}^i := (-1)^n d_X^{i+n}$. Alternatively, we may consider f as a double cochain complex concentrated horizontally in degree 0 and 1 and form the total complex (as in [82, Tag 012Z]). Then the associated total complex coincides with $\text{Fib}(-f)$.

For a complex (X^\bullet, d_X) of topological L -vector spaces we define its L -dual $((X^*)^\bullet, d_{X^*})$ to be the complex with

$$(X^*)^i := \text{Hom}_{L, \text{cont}}(X^{-i}, L)$$

and

$$d_{X^*}(f) := (-1)^{\deg(f)-1} f \circ d_X.$$

More generally, for two complexes (X^\bullet, d_X) and (Y^\bullet, d_Y) of topological L -vector spaces we define the complex $\text{Hom}_{L,\text{cont}}^\bullet(X^\bullet, Y^\bullet)$ by

$$\text{Hom}_{L,\text{cont}}^n(X^\bullet, Y^\bullet) = \prod_{i \in \mathbb{Z}} \text{Hom}_{L,\text{cont}}(X^i, Y^{i+n})$$

with differentials $df = d \circ f + (-1)^{\deg(f)-1} f \circ d$. Note that the canonical isomorphism

$$\text{Hom}^\bullet(X^\bullet, Y^\bullet)[n] \xrightarrow{\cong} \text{Hom}^\bullet(X^\bullet, Y^\bullet[n])$$

does not involve any sign; i.e., it is given by the identity map in all degrees.

We also recall that the tensor product of two complexes X^\bullet and Y^\bullet is given by

$$(X^\bullet \otimes_L Y^\bullet)^i := \bigoplus_n X^n \otimes_L Y^{i-n}$$

and

$$d(x \otimes y) = dx \otimes y + (-1)^{\deg(x)} x \otimes dy.$$

The adjunction morphism on the level of complexes

$$\text{adj} : \text{Hom}_{L,\text{cont}}^\bullet(X^\bullet \otimes_L Y^\bullet, Z^\bullet) \rightarrow \text{Hom}_{L,\text{cont}}^\bullet(Y^\bullet, \text{Hom}_{L,\text{cont}}^\bullet(X^\bullet, Z^\bullet))$$

sends u to $(y \mapsto (x \mapsto (-1)^{\deg(x)} \deg(y) u(x \otimes y)))$. It is well defined and continuous with respect to the projective tensor product topology and the strong topology for the Homs. Furthermore, by definition we have the following commutative diagram:

$$\begin{array}{ccc} X^\bullet \otimes_L Y^\bullet & \xrightarrow{u} & L[-2] \\ \text{id} \otimes \text{adj}(u) \downarrow & & \parallel \\ X^\bullet \otimes_L \text{Hom}_{L,\text{cont}}^\bullet(X^\bullet, L[-2]) & \xrightarrow{\text{ev}_2} & L[-2], \end{array} \quad (5.10)$$

where ev_2 sends (x, f) to $(-1)^{\deg(x)} \deg(f) f(x)$.

Lemma 5.2.4. *Let $(\mathcal{C}^\bullet, d^\bullet)$ be a complex in the category of locally convex topological L -vector spaces.*

- (i) *If \mathcal{C} consists of Fréchet spaces and $h^i(\mathcal{C}^\bullet)$ is finite dimensional over L , then d^{i-1} is strict and has closed image.*
- (ii) *If d^i is strict, then $h^{-i}(\mathcal{C}^*) \cong h^i(\mathcal{C})^*$.*

Proof. (i) Apply the argument from [13, §IX, Lem. 3.4] and use the open mapping theorem [71, Prop. 8.8]. (ii) If

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C$$

forms part of the complex with B in degree i , one immediately obtains a map

$$\ker(\alpha^*)/\text{im}(\beta^*) \rightarrow (\ker(\beta)/\text{im}(\alpha))^*,$$

where $\ker(\beta)$ carries the subspace topology and $\ker(\beta)/\text{im}(\alpha)$ the quotient topology. Now use the Hahn–Banach theorem [71, Cor. 9.4] for the strict maps $B/\ker(\beta) \hookrightarrow C$ (induced from β) and $\ker(\beta) \hookrightarrow B$ in order to show that this map is an isomorphism. ■

Definition 5.2.5. A locally convex topological vector space is called an LF-space if it is the direct limit of a countable family of Fréchet spaces, the limit being formed in the category of locally convex vector spaces.

Remark 5.2.6. If one replaces Fréchet spaces by LF-spaces, the situation of Lemma 5.2.4 is more subtle:

- (i) If $V \xrightarrow{\alpha} W$ is a continuous linear map of Hausdorff LF-spaces with finite-dimensional cokernel, then α is strict and has closed image by the same argument used in (i) of the previous lemma. However, since a closed subspace of an LF-space need not be an LF-space, we cannot achieve the same conclusion for complexes by this argument as $\ker(d^i)$ may fail to be an LF-space, whence one cannot apply the open mapping theorem, in general. But consider the following special situation. Assume that the complex \mathcal{C}^\bullet consists of LF-spaces and $h^i(\mathcal{C}^\bullet)$ is finite dimensional. If moreover $\mathcal{C}^{i+1} = 0$, i.e., $\mathcal{C}^i = \ker(d^i)$, then d^{i-1} is strict and $h^{1-i}(\mathcal{C}^*) \cong h^{i-1}(\mathcal{C})^*$.
- (ii) If d^i is not strict, the above proof still shows that we obtain a surjection $h^{-i}(\mathcal{C}^*) \twoheadrightarrow h^i(\mathcal{C})^*$.

However, for a special class of LF-spaces and under certain conditions we can say more about how forming duals and cohomology interacts.

Lemma 5.2.7. Let $(\mathcal{C}^\bullet, d^\bullet) = \varinjlim_r (\mathcal{C}_r^\bullet, d_r^\bullet)$ be a complex in the category of locally convex topological L -vector spaces arising as regular inductive limit of complexes of Fréchet spaces; i.e., in each degree i the transition maps in the countable sequence $(\mathcal{C}_r^i)_r$ are injective and for each bounded subset $B \subseteq \mathcal{C}^i$ there exists an $r \geq 1$ such that B is contained in \mathcal{C}_r^i and is bounded as a subset of the Fréchet space \mathcal{C}_r^i . Then,

- (i) we have topological isomorphisms $(\mathcal{C}^\bullet)^* \cong \varprojlim_r (\mathcal{C}_r^\bullet)^*$,
- (ii) if, in addition, $\varprojlim_{r \geq 0}^1 h^i((\mathcal{C}_r^\bullet)^*) = 0$ for all i , we have a long exact sequence

$$\cdots \rightarrow h^i((\mathcal{C}^\bullet)^*) \rightarrow \varprojlim_{r \geq 0}^1 h^i((\mathcal{C}_r^\bullet)^*) \rightarrow h^{i-1}(\varprojlim_{r \geq 0}^1 (\mathcal{C}_r^\bullet)^*) \rightarrow h^{i+1}((\mathcal{C}^\bullet)^*) \rightarrow \cdots,$$

- (iii) if, in addition to (ii), the differentials d_r^\bullet are strict, e.g., if all $h^i(\mathcal{C}_r^\bullet)$ have finite dimension over L , and $\varprojlim_{r \geq 0}^1 (\mathcal{C}_r^\bullet)^* = 0$, we have isomorphisms

$$h^i((\mathcal{C}^\bullet)^*) \cong \varprojlim_{r \geq 0} h^{-i}(\mathcal{C}_r^\bullet)^*.$$

Proof. (i) is [61, Thm. 11.1.13] while (ii) and (iii) follow from (i) and [53, Chap. 3, Prop. 1] applied to the inverse system $((\mathcal{C}_r^\bullet)^*)_r$ combined with Lemma 5.2.4. ■

5.2.2 Koszul complexes

In this paragraph, we restrict to the situation $U \cong \mathbb{Z}_p^d$ and fix topological generators $\gamma_1, \dots, \gamma_d$ of U and we set $\Lambda := \Lambda(U)$. Furthermore, let M be any complete linearly topologized o_L -module with a continuous U -action. Then by [49, Thm. II.2.2.6] this actions extends to continuous Λ -action and one has

$$\mathrm{Hom}_{\Lambda, \mathrm{cont}}(\Lambda, M) = \mathrm{Hom}_{\Lambda}(\Lambda, M).$$

Consider the (homological) complexes $K_\bullet(\gamma_i) := [\Lambda \xrightarrow{\gamma_i - 1} \Lambda]$ concentrated in degrees 1 and 0 and define

$$K_\bullet := K_\bullet^U := K_\bullet(\boldsymbol{\gamma}) := \bigotimes_{i=1}^d K_\bullet(\gamma_i),$$

$$K^\bullet(M) := K_U^\bullet(M) := \mathrm{Hom}_\Lambda^\bullet(K_\bullet, M) \cong \mathrm{Hom}_\Lambda^\bullet(K_\bullet, \Lambda) \otimes_\Lambda M = K^\bullet(\Lambda) \otimes_\Lambda M,$$

$$K_\bullet(M) := K_\bullet \otimes_\Lambda M \quad (\text{homological complex}),$$

$$K_\bullet(M)^\bullet := (K_\bullet \otimes_\Lambda M)^\bullet \quad (\text{the associated cohomological complex}).$$

If we want to indicate the dependence on $\boldsymbol{\gamma} = (\gamma_1, \dots, \gamma_d)$, we also write $K^\bullet(\boldsymbol{\gamma}, M)$ instead of $K^\bullet(M)$ and similarly for other notation; moreover, we shall use the notation $\boldsymbol{\gamma}^{-1} = (\gamma_1^{-1}, \dots, \gamma_d^{-1})$ and $\boldsymbol{\gamma}^{p^n} = (\gamma_1^{p^n}, \dots, \gamma_d^{p^n})$. Note that in each degree these complexes consist of a direct sum of finitely many copies of M and will be equipped with the corresponding direct product topology.

The complex K_\bullet will be identified with the exterior algebra complex $\bigwedge_\Lambda^\bullet \Lambda^d$ of the free Λ -module with basis e_1, \dots, e_d , for which the differentials $d_q : \bigwedge_\Lambda^q \Lambda^d \rightarrow \bigwedge_\Lambda^{q-1} \Lambda^d$ with respect to the standard basis $e_{i_1, \dots, i_q} = e_{i_1} \wedge \dots \wedge e_{i_q}$, $1 \leq i_1 < \dots < i_q \leq d$, are given by the formula

$$d_q(a_{i_1, \dots, i_q}) = \sum_{k=1}^q (-1)^{k+1} (\gamma_{i_k} - 1) a_{i_1, \dots, \widehat{i_k}, \dots, i_q}.$$

Then the well-known self-duality (compare [28, Prop. 17.15] although the claim there is not precisely the same) of the Koszul complex, i.e., the isomorphism of complexes

$$K_\bullet(\Lambda)^\bullet \cong K^\bullet(\Lambda)[d], \tag{5.11}$$

can be explicitly described in degree $-q$ as follows (by identifying $\bigwedge_{\Lambda}^d \Lambda^d = \Lambda e_1 \wedge \cdots \wedge e_d = \Lambda$):

$$\begin{aligned} \bigwedge_{\Lambda}^q \Lambda^d &\xrightarrow{\alpha-q} \text{Hom}_{\Lambda} \left(\bigwedge_{\Lambda}^{d-q} \Lambda^d, \Lambda \right) \\ e_{i_1, \dots, i_q} &\longmapsto \text{sign}(I, J) e_{j_1, \dots, j_{d-q}}^*, \end{aligned}$$

where e_1^*, \dots, e_d^* denotes the dual basis of e_1, \dots, e_d , the elements $e_{j_1, \dots, j_{d-q}}^* = e_{j_1}^* \wedge \cdots \wedge e_{j_{d-q}}^*$, $1 \leq j_1 < \cdots < j_{d-q} \leq d$, form a (dual) basis of $\text{Hom}_{\Lambda}(\bigwedge_{\Lambda}^{d-q} \Lambda^d, \Lambda)$, the indices $J = (j_k)_k$ are complementary to $I = (i_n)_n$ in the following sense $\{i_1, \dots, i_q\} \cup \{j_1, \dots, j_{d-q}\} = \{1, \dots, d\}$, and $\text{sign}(I, J)$ denotes the sign of the permutation $[i_1, \dots, i_q, j_1, \dots, j_{d-q}]$. Indeed, the verification that the induced diagram involving the differentials from cohomological degree $-q$ to $-q + 1$

$$\begin{array}{ccc} \bigwedge_{\Lambda}^q \Lambda^d & \xrightarrow{\alpha-q} & \text{Hom}_{\Lambda} \left(\bigwedge_{\Lambda}^{d-q} \Lambda^d, \Lambda \right) \\ \downarrow d_q & & \downarrow (-1)^d (-1)^{d-q-1} d_{d-q+1}^* \\ \bigwedge_{\Lambda}^{q-1} \Lambda^d & \xrightarrow{\alpha-q+1} & \text{Hom}_{\Lambda} \left(\bigwedge_{\Lambda}^{d-q+1} \Lambda^d, \Lambda \right) \end{array}$$

commutes² relies on the observation that

$$\text{sign}(I, J) \text{sign}(I_{\hat{k}}, J_k)^{-1} = (-1)^{q-k+l-1},$$

where $I_{\hat{k}} := (i_1, \dots, \widehat{i_k}, \dots, i_q)$ denotes the sequence which results from I by omitting i_k , while $J_k = (j_1, \dots, j_{l-1}, i_k, j_l, \dots, j_{d-q})$ denotes the sequence which arises from J by inserting i_k at position l with regard to the strict increasing ordering: The permutations $[i_1, \dots, i_q, j_1, \dots, j_{d-q}]$ and $[i_1, \dots, \widehat{i_k}, \dots, i_q, j_1, \dots, j_{l-1}, i_k, j_l, \dots, j_{d-q}]$ differ visibly by $q - k + l - 1$ transpositions.

Now we assume that M is any complete locally convex L -vector space with continuous U -action such that its strong dual is again complete with continuous U -action. Then we obtain isomorphisms of complexes

$$\begin{aligned} K^{\bullet}(\gamma, M)^* &= \text{Hom}_{L, \text{cont}}^{\bullet} \left(\text{Hom}_{\Lambda}^{\bullet} (K_{\bullet}(\gamma), \Lambda) \otimes_{\Lambda} M, L \right) \\ &\cong \text{Hom}_{\Lambda}^{\bullet} \left(\text{Hom}_{\Lambda}^{\bullet} (K_{\bullet}(\gamma^{-1}), \Lambda), \text{Hom}_{L, \text{cont}}(M, L) \right) \\ &\cong \text{Hom}_{\Lambda}^{\bullet} \left(\text{Hom}_{\Lambda}^{\bullet} (K_{\bullet}(\gamma^{-1}), \Lambda), \Lambda \right) \otimes_{\Lambda} \text{Hom}_{L, \text{cont}}(M, L) \\ &\cong K_{\bullet}(\gamma^{-1}, \Lambda)^{\bullet} \otimes_{\Lambda} \text{Hom}_{L, \text{cont}}(M, L) \\ &\cong K^{\bullet}(\gamma^{-1}, \Lambda)[d] \otimes_{\Lambda} M^* \\ &\cong K^{\bullet}(\gamma^{-1}, M^*)[d], \end{aligned} \tag{5.12}$$

²The signs $(-1)^d$ and $(-1)^{d-q-1}$ result from the shift by d and the sign rule for complex homomorphisms, respectively.

where in the second line we use the adjunction morphism; the isomorphism in the fourth line being the biduality morphism (according to [58, (1.2.8)])

$$K_{\bullet}(\Lambda)^{\bullet} \xrightarrow{\cong} \text{Hom}_{\Lambda}^{\bullet}(\text{Hom}_{\Lambda}^{\bullet}(K_{\bullet}, \Lambda), \Lambda)$$

$$x \mapsto (-1)^i x^{**}$$

with the usual biduality of modules

$$K_{\bullet}(\Lambda)^i \xrightarrow{\cong} \text{Hom}_{\Lambda}(\text{Hom}_{\Lambda}(K_{-i}, \Lambda), \Lambda)$$

$$x \mapsto (x^{**} : f \mapsto f(x))$$

involves a sign, while the isomorphism in the third last line stems from (5.11) together with Lemma 4.5.1 (i). Note that the isomorphism in the second last line does not involve any further signs by [58, (1.2.15)].

We finish this subsection by introducing restriction and corestriction maps concerning the change of group for Koszul complexes. To this end let $U_1 \subseteq U$ be the open subgroup generated by $\gamma_1^{p^n}, \dots, \gamma_d^{p^n}$. Then $\text{Hom}_{\Lambda}^{\bullet}(-, M)$ applied to the tensor product of the diagrams

$$\begin{array}{ccc} \Lambda(U) & \xrightarrow{\gamma_i^{p^n-1}} & \Lambda(U) \\ \parallel & & \uparrow \sum_{k=0}^{p^n-1} \gamma_i^k \\ \Lambda(U) & \xrightarrow{\gamma_i-1} & \Lambda(U) \end{array}$$

gives a map

$$\text{cor}_{U_1}^U : K_{U_1}^{\bullet}(\gamma^{p^n})(M) \rightarrow K_U^{\bullet}(\gamma)(M)$$

which we call corestriction map and which is compatible under (5.20) below with the corestriction map on cocycles (for appropriate choices of representatives in the definition of the latter). Using the diagram

$$\begin{array}{ccc} \Lambda(U) & \xrightarrow{\gamma_i^{p^n-1}} & \Lambda(U) \\ \sum_{k=0}^{p^n-1} \gamma_i^k \downarrow & & \parallel \\ \Lambda(U) & \xrightarrow{\gamma_i-1} & \Lambda(U) \end{array}$$

instead, one obtains the restriction map

$$\text{res}_{U_1}^U : K_U^{\bullet}(\gamma)(M) \rightarrow K_{U_1}^{\bullet}(\gamma^{p^n})(M),$$

again compatible under (5.20) with the restriction map on cocycles.

5.2.3 Continuous and analytic cohomology

For any profinite group G and topological abelian group M with continuous G -action we write $\mathcal{C}^\bullet := \mathcal{C}^\bullet(G, M)$ for the continuous (inhomogeneous) cochain complex of G with coefficients in M and $H^*(G, M) := h^*(\mathcal{C}^\bullet(G, M))$ for continuous group cohomology. Note that $\mathcal{C}^0(G, M) = M$.

Assume that G is moreover an L -analytic group and $M = \lim_{\substack{\rightarrow s \\ \leftarrow r}} M^{[r,s]}$ with Banach spaces $M^{[r,s]}$, an LF space with a pro- L -analytic action of G , i.e., a locally analytic action on each $M^{[r,s]}$, which means that for all $m \in M^{[r,s]}$ there exists an open L -analytic subgroup $\Gamma_n \subseteq \Gamma$ in the notation of Section 4.3.4 such that the orbit map of m restricted to Γ_n is a power series of the form $g(m) = \sum_{k \geq 0} \ell(g)^k m_k$ for a sequence m_k of elements in $M^{[r,s]}$ with $\pi_L^{nk} m_k$ converging to zero. Following [25, §5], we write

$$\mathcal{C}_{\text{an}}^\bullet := \mathcal{C}_{\text{an}}^\bullet(G, M)$$

for the locally L -analytic cochain complex of G with coefficients in M and

$$H_{\text{an}}^*(G, M) := h^*(\mathcal{C}_{\text{an}}^\bullet(G, M))$$

for locally L -analytic group cohomology. More precisely, if $\text{Maps}_{\text{loc } L\text{-an}}(G, M^{[r,s]})$ denotes the space of locally L -analytic maps from G to $M^{[r,s]}$, then

$$C_{\text{an}}^n(G, M) = \lim_{\substack{\rightarrow s \\ \leftarrow r}} \text{Maps}_{\text{loc } L\text{-an}}(G^n, M^{[r,s]})$$

is the space of locally L -analytic functions (locally with values in $\lim_{\leftarrow r} M^{[r,s]}$ for some s and such that the composite with the projection onto $M^{[r,s]}$ is locally L -analytic for all r). Note that again $\mathcal{C}_{\text{an}}^0(G, M) = M$ and that there are canonical homomorphisms

$$\begin{aligned} \mathcal{C}_{\text{an}}^\bullet(G, M) &\hookrightarrow \mathcal{C}^\bullet(G, M), \\ H_{\text{an}}^\bullet(G, M) &\rightarrow H^\bullet(G, M). \end{aligned}$$

Let f be any continuous endomorphism of M which commutes with the G -action. We define

$$H^0(f, M) := M^{f=1} \quad \text{and} \quad H^1(f, M) := M_{f=1}$$

as the kernel and cokernel of the map $M \xrightarrow{f-1} M$, respectively.

The endomorphism f induces an operator on \mathcal{C}^\bullet or $\mathcal{C}_{\text{an}}^\bullet$ and we denote by $\mathcal{T} := \mathcal{T}_{f,G}(M)$ and $\mathcal{T}^{\text{an}} := \mathcal{T}_{f,G}^{\text{an}}(M)$ the mapping fiber of $\mathcal{C}^\bullet(G, f)$ and $\mathcal{C}_{\text{an}}^\bullet(G, f)$, respectively.

Again there are canonical homomorphisms

$$\begin{aligned} \mathcal{T}_{f,G}^{\text{an}}(M) &\hookrightarrow \mathcal{T}_{f,G}(M), \\ h^\bullet(\mathcal{T}_{f,G}^{\text{an}}(M)) &\rightarrow h^\bullet(\mathcal{T}_{f,G}(M)). \end{aligned} \tag{5.13}$$

For \mathcal{T} either empty or an , one of the corresponding double complex spectral sequences is

$${}_{II}E_2^{i,j} = H^i(f, H^j_\mathcal{T}(G, M)) \implies h^{i+j}(\mathcal{T}^?).^3$$

It degenerates into the short exact sequences

$$0 \rightarrow H^{i-1}_\mathcal{T}(G, M)_{f=1} \rightarrow h^i(\mathcal{T}_{f,G}^?) \rightarrow H^i_\mathcal{T}(G, M)^{f=1} \rightarrow 0.$$

In loc. cit. as well as in [7], analytic cohomology is also defined for the semi-groups $\Gamma_L \times \Phi$ and $\Gamma_L \times \Psi$ with $\Phi = \{\varphi_L^n \mid n \geq 0\}$ and $\Psi = \{(\frac{\pi}{q}\psi_L)^n \mid n \geq 0\}$ if M denotes an L -analytic (φ_L, Γ_L) -module over the Robba ring \mathcal{R} .

Remark 5.2.8. Any L -analytic (φ_L, Γ_L) -module M over the Robba ring \mathcal{R} is a pro- L -analytic Γ_L -module by the discussion at the end of the proof of [8, Prop. 2.25], whence it is also an L -analytic $\Gamma_L \times \Phi$ - and $\Gamma_L \times \Psi$ -module as Φ and Ψ possess the discrete structure as L -analytic manifolds.

Proposition 5.2.9. *We have canonical isomorphisms*

$$h^i(\mathcal{T}_{\varphi_L, \Gamma_L}^{\text{an}}(M)) \cong H_{\text{an}}^i(\Gamma_L \times \Phi, M) \cong H_{\text{an}}^i(\Gamma_L \times \Psi, M) \cong h^i(\mathcal{T}_{\frac{\pi}{q}\psi_L, \Gamma_L}^{\text{an}}(M))$$

and an exact sequence

$$0 \rightarrow H_{\text{an}}^1(\Gamma_L, M^{\psi_L = \frac{q}{\pi}}) \rightarrow h^i(\mathcal{T}_{\frac{\pi}{q}\psi_L, \Gamma_L}^{\text{an}}(M)) \rightarrow (M_{\psi_L = \frac{q}{\pi}})^{\Gamma_L} \rightarrow H_{\text{an}}^2(\Gamma_L, M^{\psi_L = \frac{q}{\pi}}) \rightarrow h^2(\mathcal{T}_{\frac{\pi}{q}\psi_L, \Gamma_L}^{\text{an}}(M)). \tag{5.14}$$

Proof. The isomorphism in the middle is [7, Cor. 2.2.3]. For the two outer isomorphisms we refer the reader to [87, Thm. 3.7.6]. The exact sequence is the extension [87, Thm. 5.1.5] of [7, Thm. 2.2.4]. ■

³Naively, one would expect that the second corresponding double complex spectral sequence looks like

$${}_{II}E_2^{i,j} = H^j_\mathcal{T}(G, H^i(f, M)) \implies h^{i+j}(\mathcal{T}^?).$$

But this would require to first of all give sense to the required structure of $H^j(f, M)$ as topological/analytic G -module! In low degrees this can be achieved and we obtain an exact sequence

$$0 \rightarrow H^1_\mathcal{T}(G, M^{f=1}) \rightarrow h^1(\mathcal{T}^?) \rightarrow (M_{f=1})^G \xrightarrow{\delta} H^2_\mathcal{T}(G, M^{f=1}).$$

See [87–89]. If $M^{f=1}$ is again an LF-space with pro- L -analytic G -operation, one might be able to interpret the second spectral sequence in low degrees.

Note that, for an inclusion $U \subseteq U'$, the restriction and corestriction homomorphisms $\mathcal{C}^\bullet(U', M) \xrightarrow{\text{res}} \mathcal{C}^\bullet(U, M)$ and $\mathcal{E}^\bullet(U, M) \xrightarrow{\text{cor}} \mathcal{E}^\bullet(U', M)$ induce maps on $\mathcal{T}_{f,U'}(M) \xrightarrow{\text{res}} \mathcal{T}_{f,U}(M)$ and $\mathcal{T}_{f,U}(M) \xrightarrow{\text{cor}} \mathcal{T}_{f,U'}(M)$, respectively.

We write $\text{Ext}_{\mathcal{C}}^1(A, B)$ for isomorphism classes of extensions of B by A in any abelian category \mathcal{C} . Furthermore, we denote by $\mathfrak{M}_U(R)$ ($\mathfrak{M}_U^{\text{ét}}(R)$, $\mathfrak{M}_U^\dagger(R)$) the category of all (étale, overconvergent) (φ_L, U) -modules over R , respectively, and by $\text{Rep}_L^\dagger(G_{L_\infty^U})$ the category of overconvergent representations of $G_{L_\infty^U}$ consisting of those representations V of $G_{L_\infty^U}$ such that $\dim_{\mathbf{B}_L^\dagger} D^\dagger(V) = \dim_L V$ with $D^\dagger(V) := (\mathbf{B}^\dagger \otimes_L V)^{H_L}$.

Theorem 5.2.10. *Let V be in $\text{Rep}_L(G_L)$ and $U \subseteq \Gamma_L$ any open subgroup.*

- (i) *For $D(V)$ the corresponding (φ_L, Γ_L) -module over \mathbf{B}_L we have canonical isomorphisms*

$$h^* = h_{U,V}^* : H^*(L_\infty^U, V) \xrightarrow{\cong} h^*(\mathcal{T}_{\varphi_L,U}(D(V))) \tag{5.15}$$

which are functorial in V and compatible with restriction and corestriction.

- (ii) *If V is in addition overconvergent, there are isomorphisms*

$$h^0(\mathcal{T}_{\varphi_L,U}(D_{\text{rig}}^\dagger(V))) \cong V^{G_{L_\infty^U}}, \tag{5.16}$$

$$h^1(\mathcal{T}_{\varphi_L,U}(D_{\text{rig}}^\dagger(V))) \cong H_\dagger^1(L_\infty^U, V), \tag{5.17}$$

which are functorial in V and compatible with restriction and corestriction and where by definition $H_\dagger^1(L_\infty^U, V) \subseteq H^1(L_\infty^U, V)$ classifies the overconvergent extensions of L by V . In particular, these L -vector spaces have finite dimension.

- (iii) *If V is in addition L -analytic, then we have*

$$H_{\text{an}}^1(L_\infty^U, V) \xrightarrow{\cong} h^1(\mathcal{T}_{\varphi_L,U}^{\text{an}}(D_{\text{rig}}^\dagger(V))),$$

where by definition⁴ $H_{\text{an}}^1(L_\infty^U, V) \subseteq H_\dagger^1(L_\infty^U, V) \subseteq H^1(L_\infty^U, V)$ classifies the L -analytic extensions of L by V .

Proof. (i) is [46, Thm. 5.1.11] or [45, Thm. 5.1.11]. Statement (iii) is [7, Prop. 2.2.1] combined with Proposition 5.2.9, while (ii) follows from [33] (the reference literally only covers the case $U = \Gamma_L$, but the same arguments allow us to extend the result to general U) as follows: Firstly, by Lemma 5.2.11 below one has an isomorphism $h^1(\mathcal{T}_{\varphi_L,U}(D_{\text{rig}}^\dagger(V))) \cong \text{Ext}_{\mathfrak{M}_U(\mathcal{R}_L)}^1(\mathcal{R}_L, D_{\text{rig}}^\dagger(V))$. Then use the HN-filtration à la

⁴Note that the absolute Galois group of L_∞^U is not L -analytic, so this group has not been defined earlier.

Kedlaya to see that any extension of étale (φ_L, U) -modules is étale again, whence

$$\mathrm{Ext}_{\mathcal{M}_U(\mathcal{R}_L)}^1(\mathcal{R}_L, D_{\mathrm{rig}}^\dagger(V)) = \mathrm{Ext}_{\mathcal{M}_U^{\mathrm{ét}}(\mathcal{R}_L)}^1(\mathcal{R}_L, D_{\mathrm{rig}}^\dagger(V))$$

and the latter group equals

$$\mathrm{Ext}_{\mathcal{M}_U^\dagger(\mathcal{R}_L)}^1(\mathcal{R}_L, D_{\mathrm{rig}}^\dagger(V)) \cong \mathrm{Ext}_{\mathrm{Rep}_L^\dagger(G_{L_\infty^U})}^1(L, V) = H_\dagger^1(L_\infty^U, V)$$

by [33, Props. 1.5 and 1.6]. For the claim in degree 0 one has to show that the inclusion $D^\dagger(V) \subseteq D_{\mathrm{rig}}^\dagger(V)$ induces an isomorphism on φ_L -invariants, which follows from [39, Hyp. 1.4.1, Prop. 1.2.6].⁵ ■

Lemma 5.2.11. *Let M be in $\mathfrak{M}_U(\mathcal{R})$. Then we have a canonical isomorphism*

$$h^1(\mathfrak{T}_{\varphi_L, U}(M)) \cong \mathrm{Ext}_{\mathfrak{M}_U(\mathcal{R}_L)}^1(\mathcal{R}_L, M).$$

Proof. Starting with a class $z = [(c_1, -c_0)]$ in $h^1(\mathcal{T}_{\varphi_L, U}(M))$ with $c_1 \in C^1(M)$ and $c_0 \in C^0(M) = M$ (i.e., we work with *inhomogeneous* continuous cocycles) satisfying the cocycle property

$$\begin{aligned} c_1(\sigma\tau) &= \sigma c_1(\tau) + c_1(\sigma) \quad \text{for all } \sigma, \tau \in U, \\ (\varphi_L - 1)c_1(\tau) &= (\tau - 1)c_0 \quad \text{for all } \tau \in U, \end{aligned} \tag{5.18}$$

we define an extension of (φ_L, U) -modules

$$0 \rightarrow M \rightarrow E_c \rightarrow \mathcal{R}_L \rightarrow 0$$

with $E_c := M \times \mathcal{R}_L$ as \mathcal{R}_L -module, $g(m, r) := (gm + gr \cdot c_1(g), gr)$ for $g \in U$ and $\varphi_{E_c}((m, r)) := (\varphi_M(m) + \varphi_L(r)c_0, \varphi_L(r))$; note that this defines a (continuous) group action by the first identity in (5.18), while the U - and φ_L -action commute by the second identity in (5.18). If we change the representatives $(c_1, -c_0)$ by the coboundary induced by $m_0 \in M$, then sending $(0, 1)$ to $(-m_0, 1)$ induces an isomorphism of extensions from the first to the second one, whence our map is well defined.

Conversely, if E is any such extension, choose a lift $e \in E$ of $1 \in \mathcal{R}_L$ and define

$$c_1(\tau) := (\tau - 1)e \in M, \quad c_0 := (\varphi_E - 1)e,$$

which evidently satisfy the cocycle conditions (5.18). Choosing another lift \tilde{e} leads to a cocycle which differs from the previous one by the coboundary induced by $\tilde{e} - e \in M$, whence the inverse map is well defined.

One easily verifies that these maps are mutually inverse to each other. ■

⁵Since the *strong hypothesis* holds by [39, Hyp. 1.4.1, Prop. 1.2.6], we also obtain an isomorphism on the φ_L -coinvariants $H^1(\varphi_L, -)$. Then the second spectral sequence above or a similar argument via the Koszul complexes as in Corollary A.8 implies that the canonical base change map induces an isomorphism $h^*(\mathcal{T}_{\varphi_L, U}(D^\dagger(V))) \cong h^*(\mathcal{T}_{\varphi_L, U}(D_{\mathrm{rig}}^\dagger(V)))$. Cf. [50, proof of Prop. 2.7].

Question 5.2.12. Can one show that $h^2(\mathcal{T}_{\varphi_L, U}(D_{\text{rig}}^\dagger(V)))$ is finite dimensional (and related to $H^2(L_\infty^U, V)$) and that the groups $h^i(\mathcal{T}_{\varphi_L, U}(D_{\text{rig}}^\dagger(V)))$ vanish for $i \geq 3$?

Remark 5.2.13. By [33, Thm. 0.2, Rem. 5.21], it follows that the inclusions

$$H_{\text{an}}^1(L_\infty^U, V) \subseteq H_{\dagger}^1(L_\infty^U, V) \subseteq H^1(L_\infty^U, V)$$

are in general strict. More precisely, the codimension of the left most term equals $([L_\infty^U : \mathbb{Q}_p] - 1) \dim_L V^{G_{L_\infty^U}}$.

Let us recall Tate’s local duality in this context.

Proposition 5.2.14 (Local Tate duality). *Let V be an object in $\text{Rep}_L(G_L)$, and K any finite extension of L . Then the cup product and the local invariant map induce perfect pairings of finite-dimensional L -vector spaces*

$$\begin{aligned} H^i(K, V) \times H^{2-i}(K, \text{Hom}_{\mathbb{Q}_p}(V, \mathbb{Q}_p(1))) &\rightarrow H^2(K, \mathbb{Q}_p(1)) = \mathbb{Q}_p, \\ H^i(K, V) \times H^{2-i}(K, \text{Hom}_L(V, L(1))) &\rightarrow H^2(K, L(1)) = L, \end{aligned}$$

where $-(1)$ denotes the Galois twist by the cyclotomic character. In other words, there are canonical isomorphisms

$$H^i(K, V) \cong H^{2-i}(K, V^*(1))^*.$$

Proof. This is well known. For lack of a reference (with proof) we sketch the second claim (the first being proved similarly). Choose a Galois stable o_L -lattice $T \subseteq V$ and denote by $\pi_L^n A$ the kernel of multiplication by π_L^n on any o_L -module A . Observe that we have short exact sequences

$$0 \rightarrow H^i(K, T)/\pi_L^n \rightarrow H^i(K, T/\pi_L^n T) \rightarrow \pi_L^n H^{i+1}(K, T) \rightarrow 0$$

for $i \geq 0$ and similarly for T replaced by $T^*(1) = \text{Hom}_{o_L}(T, o_L(1))$. By [80, Prop. 5.7] (remember the normalization given there!) the cup product induces isomorphism

$$H^i(K, T/\pi_L^n T) \cong \text{Hom}_{o_L}(H^{2-i}(K, T^*(1)/\pi_L^n T^*(1)), o_L/\pi_L^n)$$

such that we obtain altogether canonical maps

$$\begin{aligned} H^i(K, T)/\pi_L^n &\rightarrow \text{Hom}_{o_L}(H^{2-i}(K, T^*(1))/\pi_L^n, o_L/\pi_L^n) \\ &\cong \text{Hom}_{o_L}(H^{2-i}(K, T^*(1)), o_L)/\pi_L^n. \end{aligned}$$

Using that the cohomology groups are finitely generated o_L -modules and isomorphic to the inverse limits of the corresponding cohomology groups with coefficients modulo π_L^n , we see that the inverse limit of the above maps induces a surjective map

$$H^i(K, T) \twoheadrightarrow \text{Hom}_{o_L}(H^{2-i}(K, T^*(1)), o_L)$$

with finite kernel, whence the claim after tensoring with L over o_L using the isomorphism $H^i(K, T) \otimes_{o_L} L \cong H^i(K, V)$ and analogously for $T^*(1)$. ■

Now let W be an L -analytic representation of G_L and set

$$H_{/\dagger}^1(L_\infty^U, W^*(1)) := H_{\dagger}^1(L_\infty^U, W)^*,$$

which, by local Tate duality and Theorem 5.2.10, is a quotient of $H^1(L_\infty^U, W^*(1))$. By definition, the local Tate pairing induces a non-degenerate pairing

$$\langle \cdot, \cdot \rangle_{\text{Tate}, L, \dagger} : H_{\dagger}^1(L_\infty^U, W) \times H_{/\dagger}^1(L_\infty^U, W^*(1)) \rightarrow H^2(L, L(1)) \cong L. \quad (5.19)$$

In order to compute this pairing more explicitly in certain situations, we shall use Koszul complexes. For this we have to assume first that U is torsion-free. Following [26, §4.2], we obtain for any complete linearly topologized \mathcal{o}_L -module M with continuous U -action a quasi-isomorphism⁶

$$K_U^\bullet(M) \xrightarrow{\sim} \mathcal{C}^\bullet(U, M) \quad (5.20)$$

which arises as follows: Let $X_\bullet := X_\bullet(U)$ and $Y_\bullet = Y_\bullet(U)$ denote the completed standard complex [49, §V.1.2.1], i.e., $X_n = \mathbb{Z}_p[[U]]^{\widehat{\otimes}(n+1)}$, and the standard complex computing group cohomology, i.e., $Y_n = \mathbb{Z}_p[U]^{\otimes(n+1)}$. Then, by [49, Lem. V.1.1.5.1], we obtain a diagram of complexes

$$\begin{array}{ccc} Y_\bullet(U) & \xrightarrow{\Delta} & Y_\bullet(U \times U) \cong Y_\bullet(U) \otimes_{\mathbb{Z}_p} Y_\bullet(U) \\ \downarrow & & \downarrow \\ X_\bullet(U) & \xrightarrow{\Delta} & X_\bullet(U \times U) \cong X_\bullet(U) \widehat{\otimes}_{\mathbb{Z}_p} X_\bullet(U) \\ \downarrow & & \downarrow \\ K_\bullet^U & \xrightarrow{\Delta} & K_\bullet^{U \times U} \cong K_\bullet^U \widehat{\otimes}_{\mathbb{Z}_p} K_\bullet^U, \end{array} \quad (5.21)$$

which commutes up to homotopy (of filtered Λ -modules). Here the maps Δ are induced by the diagonal maps $U \rightarrow U \times U$, e.g.,

$$\mathbb{Z}_p[[U]] \rightarrow \mathbb{Z}_p[[U \times U]] \cong \mathbb{Z}_p[[U]] \widehat{\otimes}_{\mathbb{Z}_p} \mathbb{Z}_p[[U]].$$

The first column induces a morphism

$$\text{Hom}_\Lambda(K_\bullet^U, M) \rightarrow \text{Hom}_{\Lambda, \text{cont}}(X_\bullet(U), M) \rightarrow \text{Hom}_{\mathbb{Z}_p[[U]], \text{cont}}(Y_\bullet(U), M),$$

which is (5.20). The upper line induces as usual the cup product on continuous group cohomology

$$H^r(U, M) \times H^s(U, N) \xrightarrow{\cup_U} H^{r+s}(U, M \otimes N)$$

⁶This quasi-isomorphism is unique up to homotopy, i.e., unique in the derived category of \mathcal{o}_L -linear topological U -modules. We have not yet defined any topology on the cocycles nor do we know whether the references say anything about it! M is allowed to be any complete linearly topologized \mathcal{o}_L -module with continuous U -action by [49, Prop. V.1.2.6].

via

$$\begin{aligned} & \text{Hom}_{\mathbb{Z}_p[U], \text{cont}}(Y_\bullet(U), M) \times \text{Hom}_{\mathbb{Z}_p[U], \text{cont}}(Y_\bullet(U), N) \\ & \xrightarrow{\times} \text{Hom}_{\mathbb{Z}_p[U] \otimes \mathbb{Z}_p[U], \text{cont}}(Y_\bullet(U) \otimes_{\mathbb{Z}_p} Y_\bullet(U), M \otimes N) \\ & \xrightarrow{\Delta^*} \text{Hom}_{\mathbb{Z}_p[U], \text{cont}}(Y_\bullet(U), M \otimes N). \end{aligned}$$

The lower line induces analogously the Koszul product

$$K_U^r(M) \times K_U^s(N) \xrightarrow{\cup_K} K_U^{r+s}(M \otimes N).$$

By diagram (5.21) both products are compatible with each other.

Let f be any continuous endomorphism of M which commutes with the U -action; it induces an operator on $K^\bullet(M)$ and we denote by

$$K_{f,U}(M) := \text{cone}(K^\bullet(M) \xrightarrow{f - \text{id}} K^\bullet(M))[-1]$$

the mapping fiber of $K^\bullet(f)$. Then the quasi-isomorphism (5.20) induces a quasi-isomorphism

$$K_{\varphi,U}(M) \xrightarrow{\cong} \mathcal{T}_{\varphi,U}(M).$$

Remark 5.2.15. By a standard procedure cup products can be extended to hypercohomology (defined via total complexes), we follow [58, (3.4.5.2)], but for the special case of a cone, see also [60, Prop. 3.1]. In particular, we obtain compatible cup products \cup_K and \cup_U for $K_{\varphi,U}(M)$ and $\mathcal{T}_{\varphi,U}(M)$, respectively.

Now we allow some arbitrary open subgroup $U \subseteq \Gamma_L$ and let $L' = L_\infty^U$. Note that we obtain a decomposition $U \cong \Delta \times U'$ with a subgroup $U' \cong \mathbb{Z}_p^d$ of U and Δ the torsion subgroup of U . By Lemma A.1 we obtain a canonical isomorphism

$$K_{\varphi,U'}(M^\Delta) \xrightarrow{\cong} \mathcal{T}_{\varphi,U}(M). \tag{5.22}$$

Now let M be a finitely generated projective \mathcal{R} -module M with continuous U -action. Then $M^* = \check{M}$ is again a finitely generated projective \mathcal{R} -module M with continuous U -action by Lemma 4.5.1 (i). Hence, M as well as M^Δ satisfies the assumptions of (5.12) and we have isomorphisms⁷

$$\begin{aligned} K_{\varphi,U}(M^\Delta)^* & \cong \text{cone}(K^\bullet(M^\Delta)^* \xrightarrow{\varphi^{*-1}} K^\bullet(M^\Delta)^*) \\ & = \text{cone}(K^\bullet((M^\Delta)^*)[d] \xrightarrow{\varphi^{*-1}} K^\bullet((M^\Delta)^*)[d]) \\ & = K_{\varphi^*,U}((M^*)_\Delta)[d + 1] = K_{\psi,U}(\check{M}_\Delta)[d + 1] \\ & = K_{\psi,U}(\check{M}^\Delta)[d + 1]. \end{aligned} \tag{5.23}$$

The last isomorphism is induced by the canonical isomorphism $\check{M}^\Delta \cong \check{M}_\Delta$.

⁷For $X \xrightarrow{f} Y$ we have $\text{cone}(f)^* \cong \text{cone}(f^*)[-1]$, the isomorphism being realized by multiplying with $(-1)^i$ on $(X^*)^i$ and $\text{cone}(f[n]) = \text{cone}((-1)^n f)[n]$.

Now note that

$$D_{\text{rig}}^\dagger(W)^\vee \cong D_{\text{rig}}^\dagger(W^*(\chi_{\text{LT}})) \tag{5.24}$$

for any L -analytic representation W by the fact that the functor D_{rig}^\dagger respects inner homs (cf. [79, Rem. 5.6] for the analogous case D_{LT}). Hence, the tautological pairing ev_2 from (5.10) together with the above isomorphism (5.23) induces the following pairing:

$$\cup_{K,\psi} : h^1(K_{\varphi,U'}(D_{\text{rig}}^\dagger(W)^\Delta)) \times h^1(K_{\psi,U'}(D_{\text{rig}}^\dagger(W^*(\chi_{\text{LT}})^\Delta)[d-1])) \rightarrow L.$$

Remark 5.2.16. For $U = U'$ and $M = D_{\text{rig}}^\dagger(W)$, on the level of cochains, this pairing is given as follows:

$$\check{M} \oplus K^{d-1}(\check{M}) \times K^1(M) \oplus M \rightarrow L, \quad ((x, y), (x', y')) \mapsto \{y', x\} - y(x'),$$

where we again use that $K^{d-1}(\check{M}) \cong K^1(M)^*$ and where $\{, \}$ denotes the pairing (4.73). More generally, we have a diagram as in Figure 5.3.

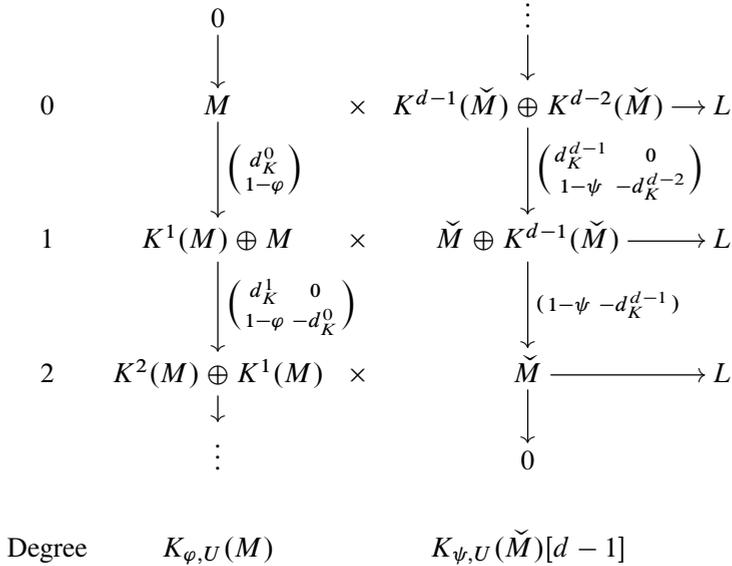


Figure 5.3. Duality for generalized Herr complexes.

Recalling that $W = V^*(1)$ is L -analytic, set $M = D_{\text{rig}}^\dagger(W)$ and $\check{M} = D_{\text{rig}}^\dagger(V(\tau^{-1})) = D_{\text{rig}}^\dagger(W^*(\chi_{\text{LT}}))$. We obtain a Fontaine-style, explicit map

$$\text{pr}_U : D_{\text{rig}}^\dagger(V(\tau^{-1}))^{\psi=1} \rightarrow h^1(K_{\psi,U'}(\check{M}^\Delta)[d-1]), \quad m \mapsto [(\bar{m}, 0)], \tag{5.25}$$

where $\bar{m} = \frac{1}{\#\Delta} \sum_{\delta \in \Delta} \delta m$ denotes the image of m under the map $\check{M} \twoheadrightarrow \check{M}_\Delta \cong \check{M}^\Delta$.

Remark 5.2.17. Let $U_1 \subseteq U$ be an open subgroup with torsion subgroups Δ_1 and Δ , respectively. Assume that the torsion-free parts U'_1 and U' are generated by $\gamma_1^{p^n}, \dots, \gamma_d^{p^n}$ and $\gamma_1, \dots, \gamma_d$, respectively. Then, for M any complete locally convex L -vector space with continuous U -action, the restriction and corestriction maps of Koszul complexes from Section 5.2.2 extend by functoriality to the mapping fiber

$$\begin{aligned} \text{cor}_{U_1}^{U_1} &:= \text{cor}_{U'_1}^{U'_1} \circ K_{\varphi, U'_1}(N_{\Delta/\Delta_1}) : K_{\varphi, U'_1}(M^{\Delta_1}) \rightarrow K_{\varphi, U'}(M^\Delta), \\ \text{res}_{U_1}^U &:= K_{\varphi, U'_1}(\iota) \circ \text{res}_{U'_1}^{U'} : K_{\varphi, U'}(M^\Delta) \rightarrow K_{\varphi, U'_1}(M^{\Delta_1}). \end{aligned}$$

Here $N_{\Delta/\Delta_1} : M^{\Delta_1} \rightarrow M^\Delta$ denotes the norm/trace map sending m to $\sum_{\delta \in \Delta/\Delta_1} \delta m$, while $\iota : M^\Delta \rightarrow M^{\Delta_1}$ is the inclusion. Taking duals as in (5.23), we also obtain

$$\begin{aligned} \text{cor}_U^{U_1} &:= (\text{res}_{U_1}^U)^*[1-d] : K_{\psi, U'_1}(M^{\Delta_1}) \rightarrow K_{\psi, U'}(M^\Delta), \\ \text{res}_{U_1}^U &:= (\text{cor}_{U_1}^U)^*[1-d] : K_{\psi, U'}(M^\Delta) \rightarrow K_{\psi, U'_1}(M^{\Delta_1}) \end{aligned}$$

(co)restriction maps for the ψ -Herr complexes.

Since inflation is compatible with restriction and corestriction, one checks that the above maps are compatible under the isomorphism (5.15) with the usual maps in Galois cohomology. Moreover, they define such maps on H_{\dagger}^1 and $H_{/\dagger}^1$ via (5.17) and

$$h^1(K_{\psi, U'}(D_{\text{rig}}^\dagger(W^*(\chi_{\text{LT}}))^\Delta[d-1])) \cong H_{/\dagger}^1(L', W^*(1)).$$

By the discussion at the end of Section 5.2.2 the restriction map

$$K_{\varphi, U'}(M^\Delta) \xrightarrow{\text{res}_{U_1}^U} K_{\varphi, U'_1}(M^{\Delta_1})$$

and corestriction map $K_{\varphi, U'_1}(M^{\Delta_1}) \xrightarrow{\text{cor}_{U_1}^U} K_{\varphi, U'}(M^\Delta)$ in degree 0 are given as inclusion $M^\Delta \hookrightarrow M^{\Delta_1}$ and norm $M^{\Delta_1} \xrightarrow{N_{U', U'_1} \circ N_{\Delta/\Delta_1}} M^\Delta$, respectively, where

$$N_{U', U'_1} := \prod_{i=1}^d \sum_{k=0}^{p^n-1} \gamma_i^k \in \Lambda(U').$$

Hence, by duality the restriction map

$$K_{\psi, U}(\check{M}^\Delta)[d-1]^2 \xrightarrow{\text{res}_{U_1}^U} K_{\psi, U_1}(\check{M}^{\Delta_1})[d-1]^2$$

and corestriction map $K_{\psi, U_1}(\check{M}^{\Delta_1})[d-1]^2 \xrightarrow{\text{cor}_{U_1}^U} K_{\psi, U}(\check{M}^\Delta)[d-1]^2$ are given by

the norm $\check{M}^\Delta \xrightarrow{(\Delta:\Delta_1)\iota(N_{U',U'_1})} \check{M}^{\Delta_1}$ and projection map $\check{M}^{\Delta_1} \xrightarrow{\frac{1}{(\Delta:\Delta_1)}N_{\Delta/\Delta_1}} \check{M}^\Delta$, respectively. Here ι denotes the involution of $\Lambda(U)$ sending u to u^{-1} . Note that the latter two descriptions also hold for the first components of

$$\begin{aligned} K_{\psi,U}(\check{M}^\Delta)[d-1]^1 &\xrightarrow{\text{res}_{U'_1}^U} K_{\psi,U_1}(\check{M}^{\Delta_1})[d-1]^1, \\ K_{\psi,U_1}(\check{M}^{\Delta_1})[d-1]^1 &\xrightarrow{\text{cor}_{U'}^{U_1}} K_{\psi,U}(\check{M}^\Delta)[d-1]^1, \end{aligned}$$

respectively. Hence, we obtain

$$\text{cor}_{U'}^{U_1} \circ \text{pr}_{U_1} = \text{pr}_U \quad \text{and} \quad \text{res}_{U'_1}^U \circ \text{pr}_U = \text{pr}_{U_1} \circ N_{\Delta/\Delta_1} \circ \iota(N_{U',U'_1}).$$

Berger and Fourquaux by contrast define a different Fontaine-style map⁸ in [7, Thm. 2.5.8] for an L -analytic representation Z and $N = D_{\text{rig}}^\dagger(Z)$

$$\begin{aligned} h_{L_\infty,Z}^1 : D_{\text{rig}}^\dagger(Z)^{\psi_L = \frac{q}{\pi_L}} &\rightarrow H_{\text{an}}^1(U, D_{\text{rig}}^\dagger(Z)^{\psi_L = \frac{q}{\pi_L}}) \\ &\rightarrow h^1(\mathcal{J}_{\varphi_L,U}(N)) \xrightarrow{\cong} h^1(K_{\varphi_L,U}(N^\Delta)) \quad (5.26) \\ y \mapsto [c_b(y)] &\mapsto [(c_b(y), -m_c)] \mapsto [(\tilde{c}_b(y), -\tilde{m}_c)], \end{aligned}$$

in which the cocycle $h_{L_\infty,Z}^1(y)$ is given in terms of the pair $(c_b(y), -m_c)$ in the notation of loc. cit.: m_c is the unique element in $D_{\text{rig}}^\dagger(Z)^{\psi_L=0}$ such that

$$(\varphi_L - 1)c_b(y)(\gamma) = (\gamma - 1)m_c \quad (5.27)$$

for all $\gamma \in U$ and this pair defines the extension class in the sense of Lemma 5.2.11. Here, the first map is implicitly given by [7, Prop. 2.5.1], the second one is the composite from maps arising in [7, Cor. 2.2.3, Thm. 2.2.4] with the natural map from analytic to continuous cohomology

$$\begin{aligned} H_{\text{an}}^1(U, D_{\text{rig}}^\dagger(Z)^{\psi_L = \frac{q}{\pi_L}}) &\rightarrow H_{\text{an}}^1(U \times \Psi, D_{\text{rig}}^\dagger(Z)) \\ &\cong H_{\text{an}}^1(U \times \Phi, D_{\text{rig}}^\dagger(Z)) \rightarrow H^1(U \times \Phi, D_{\text{rig}}^\dagger(Z)) \end{aligned}$$

⁸We do not know whether this map coincides with the following composite we had used in older versions and which uses the shuffle maps from Proposition 5.2.9:

$$\begin{aligned} h_{L_\infty,Z}^1 : D_{\text{rig}}^\dagger(Z)^{\psi_L = \frac{q}{\pi_L}} &\rightarrow H_{\text{an}}^1(U, D_{\text{rig}}^\dagger(Z)^{\psi_L = \frac{q}{\pi_L}}) \\ &\rightarrow h^1(\mathcal{J}_{\frac{\pi}{q}\psi_L,U}^{\text{an}}(N)) \cong h^1(\mathcal{J}_{\varphi_L,U}^{\text{an}}(M)) \\ &\rightarrow h^1(\mathcal{J}_{\varphi_L,U}(N)) \cong H_{\dagger}^1(L_\infty^U, Z). \end{aligned}$$

Here the second map is stemming from the spectral sequence (5.14), the third from Proposition 5.2.9, the fourth is the natural map (5.13), and the last one is (5.17).

combined with the interpretation of extension classes (see [7, §1.4] and Lemma 5.2.11), and the last one is (5.22) (the concrete image $(\widetilde{c}_b(y), -\widetilde{m}_c)$ will be of interest for us only in the situation where Δ is trivial, when $\widetilde{m}_c = m_c$).

According to [7, Prop. 2.5.6, Rem. 2.5.7], this map also satisfies

$$\text{cor}_{U'}^U \circ h_{L_{\infty, Z}^{U'}}^1 = h_{L_{\infty, Z}^U}^1. \quad (5.28)$$

Since $D_{\text{rig}}^\dagger(V(\tau^{-1}))^\vee \cong D_{\text{rig}}^\dagger(V^*(1))$ by (5.24), concerning the Iwasawa pairing, we have the following.

Proposition 5.2.18. *For a G_L -representation V such that $V^*(1)$ is L -analytic the following diagram consisting of $D(\Gamma_L, K)$ - ι_* -sesquilinear pairings (in the sense of (4.87)) is commutative:*

$$\begin{array}{ccc} h^1(K_{\varphi, U'}(D_{\text{rig}}^\dagger(V^*(1)(\chi_{\text{LT}}^j)^\Delta))) \times h^1(K_{\psi, U'}(D_{\text{rig}}^\dagger(V(\chi_{\text{LT}}^{-j})^\Delta)[d-1])) & \xrightarrow{\cup_{K, \psi}} & L \subseteq \mathbb{C}_p \\ \uparrow h_{L, V^*(1)(\chi_{\text{LT}}^j)}^1 \circ \text{otw}_{\chi_{\text{LT}}^j} & & \uparrow \text{pr}_U \circ \text{otw}_{\chi_{\text{LT}}^{-j}} \\ D_{\text{rig}}^\dagger(V^*(1))^{\psi_L = \frac{q}{\pi L}} & \times & D_{\text{rig}}^\dagger(V(\tau^{-1}))^{\psi_L = 1} \xrightarrow{\text{res}_{\Gamma_L}} D(\Gamma_L, \mathbb{C}_p) \end{array}$$

taking $U' \times \Delta = U = \Gamma_L$.

Proof. By Lemma 4.5.22, it suffices to show the case $j = 0$, i.e., the trivial character χ_{triv} . Furthermore, it suffices to show the following statement for any subgroup of the form Γ_n without any p -torsion:

$$q^{-n} \text{ev}_{L_n, \chi_{\text{triv}|_{\Gamma_n}}} \circ \{x, y\}_{\text{Iw}, \Gamma_n} = h_{L_n, V^*(1)}^1(x) \cup_{K, \psi} \text{pr}_{\Gamma_n}(y) \quad (5.29)$$

for $x \in D_{\text{rig}}^\dagger(V^*(1))^{\psi_L = \frac{q}{\pi L}}$, $y \in D_{\text{rig}}^\dagger(V(\tau^{-1}))^{\psi_L = 1}$.

Indeed, by Remark 5.2.17, for every such n , we have the commutative diagram

$$\begin{array}{ccc} h^1(K_{\varphi, \Gamma_n}(D_{\text{rig}}^\dagger(V^*(1)))) & \times & h^1(K_{\psi, \Gamma_n}(D_{\text{rig}}^\dagger(V)[d-1])) \xrightarrow{\cup_{K, \psi}} L \\ \downarrow \text{cor} & & \uparrow \text{res} \\ h^1(K_{\varphi, U'}(D_{\text{rig}}^\dagger(V^*(1))^\Delta)) & \times & h^1(K_{\psi, U'}(D_{\text{rig}}^\dagger(V)^\Delta)[d-1]) \xrightarrow{\cup_{K, \psi}} L \end{array}$$

Hence, we obtain using (5.28)

$$\begin{aligned} h_{L', V^*(1)}^1(x) \cup_{K, \psi} \text{pr}_U(y) &= (\text{cor} \circ h_{L_n, V^*(1)}^1(x)) \cup_{K, \psi} \text{pr}_U(y) \\ &= h_{L_n, V^*(1)}^1(x) \cup_{K, \psi} (\text{res} \circ \text{pr}_U(y)) \\ &= h_{L_n, V^*(1)}^1(x) \cup_{K, \psi} (\text{pr}_{\Gamma_n}(N_\Delta \circ \iota(N_{U', \Gamma_n})y)), \end{aligned}$$

where we use Remark 5.2.17 for the last equality. On the other hand, one easily checks that⁹

$$\mathrm{ev}_{L_n, \chi_{\mathrm{triv}}|\Gamma_n} \circ \mathrm{pr}_{U, \Gamma_n} \circ N_\Delta \circ \iota(N_{U', \Gamma_n}) = \mathrm{ev}_{L, \chi_{\mathrm{triv}}} : D(U, \mathbb{C}_p) \rightarrow \mathbb{C}_p,$$

whence

$$\begin{aligned} \mathrm{ev}_{L, \chi_{\mathrm{triv}}} \circ \frac{q-1}{q} \{x, y\}_{\mathrm{Iw}, U} &= \frac{q-1}{q} \mathrm{ev}_{L_n, \chi_{\mathrm{triv}}|\Gamma_n} \circ \mathrm{pr}_{U, \Gamma_n} (N_\Delta \circ \iota(N_{U', \Gamma_n}) \{x, y\}_{\mathrm{Iw}, U}) \\ &= \frac{q-1}{q} \mathrm{ev}_{L_n, \chi_{\mathrm{triv}}|\Gamma_n} \circ \mathrm{pr}_{U, \Gamma_n} (\{x, N_\Delta \circ \iota(N_{U', \Gamma_n})y\}_{\mathrm{Iw}, U}) \\ &= \frac{q-1}{q} [U : \Gamma_n]^{-1} \mathrm{ev}_{L_n, \chi_{\mathrm{triv}}|\Gamma_n} \circ \{x, N_\Delta \circ \iota(N_{U', \Gamma_n})y\}_{\mathrm{Iw}, \Gamma_n} \\ &= q^{-n} \mathrm{ev}_{L_n, \chi_{\mathrm{triv}}|\Gamma_n} \circ \{x, N_\Delta \circ \iota(N_{U', \Gamma_n})y\}_{\mathrm{Iw}, \Gamma_n}, \end{aligned}$$

where we have used Remark 4.5.21 for the last equation.

In order to prove (5.29), choose $n = n_0$ (see Section 4.3.4). As recalled in (5.26), the map

$$h_{L_{n_0}, V^*(1)}^1 : D_{\mathrm{rig}}^\dagger(V^*(1))^{\psi_L = \frac{q}{\pi_L}} \rightarrow h^1(K_{\varphi_L, \Gamma_{n_0}}(D_{\mathrm{rig}}^\dagger(V^*(1))))$$

is given by the cocycle $h_{L_{n_0}, V^*(1)}^1(x)$ in terms of the pair $(\widetilde{c}_b(x), -m_c)$. Note that we have

$$m_c = \widehat{\Xi}_b(\varphi_L - 1)x.$$

Indeed, by [7, Thm. 2.5.8] we have $c_b(x)(b_j^k) = (b_j^k - 1)\widehat{\Xi}_b x$ for all $j, k \geq 0$, which together with (5.27) and the uniqueness of m_c (cf. loc. cit.) implies the claim. On the other hand, we have the map (5.25)

$$\begin{aligned} \mathrm{pr}_{\Gamma_{n_0}} : D_{\mathrm{rig}}^\dagger(V(\tau^{-1}))^{\psi_L = 1} &\rightarrow h^1(K_{\psi, \Gamma_{n_0}}(D_{\mathrm{rig}}^\dagger(V(\tau^{-1}))) [d-1]) \\ y &\mapsto \text{class of } (y, 0). \end{aligned}$$

Thus, the pairing $\cup_{K, \psi}$ sends by construction (see diagram (5.3)) the above classes to

$$\begin{aligned} h_{L_n, V^*(1)}^1(x) \cup_{K, \psi} \mathrm{pr}_{\Gamma_n}(y) &= 0(\widetilde{c}_b(x)) + \{-\widehat{\Xi}_b(\varphi_L - 1)x, y\} \\ &= \{\widehat{\Xi}_b(\varphi_L - 1)x, (\frac{\pi_L}{q}\varphi_L - 1)y\} \\ &= \langle \widehat{\Xi}_b, \{x, y\}_{\mathrm{Iw}, \Gamma_n} \rangle_{\Gamma_n} \\ &= q^{-n} \mathrm{aug}(\{x, y\}_{\mathrm{Iw}, \Gamma_n}). \end{aligned}$$

Here the second equality holds due to Lemma 4.5.1 because the left-hand side belongs to $D_{\mathrm{rig}}^\dagger(V^*(1))^{\psi_L = 0}$, the third one is (4.89), and the last one comes from (4.81). ■

⁹This is obvious if you decompose $D(U, \mathbb{C}_p) = \bigoplus_{g \in U/\Gamma_n} D(\Gamma_n, \mathbb{C}_p)g$ with respect to the inverses of the representatives used in the definition of $N_\Delta \circ \iota(N_{U', \Gamma_n})$.

Proposition 5.2.19. *For W an L -analytic representation we have a canonical commutative diagram*

$$\begin{array}{ccc}
 \cup_{K,\psi} : h^1(K_{\varphi,U'}(D_{\text{rig}}^\dagger(W)^\Delta)) \times h^1(K_{\psi,U'}(D_{\text{rig}}^\dagger(W^*(\chi_{\text{LT}})^\Delta)[d-1]) \rightarrow L & & \\
 \downarrow b \cong & & \downarrow a \cong \\
 \langle \cdot, \cdot \rangle_{\text{Tate},L,\dagger} : H_{\dagger}^1(L', W) \times H_{/\dagger}^1(L', W^*(1)) \rightarrow H^2(L', L(1)) \cong L & & \\
 \downarrow & & \uparrow \text{pr} \\
 \langle \cdot, \cdot \rangle_{\text{Tate},L'} : H^1(L', W) \times H^1(L', W^*(1)) \rightarrow H^2(L', L(1)) \cong L. & &
 \end{array}$$

Moreover, the isomorphism a is compatible with the middle map of the diagram (A.5).

Proof. The lower square of pairings comes from Tate duality as in Proposition 5.2.14 and (5.19). Its commutativity holds by definition. In the upper square of pairings the left upper vertical isomorphism b arises from (5.17) combined with (5.22), while the middle vertical isomorphism a is uniquely determined as adjoint of the latter because both pairings are non-degenerate: the middle one by definition of $H_{/\dagger}^1$ while the upper one due to Corollary A.4 (ii) with $W = V^*(1)$. Therefore, one immediately checks that $a^{-1} \circ \text{pr}$ is induced by the cohomology of the middle map (going down) in diagram (A.5) (being the same as the middle map (going from right to left) of diagram (A.6) upon identifying $h^1(K_{\psi,U'}^\bullet(D(V(\tau^{-1}))^\Delta)[d-1])$ and $H^1(L', V)$ by the isomorphism described there). ■

Combining the last two propositions, we get the following result.

Corollary 5.2.20. *For a G_L -representation V such that $V^*(1)$ is L -analytic the following diagram is commutative:*

$$\begin{array}{ccc}
 H_{\dagger}^1(L, V^*(1)(\chi_{\text{LT}}^j)) \times H_{/\dagger}^1(L, V(\chi_{\text{LT}}^{-j})) \xrightarrow{\langle \cdot, \cdot \rangle_{\text{Tate},L,\dagger}} H^2(L, L(1)) \cong L \subseteq \mathbb{C}_p & & \\
 \uparrow h_{L,V^*(1)}^{\text{otw}} \chi_{\text{LT}}^j & & \uparrow \text{pr}_L \text{ otw } \chi_{\text{LT}}^{-j} \\
 D_{\text{rig}}^\dagger(V^*(1))^{\psi_L = \frac{q}{\pi_L}} \times D_{\text{rig}}^\dagger(V(\tau^{-1}))^{\psi_L = 1} \xrightarrow{\frac{q-1}{q} \langle \cdot, \cdot \rangle_{\text{Iw}}} D(\Gamma_L, \mathbb{C}_p). & & \uparrow \text{ev } \chi_{\text{LT}}^{-j}
 \end{array}$$

Remark 5.2.21. For applications it might be useful to renormalize $\cup_{K,\psi}$ by the factor $\frac{q}{q-1}$, i.e., setting $\cup'_{K,\psi} := \frac{q}{q-1} \cup_{K,\psi}$. Then we would get rid of the factor $\frac{q-1}{q}$ in front of the Iwasawa pairing in the above results. Moreover and more importantly, the new normalization would be compatible with the cyclotomic situation taking $L = \mathbb{Q}_p$, $\pi_L = p = q$; i.e., the upper pairing in Proposition 5.2.18 would coincide – at least up to a sign – with the cup product pairing of Galois cohomology

$$H^1(\mathbb{Q}_p, V^*(j+1)) \times H^1(\mathbb{Q}_p, V(-j)) \longrightarrow H^2(\mathbb{Q}_p, \mathbb{Q}_p(1)) \xrightarrow[\cong]{\text{inv}} \mathbb{Q}_p,$$

using Tate’s trace map

$$\text{inv} : H^2(\mathbb{Q}_p, \mathbb{Q}_p(1)) \cong \mathbb{Q}_p$$

given by class field theory, if one chooses $Z = \gamma - 1$ for a topological generator γ of $\Gamma_{\mathbb{Q}_p}$ satisfying $\log \chi_{\text{cyc}}(\gamma) = 1$. This follows from [3, Prop. 1.3.4, Thm. 2.2.6], [38, Thm. 5.2, Rem. 5.3], and [42, Rem. 2.3.11/12]: they claim that $-\frac{p-1}{p}\text{inv}$ corresponds to the trace map from the second cohomology group of the φ -Herr complex induced by sending $f \otimes \eta$ to $\frac{1}{\log \chi_{\text{cyc}}(\gamma)} \text{res}_{\omega_{\text{cyc}}} (f \frac{d\omega_{\text{cyc}}}{1+\omega_{\text{cyc}}})$.

With respect to evaluating at a character, we have the following analogue of Corollary 5.2.20.

Proposition 5.2.22. *For a G_L -representation V such that $V^*(1)$ is L -analytic the following diagram is commutative:*

$$\begin{CD} \mathbb{C}_p \otimes_L D_{\text{cris},L}(V^*(1)(\chi_{\text{LT}}^j)) @\times\ \mathbb{C}_p \otimes_L D_{\text{cris},L}(V(\tau^{-1})(\chi_{\text{LT}}^{-j})) @>[\cdot,]_{\text{cris}}>> \mathbb{C}_p \otimes_L D_{\text{cris},L}(L(\chi_{\text{LT}})) \\ @V \uparrow \text{ev}_{\chi_{\text{LT}}^{-j}} \otimes t_{\text{LT}}^{-j} \eta^{\otimes j} VV @V \uparrow \text{ev}_{\chi_{\text{LT}}^j} \otimes t_{\text{LT}}^j \eta^{\otimes -j} VV @V \uparrow \text{ev}_{\chi_{\text{LT}}^{-j}} VV \\ D(\Gamma_L, \mathbb{C}_p) \otimes_L D_{\text{cris},L}(V^*(1)) \times D(\Gamma_L, \mathbb{C}_p) \otimes_L D_{\text{cris},L}(V(\tau^{-1})) @>[\cdot,]^0>> D(\Gamma_L, \mathbb{C}_p), \end{CD}$$

where in the right upper corner we identify $\mathbb{C}_p \otimes_L D_{\text{cris},L}(L(\chi_{\text{LT}})) \cong \mathbb{C}_p$ by choosing $t_{\text{LT}}^{-1} \otimes \eta$ as a basis.

Proof. Using Lemma 5.2.23 below, the statement is reduced to $j = 0$. Evaluation of (5.9) implies the claim in this case. ■

Lemma 5.2.23. *There is a commutative diagram*

$$\begin{CD} D(\Gamma_L, \mathbb{C}_p) \otimes_L D_{\text{cris},L}(V^*(1)(\chi_{\text{LT}}^j)) @\times\ D(\Gamma_L, \mathbb{C}_p) \otimes_L D_{\text{cris},L}(V(\tau^{-1})(\chi_{\text{LT}}^{-j})) @>[\cdot,]^0>> D(\Gamma_L, \mathbb{C}_p) \\ @V \uparrow \text{Tw}_{\chi_{\text{LT}}^{-j}} \otimes t_{\text{LT}}^{-j} \eta^{\otimes j} VV @V \uparrow \text{Tw}_{\chi_{\text{LT}}^j} \otimes t_{\text{LT}}^j \eta^{\otimes -j} VV @V \uparrow \text{Tw}_{\chi_{\text{LT}}^{-j}} VV \\ D(\Gamma_L, \mathbb{C}_p) \otimes_L D_{\text{cris},L}(V^*(1)) @\times\ D(\Gamma_L, \mathbb{C}_p) \otimes_L D_{\text{cris},L}(V(\tau^{-1})) @>[\cdot,]^0>> D(\Gamma_L, \mathbb{C}_p). \end{CD}$$

Proof. The claim follows immediately from (5.9), the compatibility of the usual D_{cris} -pairing with twists, and the fact that $\text{Tw}_{\chi_{\text{LT}}^j}(\lambda)_*(\mu) = \text{Tw}_{\chi_{\text{LT}}^j}(\lambda) \iota_*(\text{Tw}_{\chi_{\text{LT}}^{-j}}(\mu))$ holds. ■

5.2.4 The interpolation formula for the regulator map

In this subsection, we are going to prove the interpolation property for \mathcal{L}_V . First, recall that we introduced in Section 3.1 the notation $D_{\text{dR},L'}(V) := (B_{\text{dR}} \otimes_{\mathbb{Q}_p} V)^{G_{L'}}$. Since B_{dR} contains the algebraic closure \bar{L} of L , we have the isomorphism

$$B_{\text{dR}} \otimes_{\mathbb{Q}_p} V = (B_{\text{dR}} \otimes_{\mathbb{Q}_p} L) \otimes_L V \cong \prod_{\sigma \in G_{\mathbb{Q}_p}/G_L} B_{\text{dR}} \otimes_{\sigma,L} V$$

which sends $b \otimes v$ to $(b \otimes v)_\sigma$. The tensor product in the factor $B_{\text{dR}} \otimes_{\sigma,L} V$ is formed with respect to L acting on B_{dR} through σ . With respect to the G_L -action, the right-hand side decomposes according to the double cosets in $G_L \setminus G_{\mathbb{Q}_p}/G_L$. It follows, in

particular, that $D_{\text{dR}}^{\text{id}}(V) := (B_{\text{dR}} \otimes_L V)^{G_L}$ is a direct summand of $D_{\text{dR},L}(V)$ and we denote by pr^{id} the corresponding projection. Similarly, $\text{tan}_{L,\text{id}}(V) := (B_{\text{dR}}/B_{\text{dR}}^+ \otimes_L V)^{G_L}$ is a direct summand of $\text{tan}_L(V) := (B_{\text{dR}} \otimes_L V)^{G_L}$. More generally, also the filtration $D_{\text{dR},L}^i(V)$ decomposes into direct summands.

According to [80, Appx. A], the dual Bloch–Kato exponential map is uniquely determined by the commutativity of the following diagram, in which all pairings are perfect:

$$\begin{array}{ccccc}
 H^1(L', W) & \times & H^1(L', W^*(1)) & \xrightarrow{\langle \cdot, \cdot \rangle_{\text{Tate}, L'}} & L \\
 \downarrow \text{exp}_{L', W}^* & & \uparrow \text{exp}_{L', W^*(1)} & & \downarrow \\
 D_{\text{dR}, L'}^0(W) & \times & \text{tan}_{L'}(W^*(1)) & \longrightarrow & D_{\text{dR}, L'}(\mathbb{Q}_p(1)) \xrightarrow{\cong} L' \\
 \downarrow & & \uparrow \text{pr} & & \parallel \\
 D_{\text{dR}, L'}(W) & \times & D_{\text{dR}, L'}(W^*(1)) & \longrightarrow & D_{\text{dR}, L'}(\mathbb{Q}_p(1)) \xrightarrow{\cong} L'.
 \end{array}$$

Note that we have the compatibility of the following pairings:

$$\begin{array}{ccccc}
 D_{\text{cris}, L}(V^*(1)) & \times & D_{\text{cris}, L}(V(\tau^{-1})) & \xrightarrow{[\cdot, \cdot]_{\text{cris}}} & D_{\text{cris}, L}(L(\chi_{\text{LT}})) \xrightarrow{\cong} L \\
 \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\
 D_{\text{dR}}^{\text{id}}(V^*(1)) & \times & D_{\text{dR}}^{\text{id}}(V(\tau^{-1})) & \xrightarrow{[\cdot, \cdot]_{\text{dR}}^{\text{id}}} & D_{\text{dR}}^{\text{id}}(L(\chi_{\text{LT}})) \xrightarrow{\cong} L \\
 \downarrow \text{injective} & & \uparrow \text{pr}^{\text{id}} \text{ surjective} & & \swarrow \text{pr}^{\text{id}} \\
 D_{\text{dR}}(V^*(1)) & \times & D_{\text{dR}}(V(\tau^{-1})) & \longrightarrow & D_{\text{dR}}(L(\chi_{\text{LT}})) \xrightarrow{\cong} L \otimes_{\mathbb{Q}_p} L \\
 \uparrow = & & \uparrow \text{id} \otimes t_{\mathbb{Q}_p} t_{\text{LT}}^{-1} \otimes \eta \otimes (\eta_{\text{cyc}})^{-1} & & \searrow \text{pr}^{\text{id}} \\
 D_{\text{dR}}(V^*(1)) & \times & D_{\text{dR}}(V) & \xrightarrow{[\cdot, \cdot]_{\text{dR}}} & D_{\text{dR}}(\mathbb{Q}_p(1)) \xrightarrow{\cong} L
 \end{array}$$

(cf. [80, (57) in the appendix]). Therefore, in the Lubin–Tate setting we can also consider the dual of the identity component $\text{exp}_{L', W^*(1), \text{id}}$ of $\text{exp}_{L', W^*(1)}$:

$$\begin{array}{ccccc}
 H^1(L', W) & \times & H^1(L', W^*(1)) & \xrightarrow{\langle \cdot, \cdot \rangle_{\text{Tate}, L'}} & L \\
 \downarrow \widetilde{\text{exp}}_{L', W, \text{id}}^* & & \uparrow \text{exp}_{L', W^*(1), \text{id}} & & \downarrow \\
 D_{\text{dR}, L'}^{\text{id}, 0}(W(\tau^{-1})) & \times & \text{tan}_{L', \text{id}}(W^*(1)) & \longrightarrow & D_{\text{dR}, L'}^{\text{id}}(L(\chi_{\text{LT}})) \xrightarrow{\cong} L' \\
 \downarrow & & \uparrow \text{pr} & & \parallel \\
 D_{\text{dR}, L'}^{\text{id}}(W(\tau^{-1})) & \times & D_{\text{dR}, L'}^{\text{id}}(W^*(1)) & \longrightarrow & D_{\text{dR}, L'}^{\text{id}}(L(\chi_{\text{LT}})) \xrightarrow{\cong} L'.
 \end{array} \tag{5.30}$$

Upon noting that under the identifications

$$D_{\mathrm{dR},L'}(\mathbb{Q}_p(1)) \cong L' \quad \text{and} \quad D_{\mathrm{dR},L'}^{\mathrm{id}}(\mathbb{Q}_p(1)) \cong L'$$

the elements $t_{\mathbb{Q}_p} \otimes \eta_{\mathrm{cyc}}$ and $t_{\mathrm{LT}} \otimes \eta$ are sent to 1, one easily checks that if $W^*(1)$ is L -analytic, whence the inclusion $\mathrm{tan}_{L',\mathrm{id}}(W^*(1)) \subseteq \mathrm{tan}_{L'}(W^*(1))$ is an equality and $\exp_{L',W^*(1),\mathrm{id}} = \exp_{L',W^*(1)}$, then it holds that

$$\mathbb{T}_{\tau^{-1}} \circ \exp_{L',W}^* = \widetilde{\exp}_{L',W,\mathrm{id}}^*$$

where

$$\mathbb{T}_{\tau^{-1}} : D_{\mathrm{dR},L'}^0(W) \rightarrow D_{\mathrm{dR},L'}^{\mathrm{id},0}(W(\tau^{-1}))$$

is the isomorphism which sends $b \otimes v$ to $b \frac{t_{\mathbb{Q}_p}}{t_{\mathrm{LT}}} \otimes v \otimes \eta \otimes (\eta_{\mathrm{cyc}})^{\otimes -1}$; note that $\frac{t_{\mathbb{Q}_p}}{t_{\mathrm{LT}}} \in (B_{\mathrm{dR}}^+)^{\times}$, whence the filtration is preserved.

Now let W be an L -analytic, crystalline L -linear representation of G_L . Recall that $\eta = (\eta_n)_n$ denotes a fixed generator of T_π and that the map

$$\mathrm{tw}_{\chi_{\mathrm{LT}}^j} : D_{\mathrm{rig}}^\dagger(W) \rightarrow D_{\mathrm{rig}}^\dagger(W(\chi_{\mathrm{LT}}^j))$$

has been defined before Lemma 4.5.22. For D_{cris} the twisting map

$$D_{\mathrm{cris},L}(W) \xrightarrow{-\otimes e_j} D_{\mathrm{cris},L}(W(\chi_{\mathrm{LT}}^j))$$

sends d to $d \otimes e_j$ with the element e_j given as $e_j = t_{\mathrm{LT}}^{-j} \otimes \eta^{\otimes j} \in D_{\mathrm{cris},L}(L(\chi_{\mathrm{LT}}^j))$.

If we assume, in addition, that

- (i) W has Hodge–Tate weights ≤ 0 , whence $W^*(1)$ has Hodge–Tate weights ≥ 1 and $D_{\mathrm{dR},L}^0(W^*(1)) = 0$, and
- (ii) $D_{\mathrm{cris},L}(W^*(\chi_{\mathrm{LT}}))^{\varphi_L = \frac{q}{\pi L}} = 0$,

then by our assumption $\exp_{L,W^*(1)} : D_{\mathrm{dR},L}(W^*(1)) \hookrightarrow H^1(L, W^*(1))$ is injective with image $H_e^1(L, W^*(1)) = H_f^1(L, W^*(1))$ (see [12, Cor. 3.8.4]). We denote its inverse by

$$\log_{L,W^*(1)} : H_f^1(L, W^*(1)) \rightarrow D_{\mathrm{dR},L}(W^*(1))$$

and define

$$\widetilde{\log}_{L,W^*(1)} : H_f^1(L, W^*(1)) \rightarrow D_{\mathrm{dR},L}(W^*(1)) \xrightarrow{\mathbb{T}_{\tau^{-1}}} D_{\mathrm{cris},L}(W^*(\chi_{\mathrm{LT}})),$$

where (by abuse of notation) we also write

$$\mathbb{T}_{\tau^{-1}} : D_{\mathrm{dR},L}(W^*(1)) \rightarrow D_{\mathrm{dR},L}^{\mathrm{id}}(W^*(\chi_{\mathrm{LT}})) = D_{\mathrm{cris},L}(W^*(\chi_{\mathrm{LT}}))$$

for the isomorphism, which sends $b \otimes v$ to $b \frac{t_{\mathbb{Q}_p}}{t_{\text{LT}}} \otimes v \otimes \eta \otimes (\eta_{\text{cyc}})^{\otimes -1}$. We obtain the following commutative diagram, which defines the dual map $\log_{L,W}^*$ being inverse to $\exp_{L,W}^*$ (up to factorization over $H^1(L, W)/H_f^1(L, W)$):

$$\begin{array}{ccc} H^1(L, W)/H_f^1(L, W) \times H_f^1(L, W^*(1)) & \xrightarrow{\langle \cdot, \cdot \rangle_{\text{Tate}, L}} & L \\ \log_{L,W}^* \uparrow & & \parallel \\ D_{\text{dR},L}(W) \times D_{\text{dR},L}(W^*(1)) & \longrightarrow & D_{\text{dR},L}(\mathbb{Q}_p(1)) \xrightarrow{\cong} L \end{array}$$

Similarly as above, we obtain a commutative diagram more convenient for the Lubin-Tate setting:

$$\begin{array}{ccc} H^1(L, W)/H_f^1(L, W) \times H_f^1(L, W^*(1)) & \xrightarrow{\langle \cdot, \cdot \rangle_{\text{Tate}, L}} & L \\ \log_{L,W,\text{id}}^* \uparrow & & \parallel \\ D_{\text{dR},L}^{\text{id}}(W) \times D_{\text{dR},L}^{\text{id}}(W^*(\chi_{\text{LT}})) & \longrightarrow & D_{\text{dR},L}^{\text{id}}(L(\chi_{\text{LT}})) \xrightarrow{\cong} L \end{array}$$

We write $\text{Ev}_{W,n} : \mathcal{O}_L \otimes_L D_{\text{cris},L}(W) \rightarrow L_n \otimes_L D_{\text{cris},L}(W)$ for the composite map $\partial_{D_{\text{cris},L}(W)} \circ \varphi_q^{-n}$ from the introduction of [7], which actually sends $f(Z) \otimes d$ to $f(\eta_n) \otimes \varphi_L^{-n}(d)$. By abuse of notation we also use $\text{Ev}_{W,0}$ for the analogous map

$$\mathcal{O}_K \otimes_L D_{\text{cris},L}(W) \rightarrow K \otimes_L D_{\text{cris},L}(W).$$

For $x \in D(\Gamma_L, K) \otimes_L D_{\text{cris},L}(W)$ we denote by $x(\chi_{\text{LT}}^j)$ the image under the map

$$D(\Gamma_L, K) \otimes_L D_{\text{cris},L}(W) \rightarrow K \otimes_L D_{\text{cris},L}(W), \quad \lambda \otimes d \mapsto \lambda(\chi_{\text{LT}}^j) \otimes d.$$

Lemma 5.2.24. *Assume that Ω is contained in K . Then there are commutative diagrams*

$$\begin{array}{ccccc} D(\Gamma_L, K) \otimes_L D_{\text{cris},L}(W) & \xrightarrow{\mathfrak{M} \otimes \text{id}} & \mathcal{O}_K \otimes_L D_{\text{cris},L}(W) & \xleftarrow{1 - \varphi_L \otimes \varphi_L} & \mathcal{O}_K \otimes_L D_{\text{cris},L}(W) \\ \text{ev}_{\text{triv}} \downarrow & & \downarrow \text{Ev}_{W,0} & & \downarrow \text{Ev}_{W,0} \\ K \otimes_L D_{\text{cris},L}(W) & \xlongequal{\quad} & K \otimes_L D_{\text{cris},L}(W) & \xleftarrow{1 - \text{id} \otimes \varphi_L} & K \otimes_L D_{\text{cris},L}(W) \end{array}$$

and

$$\begin{array}{ccccc} D(\Gamma_L, K) \otimes_L D_{\text{cris},L}(W) & \xrightarrow{\mathfrak{M} \otimes \text{id}} & \mathcal{O}_K \otimes_L D_{\text{cris},L}(W) & \xleftarrow{1 - \varphi_L \otimes \varphi_L} & \mathcal{O}_K \otimes_L D_{\text{cris},L}(W) \\ \text{Tw}_{\chi_{\text{LT}}^{-j}} \otimes e_j \downarrow & & \downarrow (\frac{\partial}{\partial \Omega})^{-j} \otimes e_j & & \downarrow (\frac{\partial}{\partial \Omega})^{-j} \otimes e_j \\ D(\Gamma_L, K) \otimes_L D_{\text{cris},L}(W(\chi_{\text{LT}}^j)) & \xrightarrow{\mathfrak{M} \otimes \text{id}} & \mathcal{O}_K \otimes_L D_{\text{cris},L}(W(\chi_{\text{LT}}^j)) & \xleftarrow{1 - \varphi_L \otimes \varphi_L} & \mathcal{O}_K \otimes_L D_{\text{cris},L}(W(\chi_{\text{LT}}^j)). \end{array}$$

In the latter we follow the (for $j > 0$) abusive notation ∂^{-j} from [7, Rem. 3.5.5].

Proof. For the upper diagram note that $\eta_0 = 0$ and $(\delta_g \cdot \eta(1, Z))|_{Z=0} = 1$, from which the claim follows for Dirac distributions, whence in general. For the right square we observe that $\varphi_L(g(Z))|_{Z=0} = g(0)$. Regarding the lower diagram, we use Lemma 4.3.25 and the relation $\partial_{\text{inv}} \circ \varphi_L = \pi_L \varphi_L \circ \partial$. \blacksquare

With this notation, Berger's and Fourquaux's interpolation property reads as follows.

Theorem 5.2.25 (Berger–Fourquaux [7, Thm. 3.5.3]). *Let W be L -analytic and $h \geq 1$ such that $\text{Fil}^{-h} D_{\text{cris},L}(W) = D_{\text{cris},L}(W)$. For any $f \in (\mathcal{O}^{\psi=0} \otimes_L D_{\text{cris},L}(W))^{\Delta=0}$ and $y \in (\mathcal{O} \otimes_L D_{\text{cris},L}(W))^{\psi=\frac{q}{\pi_L}}$ with $f = (1 - \varphi_L)y$, we have: If $h + j \geq 1$, then*

$$\begin{aligned} & h_{L_n, W(\chi_{\text{LT}}^j)}^1(\text{tw}_{\chi_{\text{LT}}^j}(\Omega_{W,h}(f))) \\ &= (-1)^{h+j-1}(h+j-1)! \\ & \quad \times \begin{cases} \exp_{L_n, W(\chi_{\text{LT}}^j)}(q^{-n} \text{Ev}_{W(\chi_{\text{LT}}^j),n}(\partial_{\text{inv}}^{-j} y \otimes e_j)) & \text{if } n \geq 1; \\ \exp_{L, W(\chi_{\text{LT}}^j)}((1 - q^{-1} \varphi_L^{-1}) \text{Ev}_{W(\chi_{\text{LT}}^j),0}(\partial_{\text{inv}}^{-j} y \otimes e_j)) & \text{if } n = 0. \end{cases} \end{aligned}$$

If $h + j \leq 0$, then¹⁰

$$\begin{aligned} & - \exp_{L_n, W(\chi_{\text{LT}}^j)}^*(h_{L_n, W(\chi_{\text{LT}}^j)}^1(\text{tw}_{\chi_{\text{LT}}^j}(\Omega_{W,h}(f)))) \\ &= \frac{1}{(-h-j)!} \begin{cases} q^{-n} \text{Ev}_{W(\chi_{\text{LT}}^j),n}(\partial_{\text{inv}}^{-j} y \otimes e_j) & \text{if } n \geq 1; \\ (1 - q^{-1} \varphi_L^{-1}) \text{Ev}_{W(\chi_{\text{LT}}^j),0}(\partial_{\text{inv}}^{-j} y \otimes e_j) & \text{if } n = 0. \end{cases} \end{aligned}$$

By abuse of notation, we shall denote the base change $K \otimes_L -$ of the (dual) Bloch–Kato exponential map by the same expression. Using Lemma 5.2.24, we deduce the following interpolation property for the modified big exponential map with $x \in D(\Gamma_L, K) \otimes_L D_{\text{cris},L}(W)$: If $j \geq 0$, then

$$\begin{aligned} & h_{L, W(\chi_{\text{LT}}^j)}^1(\text{tw}_{\chi_{\text{LT}}^j}(\Omega_{W,1}(x))) \\ &= (-1)^j j! \Omega^{-j-1} \exp_{L, W(\chi_{\text{LT}}^j)}((1 - q^{-1} \varphi_L^{-1})(1 - \varphi_L)^{-1}(x(\chi_{\text{LT}}^{-j}) \otimes e_j)); \end{aligned} \tag{5.31}$$

if $j < 0$, then

$$\begin{aligned} & h_{L, W(\chi_{\text{LT}}^j)}^1(\text{tw}_{\chi_{\text{LT}}^j}(\Omega_{W,1}(f))) \\ &= \frac{-\Omega^{-j-1}}{(-1-j)!} \log_{L, W(\chi_{\text{LT}}^j)}^*((1 - q^{-1} \varphi_L^{-1})(1 - \varphi_L)^{-1}(x(\chi_{\text{LT}}^{-j}) \otimes e_j)), \end{aligned}$$

¹⁰Note that the definition of the dual exponential map in [7] coincides with (5.30) only up to reversing the order of the cohomology groups in degree one, which induces a sign due to the skew symmetry of the cup product.

assuming in both cases that the operator $1 - \varphi_L$ is invertible on $D_{\text{cris},L}(W(\chi_{\text{LT}}^j))$ and for $j < 0$ also that the operator $1 - q^{-1}\varphi_L^{-1}$ is invertible on $D_{\text{cris},L}(W(\chi_{\text{LT}}^j))$ (in order to grant the existence of $\log_{L,W(\chi_{\text{LT}}^j)}$).

Recall that the generalized Iwasawa cohomology of $T \in \text{Rep}_{o_L}(G_L)$ is defined by

$$H_{\text{Iw}}^*(L_\infty/L, T) := \varprojlim_{L'} H^*(L', T),$$

where L' runs through the finite Galois extensions of L contained in L_∞ and the transition maps in the projective system are the cohomological corestriction maps. For $V := T \otimes_{o_L} L \in \text{Rep}_L(G_L)$ we define

$$H_{\text{Iw}}^*(L_\infty/L, V) := H_{\text{Iw}}^*(L_\infty/L, T) \otimes_{o_L} L,$$

which is independent of the choice of T . As usual, we denote by

$$\text{cor} : H_{\text{Iw}}^*(L_\infty/L, T) \rightarrow H^*(L', T)$$

the projection map and analogously for rational coefficients. In the same way as in (5.25), we have a map

$$\begin{aligned} \text{pr}_U : D(V(\tau^{-1}))^{\psi=1} &\rightarrow h^1(K_{\psi,U'}(D(V(\tau^{-1}))^\Delta)[d-1]) \cong H^1(L', V) \\ m &\mapsto [(\bar{m}, 0)], \end{aligned}$$

where $\bar{m} = \frac{1}{\#\Delta} \sum_{\delta \in \Delta} \delta m$ denotes the image of m under the map $\check{M} \twoheadrightarrow \check{M}_\Delta \cong \check{M}^\Delta$. Note that under the assumptions of Lemma 3.3.6 for $V(\tau^{-1})$, there is a commutative diagram

$$\begin{array}{ccccc} H_{\text{Iw}}^1(L_\infty/L, T) & \xrightarrow{\cong} & D_{\text{LT}}(T(\tau^{-1}))^{\psi=1} & \hookrightarrow & D_{\text{rig}}^\dagger(V(\tau^{-1}))^{\psi=1} \\ \text{cor} \downarrow & & \text{pr}_U \downarrow & & \downarrow \text{pr}_U \\ H^1(L', V) & \xlongequal{\quad} & H^1(L', V) & \xrightarrow{\quad} & H_{\dagger}^1(L', V), \end{array} \tag{5.32}$$

where the right vertical map is induced by (5.25). Indeed, for the commutativity of the left rectangle and the right rectangle we refer the reader to (B.5) and (A.6), respectively. Let $y_{\chi_{\text{LT}}^{-j}}$ denote the image of y under the map

$$\begin{aligned} H_{\text{Iw}}^1(L_\infty/L, T) &\xrightarrow{\cdot \otimes \eta^{\otimes -j}} H_{\text{Iw}}^1(L_\infty/L, T(\chi_{\text{LT}}^{-j})) \xrightarrow{\text{cor}} H^1(L, T(\chi_{\text{LT}}^{-j})) \\ &\rightarrow H^1(L, V(\chi_{\text{LT}}^{-j})). \end{aligned}$$

The following result generalizes [51, Thm. A.2.3] and [52, Thm. B.5] from the cyclotomic case.

Theorem 5.2.26. *Assume that $V^*(1)$ is L -analytic and fulfills*

$$\begin{aligned} \mathrm{Fil}^{-1} D_{\mathrm{cris},L}(V^*(1)) &= D_{\mathrm{cris},L}(V^*(1)), \\ D_{\mathrm{cris},L}(V^*(1))^{\varphi_L = \pi_L^{-1}} &= D_{\mathrm{cris},L}(V^*(1))^{\varphi_L = 1} = 0. \end{aligned}$$

Then, for all $y \in H_{\mathrm{Iw}}^1(L_\infty/L, T)$, it holds that for $j \geq 0$

$$\begin{aligned} \Omega^j \mathbf{L}_V(y)(\chi_{\mathrm{LT}}^j) &= -j! \left((1 - \pi_L^{-1} \varphi_L^{-1})^{-1} \left(1 - \frac{\pi_L}{q} \varphi_L \right) \widetilde{\mathrm{exp}}_{L, V(\chi_{\mathrm{LT}}^{-j}), \mathrm{id}}^*(y_{\chi_{\mathrm{LT}}^{-j}}) \right) \otimes e_j \\ &= -j! \left(1 - \pi_L^{-1-j} \varphi_L^{-1} \right)^{-1} \left(1 - \frac{\pi_L^{j+1}}{q} \varphi_L \right) \left(\widetilde{\mathrm{exp}}_{L, V(\chi_{\mathrm{LT}}^{-j}), \mathrm{id}}^*(y_{\chi_{\mathrm{LT}}^{-j}}) \otimes e_j \right) \end{aligned}$$

and for $j \leq -1$

$$\begin{aligned} \Omega^j \mathbf{L}_V(y)(\chi_{\mathrm{LT}}^j) &= \frac{(-1)^{j+1}}{(-1-j)!} \left((1 - \pi_L^{-1} \varphi_L^{-1})^{-1} \left(1 - \frac{\pi_L}{q} \varphi_L \right) \widetilde{\mathrm{log}}_{L, V(\chi_{\mathrm{LT}}^{-j}), \mathrm{id}}(y_{\chi_{\mathrm{LT}}^{-j}}) \right) \otimes e_j \quad (5.33) \\ &= \frac{(-1)^{j+1}}{(-1-j)!} \left(1 - \pi_L^{-1-j} \varphi_L^{-1} \right)^{-1} \left(1 - \frac{\pi_L^{j+1}}{q} \varphi_L \right) \left(\widetilde{\mathrm{log}}_{L, V(\chi_{\mathrm{LT}}^{-j}), \mathrm{id}}(y_{\chi_{\mathrm{LT}}^{-j}}) \otimes e_j \right) \end{aligned}$$

if the operators $1 - \pi_L^{-1-j} \varphi_L^{-1}$, $1 - \frac{\pi_L^{j+1}}{q} \varphi_L$ or equivalently $1 - \pi_L^{-1} \varphi_L^{-1}$, $1 - \frac{\pi_L}{q} \varphi_L$ are invertible on $D_{\mathrm{cris},L}(V(\tau^{-1}))$ and $D_{\mathrm{cris},L}(V(\tau^{-1} \chi_{\mathrm{LT}}^j))$, respectively.

Proof. From the reciprocity formula in Corollaries 5.2.2 and 5.2.20 and Proposition 5.2.22 we obtain for $x \in D(\Gamma_L, \mathbb{C}_p) \otimes_L D_{\mathrm{cris},L}(V^*(1))$, $y \in D(V(\tau^{-1}))^{\psi_L = 1}$ and $j \geq 0$ using (5.32)

$$\begin{aligned} &[x(\chi_{\mathrm{LT}}^{-j}) \otimes e_j, (-1)^j \mathbf{L}_V(y)(\chi_{\mathrm{LT}}^j) \otimes e_{-j}]_{\mathrm{cris}} \\ &= \Omega \left[x, \frac{\sigma_{-1} \mathbf{L}_V(y)}{\Omega} \right]^0 (\chi_{\mathrm{LT}}^{-j}) \\ &= \Omega \frac{q-1}{q} \left\{ \Omega_{V^*(1), 1}(x), y \right\}_{\mathrm{Iw}} (\chi_{\mathrm{LT}}^{-j}) \\ &= \Omega \langle h_L^1 \circ \mathrm{tw}_{\chi_{\mathrm{LT}}^j} (\Omega_{V^*(1), 1}(x)), y_{\chi_{\mathrm{LT}}^{-j}} \rangle_{\mathrm{Tate}} \\ &= \Omega \left((-1)^j j! \Omega^{-j-1} \mathrm{exp}_{L, V^*(1)(\chi_{\mathrm{LT}}^j)} \left((1 - q^{-1} \varphi_L^{-1}) (1 - \varphi_L)^{-1} (x(\chi_{\mathrm{LT}}^{-j}) \otimes e_j) \right), y_{\chi_{\mathrm{LT}}^{-j}} \right)_{\mathrm{Tate}} \\ &= -(-1)^j \Omega^{-j} j! \left[(1 - q^{-1} \varphi_L^{-1}) (1 - \varphi_L)^{-1} (x(\chi_{\mathrm{LT}}^{-j}) \otimes e_j), \widetilde{\mathrm{exp}}_{L, V(\chi_{\mathrm{LT}}^{-j}), \mathrm{id}}^*(y_{\chi_{\mathrm{LT}}^{-j}}) \right]_{\mathrm{cris}} \\ &= [x(\chi_{\mathrm{LT}}^{-j}) \otimes e_j, -(-1)^j \Omega^{-j} j! (1 - \pi_L^{-1} \varphi_L^{-1})^{-1} \left(1 - \frac{\pi_L}{q} \varphi_L \right) \widetilde{\mathrm{exp}}_{L, V(\chi_{\mathrm{LT}}^{-j}), \mathrm{id}}^*(y_{\chi_{\mathrm{LT}}^{-j}})]_{\mathrm{cris}}. \end{aligned}$$

Here we used (5.31) in the fourth equation for the interpolation property of $\Omega_{V^*(1), 1}$. The fifth equation is the defining equation for the dual exponential map resulting from (5.30) upon taking the skew-symmetry of the cup product into account. Furthermore, for the last equality we use that $\pi_L^{-1} \varphi_L^{-1}$ is adjoint to φ_L under the lower pairing. The claim follows since the evaluation map is surjective and $[,]_{\mathrm{cris}}$ is non-degenerated.

Now assume that $j < 0$:

$$\begin{aligned}
 & [x(\chi_{\text{LT}}^{-j}) \otimes e_j, (-1)^j \mathbf{L}_V(y)(\chi_{\text{LT}}^j) \otimes e_{-j}]_{\text{cris}} \\
 &= \Omega \left[x, \frac{\sigma_{-1} \mathbf{L}_V(y)}{\Omega} \right]^0 (\chi_{\text{LT}}^{-j}) \\
 &= \Omega^{\frac{q-1}{q}} \{ \mathbf{\Omega}_{V^*(1),1}(x), y \}_{\text{Iw}} (\chi_{\text{LT}}^{-j}) \\
 &= \Omega (h_L^1 \circ \text{tw}_{\chi_{\text{LT}}^j} (\mathbf{\Omega}_{V^*(1),1}(x)), y_{\chi_{\text{LT}}^{-j}})_{\text{Tate}} \\
 &= \Omega \left(\frac{-\Omega^{-j-1}}{(-1-j)!} \log_{L,W(\chi_{\text{LT}}^j)}^* ((1 - q^{-1} \varphi_L^{-1})(1 - \varphi_L)^{-1} (x(\chi_{\text{LT}}^{-j}) \otimes e_j)), y_{\chi_{\text{LT}}^{-j}} \right)_{\text{Tate}} \\
 &= \frac{-\Omega^{-j}}{(-1-j)!} [(1 - q^{-1} \varphi_L^{-1})(1 - \varphi_L)^{-1} (x(\chi_{\text{LT}}^{-j}) \otimes e_j), \widetilde{\log}_{L,V(\chi_{\text{LT}}^{-j}),\text{id}}(y_{\chi_{\text{LT}}^{-j}})]_{\text{cris}} \\
 &= [x(\chi_{\text{LT}}^{-j}) \otimes e_j, \frac{-\Omega^{-j}}{(-1-j)!} (1 - \pi_L^{-1} \varphi_L^{-1})^{-1} (1 - \frac{\pi_L}{q} \varphi_L) \widetilde{\log}_{L,V(\chi_{\text{LT}}^{-j}),\text{id}}(y_{\chi_{\text{LT}}^{-j}})]_{\text{cris}}. \blacksquare
 \end{aligned}$$

Now consider $V = L(\tau \chi_{\text{LT}})$ and $W = V(\chi_{\text{LT}})$. Then the latter satisfies the condition of Theorem 5.2.26 and using Proposition 5.1.1 and Lemma 5.1.2 one easily derives the following interpolation property concerning the former for $y = \kappa(u)$, $u \in U$ for all $r \geq 1$:

$$\begin{aligned}
 \mathbf{L}_V(y)(\chi_{\text{LT}}^r) &= -\frac{r!}{\Omega^r} ((1 - \pi_L^{-1} \varphi_L^{-1})^{-1} (1 - \frac{\pi_L}{q} \varphi_L) \widetilde{\exp}_{L,V(\chi_{\text{LT}}^r),\text{id}}^*(y_{\chi_{\text{LT}}^r})) \otimes e_r \\
 &= -\frac{r!}{\Omega^r} (1 - \pi_L^{-r})^{-1} (1 - \frac{\pi_L^r}{q}) \widetilde{\exp}_{L,V(\chi_{\text{LT}}^r),\text{id}}^*(y_{\chi_{\text{LT}}^r}) \otimes e_r.
 \end{aligned}$$

On the other hand, we have $\mathbf{L}_V(y) \otimes \mathbf{d}_1 = \mathcal{L}_V(y)$ and hence by the claim concerning (5.4)

$$\mathbf{L}_V(y)(\chi_{\text{LT}}^r) \otimes \mathbf{d}_1 \otimes \eta^{-1} \otimes t_{\text{LT}} = \mathcal{L}_V(y)(\chi_{\text{LT}}^r) \otimes \eta^{-1} \otimes t_{\text{LT}} = \mathcal{L}(u \otimes \eta^{-1})(\chi_{\text{LT}}^r),$$

whence

$$\mathcal{L}(-u \otimes \eta^{-1})(\chi_{\text{LT}}^r) \otimes e_{1-r} = \frac{r!}{\Omega^r} (1 - \pi_L^{-r})^{-1} (1 - \frac{\pi_L^r}{q}) \exp_{L,V(\chi_{\text{LT}}^r),\text{id}}^*(y_{\chi_{\text{LT}}^r}).$$

This is (5.2); i.e., together with (5.3) we have just obtained a new proof of Kato's reciprocity law (Theorem 5.1.3) and we may consider Theorem 5.2.26 as a vast generalization of it. In [66], the general version of Kato's reciprocity will be derived from a refined version of Theorem 5.2.26 describing also the interpolation at Artin characters.