Appendix A

Cup products and local Tate duality

The aim of this subsection is to discuss cup products and to prove Proposition 5.2.19. We fix some open subgroup $U \subseteq \Gamma_L$ and let $L' = L_\infty^U$. Note that we obtain a decomposition $U \cong \Delta \times U'$ with a subgroup $U' \cong \mathbb{Z}_p^d$ of U and Δ the torsion subgroup of U.

Lemma A.1. *Let M_0 be a complete linearly topologized o_L-module with continuous U-action and with a continuous U-equivariant endomorphism f. Then there is a canonical quasi-isomorphism*

$$\mathcal{T}_{f,U}(M_0)\big[\tfrac{1}{\pi_L}\big] \simeq K^\bullet_{f,U'}\big(M_0\big[\tfrac{1}{\pi_L}\big]^\Delta\big).$$

If M_0 is an L-vector space, the inversion of π_L can be omitted on both sides.

Proof. Let $\mathcal{C}_n^\bullet(U, M_0) \subseteq \mathcal{C}^\bullet(U, M_0)$ denote the subcomplex of normalized cochains. Since Δ is finite, [87, Thm. 3.7.6] gives a canonical quasi-isomorphism:

$$\mathcal{C}_n^\bullet(U, M_0) = \mathcal{C}_n^\bullet(\Delta \times U', M_0) \xrightarrow{\simeq} \mathcal{C}_n^\bullet\big(\Delta, \mathcal{C}_n^\bullet(U', M_0)\big).$$

Here we understand the above objects in the sense of hypercohomology as total complexes of the obvious double complexes involved. After inverting π_L, we may compute the right-hand side further as

$$\begin{array}{ccc} \mathcal{C}_n^\bullet\big(\Delta, \mathcal{C}_n^\bullet(U', M_0)\big)\big[\tfrac{1}{\pi_L}\big] & & \mathcal{C}_n^\bullet(U', M_0^\Delta)\big[\tfrac{1}{\pi_L}\big] \\ \| & & \| \\ \mathcal{C}_n^\bullet\big(\Delta, \mathcal{C}_n^\bullet(U', M_0)\big[\tfrac{1}{\pi_L}\big]\big) & \xleftarrow{\simeq} & \mathcal{C}_n^\bullet(U', M_0)\big[\tfrac{1}{\pi_L}\big]^\Delta. \end{array}$$

Here the middle quasi-isomorphism comes from the fact that a finite group has no cohomology in characteristic zero. The right-hand equality is due to the fact that Δ acts on the cochains through its action on M_0. Altogether we obtain a natural quasi-isomorphism

$$\mathcal{C}_n^\bullet(U, M_0)\big[\tfrac{1}{\pi_L}\big] \cong \mathcal{C}_n^\bullet(U', M_0^\Delta)\big[\tfrac{1}{\pi_L}\big].$$

By using [87, Prop. 3.3.3] we may replace the normalized cochains again by general cochains obtaining the left-hand quasi-isomorphism in

$$\mathcal{C}^\bullet(U, M_0)\big[\tfrac{1}{\pi_L}\big] \cong \mathcal{C}^\bullet(U', M_0^\Delta)\big[\tfrac{1}{\pi_L}\big] \cong K^\bullet_{U'}(M_0^\Delta)\big[\tfrac{1}{\pi_L}\big] = K^\bullet_{U'}\big(M_0\big[\tfrac{1}{\pi_L}\big]^\Delta\big).$$

The middle quasi-isomorphism is (5.20). The claim follows by taking mapping fibers of the attached map $f - 1$ of complexes. ∎

Proposition A.2. *Let M be a φ_L-module over $\mathcal{R} = \mathcal{R}_K$, Definition 4.3.3, and $c \in K^\times$. Then $M/(\psi - c)(M)$ is finite dimensional over K.*

Proof. (The proof follows closely the proof of [42, Prop. 3.3.2] in the cyclotomic situation.) We set $\psi_c := c^{-1}\psi$ and show that $M/(\psi_c - 1)(M)$ is finite dimensional over K.

Choose a model $M^{[r_0,1)}$ of M with $1 > r_0 > p^{\frac{-1}{(q-1)e}}$ and put $r = r_0^{\frac{1}{q^2}}$. Recall that for all $1 > s \geq r$ we have maps $M^{[s,1)} \xrightarrow{\psi_c - 1} M^{[s,1)}$ (where strictly speaking we mean ψ_c followed by the corresponding restriction). We first show that it suffices to prove that $\mathrm{coker}(M^{[r,1)} \xrightarrow{\psi_c - 1} M^{[r,1)})$ has finite dimension over K. Indeed, given any $m \in M$ we have $m \in M^{[s,1)}$ for some $1 > s \geq r$. Then there exists $k \geq 0$ such that $r \geq s^{q^k} \geq r_0$, whence $\psi_c^k(m)$ belongs to $M^{[r,1)}$ and represents the same class in $M/(\psi_c - 1)(M)$ as m.

Choose a basis $\mathbf{e}_1', \ldots, \mathbf{e}_n'$ of $M^{[r_0,1)}$ and take $\mathbf{e}_i := \varphi(\mathbf{e}_i') \in M^{[r^q,1)}$; by the φ-module property the latter elements also form a basis of $M^{[r^q,1)}$. Note that by base change these two bases also give rise to bases in $M^{[s,1)}$ for $1 > s \geq r^q$. Thus, we find a matrix F' with entries in $\mathcal{R}^{[r^q,1)}$ such that $\mathbf{e}_j = \sum_i F_{ij}'\mathbf{e}_i'$ and we put $F = \varphi(F')$ with entries in $\mathcal{R}^{[r,1)}$, i.e., $\varphi(\mathbf{e}_j) = \sum F_{ij}\mathbf{e}_i$. Similarly, let G be a matrix with values in $\mathcal{R}^{[r^q,1)} \subseteq \mathcal{R}^{[r,1)}$ such that $\mathbf{e}_j' = \sum_i G_{ij}\mathbf{e}_i$ and hence $\mathbf{e}_j = \varphi(\sum_i G_{ij}\mathbf{e}_i)$.

We identify $M^{[r,1)}$ with $(\mathcal{R}^{[r,1)})^n$ by sending $(\lambda_i)_i$ to $\sum_i \lambda_i \mathbf{e}_i$ and endow it for each $r \leq s < 1$ with the norm given by $\max_i |\lambda_i|_s$. Note that then the “semilinear” map ψ_c (followed by the corresponding restriction) on $(\mathcal{R}^{[r,1)})^n$ is given by the matrix G as follows from the projection formula (4.40):

$$\psi_c\Big(\sum_j \lambda_j \mathbf{e}_j\Big) = \sum_j \psi_c\Big(\lambda_j \varphi\Big(\sum_i G_{ij}\mathbf{e}_i\Big)\Big) = \sum_{i,j} \psi_c(\lambda_j) G_{ij}\mathbf{e}_i.$$

Moreover, the restriction of $\varphi : M^{[r,1)} \to M^{[r^{\frac{1}{q}},1)}$ to $\sum_I \mathcal{O}_K(\mathbf{B})e_i$ becomes the semi-linear map $(\mathcal{O}_K(\mathbf{B}))^n \to (\mathcal{R}^{[r,1)})^n$ given by the matrix F.

Consider, for I any subset of the reals $\mathbf{R}$, the K-linear map $P_I : \mathcal{R}^{[r,1)} \to \mathcal{R}^{[r,1)}$, $\sum a_i Z^i \mapsto \sum_{i \in \mathbf{Z} \cap I} a_i Z^i$. We then introduce K-linear operators P_I and Q_k, $k \geq 0$, on $M^{[r,1)}$ by

$$P_I\big((\lambda)_i\big) := \big(P_I(\lambda_i)\big)_i, \quad Q_k := P_{(-\infty,-k)} - \tfrac{c\pi_L}{q}\varphi \circ P_{(k,\infty)},$$

i.e.,

$$Q_k\big((\lambda)_i\big) = \big(P_{(-\infty,-k)}(\lambda)_i\big)_i - \tfrac{c\pi_L}{q} F \cdot \big(\varphi\big(P_{(k,\infty)}(\lambda_i)\big)\big)_i,$$

because $P_{(k,\infty)}$ factorizes through $\mathcal{O}_K(\mathbf{B})$. Note then that the K-linear operator $\Psi_k := \mathrm{id} - P_{[-k,k]} + (\psi_c - 1)Q_k$ of $M^{[r,1)}$ satisfies

$$\Psi_k = \psi_c \circ P_{(-\infty,-k)} + \tfrac{c\pi_L}{q}\varphi \circ P_{(k,\infty)},$$

i.e.,

$$\Psi_k\big((\lambda_i)_i\big) = G \cdot \big(\psi_c\big(P_{(-\infty,-k)}(\lambda_i)\big)\big)_i + F \cdot \big(\tfrac{c\pi_L}{q}\varphi\big(P_{(k,\infty)}(\lambda_i)\big)\big)_i,$$

whence its operator norm satisfies

$$\|\Psi_k\|_s \le \max\left\{\|G\|_s\|\psi_c \circ P_{(-\infty,-k)}\|_s, \tfrac{c\pi_L}{q}\|F\|_s\|\varphi \circ P_{(k,\infty)}\|_s\right\}.$$

It is easy to check that, for $1 > s > q^{\frac{-1}{q-1}}$, we have $\|\varphi \circ P_{(k,\infty)}\|_s \le |Z|_s^{(q-1)k} = s^{(q-1)k}$ (using the norm relation after (4.28)) and $\|\psi_c \circ P_{(-\infty,-k)}\|_s \le C_s s^{k(1-q^{-1})}$ for some constant $C_s > 0$. For example, for the latter we have for $\lambda = \sum_i a_i Z^i \in \mathcal{R}^{[r,1)}$

$$\begin{aligned}\Big|\sum_{i<-k}\psi_c(a_i Z^i)\Big|_s &\le \sup_{i<-k}|a_i|\big|\psi_c(Z^i)\big|_s\\ &\le \sup_{i<-k}|a_i| C_s s^{\frac{i}{q}}\\ &\le C_s \sup_{i<-k}|a_i||Z|_s^i s^{i(q^{-1}-1)}\\ &\le C_s|\lambda|_s s^{-k(q^{-1}-1)},\end{aligned}$$

where we use that by continuity of ψ_c there exists C_s such that

$$\big|\psi_c(Z^i)\big|_s \le C_s|Z^i|_{s^{\frac{1}{q}}} = C_s s^{\frac{i}{q}}.$$

Thus, we may and do choose k sufficiently big such that $\|\Psi_k\|_r \le \frac{1}{2}$. Given $m_0 \in M^{[r,1)}$ we define inductively $m_{i+1} := \Psi_k(m_i)$. This sequence obviously converges to zero with respect to the r-Gauss norm. We shall show below that also for all $s \in (r^{\frac{1}{q}}, 1)$ the series $(m_i)_i$ tends to zero with respect to the Gauss norm $|\ |_s$; i.e., by cofinality the sum $m := \sum_{i\ge 0} m_i$ converges in $M^{[r,1)}$ for the Fréchet topology and satisfies

$$m - m_0 = m - P_{[-k,k]}(m) + (\psi_c - 1)Q_k(m);$$

i.e., $P_{[-k,k]}(m)$ represents the same class as m_0 in $M^{[r,1)}/(\psi_c - 1)(M)$. Since the image of $P_{[-k,k]}$ has finite dimension, the proposition follows, once we have shown the following.

Claim. For all $s \in (r^{\frac{1}{q}}, 1)$ we have

$$\big|\Psi_k(m)\big|_s \le \max\left\{\tfrac{1}{2}|m|_s, C_s\|G\|_s\Big(\tfrac{s^{\frac{1}{q}}}{r}\Big)^{-k}|m|_r, \Big|\tfrac{c\pi_L}{q}\Big|\|F\|_s\Big(\tfrac{s^q}{r}\Big)^{k'}|m|_r\right\}.$$

Indeed, we fix such s and may choose $k' \ge k$ such that $\|\Psi_{k'}\|_s \le \frac{1}{2}$. Then $\Psi_k = \Psi_{k'} - \psi_c \circ P_{[-k',-k)} - \frac{c\pi_L}{q}\varphi \circ P_{(k,k']}$, whence the claim as for $\lambda \in \mathcal{R}^{[r,1)}$

$$\big|\psi_c \circ P_{[-k',-k)}(\lambda)\big|_s \le C_s\Big(\tfrac{s^{\frac{1}{q}}}{r}\Big)^{-k}|\lambda|_r$$
$$\big|\varphi \circ P_{(k,k']}(\lambda)\big|_s \le \Big(\tfrac{s^q}{r}\Big)^{k'}|\lambda|_r$$

by similar estimations as above. ∎

Remark A.3. This result answers the expectation from [7, Rem. 2.3.7] positively.

Corollary A.4. *Let $V^*(1)$ be L-analytic and $M := D^\dagger_{\mathrm{rig}}(V^*(1))$.*

(i) *The cohomology group $h^2(K^\bullet_{\psi,U'}((M)^\Delta)[d-1])$ is finite dimensional over L.*

(ii) *We have isomorphisms*

$$h^1\big(K^\bullet_{\psi,U'}\big(D^\dagger_{\mathrm{rig}}\big(V(\tau^{-1})\big)^\Delta\big)[d-1]\big)^* \cong h^1\big(K^\bullet_{\varphi,U'}(M^\Delta)\big) \cong H^1_\dagger\big(L', V^*(1)\big),$$
$$h^2\big(K^\bullet_{\psi,U'}\big(D^\dagger_{\mathrm{rig}}\big(V(\tau^{-1})\big)^\Delta\big)[d-1]\big)^* \cong h^0\big(K^\bullet_{\varphi,U'}(M^\Delta)\big) = \big(V^*(1)\big)^{G_{L'}}.$$

Proof. (i) Since $h^2(K^\bullet_{\psi,U'}((M)^\Delta)[d-1])$ is a quotient of $(M/(\psi-1)(M))^\Delta$ by (5.3), this follows from the proposition. (ii) We are in the situation of Remark 5.2.6 (i) with regard to $\mathcal{C} = K^\bullet_{\psi,U'}(D^\dagger_{\mathrm{rig}}(V(\tau^{-1}))^\Delta)[d-1]$ and $i = 2, 3$ in the notation of the remark. Indeed, regarding $i = 3$ we have $h^3 = \mathcal{C}^4 = 0$ by construction; regarding $i = 2$ we have $\mathcal{C}^3 = 0$ by construction and h^2 is finite by (i). Hence, the first isomorphism follows in both cases from (5.23) using the reflexivity of M. The second isomorphisms arise by Lemma A.1 together with (5.17) and (5.16), respectively. ∎

We quickly discuss the analogues of some results in [19, §I.6]. First, we remind the reader of some definitions: $\widetilde{\mathbf{A}} := W(\mathbb{C}_p^\flat)_L$,

$$\widetilde{\mathbf{A}}^\dagger := \Big\{x = \sum_{n\geq 0} \pi_L^n[x_n] \in \widetilde{\mathbf{A}} : |\pi_L^n||x_n|_\flat^r \xrightarrow{n\to\infty} 0 \text{ for some } r > 0\Big\},$$

$\boldsymbol{A}^\dagger := \widetilde{\boldsymbol{A}}^\dagger \cap \boldsymbol{A}$ and $\boldsymbol{A}_L^\dagger := (\boldsymbol{A}^\dagger)^{H_L}$.[1]

Remark A.5. There is also the following more concrete description for $\boldsymbol{A}_L^\dagger$ in terms of Laurent series in ω_{LT}:

$$\boldsymbol{A}_L^\dagger = \big\{F(\omega_{\mathrm{LT}}) \in \mathbf{A}_L \mid F(Z) \text{ converges on } \rho \leq |Z| < 1 \text{ for some } \rho \in (0,1)\big\} \subseteq \mathbf{A}_L.$$

Indeed, this follows from the analogue of [18, Lem. II.2.2] upon noting that the latter holds with and without the integrality condition: "$r v_p(a_n) + n \geq 0$ for all $n \in \mathbb{Z}$"

[1] In the literature, one also finds the subring $\widetilde{\mathbf{A}}^\dagger_{\leq 1} := \bigcup_{r>0} W^r_{\leq 1}(\mathbb{C}_p^\flat)_L$ of $\widetilde{\mathbf{A}}^\dagger$, where $W^r_{\leq 1}(\mathbb{C}_p^\flat)_L = \{x \in W^r(\mathbb{C}_p^\flat)_L \mid |x|_r \leq 1\}$ consists of those $x \in \widetilde{\mathbf{A}}$ such that $|\pi_L^n||x_n|_\flat^r \xrightarrow{n\to\infty} 0$ and $|\pi_L^n||x_n|_\flat^r \leq 1$ for all n. Denoting by $\widetilde{\mathbf{A}}^{\dagger,s}_{\mathrm{St}}$ the ring defined in [83, Def. 3.4], we have the equality $W^r_{\leq 1}(\mathbb{C}_p^\flat)_L = \widetilde{\boldsymbol{A}}^{\dagger,\frac{q-1}{qr}}_{\mathrm{St}}$ corresponding to $\widetilde{\boldsymbol{A}}^\dagger_{(\frac{q-1}{qr})-}$ in the notation of [18, §II.1] for $q = p$. For these relations use the following normalizations compatible with [83]: $|\pi_L| = \frac{1}{q}$, $v_{\mathbb{C}_p^\flat}(\omega) = \frac{q}{q-1}$, $v_{\pi_L}(\pi_L) = 1$, $|x|_\flat = q^{-v_{\mathbb{C}_p^\flat}}$, $|\omega|_\flat = q^{-\frac{q}{q-1}} = |\pi_L|^{\frac{q}{q-1}}$, where $\omega = \omega_{\mathrm{LT}}$ mod π_L as in Section 3.1. Furthermore, $|x|_r = |\pi_L|^{V(x,\frac{q-1}{rq})}$ and $|\omega_{\mathrm{LT}}|_r = |\pi_L|^{\frac{rg}{q-1}} = |\omega|_\flat^r$, where $V(x,r) := \inf_k(v_{\mathbb{C}_p^\flat}(x_k)\frac{q-1}{rq} + k)$ for $x = \sum_{k\geq 0} \pi_L^k[x_k] \in \widetilde{\boldsymbol{A}}$. For $x \in \widetilde{\mathbf{A}}^\dagger$ we have $V(x,r) = \frac{q-1}{rq} V_{\mathrm{St}}(x,r)$, where $V_{\mathrm{St}}(x,r)$ uses the notation in [83]. Note also that $\omega_{\mathrm{LT}}^{-1}$ is contained in $W^{\frac{q-1}{q}}_{\leq 1}(\widehat{L_\infty}^\flat)_L$ by [83, Lem. 3.10] (in analogy with [18, Cor. II.1.5]).

(for $r \in \bar{\mathbf{R}} \setminus \mathbf{R}$) in the notation of that article.[2] In particular, we obtain canonical embeddings $\boldsymbol{A}_L^\dagger \subseteq \boldsymbol{B}_L^\dagger \hookrightarrow \mathcal{R}_L$ of rings.

Now consider the subring

$$A = \mathbf{A}_L^+[\![\tfrac{\pi_L}{Z^{q-1}}]\!] = \Big\{x = \sum_k a_k Z^k \in \mathbf{A}_L \mid v_{\pi_L}(a_k) \geq -\tfrac{k}{q-1}\Big\} \subseteq \mathbf{A}_L.$$

For $x \in \mathbf{A}_L$ and each inter $n \geq 0$, we define $w_n(x)$ to be the smallest integer $k \geq 0$ such that $x \in Z^{-k}A + \pi_L^{n+1}\mathbf{A}_L$. It satisfies $w_n(x+y) \leq \max\{w_n(x), w_n(y)\}$ and $w_n(xy) \leq w_n(x) + w_n(y)$ (since A is a ring) and $w_n(\varphi(x)) \leq q w_n(x)$ (use that $\frac{\varphi(Z)}{Z^q} \in A^\times$, whence $\varphi(Z^{-k})A = Z^{-qk}A$). Set for $n \geq 2$, $m \geq 0$ the integers $r(n) := (q-1)q^{n-1}$, $l(m,n) = m(q-1)(q^{n-1}-1) = m(r(n)-(q-1))$ and define $\mathbf{A}_L^{\dagger,n} = \{x = \sum_k a_k Z^k \in \mathbf{A}_L | v_{\pi_L}(a_k) + \frac{k}{r(n)} \to \infty \text{ for } k \to -\infty\}$. Then, by Remark A.5 and Footnote 2, we obtain that $\mathbf{A}_L^\dagger = \bigcup_{n\geq 2} \mathbf{A}_L^{\dagger,n}$.

Lemma A.6. *Let* $x = \sum_k a_k Z^k \in \mathbf{A}_L$ *and* $l \geq 0$, $n \geq 2$. *Then*

(i) *we have*

$$w_m(x) \leq l \Leftrightarrow v_{\pi_L}(a_k) \geq \min\big\{m+1, -\tfrac{k+l}{q-1}\big\} \quad \textit{for } k < -l.$$

(ii) $x \in \mathbf{A}_L^{\dagger,n}$ *if and only if* $w_m(x) - l(m,n)$ *goes to* $-\infty$ *when* m *runs to* ∞.

Item (ii) of the lemma is an analogue of [19, Prop. III 2.1 (ii)] for $\mathbf{A}_L^{\dagger,n}$ instead of

$$\mathbf{A}_{L,\leq 1}^{\dagger,n} = \Big\{x = \sum_k a_k Z^k \in \mathbf{A}_L^{\dagger,n} \mid v_{\pi_L}(a_k) + \tfrac{k}{r(n)} \geq 0 \text{ for all } k \leq 0\Big\}.$$

Proof. (i) follows from the fact that $x \in Z^{-l}A$ if and only if $v_{\pi_L}(a_k) \geq -\frac{k+l}{q-1}$ for $k < -l$. (ii) Let $M, N = M(q-1) \gg 0$ be arbitrary huge integers and assume first that $x \in \mathbf{A}_L^{\dagger,n}$. Then

$$w_m(x) - l(m,n) \leq -N \tag{A.1}$$

is equivalent to

$$v_{\pi_L}(a_k) \geq \min\big\{m+1, -\tfrac{k+l(m,n)-N}{q-1}\big\} \quad \text{for } k < -l(m,n) + N \tag{A.2}$$

[2] This description does not require any completeness property! A similar result holds for $\mathbf{A}_{\leq 1,L}^\dagger$ when requiring for the Laurent series in ω_{LT} in addition that $F(Z)$ takes values on $\rho \leq |Z| < 1$ of norm at most 1. More precisely, for $r < 1$ (or equivalently $s(r) := \frac{q-1}{rq} > \frac{q-1}{q}$) $W^r(\mathbf{C}_p^\flat)_L$ and $W^r_{\leq 1}(\mathbf{C}_p^\flat)_L$ correspond to $\{F(\omega_{\mathrm{LT}}) \in \mathbf{A}_L \mid F(Z) \text{ converges on } |\omega|^r \leq |Z| < 1\}$ and $\{F(\omega_{\mathrm{LT}}) \in \mathbf{A}_L \mid F(Z) \text{ converges on } |\omega|^r = q^{-\frac{1}{s(r)}} \leq |Z| < 1 \text{ with values } |F(z)| \leq 1\}$, respectively. The latter condition on the values can also be rephrased as $s(r)v_{\pi_L}(a_m) + m \geq 0$ for all $m \in \mathbb{Z}$ corresponding to $V(x, s(r)) \geq 0$ on the Witt vector side if $F(Z) = \sum_m a_m Z^m \in \mathbf{A}_L$.

by (i). To verify this relation for m sufficiently huge, we choose a $k_0 \in \mathbb{Z}$ such that $v_{\pi_L}(a_k) + \frac{k}{r(n)} \geq N \geq 0$ for all $k \leq k_0$. Now choose m_0 with $-l(m_0, n) < k_0$ and fix $m \geq m_0$. For $\frac{-k}{r(n)} > m$ we obtain $v_{\pi_L}(a_k) \geq m + 1$ because $k < k_0$ holds. For k with

$$k \geq -r(n)m \Leftrightarrow \frac{k}{r(n)} - \frac{k + l(m,n)}{q-1} \leq 0 \tag{A.3}$$

we obtain $v_{\pi_L}(a_k) \geq -\frac{k+l(m,n)-N}{q-1}$. Thus, the above relation holds true.

Vice versa choose m_0 such that (A.1) holds for all $m \geq m_0$, and fix

$$k \leq k_0 := -r(n) \max\{Mq^{n-1}, m_0\}.$$

Let m_1 be the unique integer satisfying $r(n)M - k \geq r(n)m_1 \geq r(n)M - k - r(n)$. In particular, we have $m_1 + 1 + \frac{k}{r(n)} \geq M$ and $k \geq -r(n)m_1$, which implies

$$-\tfrac{k+l(m_1,n)-N}{q-1} + \tfrac{k}{r(n)} \geq M$$

by (A.3). Moreover, it holds that $m_1 \geq m_0$ and $k < -l(m_1, n) + N$ (using $k \leq r(n)M - r(n)m_1 = -l(m_1, n) + q^{n-1}N - (q-1)m_1$ and $m_1 > (q^{n-1} - 1)M$ by our assumption on k). Hence, we can apply (A.2) to conclude $v_{\pi_L}(a_k) + \frac{k}{r(n)} \geq M$ as desired. ■

The analogue of [19, Lem. I.6.2] holds by the discussion following [80, Rem. 2.1]. This can be used to show the analogue of [19, Cor. I.6.3], i.e., $w_n(\psi(x)) \leq 1 + \frac{w_n(x)}{q}$. Now fix a basis $(e_1, \ldots, e_d)$ of $D(T)$ over $\mathbf{A}_L$ and denote by $\Phi = (a_{ij})$ the matrix defined by $e_j = \sum_{i=1}^d a_{ij}\varphi(e_i)$. The proof of [19, Lem. I.6.4] then carries over to show that for $x = \psi(y) - y$ with $x, y \in D(T)$ we have

$$w_n(y) \leq \max\left\{w_n(x), \tfrac{q}{q-1}\big(w_n(\Phi) + 1\big)\right\}, \tag{A.4}$$

where $w_n(\Phi) = \max_{ij} w_n(a_{ij})$ and $w_n(a) = \max_i w_n(a_i)$ for $a = \sum_{i=1}^d a_i\varphi(e_i)$ with $a_i \in \mathbf{A}_L$.

Lemma A.7. *Let $T \in \mathrm{Rep}_{o_L}(G_L)$ such that T is free over o_L and $V = T \otimes_{o_L} L$ is overconvergent. Then the canonical map $D^\dagger(T) \to D(T)$ induces an isomorphism $D^\dagger(T)/(\psi - 1)(D^\dagger(T)) \cong D(T)(\psi - 1)(D(T))$.*

Proof. We follow closely the proof of [50, Lem. 2.6], but note that he claims the statement for $D^\dagger_{\leq 1}(T)$.

Choose a basis $e_1, \ldots, e_d$ of the $A_L^\dagger$-module $D^\dagger(T)$, which is free because $A_L^\dagger$ is a henselian discrete valuation ring with respect to the uniformizer π_L (cf. [40, Def. 2.1.4]). Since V is overconvergent, it is also a basis of $D(T)$. Due to étaleness and since $(\mathbf{A}_L^\dagger) \cap \mathbf{A}_L^\times = (\mathbf{A}_L^\dagger)^\times$, also $\varphi(e_1), \ldots, \varphi(e_d)$ is a basis of all these modules. Given $x = \psi(y) - y$ with $x \in D^\dagger(T)$ and $y \in D(T)$, there is an $m > 0$ such that all x_i, a_{ij} lie in $\mathbf{A}_L^{\dagger,m}$ for some m. Since $q \geq 2$, it follows from the criterion in Lemma A.6 (ii) combined with (A.4) that all y_i belong to $\mathbf{A}_L^{\dagger,m+1}$, whence $y \in D^\dagger(T)$. This shows injectivity.

In order to show surjectivity, we apply Nakayama's lemma with regard to the ring o_L upon recalling that $D(T)/(\psi-1)$ is of finite type over it. Indeed, by left exactness of $D^\dagger$ we obtain $D^\dagger(T)/\pi_L D^\dagger(T) \subseteq D^\dagger(T/\pi_{\mathrm{LT}}) = D(T/\pi_{\mathrm{LT}})$. Since these are vector spaces over $\mathbf{E}_L$ of the same dimension, they are equal, whence

$$\begin{aligned}\big(D^\dagger(T)/(\psi-1)\big)/(\pi_L) = \big(D^\dagger(T)/(\pi_L)\big)/(\psi-1) &= \big(D(T)/(\pi_L)\big)/(\psi-1)\\ &= \big(D(T)/(\psi-1)\big)/(\pi_L).\end{aligned}$$ ∎

Corollary A.8. *Under the assumption of Lemma 3.3.6 for $V(\tau^{-1})$, the inclusion of complexes*

$$K^\bullet_{\psi,U'}\big(D^\dagger\big(V(\tau^{-1})\big)^\Delta\big) \subseteq K^\bullet_{\psi,U'}\big(D\big(V(\tau^{-1})\big)^\Delta\big)$$

is a quasi-isomorphism.

Proof. Forming Koszul complexes with regard to U', we obtain a diagram of (double) complexes with exact columns as illustrated in Figure A.1 in which the bottom line is an isomorphism of complexes because under the assumptions $\psi - 1$ induces an automorphism of $D(V(\tau^{-1}))/D^\dagger(V(\tau^{-1}))$ and as the action of Δ commutes with ψ. Hence, going over to total complexes gives an exact sequence

$$\begin{aligned}0 \to K^\bullet_{\psi,U'}\big(D^\dagger\big(V(\tau^{-1})\big)^\Delta\big) &\to K^\bullet_{\psi,U'}\big(D\big(V(\tau^{-1})\big)^\Delta\big)\\ &\to K^\bullet_{\psi,U'}\Big(\big(D\big(V(\tau^{-1})\big)/D^\dagger\big(V(\tau^{-1})\big)\big)^\Delta\Big) \to 0,\end{aligned}$$

in which $K^\bullet_{\psi,U'}((D(V(\tau^{-1}))/D^\dagger(V(\tau^{-1})))^\Delta)$ is acyclic, whence the statement follows.

$$\begin{array}{ccc}
0 & & 0\\
\downarrow & & \downarrow\\
K^\bullet\big(D^\dagger\big(V(\tau^{-1})\big)^\Delta\big) & \xrightarrow{\psi-1} & K^\bullet\big(D^\dagger\big(V(\tau^{-1})\big)^\Delta\big)\\
\downarrow & & \downarrow\\
K^\bullet\big(D\big(V(\tau^{-1})\big)^\Delta\big) & \xrightarrow{\psi-1} & K^\bullet\big(D\big(V(\tau^{-1})\big)^\Delta\big)\\
\downarrow & & \downarrow\\
K^\bullet\Big(\big(D\big(V(\tau^{-1})\big)/D^\dagger\big(V(\tau^{-1})\big)\big)^\Delta\Big) & \underset{\cong}{\xrightarrow{\psi-1}} & K^\bullet\Big(\big(D\big(V(\tau^{-1})\big)/D^\dagger\big(V(\tau^{-1})\big)\big)^\Delta\Big)\\
\downarrow & & \downarrow\\
0 & & 0
\end{array}$$

Figure A.1. Comparison of Herr complexes of D and $D^\dagger$. ∎

Remark A.9. Instead of using Lemma 3.3.6 (for crystalline, analytic representations) one can probably show by the same techniques as in [19, Prop. III.3.2 (ii)] that for any overconvergent representation V we have

$$D^\dagger(V)^{\psi=1} = D(V)^{\psi=1}.$$

Recall from [58, (5.2.1)] the quasi-isomorphism

$$L[-2] \xrightarrow{\iota} \tau_{\geq 2}\mathcal{C}^\bullet\big(G_{L'}, L(1)\big)$$

which allows us to define a trace map

$$\mathrm{tr}_{\mathcal{C}} : \mathcal{C}^\bullet\big(G_{L'}, L(1)\big) \to L[-2]$$

in the derived category of L-vector spaces as

$$\mathcal{C}^\bullet\big(G_{L'}, L(1)\big) \to \tau_{\geq 2}\mathcal{C}^\bullet\big(G_{L'}, L(1)\big) \xleftarrow{\iota} L[-2].$$

Then local Tate duality is induced by the following pairing on cocycles:

$$\mathcal{C}^\bullet\big(G_{L'}, V^*(1)\big) \times \mathcal{C}^\bullet(G_{L'}, V) \xrightarrow{\cup_{G_{L'}}} \mathcal{C}^\bullet\big(G_{L'}, L(1)\big) \xrightarrow{\mathrm{tr}_{\mathcal{C}}} L[-2].$$

The interest in the following diagram, the commutativity of which is shown before Lemma B.5, stems from the discrepancy that the reciprocity law has been formulated and proved in the setting of $K^\bullet_{\psi,U'}(D^\dagger_{\mathrm{rig}}(V(\tau^{-1}))^\Delta)[d-1]$ while the regulator map originally lives in the setting of $K^\bullet_{\psi,U'}(D(V(\tau^{-1}))^\Delta)[d-1]$:

$$\begin{array}{ccccc}
\mathcal{C}^\bullet\big(G_{L'},V^*(1)\big)\times & \mathcal{C}^\bullet(G_{L'},V) & \xrightarrow{\cup_{G_{L'}}} \mathcal{C}^\bullet\big(L',L(1)\big) & \xrightarrow{\mathrm{tr}_{\mathcal{C}}} & L[-2] \\
\downarrow \simeq & e \downarrow \simeq & & & \| \\
K^\bullet_{\varphi,U'}(M^\Delta) \times & K^\bullet_{\psi,U'}\big(D\big(V(\tau^{-1})\big)^\Delta\big)[d-1] & & \xrightarrow{\cup_{K,\psi}} & L[-2] \\
\uparrow & \uparrow & & & \| \\
K^\bullet_{\varphi,U'}\big((M^\dagger)^\Delta\big) \times & K^\bullet_{\psi,U'}\big(D^\dagger\big(V(\tau^{-1})\big)^\Delta\big)[d-1] & & \xrightarrow{\cup_{K,\psi}} & L[-2] \\
\downarrow \simeq & \downarrow & & & \| \\
K^\bullet_{\varphi,U'}\big((M^\dagger_{\mathrm{rig}})^\Delta\big) \times & K^\bullet_{\psi,U'}\big(D^\dagger_{\mathrm{rig}}\big(V(\tau^{-1})\big)^\Delta\big)[d-1] & & \xrightarrow{\cup_{K,\psi}} & L[-2],
\end{array} \tag{A.5}$$

which in turn induces the commutativity of the lower rectangle in the following diagram (the upper quadrangles commute obviously):

$$\begin{array}{ccccc}
D^{\dagger}_{\mathrm{rig}}\big(V(\tau^{-1})\big)^{\psi=1} & \hookleftarrow & D^{\dagger}\big(V(\tau^{-1})\big)^{\psi=1} & \overset{a}{\hookrightarrow} & D\big(V(\tau^{-1})\big)^{\psi=1} \\
\downarrow{\scriptstyle \mathrm{pr}_U} & & \downarrow{\scriptstyle \mathrm{pr}_U} & & \downarrow{\scriptstyle \mathrm{pr}_U} \\
h^1\big(K^{\bullet}_{\psi,U'}\big(D^{\dagger}_{\mathrm{rig}}\big(V(\tau^{-1})\big)^{\Delta}\big)[d-1]\big) & \longleftarrow & h^1\big(K^{\bullet}_{\psi,U'}\big(D^{\dagger}\big(V(\tau^{-1})\big)^{\Delta}\big)[d-1]\big) & \overset{b}{\longrightarrow} & h^1\big(K^{\bullet}_{\psi,U'}\big(D\big(V(\tau^{-1})\big)^{\Delta}\big)[d-1]\big) \\
\cong\downarrow{\scriptstyle a} & & & & {\scriptstyle c}\downarrow\cong \\
H^1_{/\dagger}(L',V) & & \overset{\mathrm{pr}}{\twoheadleftarrow} & & H^1(L',V).
\end{array} \tag{A.6}$$

Here the vertical maps pr_U are defined as in (5.25), a and pr are taken from Proposition 5.2.19, and the isomorphism c stems from (B.6). The map a is bijective under the assumption of Lemma 3.3.6, which extends to the map b by Corollary A.8.