Abstract

The purpose of this memoir is to investigate the well-posedness of several linear and nonlinear equations with a parabolic forward-backward structure, and to highlight the similarities and differences between them. The epitomal linear example will be the stationary Kolmogorov equation $y\partial_x u - \partial_{yy} u = f$ in a rectangle. We first prove that this equation admits a finite number of singular solutions, of which we provide an explicit construction. Hence, the solutions to the Kolmogorov equation associated with a smooth source term are regular if and only if f satisfies a finite number of orthogonality conditions.

We then extend this theory to a Vlasov–Poisson–Fokker–Planck system, and to two quasilinear equations: the Burgers-type equation $u\partial_x u - \partial_{yy} u = f$ in the vicinity of the linear shear flow, and the Prandtl system in the vicinity of a recirculating solution, close to the line where the horizontal velocity changes sign. We therefore revisit part of a recent work by Iyer and Masmoudi. For the two latter quasilinear equations, we introduce a geometric change of variables which simplifies the analysis. In these new variables, the linear differential operator is very close to the Kolmogorov operator $y\partial_x - \partial_{yy}$. Stepping on the linear theory, we prove existence and uniqueness of regular solutions for data within a manifold of finite codimension, corresponding to some nonlinear orthogonality conditions.

Keywords. mixed-type boundary value problems, forward-backward equations, nonlinear orthogonality conditions, free boundary problems in fluid mechanics, recirculation in boundary layers

Mathematics Subject Classification (2020). Primary 35M12; Secondary 76D10, 35K65, 35R35

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