Chapter 1

Introduction

This memoir is devoted to the well-posedness of linear and nonlinear equations having a parabolic forward-backward structure. In the linear case, our main example will be the Kolmogorov equation $y\partial_x u - \partial_{yy} u = f$ in the rectangular domain $\Omega := (x_0, x_1) \times (-1, 1)$, where $x_0 < x_1$ and f is an external source term. We will also consider nonlinear perturbations of this linear setting. The easiest nonlinear perturbation we consider is a Vlasov–Poisson–Fokker–Planck-type system of the form $y\partial_x u + E[u]\partial_y u - \partial_{yy} u = f$, where $E[u] = \partial_x^{-1} \int u \, \mathrm{d}y$. In this case the nonlinearity does not perturb the geometry of the problem, which remains forward parabolic in the region $y > 0$, and backward parabolic in the region $y < 0$.

We will also investigate the existence and uniqueness of sign-changing solutions to the Burgers-type equation

$$u\partial_x u - \partial_{yy} u = f \tag{1.1}$$

and to the Prandtl system

$$\begin{aligned} u\partial_x u + v\partial_y u - \partial_{yy} u &= -\partial_x p, \\ \partial_x u + \partial_y v &= 0. \end{aligned} \tag{1.2}$$

A natural solution to (1.1) with a null source term $\mathbb{f} = 0$ is the linear shear flow $\mathbb{u}(x, y) := y$, which changes sign across the horizontal line $\{y = 0\}$. In a similar way, semi-explicit solutions $(\mathbb{u}_\mathrm{P}, \mathbb{v}_\mathrm{P})$ of the Prandtl system (1.2) such that $\mathbb{u}_\mathrm{P}$ changes sign have been exhibited, see the discussion in Section 1.3 below. We are interested in strong solutions to (1.1) (resp. (1.2)) which are close in an appropriate norm to this linear shear flow $\mathbb{u}$ (resp. to the reference solution $(\mathbb{u}_\mathrm{P}, \mathbb{v}_\mathrm{P})$). Our purpose is to construct such solutions by perturbing the lateral boundary data or the source term.

Forward-backward nature. Since solutions to (1.1) and (1.2) will change sign across a curve $\{u = 0\}$ lying within Ω, a key feature of this work these problems must be seen as quasilinear forward-backward parabolic equations in the horizontal direction. Thus, to ensure the existence of a solution, one must be particularly careful as to how one enforces the lateral perturbations. More precisely, the problem is forward parabolic in the domain above the curve $\{u = 0\}$, in which $u > 0$, and therefore we shall prescribe a boundary condition on $\Sigma_0 := \{x = x_0\} \cap \{u > 0\}$; and backward parabolic in the domain below the curve $\{u = 0\}$, and we shall prescribe a boundary condition on $\Sigma_1 := \{x = x_1\} \cap \{u < 0\}$.

We will construct solutions to these problems thanks to an abstract implicit function theorem taking into account the geometry of the problem. More precisely, we will

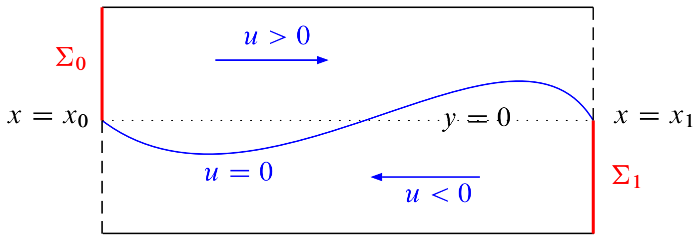

Figure 1.1. Fluid domain Ω and inflow boundaries $\Sigma_0 \cup \Sigma_1$.

first straighten the free boundary $\{u = 0\}$ by introducing as a new vertical variable $z = u(x, y)$. A suitable change of unknown function will then transform (1.1) and (1.2) into quasilinear equations with an easier-to-handle nonlinearity (see Remark 4.2).

Orthogonality conditions. Because of the nonlinearity, we need to work in a high enough regularity space in order to have a suitable control of the derivatives. However, one key difficulty of our work lies in the fact that, even when the source term f is smooth, say in $C_0^\infty(\Omega)$, *solutions to* (1.1) *and* (1.2) *have singularities in general*. Actually, this feature is already present at the linear level, i.e., for the equation $y\partial_x u - \partial_{xx} u = f$. We prove that if f is smooth, the associated weak solution to the linear system inherits the regularity of f if and only if f satisfies *orthogonality conditions* (i.e., the scalar products of f with some identified profiles must vanish). We also describe the singularities that appear when these orthogonality conditions are not satisfied. At the nonlinear level, these orthogonality conditions become a finiteness assumption on the codimension of the data manifold.

All of the features described above (orthogonality conditions for linear forward-backward equations, description of the potential singularities, handling of orthogonality conditions for quasilinear systems) appear to be new. We believe that the strategy we use could be extended to other nonlinear settings in which orthogonality conditions appear (elliptic equations in domains with corners, problems in which the linearized operator is Fredholm with negative index, ...).

1.1 Statement of the main results

1.1.1 Linear theory

Due to the forward-backward nature of the problem, we must choose the lateral perturbations and the source term in a particular product space. We therefore introduce the vector space

$$\mathcal{X}_B := \big\{(f,\delta_0,\delta_1) \in H^1_x H^2_y \times H^5(0,1) \times H^5(-1,0);\ f|_{\Sigma_0\cup\Sigma_1} = 0 \\ \text{and } \delta_i(0) = \delta_i((-1)^i) = \delta_i''(0) = \delta_i''((-1)^i) = 0\big\}, \tag{1.3}$$

where $\Sigma_0 = \{x_0\} \times (0,1)$ and $\Sigma_1 = \{x_1\} \times (-1,0)$ are the lateral boundaries on which we prescribe boundary conditions. We endow $\mathcal{X}_B$ with its canonical norm

$$\|(f,\delta_0,\delta_1)\|_{\mathcal{X}_B} := \|f\|_{H^1_x H^2_y} + \|\delta_0\|_{H^5} + \|\delta_1\|_{H^5}. \tag{1.4}$$

We establish existence and uniqueness of solutions to our problems in the following anisotropic Sobolev space:

$$Q^1 := H^{5/3}((x_0,x_1);L^2(-1,1)) \cap H^1((x_0,x_1);H^2(-1,1)). \tag{1.5}$$

In particular, for solutions with such regularity, equation (1.1) or its linear version $y\partial_x u - \partial_{yy} u = f$ hold in a strong sense, almost everywhere and the various boundary conditions hold in the usual sense of traces. We first state a result concerning the well-posedness in Q^1 of the stationary Kolmogorov equation (see (1.6) below), *up to two orthogonality conditions* (see comments below). Although equation (1.6) has been thoroughly investigated, as we recall in Section 1.2, we could not find this statement in the existing literature.

Theorem 1 (Orthogonality conditions for linear forward-backward parabolic equations). *There exists a vector subspace $\mathcal{X}^{\perp}_{B,\mathrm{sg}} \subset \mathcal{X}_B$ of codimension 2 such that, for each $(f,\delta_0,\delta_1) \in \mathcal{X}_B$, there exists a solution $u \in Q^1$ to the problem*

$$\begin{cases} y\partial_x u - \partial_{yy} u = f, \\ u_{|\Sigma_i} = \delta_i, \\ u_{|y=\pm 1} = 0, \end{cases} \tag{1.6}$$

if and only if $(f,\delta_0,\delta_1) \in \mathcal{X}^{\perp}_{B,\mathrm{sg}}$. Such a solution is unique and satisfies

$$\|u\|_{Q^1} \lesssim \|(f,\delta_0,\delta_1)\|_{\mathcal{X}_B}. \tag{1.7}$$

We emphasize that this result implies that there exist triplets (f,δ_0,δ_1) that can be chosen arbitrarily smooth and compactly supported, and for which there are no Q^1 solutions to (1.6). Furthermore, the vector space $\mathcal{X}^{\perp}_{B,\mathrm{sg}}$ can be fully characterized: classically, $\mathcal{X}^{\perp}_{B,\mathrm{sg}} = \ker \overline{\ell^0} \cap \ker \overline{\ell^1}$, where $\overline{\ell^0}$ and $\overline{\ell^1}$ are two linear forms on $\mathcal{X}_B$ which we shall write explicitly. If the data do not belong to $\mathcal{X}^{\perp}_{B,\mathrm{sg}}$, the solution has singularities, which we can describe completely.

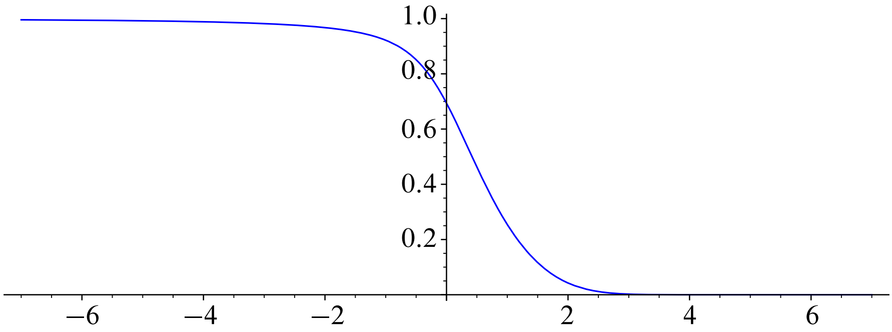

Figure 1.2. Plot of $t \mapsto \Lambda_0(t)$ for $t \in (-7, 7)$, highlighting the main properties: Λ_0 is a smooth, monotone decreasing function on $\mathbb{R}$, such that $\Lambda_0(-\infty) = 1$ and $\Lambda_0(+\infty) = 0$.

Theorem 2 (Decomposition of solutions as a sum of singular profiles and a smooth remainder). *Let $(f, \delta_0, \delta_1) \in \mathcal{X}_B$. There exists a unique solution $u \in H^{2/3}_x L^2_y \cap L^2_x H^2_y$ to equation (1.6). Furthermore, this solution admits the following decomposition: there exist $c_0, c_1 \in \mathbb{R}$, and $u_{\text{reg}} \in Q^1$, such that*

$$u = c_0 \bar{u}^0_{\text{sing}} + c_1 \bar{u}^1_{\text{sing}} + u_{\text{reg}}.$$

Each profile $\bar{u}^i_{\text{sing}}$ is supported in the vicinity of $(x_i, 0)$ and is smooth on $\bar{\Omega} \setminus \{(x_i, 0)\}$. Furthermore, for $|x - x_i| \ll 1$ and $|y| \ll 1$,

$$\bar{u}^i_{\text{sing}}(x, y) = \left(|y|^2 + |x - x_i|^{\frac{2}{3}}\right)^{\frac{1}{4}} \Lambda_0\left((-1)^i \frac{y}{|x - x_i|^{\frac{1}{3}}}\right),$$

where $\Lambda_0 \in C^\infty(\mathbb{R})$ is such that $\Lambda_0(-\infty) = 1$ and $\Lambda_0(+\infty) = 0$ (see Figure 1.2).

The existence of a weak solution was already known, see in particular [22,52,53]. The novelty of the above theorem lies in the identification of the singular profiles $\bar{u}^i_{\text{sing}}$, and in the decomposition of any weak solution. The function Λ_0 is in fact the solution to an ODE, and can be characterized in terms of special functions (namely confluent hypergeometric functions of the second kind, or Tricomi's functions).

The assumptions on the data (f, δ_0, δ_1) are not optimal and can be weakened: in particular, we merely need $H^1_x L^2_y$ regularity on the source term f, together with compatibility conditions in the corners. We will state optimal versions of Theorem 1 and Theorem 2 in Chapter 2, relying on the function space $\mathcal{H}_K$ defined in (2.13) (see respectively Proposition 2.17 and Corollary 2.30). In fact, in the sequel, we will need these sharper versions to prove our nonlinear results. In this introduction, we stick with the above versions in order to avoid defining too many functional spaces.

1.1.2 A nonlinear toy model from kinetic theory

As a corollary to Theorem 1, we obtain a similar statement for a (nonlinear) Vlasov–Poisson–Fokker–Planck system in an interval. In order not to burden the introduction, we refer to Chapter 3 for the presentation of the system and to Theorem 5 for the full statement. The proof of Theorem 5 is rather straightforward since the geometry of the considered problem remains the same as for (1.6), and the nonlinearity is very weak. We nevertheless use this example to set up our nonlinear scheme in Section 3.3 and a general abstract nonlinear existence result in Section 3.5.

1.1.3 The viscous Burgers system

We then turn towards the nonlinear problem (1.1). One of the main results of this memoir is the following nonlinear generalization of Theorem 1 for small enough perturbations. More precisely, the norm of the perturbation must be smaller than some constant η depending only on the size of the domain.

Theorem 3 (Existence and uniqueness of strong solutions to (1.1) under orthogonality conditions). *There exists a Lipschitz submanifold $\mathcal{M}_B$ of $\mathcal{X}_B$ of codimension 2, containing* 0 *and included in a ball of radius $\eta \ll 1$ in $\mathcal{X}_B$, such that, for every $(f, \delta_0, \delta_1) \in \mathcal{M}_B$, there exists a strong solution $u \in Q^1$ to*

$$\begin{cases} u\partial_x u - \partial_{yy} u = f, \\ u_{|\Sigma_i} = y + \delta_i(y), \\ u_{|y=\pm 1} = \pm 1. \end{cases} \tag{1.8}$$

More precisely, $\mathcal{M}_B$ is modeled on $\mathcal{X}_{B,\mathrm{sg}}^{\perp}$ and tangent to it at 0*. Such solutions are unique in a small neighborhood of $\mathbb{u}(x,y) = y$ in Q^1 and satisfy the estimate*

$$\|u - \mathbb{u}\|_{Q^1} \lesssim \|(f, \delta_0, \delta_1)\|_{\mathcal{X}_B}.$$

In the statement above, the condition that the data (f, δ_0, δ_1) belong to the manifold $\mathcal{M}_B$ is the nonlinear equivalent of the orthogonality conditions from Theorem 1. We emphasize that this is by no means a technical restriction which could be lifted, but actually a necessary condition to solve the equation with smooth solutions, as we state in Proposition 1.1 below. A key difficulty lies in the fact that these orthogonality conditions *depend on the solution itself.*

Proposition 1.1 (Necessity of the orthogonality conditions). *There exists $\eta > 0$ such that the following result holds. Let $(f, \delta_0, \delta_1) \in \mathcal{X}_B$ with $\|(f, \delta_0, \delta_1)\|_{\mathcal{X}_B} \le \eta$. Let $u \in Q^1$ be a solution to (1.8) such that $\|u - \mathbb{u}\|_{Q^1} \le \eta$. Then $(f, \delta_0, \delta_1) \in \mathcal{M}_B$.*

Remark 1.2. By commodity, ours results are stated using the full triplet (f, δ_0, δ_1), and so is the remainder of this memoir. Nevertheless, it is possible to obtain similar results either by fixing $\delta_0 = \delta_1 = 0$ and constructing a submanifold of source terms f yielding regular solutions, or by fixing $f = 0$ and constructing a submanifold of boundary data (δ_0, δ_1). This stems from the independence of the orthogonality conditions, which can be obtained either by Proposition 2.15 or by Proposition 2.34.

1.1.4 The Prandtl system

We also prove analogous results for the Prandtl system, revisiting the work of Iyer and Masmoudi in [34, 35] (we will comment more thoroughly on the differences between our results in the next sections). Let us now present our mathematical setting; we will provide the physical motivation and background for this system in Section 1.3. We consider a reference flow $(\mathbb{u}_{\mathrm{P}}, \mathbb{v}_{\mathrm{P}}) \in C^k([x_0, x_1] \times (0, +\infty))$ for some sufficiently large k (say $k = 4$), satisfying the Prandtl system

$$\mathbb{u}_{\mathrm{P}} \partial_x \mathbb{u}_{\mathrm{P}} + \mathbb{v}_{\mathrm{P}} \partial_y \mathbb{v}_{\mathrm{P}} - \partial_{yy} \mathbb{v}_{\mathrm{P}} = -\partial_x p, \quad \partial_x \mathbb{u}_{\mathrm{P}} + \partial_y \mathbb{v}_{\mathrm{P}} = 0$$

in the whole domain $(x_0, x_1) \times (0, +\infty)$, where p is the trace of the pressure of some outer Euler flow on the boundary $\{y = 0\}$. We assume that there exists a curve $\overline{\Gamma} := \{y = \gamma_{\mathrm{P}}(x)\}$, with γ_{P} smooth and such that $\inf_{[x_0, x_1]} \gamma_{\mathrm{P}} > 0$, such that $\mathbb{u}_{\mathrm{P}}$ changes sign on the curve $\overline{\Gamma}$: $\mathbb{u}_{\mathrm{P}}(x, \gamma_{\mathrm{P}}(x)) = 0$, and $\mathbb{u}_{\mathrm{P}}(x, y) < 0$ (resp. $\mathbb{u}_{\mathrm{P}}(x, y) > 0$) for $y < \gamma_{\mathrm{P}}(x)$ (resp. $y > \gamma_{\mathrm{P}}(x)$). Our purpose is to construct a solution to the Prandtl system close to $(\mathbb{u}_{\mathrm{P}}, \mathbb{v}_{\mathrm{P}})$ and in the vicinity of the curve $\overline{\Gamma}$, by perturbing either the inflow/outflow on the lateral boundaries or the source term.

To that end, we consider $z_b < 0 < z_t$ such that there exist smooth functions $\overline{\gamma_b}, \overline{\gamma_t}$, with $0 < \overline{\gamma_b}(x) < \gamma_{\mathrm{P}}(x) < \overline{\gamma_t}(x)$ for all $x \in [x_0, x_1]$, and such that $\mathbb{u}_{\mathrm{P}}(x, \overline{\gamma_j}(x)) = z_j$ for $j \in \{b, t\}$. We set $\overline{\Gamma_j} = \{y = \overline{\gamma_j}(x)\}$ for $j \in \{b, t\}$. We consider the Prandtl system (1.2) in a domain Ω_P, which is defined by

$$\Omega_P := \left\{(x, y) \in (x_0, x_1) \times \mathbb{R}_+; \gamma_b(x) < y < \gamma_t(x)\right\},$$

where γ_b, γ_t are smooth functions, which will actually be free boundaries, corresponding to the level sets z_b and z_t of the function u. We expect these functions (which are unknowns of the problem) to be smooth functions located in the vicinity of $\overline{\gamma_b}, \overline{\gamma_t}$. We endow system (1.2) with the following boundary conditions, which we will discuss and comment in Section 1.2. See Figure 5.1 for a sketch of the geometry of the domain.

(1) *Boundary conditions on the top and bottom free boundaries:* On the bottom boundary $\Gamma_b = \{y = \gamma_b(x)\}$, we enforce

$$\begin{aligned} u|_{\Gamma_b} &= \mathbb{u}_{\mathrm{P}}|_{\overline{\Gamma_b}} = z_b, \\ \partial_y u|_{\Gamma_b} &= \partial_y \mathbb{u}_{\mathrm{P}}|_{\overline{\Gamma_b}} + \delta_b, \\ v|_{\Gamma_b} &= \mathbb{v}_{\mathrm{P}}|_{\overline{\Gamma_b}} + v_b, \end{aligned} \tag{1.9}$$

where δ_b, v_b are small, smooth perturbations.

Similarly, on the top boundary $\Gamma_t = \{y = \gamma_t(x)\}$, we enforce

$$\begin{aligned} u|_{\Gamma_t} &= z_t, \\ \partial_y u|_{\Gamma_t} &= \partial_y \mathbb{u}_{\mathrm{P}}|_{\overline{\Gamma_t}} + \delta_t, \end{aligned} \tag{1.10}$$

where δ_t is, again, a small smooth perturbation.

Remark 1.3. Note that on the top and bottom boundary, we prescribe the trace of u and of its normal derivative. Of course, it would be impossible to prescribe simultaneously these two boundary conditions if the boundaries Γ_b, Γ_t were fixed. This is only made possible by the fact that these two boundaries are free.

(2) *Lateral boundary conditions:* As in the case of (1.6) and (1.8), we enforce lateral boundary conditions on u, namely

$$u_{|\Sigma_i^P} = \mathbb{u}_{\mathrm{P}} + \delta_i, \quad i \in \{0, 1\}, \tag{1.11}$$

where $\Sigma_0^P = \{x_0\} \times (\gamma_{\mathrm{P}}(x_0), \overline{\gamma_t}(x_0))$ and $\Sigma_1^P = \{x_1\} \times (\overline{\gamma_b}(x_1), \gamma_{\mathrm{P}}(x_1))$ (see also Figure 5.1). In particular, $\mathbb{u}_{\mathrm{P}} > 0$ on Σ_0^P and $\mathbb{u}_{\mathrm{P}} < 0$ on Σ_1^P. For simplicity, we assume that $\delta_0(\gamma_{\mathrm{P}}(x_0)) = \delta_1(\gamma_{\mathrm{P}}(x_1)) = \delta_0(\overline{\gamma_t}(x_0)) = \delta_1(\overline{\gamma_b}(x_1)) = 0$, and we recall that δ_0, δ_1 are assumed to be small in some sufficiently strong Sobolev norm. Therefore, provided that $\partial_y \mathbb{u}_{\mathrm{P}}|_{\overline{\Gamma}} > 0$, the signs of $\mathbb{u}_{\mathrm{P}}(x_i, \cdot) + \delta_i$ and of $\mathbb{u}_{\mathrm{P}}(x_i, \cdot)$ are identical on Σ_i^P.

We will in fact state two different results: one in “low regularity”, under merely one orthogonality condition, and another one in higher regularity, under three orthogonality conditions. For the sake of readability, we have stated them under the same regularity and compatibility assumptions on the data, although the assumptions are not optimal in the low regularity case, and the compatibility conditions could be generalized in both cases. We will state a more general result in Chapter 5 (see Proposition 5.2). Therefore, we take our data in the function space

$$\begin{aligned} \mathcal{X}_P := \big\{(\delta_0, \delta_1, \delta_t, \delta_b, v_b) \mid \delta_i \in H^6(\Sigma_i^P) \cap H_0^4(\Sigma_i^P) \text{ for } i \in \{0, 1\}, \\ \delta_t, \delta_b \in H_0^2(x_0, x_1), v_b \in H^1(x_0, x_1)\big\}, \end{aligned}$$

which we endow with its natural norm.

Theorem 4. *There exist numbers $\eta > 0$, $z_0 > 0$, depending only on the underlying flow $(\mathbb{u}_{\mathrm{P}}, \mathbb{v}_{\mathrm{P}})$, such that if $|z_b|, z_t \le z_0$, the following results hold.*

(1) *There exists a manifold $\mathcal{M}_0$ of codimension 1 in $\mathcal{X}_P$ and included in a ball of radius η in $\mathcal{X}_P$, such that for all $(\delta_0, \delta_1, \delta_t, \delta_b, v_b) \in \mathcal{M}_0$, equation (1.2) endowed with the boundary conditions (1.9), (1.10), (1.11) has a unique continuous solution u such that $u \in L_x^2 H_y^3(\Omega_P)$, $(x - x_0)(x - x_1)u \in H_x^1 H_y^3(\Omega_P)$,*

$\partial_y u \in L^\infty(\Omega_P)$, $u\partial_x\partial_y u \in L^2(\Omega_P)$, *and*

$$\|u - \mathbb{u}_{\mathrm{P}}\|_{L^2_x H^3_y} + \|\partial_y(u - \mathbb{u}_{\mathrm{P}})\|_{L^\infty} + \|(x-x_0)(x-x_1)(u-\mathbb{u}_{\mathrm{P}})\|_{H^1_x H^3_y} \lesssim \|(\delta_0, \delta_1, \delta_t, \delta_b, v_b)\|_{\mathcal{X}_P}.$$

(2) *There exists a manifold $\mathcal{M}_1$ of codimension 3 in $\mathcal{X}_P$ and included in a ball of radius η in $\mathcal{X}_P$, such that for all $(\delta_0, \delta_1, \delta_t, \delta_b, v_b) \in \mathcal{M}_1$, equation* (1.2) *endowed with the boundary conditions* (1.9), (1.10), (1.11) *has a unique solution u such that $u \in H^{5/3}_x H^1_y \cap H^1_x H^3_y(\Omega^P)$, and*

$$\|u - \mathbb{u}_{\mathrm{P}}\|_{H^{5/3}_x H^1_y} + \|u - \mathbb{u}_{\mathrm{P}}\|_{H^1_x H^3_y} \lesssim \|(\delta_0, \delta_1, \delta_t, \delta_b, v_b)\|_{\mathcal{X}_P}.$$

1.2 Comments and previous results

We start with a few comments on our main results and recall related known results.

Problem (1.6), involving the operator $y\partial_x - \partial_{yy}$, can be seen as a particular case of the class of "degenerate second-order elliptic-parabolic linear equations", also referred to as "second-order equations with nonnegative characteristic form" (as opposed to positive definite ones), "forward-backward" or "mixed type" problems. They date back at least to Gevrey [24].

Problem (1.6) itself, as well as these wide classes of equations, has received a lot of attention and has been investigated under different aspects: with variable coefficients or other geometries [22, 47, 53], higher-order operators [42, Chapter 3, 2.6], abstract operators [9, 54], explicit representation formulas [23, 27] or with a focus on numerical analysis [3].

On weak solutions for the linear problem. It is well known since the work of Fichera [22] that weak solutions to (1.6) with $L^2_x H^1_y$ regularity exist. For general boundary-value problems for elliptic-parabolic second-order equations, one owes to Fichera the systematic separation of the boundary of the domain into three parts: a "noncharacteristic" part, where one sets either Dirichlet or Neumann boundary conditions (here $y = \pm 1$), an "inflow" part, where one sets a Dirichlet boundary condition (here $\Sigma_0 \cup \Sigma_1$) and an "outflow" part, where one cannot set a boundary condition (here, the two sets $\{x_0\} \times (-1, 0)$ and $\{x_1\} \times (0, 1)$).

Baouendi and Grisvard [8] proved the uniqueness of weak solutions to (1.6) with $L^2_x H^1_y$ regularity, by means of a trace theorem and a Green identity (see Appendix A).

On strong solutions for the linear problem. There is an extensive literature on the regularity of solutions to degenerate elliptic-parabolic linear equations, and whether weak solutions are strong. We refer the reader in particular to the book [48] by Oleĭnik and Radkevič. Generally speaking, depending on the exact setting considered, it is quite often possible to prove that the solutions to such equations are regular far from the boundaries of the domain and/or from the regions where the characteristic form is not positive definite. A nice example is Kohn and Nirenberg's work [38], which proves a very general regularity result. A key assumption of their work is that the "outflow" part of the boundary does not meet the "noncharacteristic" and "inflow" parts (i.e., they are in disjoint connected components of $\partial\Omega$). Hence, it does not apply to (1.6), and hints towards a difficulty near the points $(x_0, 0)$ and $(x_1, 0)$.

In a series of papers [51–53], Pagani proved the existence of strong solutions to (1.6) (and related equations). More precisely, Pagani proved the existence of solutions such that $y\partial_x u$ and $\partial_{yy} u$ belong to $L^2(\Omega)$. Moreover, he determined the exact regularity of the various traces of such solutions (trace of u at $x = x_i$, at $y = \pm 1$ or $y = 0$, and trace of $\partial_y u$ at $y = 0$). These maximal regularity results play a key role in our analysis and motivate the functional spaces we introduce in Section 1.5.

On orthogonality conditions for higher regularity. As noted by Pyatkov in [55], for such forward-backward problems: "*as a rule, there is no existence theorems for smooth solutions without some additional orthogonality-type conditions on the problem data*". Even for the linear problem (1.6), there have been very few works concerning higher regularity (than the one given by Pagani's framework) in the whole domain. Most of the works focused on higher regularity (such as [55]) involve weighted estimates which entail regularity within the domain but not near the critical points $(x_i, 0)$. An attempt for global regularity is Goldstein and Mazumdar's work [25, Theorem 4.2]; however the proof seems incomplete (see Proposition 2.10 below and its proof for more details).

A misleading aspect is that it is quite easy, assuming the existence of a smooth solution, to prove *a priori* estimates at any order. Such phenomena are usual in the theory of elliptic problems in domains with corners or mixed Dirichlet–Neumann boundary conditions (see for instance [28]). Let us give an illustration of such a phenomenon in a close context. For a source term $f \in C_c^\infty(\Omega)$, consider the elliptic problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u(x_i, y) = 0 & \text{for } (-1)^i y > 0, \\ \partial_x u(x_i, y) = 0 & \text{for } (-1)^i y < 0, \\ u(x, \pm 1) = 0 & \text{for } x \in (x_0, x_1). \end{cases} \tag{1.12}$$

It is classical that such a system has a unique weak solution $u \in H^1(\Omega)$. Moreover, assuming that u is smooth enough, $v := \partial_x u$ satisfies

$$\begin{cases} -\Delta v = \partial_x f & \text{in } \Omega, \\ \partial_x v(x_i, y) = 0 & \text{for } (-1)^i y > 0, \\ v(x_i, y) = 0 & \text{for } (-1)^i y < 0, \\ v(x, \pm 1) = 0 & \text{for } x \in (x_0, x_1). \end{cases} \tag{1.13}$$

For such systems, one has $\|v\|_{H^1} \lesssim \|\partial_x f\|_{L^2}$. Hence $\|\partial_{xx} u\| \lesssim \|\partial_x f\|_{L^2}$, and, using the equation, $\|u\|_{H^2} \lesssim \|f\|_{H^1}$. So one has an *a priori* estimate. However, it is known that there exist source terms for which the unique weak solution $u \in H^1$ does not enjoy H^2 regularity (see [28, Chapter 4] and Section 2.4). The key point is that, when reconstructing u from the solution v to (1.13), say by setting $u(x, y) := \int_{x_0}^{x} v(x', y)\,\mathrm{d}x'$ for $y > 0$ and $u(x, y) := \int_{x_1}^{x} v(x', y)\,\mathrm{d}x'$ for $y < 0$, there might be a discontinuity of u or $\partial_y u$ across the line $y = 0$. Such discontinuities prevent u from solving (1.12). Preventing these discontinuities requires that the source term satisfies appropriate orthogonality conditions.

Let us also emphasize that if one wishes to construct solutions of (1.1) with even stronger regularity, say $u \in H^k_x H^1_y$ with $k \geq 1$, then generically, one needs to ensure that $2k$ orthogonality conditions are satisfied by the source terms (see Lemma 2.18). This situation occurs (at a nonlinear level) in [34].

On orthogonality conditions for nonlinear problems. Of course, such orthogonality conditions make it very difficult to obtain results at a nonlinear level. Generally, one tries to avoid such difficulties when considering nonlinear problems. For instance, for elliptic problems in polygonal domains, the classical textbook [28, Section 8.1] focuses on a nonlinear case where there is no orthogonality condition at the linear level.

Nevertheless, some results are known in the *semilinear* case. For example, for semilinear Fredholm operators with negative index, a theoretical toolbox is known (see, e.g., [60, Chapter 11, Section 4.2]) and has been implemented for some reaction-diffusion semilinear systems (see, e.g., [61, Chapter 7, Section 2.2], based on [20]).

Outside of the semilinear setting, we are not aware of nonlinear results obtained despite the presence of orthogonality conditions at the linear level prior to our present work (we discuss the recent preprint [34] by Sameer Iyer and Nader Masmoudi in Section 1.3).

Problem (1.1) is only *quasilinear*, and this makes the analysis harder. In an earlier version of this memoir, we introduced a nonlinear scheme in which the orthogonality conditions changed at every step. Tracking the evolution of these orthogonality conditions was then a major difficulty.

Following a very helpful remark by several colleagues[1], we have changed our strategy. We first perform a change of unknown which allows us to keep the same linear operator throughout the scheme, and to treat the nonlinearity perturbatively. This greatly simplifies the proof. In turn, this change of variables allows us to revisit the work of Iyer and Masmoudi [34, 35] on the analysis of the Prandtl system in the vicinity of a recirculating flow, since the equation for the vorticity after the change of variables has a very simple structure, see Section 1.4. Let us also recall that at

[1] Felix Otto, Yann Brenier, and an anonymous referee, whom we warmly thank.

the nonlinear level, the orthogonality conditions are translated in Theorem 3 (resp. Theorem 4) as the fact that the data must lie within the manifold $\mathcal{M}$ (resp. $\mathcal{M}_1$), which can be pictured as a perturbation of the linear subspace $\mathcal{X}_{B,\mathrm{sg}}^{\perp}$ of data satisfying the orthogonality conditions for the linear problem.

The proof of both of our main theorems (on Burgers and Prandtl) relies on the same abstract result (see Theorem 6 in Section 3.5) concerning quasilinear equations in a perturbative regime.

On entropy solutions. An entirely different approach to solve (1.1) is to look directly for weak solutions to the nonlinear problem, for example using an entropy formulation. The regularity for such solutions is $u \in L^\infty_{x,y} \cap L^2_x H^1_y$ and they are typically obtained as limits of solutions u^ε to regularized versions of (1.1), e.g., $u^\varepsilon \partial_x u^\varepsilon - \partial_{yy} u^\varepsilon - \varepsilon \partial_{xx} u^\varepsilon = 0$. Such solutions satisfy the equation and the lateral boundary conditions only in the weak sense of appropriate inequalities linked with "entropy pairs". Given $\delta_0, \delta_1 \in L^\infty(-1,1)$, the existence of an entropy solution to

$$\begin{cases} u\partial_x u - \partial_{yy} u = 0, \\ u_{|x=x_i} = \delta_i, \\ u_{|y=\pm 1} = 0 \end{cases} \tag{1.14}$$

was first proved in [11]. More recently, Kuznetsov proved in [39] the uniqueness of the entropy solution to (1.14), determined in which sense the lateral boundary conditions were satisfied and proved a stability estimate of the form

$$\|u - \tilde{u}\|_{L^1(\Omega)} \lesssim \|\delta_0 - \tilde{\delta}_0\|_{L^1(-1,1)} + \|\delta_1 - \tilde{\delta}_1\|_{L^1(-1,1)}.$$

In particular, this stability estimate guarantees that one can construct sign-changing solutions in the vicinity of the linear shear flow.

However, an important drawback of the entropy formulation is that the boundary conditions are only satisfied in a very weak sense. Although functions in $L^\infty_{x,y} \cap L^2_x H^1_y$ do not have classical traces at $x = x_i$, one can give a weak sense to the traces using the equation (see [40] for more details). Unfortunately, it is expected that these weak traces do not coincide with the supplied boundary data on sets of positive measure.

In contrast, since the solutions we construct in this work have (at least) $H^1_x L^2_y$ regularity, they have usual traces $u_{|\Sigma_i} \in L^2(\Sigma_i)$ and the equalities $u_{|\Sigma_i} = \delta_i$ hold in $L^2(\Sigma_i)$, so almost everywhere.

On the choice of the linear shear flow for equation (1.1). We choose to study the well-posedness of (1.1) in the vicinity of the linear shear flow to lighten the computations. Nonetheless, we expect that our results and proofs can be extended to study the well-posedness of (1.1) in the vicinity of any sufficiently regular reference flow $\mathbb{u}$

changing sign across a single curve $\{\mathbb{u} = 0\}$, satisfying $\mathbb{u}_y \geq c_0 > 0$ in Ω (so that (1.6) is the correct toy model) and with either $\|\mathbb{u}_x\|_\infty$ small enough, or with a restriction on the size of the domain (to ensure *a priori* estimates). In fact, this is precisely what we do when we study the Prandtl system around a recirculating flow: the linearized equation for the vorticity then becomes a forward-backward equation with variable coefficients, see (5.12) and Proposition 5.5.

Moreover, taking a step further in the modeling of recirculation problems in fluid mechanics (see Section 1.3), we also expect that our approach could be extended to an unbounded domain of the form $(x_0, x_1) \times (0, +\infty)$, with a reference flow such that $\mathbb{u}_{|y=0} = 0$, $\mathbb{u} < 0$ below some critical curve and then $\mathbb{u} > 0$ above, with $\mathbb{u}$ having some appropriate asymptotic behavior as $y \to +\infty$. In such a setting, the Poincaré inequalities in the vertical direction that we use here should probably be replaced with well-suited Hardy inequalities. As mentioned above, one of the issues is then to obtain *a priori* estimates on the linearized system. We comment further on this point at the end of Section 5.6.

On the conditions $\delta_0(0) = \delta_1(0) = 0$ for fixed end-points. It is an important feature of our work that we are able to enforce precisely the exact endpoints of the curve $\{u = 0\}$ at $x = x_0$ and $x = x_1$. Theorem 3 and Theorem 4 are stated for perturbations which satisfy $\delta_i(0) = 0$ (see (1.3)), so that the full boundary data $\mathbb{u}(x_i, y) + \delta_i(y)$ changes sign exactly at $y = 0$, where $\mathbb{u} = y$ in Theorem 3 and $\mathbb{u} = \mathbb{u}_{\mathrm{P}}$ in Theorem 4. This choice simplifies the definition of the submanifolds $\mathcal{M}$, $\mathcal{M}_0$ and $\mathcal{M}_1$ of boundary data for which we are able to solve the problem. Nevertheless, given y_0, y_1 sufficiently close to 0 and δ_0, δ_1 such that $y + \delta_i(y)$ changes sign at $y = y_i$, we expect that similar existence results hold, provided that the perturbations are chosen in an appropriate modification of $\mathcal{M}_j$, with suitable modifications to the functional spaces and where, in (1.8), the definitions of Σ_i are generalized by setting $\Sigma_i := \{(x_i, y); (-1)^i(\mathbb{u}(x_i, y) + \delta_i(y)) > 0\}$.

On the boundary conditions (1.9) and (1.10) for the Prandtl system in the recirculation zone. The boundary conditions we choose for the Prandtl system are mostly meant to simplify the present analysis as much as possible. As stated earlier, they are slightly unconventional since we prescribe both the trace and the normal derivative of u, but we let the boundary remain free. Other choices of boundary conditions are of course possible, and may lead to additional technical difficulties. These boundary conditions are designed in order to have a nice formulation after we have performed a change of variables in order to straighten the curve $\{u = 0\}$.

Note also that we only consider here the Prandtl system in the vicinity of the curve $\{u = 0\}$, and not in the whole infinite strip $(x_0, x_1) \times (0, +\infty)$. The coupling with the outside regions $y < \gamma_b(x)$ and $y > \gamma_t(x)$ leads to additional difficulties, which have been treated by Iyer and Masmoudi in [35], albeit with a different method, since conditions (1.9) and (1.10) are not considered in [35]. We propose in Section 5.6 a possible strategy to solve the Prandtl equation in an infinite strip.

On the compatibility conditions $\delta_i((-1)^i) = 0$ and $\delta_i''(0) = \delta_i''((-1)^i) = 0$ in Theorem 3. These conditions are classical compatibility conditions for solutions to elliptic-parabolic equations. For example, the condition $\delta_0(1) = 0$ in Theorem 3 is intended to match the condition $u_{|y=1} = 0$, and is necessary to have $L^2_x H^2_y$ regularity. The condition $\delta_0''(0) = 0$ comes from the equation. Indeed, if u is a sufficiently regular solution of (1.6) with $f(x_0, 0) = 0$, the equality $\partial_{yy} u = y\partial_x u$ at $(x_0, 0)$ enforces $\partial_{yy} u(x_0, 0) = 0$, so $\delta_0''(0) = 0$. The condition $\delta_0''(1) = 0$ stems similarly from the equation and the fact that $\partial_x u_{|y=1} = 0$. It corresponds to a classical parabolic regularity compatibility condition in order to have $L^2_x H^4_y$ regularity. We have imposed similar conditions for the Prandtl system by requiring $\delta_i \in H^4_0(\Sigma^P_i)$. Note that in the Prandtl case, we actually require extra cancellation assumptions. It is possible that the latter are technical, and could be removed.

On the number of orthogonality conditions for the Prandtl system. Note that the number of orthogonality conditions in Theorem 3 and in Theorem 4 is different. The reason for this is twofold.

Firstly, as previously noted by Iyer and Masmoudi in [35], the "good unknown" for the equation is the vorticity $\partial_y u$, which satisfies an equation which is very similar to (1.6) in a suitable set of variables (see (1.17) and (1.19) below). Therefore, in a sense, (1.2) is smoother than (1.1): indeed, without assuming any orthogonality condition, one can expect the vorticity $\partial_y u$ to belong to the function space $H^{2/3}_x L^2_y \cap L^2_x H^2_y$ (see Theorem 2), and therefore the solution of the Prandtl system belongs to $H^{2/3}_x H^1_y \cap L^2_x H^3_y$, in which we have gained one vertical derivative. On the contrary, without any orthogonality condition, one cannot expect the solution u of (1.1) to have better regularity than $H^{2/3}_x L^2_y \cap L^2_x H^2_y$, which is insufficient for a fixed-point argument for (1.1). This gain of vertical regularity allows us to have a theory for weaker solutions of the Prandtl system, and therefore to get rid of two of the orthogonality conditions.

Secondly, reconstructing the velocity from the vorticity in the Prandtl system gives rise to one additional orthogonality condition, as we will explain in Chapter 5. Hence the number of orthogonality conditions in Theorem 4 is odd, while it is even in Theorem 3.

1.3 Motivation from recirculation problems in fluid mechanics

Our original motivation stems from fluid mechanics. Indeed, the stationary Prandtl equation (1.2) describes the behavior of a fluid with small viscosity in the vicinity of a wall. The pressure $p(x)$ is the trace of the pressure of an outer Euler flow. This equation is usually set in a 2d domain of the form $I \times (0, +\infty)$, where $I \subset \mathbb{R}$ is an interval, and $y = 0$ is the solid wall. The equation is endowed with the boundary conditions $u = v = 0$ on $y = 0$, and $\lim_{y\to\infty} u(x, y) = u_E(x)$, where $u_E(x)$ is the trace of the outer Euler flow on the wall, and satisfies $u_E \partial_x u_E = -\partial_x p$.

As long as u remains positive, (1.2) can be seen as a nonlocal, nonlinear diffusion type equation, the variable x being the evolution variable. Using this point of view, Oleinik (see, e.g., [49, Theorem 2.1.1]) proved the local well-posedness of a solution to (1.2) when the equation (1.2) is supplemented with a boundary data $u_{|x=0} = u_0$, such that $u_0(y) > 0$ for $y > 0$ and $u_0'(0) > 0$. Let us mention that such positive solutions exist globally when $\partial_x p \leq 0$, but are only local when $\partial_x p > 0$. More precisely, when $\partial_x p = 1$, for instance, for a large class of boundary data u_0, there exists $x^* > 0$ such that

$$\lim_{x\to x^*} u_y(x, 0) = 0.$$

Furthermore, the solution may develop a singularity at $x = x^*$, known as Goldstein singularity. The point x^* is called the separation point: intuitively, if the solution to Prandtl exists beyond x^*, then it must have a negative sign close to the boundary (and therefore change sign). We refer to the seminal works of Goldstein [26] and Stewartson [59] for formal computations on this problem. A first mathematical statement describing separation was given by Weinan E in [21] in a joint work with Luis Cafarelli, but the complete proof was never published. The first author and Nader Masmoudi then gave a complete description of the formation of the Goldstein singularity [17]. The work [58] indicates that this singularity holds for a large class of initial data.

Because of this singularity, it is actually unclear that the Prandtl system is a relevant physical model in the vicinity of the separation point x^*, because the normal velocity v becomes unbounded at $x = x^*$. Consequently, more refined models, such as the triple deck system (see [41] for a presentation of this model, and [18, 36] for a recent mathematical analysis of its time-dependent version), were designed specifically to replace the Prandtl system with a more intricate boundary layer model in the vicinity of the separation point. However, beyond the separation point, i.e., for $x > x^*$, it is expected that the Prandtl system becomes valid again, but with a changing sign solution.

The well-posedness of the Prandtl system (1.2) when the solution u is allowed to change sign has only recently been investigated. Such solutions are called “recirculating solutions”, and the zone where $u < 0$ is called a recirculation bubble, the usual convention being that $u_E(x) > 0$, so that the flow is going forward far from the boundary. In the recent preprint [35] by Sameer Iyer and Nader Masmoudi, the authors prove *a priori* estimates in high regularity norms for smooth solutions to the Prandtl equation (1.2) in a domain of the form $I \times (0, +\infty)$, with restrictions on the length of the interval I, in the vicinity of explicit self-similar recirculating flows, called Falkner–Skan profiles. The latter are given by

$$u(x,y) = x^m f'(\zeta), \tag{1.15}$$

$$v(x,y) = -y^{-1}\zeta f(\zeta) - \frac{m-1}{m+1} y^{-1}\zeta^2 f'(\zeta), \tag{1.16}$$

where $\zeta := (\frac{m+1}{2})^{\frac{1}{2}} y x^{\frac{m-1}{2}}$ is the self-similarity variable, m is a real parameter and f is the solution to the Falkner–Skan equation

$$f''' + ff'' + \beta(1 - (f')^2) = 0,$$

where $\beta = \frac{2m}{m+1}$, subject to the boundary conditions $f(0) = f'(0) = 0$, $f'(+\infty) = 1$. Such flows correspond to an outer Euler velocity field $u_E(x) = x^m$. For some particular values of m (or, equivalently, β), these formulas provide physical solutions to (1.2) which exhibit recirculation (see [12]). Obtaining high regularity *a priori* estimates for recirculating solutions to the Prandtl system (1.2) on the whole infinite strip is a difficult task. This important step was achieved by Sameer Iyer and Nader Masmoudi in [35].

In the present memoir, we have chosen to focus on a different type of difficulty, and to consider first the toy-model (1.1), which differs from (1.2) through the lack of the nonlinear transport term $v\partial_y u$ and its associated difficulties (nonlocality, loss of derivative) and the exclusion of the zones close to the wall and far from the wall. For the model (1.1), *a priori* estimates are easy to derive, see [56, Chapter 4]. The difficulty lies elsewhere, as explained previously. Indeed, in order to construct a sequence of approximate solutions satisfying the *a priori* estimates, we need to ensure that the orthogonality conditions are satisfied all along the sequence. For the Prandtl system (1.2), this difficulty has recently been tackled by Sameer Iyer and Nader Masmoudi in [34], building upon their *a priori* estimates of [35] and the ideas developed in the first version of our present work. We revisit in Chapter 5 part of their work. Performing the different steps of the analysis (straightening the boundary, linearizing, differentiating with respect to the horizontal or vertical variable), we found a way to substantially simplify the analysis of the system in the vicinity of the recirculating line, and the results that we obtain are slightly different from the ones of [34, 35]. First, we prove a result in a rather *low regularity setting*, in which u_x is not even an L^2 function in the whole domain Ω_P. This result holds under merely one orthogonality condition, whose role is rather different from the ones of Theorem 3, for instance. Indeed, the role of this additional orthonogality condition is not to ensure a certain regularity, but rather to allow for a reconstruction of the velocity from the vorticity. Moreover, we use solely one change of variables, which we present in the next section and which is identical to the one for the Burgers equation. Once the adequate change of variables is identified, we retrieve the fact that the vorticity (in these new variables) is a good unknown. In fact, in the appropriate set of variables, the equation for the vorticity becomes remarkably simple (it is a closed, quasilinear equation). We however prescribe lateral boundary conditions on the velocity (rather than the vorticity). Eventually, we do not require any condition on the *horizontal size of the domain* (i.e., on the length $x_1 - x_0$). Our understanding is that such conditions may arise when the Prandtl system in the whole infinite strip is considered. They are linked to the well-posedness of a linearized system in the whole strip. We refer to Section 5.6 for more comments regarding this point.

1.4 Scheme of proof of nonlinear theorems and plan of the memoir

The uniqueness of solutions is fairly easy to prove. For the linear problem (1.6), uniqueness already holds at the level of weak solutions (see Proposition 2.2 and Appendix A). For the nonlinear problems, uniqueness is straightforward since we are considering strong solutions. Therefore, the main subject of this memoir is the proof of the existence of solutions for the nonlinear problems (1.8) and (1.2) endowed with the boundary conditions (1.9), (1.10), (1.11).

A first natural idea would be to prove existence thanks to a nonlinear scheme relying on the linear problem (1.6). For example, concerning equation (1.1), one could wish to construct a sequence of approximate solutions $(u_n)_{n\in\mathbb{N}}$ by setting $u_0 := 0$ (or any other initial guess) and solving

$$\begin{cases} y\partial_x u_{n+1} - \partial_{yy} u_{n+1} = f - (u_n - y)\partial_x u_n, \\ (u_{n+1})_{|\Sigma_i} = \delta_i, \\ (u_{n+1})_{|y=\pm 1} = 0. \end{cases}$$

However, this strategy fails. The key point is that the right-hand side contains a full tangential derivative of u_n, whereas the operator $y\partial_x - \partial_{yy}$ only yields a gain of $2/3$ of a derivative in this direction (more precisely, see Proposition 1.7, Remark 1.8 and Proposition 2.5). Hence, this nonlinear scheme would exhibit a “loss of derivative”, preventing us from proving a uniform bound on the sequence $(u_n)_{n\in\mathbb{N}}$.

Another drawback of this scheme is that it would not translate well to a setting where one does not assume $\delta_i(0) = 0$. Indeed, in such a case, the inflow boundaries of the problem with the perturbed data $y + \delta_i(y)$ would not match the inflow boundaries of the linear problem (1.6).

Hence, we will rather construct solutions to equation (1.1) through another iterative scheme. As suggested to us by an anonymous referee and by other colleagues, we first straighten the curve $\{u = 0\}$ by setting as a new vertical variable $z = u(x, y)$. Our new unknown, both for the Burgers equation (1.1) and for the Prandtl system (1.2), is the inverse function of u, i.e., the function Y such that $u(x, Y(x, z)) = z$. In this new set of variables, the equation for Y becomes, in the case of the Burgers equation (1.1),

$$z\partial_x Y - (\partial_z Y)^{-2}\partial_z^2 Y = -\partial_z Y f(x, Y), \tag{1.17}$$

and in the case of the Prandtl system (1.2)

$$z\partial_x Y - \int_{z_b}^{z} \partial_x Y - (\partial_z Y)^{-2}\partial_z^2 Y = -\partial_z Y \partial_x p + \mathbb{v}_{\mathrm{P}}|_{\overline{\Gamma_b}} + v_b, \tag{1.18}$$

in which the nonlocal integral term on the left-hand side stems from the transport term $v\partial_y u$ in the original equation. Differentiating (1.18) with respect to z, we find that in the case of the Prandtl system, the vorticity $W = \partial_z Y$ satisfies

$$z\partial_x W + \partial_x p \partial_z W + \partial_z^2\Big(\frac{1}{W}\Big) = 0. \tag{1.19}$$

We immediately see the linearized operator associated with (1.17) around $Y(x,z) = z$ is equation (1.6). In a similar fashion, the linearized operator of equation (1.19) around the flow $\mathbb{Y}_{\mathrm{P}}$ associated with $\mathbb{u}_{\mathrm{P}}$ is a forward-backward operator with variable coefficients, of the form $z\partial_x + \beta\partial_z - \partial_z^2(\alpha\cdot)$, where $\alpha = (\partial_z \mathbb{Y}_{\mathrm{P}})^{-2}$ and $\beta = \partial_x p$. Such forward-backward operators bear strong similarities with the canonical one $z\partial_x - \partial_{zz}$, and therefore we will rely on our linear analysis to study (1.19) (see Lemma 5.3 and Proposition 5.5).

We then construct solutions of (1.17) and (1.18) thanks to an iterative scheme[2], which we now explicit in the Burgers case, the Prandtl one being similar. We define a sequence $(\widetilde{Y}_n)_{n\in\mathbb{N}}$ such that

$$z\partial_x \widetilde{Y}_{n+1} - \partial_{zz}\widetilde{Y}_{n+1} = \frac{\partial_z \widetilde{Y}_n(2 - \partial_z \widetilde{Y}_n)}{(1 - \partial_z \widetilde{Y}_n)^2}\partial_{zz}\widetilde{Y}_n + (1 - \partial_z \widetilde{Y}_n) f(x, z - \widetilde{Y}_n) + g^{n+1},$$

where the additional term g^{n+1} ensures that the orthogonality conditions are satisfied at every step. We then prove that $(\widetilde{Y}_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in the space Q^1. Passing to the limit, we obtain a solution $Y = z - \lim_{n\to\infty} \widetilde{Y}_n$ to (1.17) with an additional source term g. The manifold $\mathcal{M}$ is then defined by requiring that the limit term g is zero.

Remark 1.4. In a first version of this memoir [16], we had chosen a strategy which seemed only slightly different, but which led to substantial technical difficulties. However, we believe that this strategy is rather natural, and could be of use in other problems. Therefore we describe it here.

[2] In fact, we will state and use an abstract theorem, whose proof follows a similar scheme.

Let $(u_n)_{n\in\mathbb{N}}$ be a sequence solving the following iterative scheme:

$$\begin{cases} u_n \partial_x u_{n+1} - \partial_{yy} u_{n+1} = f^{n+1}, \\ (u_{n+1})_{|\Sigma_i} = y + \delta_i^{n+1}, \\ (u_{n+1})_{|y=\pm 1} = \pm 1. \end{cases} \tag{1.20}$$

This scheme is similar to the one used to construct solutions to quasilinear symmetric hyperbolic systems, see for instance [7, Section 4.3]. In our case, it is possible to prove a uniform bound for u_n in the space $H_x^{5/3} L_y^2 \cap L_x^2 H_y^5$ and the convergence of the sequence in an interpolation space $L_x^2 H_y^{7/2} \cap H_x^{7/6} L_y^2$.

In (1.20), the triplet $(f^{n+1}, \delta_0^{n+1}, \delta_1^{n+1})$ is an appropriate perturbation of the data (f, δ_0, δ_1) tailored to satisfy the orthogonality conditions associated with the linear operator $u_n \partial_x - \partial_{yy}$. In order to define these orthogonality conditions, it is necessary to straighten the curve $\{u_n(x, y) = 0\}$: hence this straightening step is still necessary, but performed *after* the "linearization" of the equation, rather than *before*.

The issue lies in the fact that the orthogonality conditions change at every step, which is a key difficulty. In particular, in order to allow the sequence u_n to converge, one must prove that these perturbations also converge. This amounts to proving that the linear forms associated with the operator $u_n \partial_x - \partial_{yy}$ depend continuously (and even in a Lipschitz manner) on u_n, *for the same topology as the one within which one proves the convergence of the sequence* u_n. This continuity estimate requires identifying quite precisely what the linear forms are.

We believe that this methodology is rather robust and could be applied to other nonlinear problems in which orthogonality conditions are present at the linearized level, in particular in contexts where there is no nonlinear change of variables such as the one presented above allowing to treat the nonlinearity as a perturbation. For example, the PDE $u(1 + \partial_y u)\partial_x u - \partial_{yy} u = f$ could be an example where our previous methodology applies, but not the one exposed in the present memoir.

Remark 1.5. Let us highlight some differences between the strategy of the present memoir and the one by Iyer and Masmoudi in [35]. As explained above, there are several mathematical operations which are required to complete the proof of Theorem 4:

- changing variables in order to straighten the free boundary;
- linearizing the equation around some background profile;
- differentiating the equation with respect to the vertical variable in order to obtain an equation for the vorticity;
- differentiating the equation with respect to the tangential variable in order to derive higher order estimates (under compatibility conditions).

These operations more or less commute at main order, and lead to the study of the equation $z\partial_x u - \partial_{zz} u = f$. However, their (lower order) commutators may be a source of substantial technical difficulties. Our understanding is that the authors of [35] perform the operations in the following order (see Section 3 of their paper): (1) linearize; (2) differentiate with respect to the horizontal variable; (3) straighten the free boundary; (4) differentiate with respect to the vertical variable.

We believe that the computations are much simpler, and the structure is better understood, when the straightening change of variables is performed first. This also allows us to have a more accurate comparison between the Burgers-type equation and the Prandtl one.

The plan of this work is as follows. As a preliminary, we will introduce in Section 1.5 the functional spaces we will use. First, we study the linear problem (1.6) in Chapter 2, leading to Theorem 1, and prove that the two orthogonality conditions we expose are indeed nonvoid. We also construct the singular profiles $\bar{u}^i_{\text{sing}}$ and prove Theorem 2. In order to introduce our nonlinear scheme, we extend these linear results to the Vlasov–Poisson–Fokker–Planck system as an example in Chapter 3, where we also set up our general nonlinear methodology. We then turn towards the proof of Theorem 3 in Chapter 4, and the one of Theorem 4 in Chapter 5. In order to prove the existence of weak solutions of the Prandtl system (i.e., the first point of Theorem 4), we will need an interpolation result: this rather technical step is performed in Chapter 6. Eventually, in Appendix A, we prove the uniqueness of weak solutions to various linear problems, by adapting an argument due to Baouendi and Grisvard [8]. In Appendix B, we prove various technical results of functional analysis that we use throughout the memoir. Appendix C contain the postponed proof of a lemma of Chapter 5.

As this memoir is quite long, a list of notations is provided starting page 133.

1.5 Functional spaces and interpolation results

1.5.1 Notations

Throughout this work, an assumption of the form "$A \ll 1$" will mean that there exists a constant $c > 0$, depending only on Ω such that, if $A \leq c$, the result holds. Similarly, a conclusion of the form "$A \lesssim B$" will mean that there exists a constant $C > 0$, depending only on Ω and on the underlying flow (namely $\mathbb{u}(x, y) = y$ or $\mathbb{u}_{\mathrm{P}}$), such that the estimate $A \leq CB$ holds. For ease of reading, we will not keep track of the value of these constants, mostly linked with embeddings of functional spaces. Note in particular that the sizes of these constants will depend on the length $x_1 - x_0$ (see, e.g., Proposition 2.5).

We will often use the notations $\Omega_\pm := \Omega \cap \{\pm z > 0\}$.

1.5.2 Trace spaces for the lateral boundaries

For the traces of the solutions to (1.6) or (1.8) at $x = x_0$ and $x = x_1$, we will need the following spaces, due to [52,53]. We define $\mathcal{L}^2_z(-1,1)$ as the completion of $L^2(-1,1)$ with respect to the following norm:

$$\|\psi\|_{\mathcal{L}^2_z} := \left(\int_{-1}^{1} |z|\psi^2(z)\,\mathrm{d}z\right)^{\frac{1}{2}}, \tag{1.21}$$

and $\mathcal{H}^1_z(-1,1)$ as the completion of $H^1_0(-1,1)$ with respect to the following norm:

$$\|\psi\|_{\mathcal{H}^1_z} := \|\psi\|_{\mathcal{L}^2_z} + \|\partial_z\psi\|_{\mathcal{L}^2_z}. \tag{1.22}$$

1.5.3 Pagani's weighted Sobolev spaces

Let $\mathcal{O}$ be an open subset of $\mathbb{R}^2$, and let $\Omega := (x_0, x_1) \times (-1, 1)$. In the works [52,53] (albeit with swapped variables with respect to our setting), Pagani introduced the space $Z(\mathcal{O})$ of scalar functions ϕ on $\mathcal{O}$ such that ϕ, $\partial_z\phi$, $\partial_{zz}\phi$ and $z\partial_x\phi$ belong to $L^2(\mathcal{O})$ (in the sense of distributions). In this work, we will refer to this space with the notation $Z^0(\mathcal{O})$. It is a Banach space for the following norm:

$$\|\phi\|_{Z^0} := \|z\partial_x\phi\|_{L^2} + \|\partial_{zz}\phi\|_{L^2} + \|\partial_z\phi\|_{L^2} + \|\phi\|_{L^2}. \tag{1.23}$$

We will also need the space $Z^1(\mathcal{O})$, which we define as the space of scalar functions ϕ on $\mathcal{O}$ such that ϕ and $\partial_x\phi$ belong to $Z^0(\mathcal{O})$, associated with the following norm:

$$\|\phi\|_{Z^1} := \|\phi\|_{Z^0} + \|\partial_x\phi\|_{Z^0}. \tag{1.24}$$

The omitted proofs of the results of this section are postponed to Appendix B. We start with a straightforward extension result, which allows transferring results on $Z^0(\mathbb{R}^2)$ to $Z^0(\Omega)$.

Lemma 1.6. *There exists a continuous extension operator from $Z^0(\Omega)$ to $Z^0(\mathbb{R}^2)$.*

The next embedding is the most important result concerning the spaces Z^0. Since solutions to $(z\partial_x - \partial_{zz})u = f$ for $f \in L^2(\Omega)$ belong to $Z^0(\Omega)$ (see Proposition 2.5), the following embedding entails that such solutions belong to $H^{2/3}(\Omega)$.

Proposition 1.7. *$Z^0(\mathbb{R}^2)$ is continuously embedded in $H^{2/3}_x L^2_z$.*

Remark 1.8. Proposition 1.7 can be seen as a hypoellipticity result for the operator $L = \partial_{zz} - z\partial_x$ in the full space $\mathbb{R}^2$, which is of the form $X_1^2 + X_0$, where $X_1 = \partial_z$, $X_0 = -z\partial_x$ and $[X_0, X_1] = \partial_x$, so the Lie brackets generate the full space and L satisfies Hörmander's sufficient condition of [29] for hypoellipticity. In fact, in the full space $\mathbb{R}^2$, the $H^{2/3}_x L^2_z \cap L^2_x H^2_z$ regularity of solutions to $Lu = f$ for $f \in L^2$

can be derived from the general theory of quadratic operators, which makes a link between the anisotropic gain of regularity and the number of brackets one has to take in order to generate a direction. For instance, this regularity follows from [1, Theorem 2.10] and more precisely Example 2.11 therein applied with

$$R = 0, \quad Q = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Lemma 1.9. *$Z^0(\mathbb{R}^2)$ is continuously embedded in $C^0_z(H^{1/2}_x)$.*

Proof. On the one hand, by definition, $Z^0(\mathbb{R}^2) \hookrightarrow H^2_z(L^2_x)$. On the other hand, by Proposition 1.7, $Z^0(\mathbb{R}^2) \hookrightarrow L^2_z(H^{2/3}_x)$. Therefore, by the "fractional trace theorem" [45, equation (4.7), Chapter 1], $Z^0(\mathbb{R}^2) \hookrightarrow C^0_z(H^{1/2}_x)$. ∎

Lemma 1.10. *$Z^0(\Omega)$ is continuously embedded in $C^0([x_0, x_1]; \mathcal{H}^1_z(-1, 1))$.*

Proof. This is contained in the trace result [53, Theorem 2.1]. ∎

Lemma 1.11. *For $\phi \in Z^0(\Omega)$, $\partial_z \phi|_{z=\pm 1} \in H^{1/4}(x_0, x_1)$.*

Proof. Let $\chi_+ \in C^\infty([-1, 1])$ such that $\chi \equiv 1$ in a neighborhood of $\{z = +1\}$ and $\chi(z) = 0$ for $z < 1/2$. For $\phi \in Z^0(\Omega)$, $\chi_+(z)\phi(x, z) \in H^1_x L^2_z \cap L^2_x H^2_z$. By the "fractional trace theorem" [45, equation (4.7), Chapter 1] $(\partial_z(\chi_+\phi))|_{z=+1} \in H^{1/4}(x_0, x_1)$. The result follows since $\chi_+ \equiv 1$ near $\{z = +1\}$. The same argument applies for the trace at $z = -1$. ∎

Remark 1.12. Although it is *almost* the case, there does not hold $Z^0(\mathbb{R}^2) \hookrightarrow C^0(\mathbb{R}^2)$.

- Pagani [52, Theorem 2.1] proves that the operator $\phi \mapsto \phi(\cdot, 0)$ is onto from $Z^0(\mathbb{R}^2)$ to $H^{\frac{1}{2}}(\mathbb{R})$. But $H^{\frac{1}{2}}(\mathbb{R})$ contains unbounded functions of x.
- Pagani [52, Theorem 2.3] proves that the operator $\phi \mapsto \phi(0, \cdot)$ is onto from $Z^0(\mathbb{R}^2)$ to the space $\mathcal{H}^1_z(\mathbb{R})$. But this space contains unbounded functions, for example $\psi(z) := (-\ln |z|/2)^s \chi(z)$ for $s < \frac{1}{2}$ and $\chi \in C^\infty_c(\mathbb{R})$ with $\chi \equiv 1$ in a neighborhood of $z = 0$.

1.5.4 Baouendi and Grisvard's weak space

In [8], Baouendi and Grisvard introduce the space

$$\mathcal{B} := \left\{ \phi \in L^2((x_0, x_1), H^1_0(-1, 1)); z\partial_x \phi \in L^2_x(H^{-1}_z) \right\}. \tag{1.25}$$

Baouendi and Grisvard proved the uniqueness of solutions to (1.6) in $\mathcal{B}$. They also proved that functions in $\mathcal{B}$ have traces on $\{x = x_i\}$ in $\mathcal{L}^2_z(-1, 1)$. These results are recalled in Appendix A, and will be used abundantly throughout the memoir.

The following embedding is proved in Appendix B and used in Section 5.5.

Lemma 1.13. *$\mathcal{B}$ is continuously embedded in $H_x^{1/3} L_z^2$.*

Lemma 1.14. *$\mathcal{B}$ is continuously embedded in $C_z^0([-1,1]; H_x^{1/6})$.*

Proof. By definition, $\mathcal{B} \hookrightarrow H_z^1(L_x^2)$. By Lemma 1.13, $\mathcal{B} \hookrightarrow L_z^2(H_x^{1/3})$. Hence, the result follows from the "fractional trace theorem" [45, equation (4.7), Chapter 1]. ■

1.5.5 Anisotropic Sobolev spaces

We will construct solutions to (1.6), (1.8) and (1.2) in various anisotropic Sobolev spaces such as Q^1 of (1.5). Within these spaces, one has heuristically the correspondence $\partial_x \approx \partial_z^3$, which corresponds to the appropriate scaling due to the degeneracy of $z\partial_x$ at $z = 0$.

Indeed, if u is a solution to $z\partial_x u - \partial_{zz} u = 0$ say on the whole plane $\mathbb{R}^2$, then the rescaled functions $u_\lambda(x,z) := u(\lambda^3 x, \lambda z)$ are also solutions. This is also consistent with the shape of the singular profiles $\bar{u}^i_{\text{sing}}$ from Theorem 2, and leads to the rule of thumb "one derivative in x equals three derivatives in z" (which is different from the usual parabolic scaling, because of the cancellation on the line $z = 0$).

In particular, we will use abundantly the following embeddings from the interpolated Pagani spaces to anisotropic Sobolev ones.

Lemma 1.15. *Let $\sigma \in [0,1]$ and define $Z^\sigma(\Omega) := [Z^0(\Omega), Z^1(\Omega)]_\sigma$. Then one has the embedding $Z^\sigma \hookrightarrow H_x^{2/3+\sigma} L_z^2 \cap H_x^\sigma L_z^2$. In particular, $Z^1 \hookrightarrow Q^1$ defined in (1.5).*

Proof. By Proposition 1.7, $Z^0(\Omega) \hookrightarrow H_x^{2/3} L_z^2$ and $Z^1(\Omega) \hookrightarrow H_x^{5/3} L_z^2$. Hence

$$Z^\sigma(\Omega) \hookrightarrow [H_x^{2/3} L_z^2, H_x^{5/3} L_z^2]_\sigma = H_x^{2/3+\sigma} L_z^2$$

using, e.g., [45, equation (13.4), Chapter 1].

Moreover, by definition, $Z^0(\Omega) \hookrightarrow L_x^2 H_z^2$ and $Z^1(\Omega) \hookrightarrow H_x^1 H_z^2$, so $Z^\sigma(\Omega) \hookrightarrow H_x^\sigma H_z^2$. ■

Remark 1.16. The definition (1.5) of Q^1 does not contain the "full vertical" regularity $L_x^2 H_z^5$, since we do not need it to close our nonlinear estimates. However, assuming sufficient regularity on the source terms, one can build solutions to (1.6) and (1.8) in $Q^1 \cap L_x^2 H_x^5$, and this was in fact what we did in the earlier version [16] of this work.