

Appendix B

Iwasawa cohomology and descent

In this appendix, we recall a crucial observation from [45, 46], which is based on [58] and generalizes [80, Thm. 5.13]. As before, let U be an open subgroup of Γ_L . We set $\mathbb{T} := \Lambda(U) \otimes_{o_L} T$ with actions by $\Lambda(U) := o_L[[U]]$ via left multiplication on the left factor and by $g \in G_{L'}$ given as $\lambda \otimes t \mapsto \lambda \bar{g}^{-1} \otimes g(t)$, where \bar{g} denotes the image of g in U . We write $R\Gamma_{\text{Iw}}(L_\infty/L', T)$ for the continuous cochain complex $\mathcal{C}^\bullet(U, \mathbb{T})$ and recall that its cohomology identifies with $H_{\text{Iw}}^\bullet(L_\infty/L', T)$ by [80, Lem. 5.8]. For any continuous endomorphism f of M , we set $\mathcal{T}_f(M) := [M \xrightarrow{f-1} M]$, a complex concentrated in degree 0 and 1.

The map $p : \mathbb{T} \rightarrow o_L \otimes_{\Lambda(U)} \mathbb{T} \cong T$, $t \mapsto 1 \otimes t$, and its dual $i : T^\vee(1) \rightarrow \mathbb{T}^\vee(1)$ induce on cohomology the corestriction and restriction map, respectively, and they are linked by the following commutative diagram:

$$\begin{array}{ccc} \mathcal{C}^\bullet(G_{L'}, \mathbb{T}) \times \mathcal{C}^\bullet(G_{L'}, \mathbb{T}^\vee(1)) & \xrightarrow{\cup_{G_{L'}}} \mathcal{C}^\bullet(L', L/o_L(1)) & \xrightarrow{\text{tr}_e} L/o_L[-2] \\ \downarrow p_* & \uparrow i_* & \parallel \\ \mathcal{C}^\bullet(G_{L'}, T) \times \mathcal{C}^\bullet(G_{L'}, T^\vee(1)) & \xrightarrow{\cup_{G_{L'}}} \mathcal{C}^\bullet(L', L/o_L(1)) & \xrightarrow{\text{tr}_e} L/o_L[-2]. \end{array} \quad (\text{B.1})$$

By [34, Prop. 1.6.5 (3)] (see also [58, (8.4.8.1)]) we have a canonical isomorphism

$$o_L \otimes_{\Lambda(U)}^{\mathbb{L}} R\Gamma(L', \mathbb{T}) \cong R\Gamma(L', o_L \otimes_{\Lambda(U)} \mathbb{T}) \cong R\Gamma(L', T), \quad (\text{B.2})$$

where we denote by $R\Gamma(L', -)$ the complex $\mathcal{C}^\bullet(G_{L'}, -)$ regarded as an object of the derived category. Dually, by a version of Hochschild–Serre, there is a canonical isomorphism

$$R\text{Hom}_\Lambda(o_L, R\Gamma(L', \mathbb{T}^\vee(1))) \cong R\Gamma(L', T^\vee(1)). \quad (\text{B.3})$$

It follows that the isomorphism

$$R\Gamma_{\text{Iw}}(L_\infty/L', T) \cong R\text{Hom}_{o_L}(R\Gamma(L', \mathbb{T}^\vee(1)), L/o_L)[-2]$$

induced by the upper line of (B.1) induces an isomorphism

$$\begin{aligned} o_L \otimes_{\Lambda(U)}^{\mathbb{L}} R\Gamma_{\text{Iw}}(L_\infty/L', T) \\ \cong R\text{Hom}_{o_L}(R\text{Hom}_\Lambda(o_L, R\Gamma(L', \mathbb{T}^\vee(1))), L/o_L)[-2], \end{aligned} \quad (\text{B.4})$$

which is compatible with the lower cup product pairing in (B.1) via the canonical identifications (B.2) and (B.3).

Lemma B.1. *There is a canonical isomorphism*

$$R\Gamma(L', \mathbb{T}^\vee(1)) \cong \mathcal{T}_\varphi(D(T^\vee(1)))$$

in the derived category.

Proof. See [45, Thm. 5.1.11]. ■

For the rest of this section we assume that $U \subseteq \Gamma_L$ is an open *torsion-free* subgroup.

Lemma B.2. *Let T be in $\text{Rep}_{o_L}(G_L)$ of finite length. Set $\Lambda := \Lambda(U)$ and let $\gamma_1, \dots, \gamma_d$ be topological generators of U . Then we have an up to signs canonical isomorphism of complexes*

$$\begin{aligned} \text{Hom}_\Lambda^\bullet(K_\bullet(\gamma), \mathcal{T}_\varphi(D(T^\vee(1))))^\vee[-2] \\ \cong \text{tot}(\mathcal{T}_\psi(D(T(\tau^{-1})))[-1] \otimes_\Lambda K_\bullet(\gamma^{-1})(\Lambda)^\bullet), \end{aligned}$$

where $-\vee$ denotes forming the Pontrjagin dual.

Proof. Upon noting that

$$\mathcal{T}_\varphi(D(T^\vee(1)))^\vee[-2] \cong \mathcal{T}_\psi(D(T(\tau^{-1})))[-1]$$

(canonically up to a sign!), this is easily reduced to the following statement:

$$\text{Hom}_\Lambda^\bullet(K_\bullet(\gamma), M)^\vee \cong M^\vee \otimes_\Lambda K_\bullet(\gamma^{-1})(\Lambda)^\bullet,$$

which can be proved in the same formal way as (5.12), together with a consideration of signs. ■

Remark B.3. For every $M \in \mathfrak{M}(\mathbf{A}_L)$, we have a canonical isomorphism

$$\text{Hom}_\Lambda^\bullet(K_\bullet^U, \mathcal{T}_\varphi(M)) \cong K_{\varphi,U}(M)$$

up to the sign $(-1)^n$ in degree n and a non-canonical isomorphism

$$\text{tot}(\mathcal{T}_\psi(M)[-1] \otimes_\Lambda K_\bullet^U(\Lambda)^\bullet) \cong K_{\psi,U}(M)[d-1]$$

(involving the self-duality of the Koszul complex). Here, the right-hand sides are formed with respect to the same sequence of topological generators as the left-hand sides.

Proof. By our conventions in Section 5.2.1 $K_{\varphi,U}(M)$ is the total complex of the double complex

$$\text{Hom}^\bullet(K_\bullet(\Lambda)^\bullet, M) \xrightarrow{1-\varphi_*} \text{Hom}^\bullet(K_\bullet(\Lambda)^\bullet, M).$$

A comparison with the total Hom-complex (with the same sign rules as in Section 5.2.1) shows the first claim. For the second statement we have

$$\begin{aligned} \text{tot}(\mathcal{T}_\psi(M)[-1] \otimes_\Lambda K_\bullet(\Lambda)^\bullet) &\cong \text{tot}(\mathcal{T}_\psi(M) \otimes_\Lambda K_\bullet(\Lambda)^\bullet)[-1] \\ &= \text{tot}(\mathcal{T}_\psi(M \otimes_\Lambda K_\bullet(\Lambda)^\bullet))[-1] \\ &\cong \text{tot}(\mathcal{T}_\psi(M \otimes_\Lambda K^\bullet(\Lambda)[d]))[-1] \\ &= \text{tot}(\mathcal{T}_\psi(K^\bullet(M)[d]))[-1] \\ &= \text{cone}(K_U^\bullet(M)[d] \xrightarrow{1-\psi} K_U^\bullet(M)[d])[-2] \\ &\cong K_{\psi,U}(M)[d-1]. \end{aligned}$$

The first isomorphism involves a sign on $\mathcal{T}_\psi^1(M)$. The third isomorphism stems from (5.11), while the last isomorphism again involves signs. ■

Theorem B.4. *There are canonical isomorphisms*

$$R\Gamma_{\text{Iw}}(L_\infty/L, T) \cong \mathcal{T}_\psi(D(T(\tau^{-1})))[-1], \tag{B.5}$$

$$K_{\psi,U}(D(T(\tau^{-1}))) [d-1] \xrightarrow{\cong} R\Gamma(L', T) \tag{B.6}$$

in the derived category $D_{\text{perf}}(\Lambda_{o_L}(\Gamma_L))$ of perfect complexes and in the derived category $D^+(o_L)$ of bounded below cochain complexes of o_L -modules, respectively.

Proof. The first isomorphism is [45, Thm. 5.2.54], while the second one follows from this and (B.2) as

$$\begin{aligned} R\Gamma_{\text{Iw}}(L_\infty/L, T) \otimes_{\Lambda_{o_L}(U)}^{\mathbb{L}} o_L &\cong \mathcal{T}_\psi(D(T(\tau^{-1})))[-1] \otimes_\Lambda^{\mathbb{L}} K_\bullet(\Lambda)^\bullet \\ &= \text{tot}(\mathcal{T}_\psi(D(T(\tau^{-1})))[-1] \otimes_\Lambda K_\bullet(\Lambda)^\bullet) \\ &= K_{\psi,U}(D(T(\tau^{-1}))) [d-1] \end{aligned}$$

by Remark B.3. ■

By Lemma B.2 and Remark B.3 we see that, for T be in $\text{Rep}_{o_L}(G_L)$ of finite length,

$$K_{\varphi,U}(D(T^\vee(1))) = R\text{Hom}_\Lambda(o_L, \mathcal{T}_\varphi(D(T^\vee(1))))[2]$$

is dual to

$$K_{\psi,U}(D(T(\tau^{-1}))) = o_L \otimes_{\Lambda(U)}^{\mathbb{L}} \mathcal{T}_\psi(D(T(\tau^{-1})))[-1],$$

such that the upper rectangle in the diagram (A.5) commutes by (B.4), taking inverse limits and inverting π_L .

Lemma B.5. *Let T be in $\text{Rep}_{o_L}(G_L)$. Then the left rectangle in (5.32) is commutative.*

Sketch of proof. By an obvious analogue of Remark 5.2.17, it suffices to show the statement for $U = \Gamma_n \cong \mathbb{Z}_p^d$. In this situation, we have a homological spectral sequence

$$H_{i,\text{cont}}(U, H_{\text{Iw}}^{-j}(L_\infty/L, T)) \implies H_{\text{cont}}^{-i-j}(L', T)$$

which is induced by (B.2), see [58, (8.4.8.1)] for the statement and missing notation. We may and do assume that T is of finite length. Then, on the one hand, the map $H_{\text{Iw}}^1(L_\infty/L, T) \xrightarrow{\text{cor}} H^1(L', T)$ is dual to $H^1(L', T^\vee(1)) \xrightarrow{\text{res}} H^1(L_\infty, T^\vee(1))$, which sits in the five-term exact sequence of lower degrees associated with the Hochschild–Serre spectral sequence. As explained just before this lemma, the above homological spectral sequence arises by dualizing from the latter. Hence, cor shows up in the five-term exact sequence of lower degrees associated with this homological spectral sequence. On the other hand, via the isomorphisms (B.2) and (B.6), the latter spectral sequence is isomorphic to

$$H_{i,\text{cont}}(U, h^{-j}(\mathcal{T}_\psi(D(T(\tau^{-1}))))[-1]) \implies h^{-i-j}(K_{\psi,U}(D(T(\tau^{-1}))))[d-1]$$

and one checks by inspection that cor corresponds to pr_U . ■