Chapter 2

The case of the linear shear flow

This chapter concerns the well-posedness of the linear system (1.6) which we restate here for convenience and by using z as a vertical variable rather than y to prepare for the next sections. We thus consider, in $\Omega = (x_0, x_1) \times (-1, 1)$, the system

$$\begin{cases} z\partial_x u - \partial_{zz} u = f, \\ u_{|\Sigma_i} = \delta_i, \\ u_{|z=\pm 1} = 0, \end{cases} \tag{2.1}$$

where $\Sigma_0 = \{x_0\} \times (0, 1)$ and $\Sigma_1 = \{x_1\} \times (-1, 0)$.

First, in Section 2.1, we recall the theory of weak solutions, due to Fichera for the existence, and to Baouendi and Grisvard for the uniqueness. Then, in Section 2.2, we recall the theory of strong solutions with maximal regularity, due to Pagani. Our contributions regarding this problem are contained in the following sections. In Section 2.3, we derive orthogonality conditions which are necessary to obtain higher tangential regularity and prove the existence result of Theorem 1. In Section 2.4, we construct explicit singular solutions and prove the decomposition result of Theorem 2. Eventually, in Section 2.5, we state a result concerning the well-posedness of (2.1) with fractional tangential regularity, which will be used in Chapter 5 and proved in Chapter 6.

2.1 Existence and uniqueness of weak solutions

Definition 2.1 (Weak solution). Let $f \in L^2((x_0, x_1); H^{-1}(-1, 1))$, $\delta_0, \delta_1 \in \mathcal{L}^2_z(-1, 1)$. We say that $u \in L^2((x_0, x_1); H^1_0(-1, 1))$ is a *weak solution* to (2.1) when, for all $v \in H^1(\Omega)$ vanishing on $\partial\Omega \setminus (\Sigma_0 \cup \Sigma_1)$, the following weak formulation holds:

$$-\int_\Omega zu\partial_x v + \int_\Omega \partial_z u \partial_z v = \int_\Omega fv + \int_{\Sigma_0} z\delta_0 v - \int_{\Sigma_1} z\delta_1 v.$$

Weak solutions in the above sense are known to exist since the work Fichera [22, Theorem XX] (which concerns generalized versions of (2.1), albeit with vanishing boundary data). Uniqueness dates back to [8, Proposition 2] by Baouendi and Grisvard.

Proposition 2.2. *Let* $f \in L^2((x_0, x_1); H^{-1}(-1, 1))$ *and* $\delta_0, \delta_1 \in \mathcal{L}^2_z(-1, 1)$. *There exists a unique weak solution* $u \in L^2((x_0, x_1); H^1_0(-1, 1))$ *to* (2.1). *Moreover,*

$$\|u\|_{L^2_x H^1_z} \lesssim \|f\|_{L^2_x(H^{-1}_z)} + \|\delta_0\|_{\mathcal{L}^2_z} + \|\delta_1\|_{\mathcal{L}^2_z}. \tag{2.2}$$

Proof. The proof of uniqueness is postponed to Appendix A, where we adapt Baouendi and Grisvard's arguments to prove the uniqueness of weak solutions to all the linear problems we encounter in this memoir in Lemma A.1. It relies on the proof of a trace theorem and a Green identity for the space $\mathcal{B}$ defined in (1.25).

Let us prove the existence. We introduce two Hilbert spaces $\mathcal{U}$ and $\mathcal{V}$ satisfying $\mathcal{V} \hookrightarrow \mathcal{U} \hookrightarrow L^2((x_0, x_1); H_0^1(0,1))$ as follows. Let

$$\mathcal{V} := \left\{ v \in H^1(\Omega) \mid v = 0 \text{ on } \Omega \setminus (\Sigma_0 \cup \Sigma_1) \right\}.$$

Let $\mathcal{U}$ be the completion of $H^1(\Omega) \cap L^2((x_0, x_1); H_0^1(-1,1))$ with respect to the scalar product

$$\langle u, v \rangle_{\mathcal{U}} := \int_\Omega \partial_z u \partial_z v + \int_{\Sigma_0} zuv - \int_{\Sigma_1} zuv. \tag{2.3}$$

For $u, v \in \mathcal{U} \times \mathcal{V}$, let

$$a(u,v) := -\int_\Omega zu\partial_x v + \int_\Omega \partial_z u \partial_z v, \tag{2.4}$$

$$b(v) := \int_\Omega fv + \int_{\Sigma_0} z\delta_0 v - \int_{\Sigma_1} z\delta_1 v. \tag{2.5}$$

In particular, for every $v \in \mathcal{V}$, integration by parts leads to $a(v,v) = \|v\|_{\mathcal{U}}^2$ and

$$|b(v)| \leq \left(\|f\|_{L_x^2(H_z^{-1})} + \|\delta_0\|_{\mathcal{L}_z^2} + \|\delta_1\|_{\mathcal{L}_z^2} \right) \|v\|_{\mathcal{U}}. \tag{2.6}$$

Hence, $b \in \mathcal{L}(\mathcal{V})$ can be extended as a linear form over $\mathcal{U}$ and existence follows from the Lax–Milgram-type existence principle Lemma B.2 in Appendix B, which also yields the energy estimate (2.2) thanks to (2.6) and Poincaré's inequality. ∎

Remark 2.3. Functions in $\mathcal{U}$ *a priori* do not have traces on Σ_i, so one could wonder how definition (2.5) makes sense when $v \in \mathcal{U}$. The integrals $\int_{\Sigma_i} z\delta_i v$ make sense precisely because $\mathcal{U}$ is defined as a completion with respect to (2.3). In fact, weak solutions do have traces in a strong sense, as proved in Lemma A.2, thanks to the extra regularity in x provided by the equation.

Remark 2.4. Instead of using the weak Lax–Milgram existence principle Lemma B.2, an alternate proof would be to regularize equation (2.1) by vanishing viscosity, and to obtain uniform $L_x^2 H_z^1$ estimates on the approximation. This approach will be used in Lemma 5.4 proved in the Appendix, in which we prove $H_x^1 L_z^2$ regularity of the weak solutions far from the lateral boundaries.

2.2 Strong solutions with maximal regularity

We now turn to strong solutions, i.e., solutions for which (2.1) holds almost everywhere. The main result on this topic is due to Pagani.

Proposition 2.5. *Let $f \in L^2(\Omega)$ and $\delta_0, \delta_1 \in \mathcal{H}^1_z(-1,1)$ such that $\delta_0(1) = \delta_1(-1) = 0$. The unique weak solution u to (2.1) belongs to $Z^0(\Omega)$ and satisfies*

$$\|u\|_{Z^0} \lesssim \|f\|_{L^2} + \|\delta_0\|_{\mathcal{H}^1_z} + \|\delta_1\|_{\mathcal{H}^1_z}. \tag{2.7}$$

The boundary conditions $u_{|\Sigma_i} = \delta_i$ hold as traces in $\mathcal{H}^1_z(\Sigma_i)$ (see Lemma 1.10).

Proof. This is a particular case of [53, Theorem 5.2]. Pagani's proof proceeds by localization. Far from the critical points $(x_0, 0)$ and $(x_1, 0)$, the regularity is rather straightforward. Near these critical points, the regularity stems from the regularity obtained for a similar problem set in a half-space $(0, +\infty) \times \mathbb{R}$ or $\mathbb{R} \times (0, +\infty)$. Pagani studies such half-space problems in [52], where he derives explicit representation formulas for the solutions, using the Mellin transform and the Wiener–Hopf method. We do not reproduce these arguments here for brevity.

Note that the implicit constant, say C_P, in (2.7) may depend on Ω. By scaling arguments, one can prove that $C_P \leq C(1 + |x_1 - x_0|^{-1})$ for some universal $C > 0$. ∎

2.3 Orthogonality conditions for higher tangential regularity

We now investigate whether solutions to (2.1) enjoy higher regularity in the horizontal direction. As mentioned in Section 1.2, it is quite easy to obtain *a priori* estimates in the space $Z^1(\Omega)$ (see Proposition 2.6). However, we prove in Proposition 2.10 that the weak solution enjoys such a regularity *if only if the data satisfies appropriate orthogonality conditions*. Eventually, we give statements highlighting the fact that these conditions are non-empty.

Proposition 2.6. *Let $f \in H^1((x_0, x_1); H^{-1}(-1,1))$ and $\delta_0, \delta_1 \in \mathcal{H}^1_z(-1,1)$ such that $\delta_0(1) = \delta_1(-1) = 0$ and such that $\Delta_0, \Delta_1 \in \mathcal{L}^2_z(-1,1)$, where*

$$\Delta_i(z) := \frac{f(x_i, z) + \partial_z^2 \delta_i(z)}{z}. \tag{2.8}$$

If the unique weak solution u to (2.1) belongs to $H^1((x_0, x_1); H^1_0(-1,1))$, then one has the following weak solution estimate for $\partial_x u$:

$$\|\partial_x u\|_{L^2_x H^1_z} \lesssim \|\partial_x f\|_{L^2_x(H^{-1}_z)} + \|\Delta_0\|_{\mathcal{L}^2_z(\Sigma_0)} + \|\Delta_1\|_{\mathcal{L}^2_z(\Sigma_1)}. \tag{2.9}$$

If, moreover, $f \in H^1((x_0, x_1); L^2(-1,1))$, $\Delta_0, \Delta_1 \in \mathcal{H}^1_z(-1,1)$, $\Delta_0(1) = \Delta_1(-1) = 0$, then $u \in Z^1(\Omega)$ and one has the following strong solution estimate for $\partial_x u$:

$$\|\partial_x u\|_{Z^0} \lesssim \|\partial_x f\|_{L^2} + \|\Delta_0\|_{\mathcal{H}^1_z} + \|\Delta_1\|_{\mathcal{H}^1_z}. \tag{2.10}$$

Proof. The key point is the following argument: if $\partial_x u$ enjoys $L^2_x H^1_z$ regularity, then $\partial_x u$ is the unique weak solution to

$$\begin{cases} z\partial_x w - \partial_{zz} w = \partial_x f, \\ w_{|\Sigma_i} = \Delta_i, \\ w_{|z=\pm 1} = 0. \end{cases} \tag{2.11}$$

Then estimate (2.9) follows from (2.2) and estimate (2.10) follows from (2.7).

Hence, let us prove that, if $\partial_x u \in L^2_x H^1_z$, then $\partial_x u$ is a weak solution to (2.11). Let

$$\mathcal{V} := \big\{v \in C^\infty(\bar{\Omega}); v = 0 \text{ on } \partial\Omega \setminus (\Sigma_0 \cup \Sigma_1), \\ \partial_x v = 0 \text{ on } \{x_0\} \times (-1,0) \text{ and } \{x_1\} \times (0,1)\big\}.$$

Let $v \in \mathcal{V}$. Then $\partial_x v$ is an admissible test function for Definition 2.1. Hence, since u is the weak solution to (2.1), one has

$$-\int_\Omega zu\partial_x(\partial_x v) + \int_\Omega \partial_z u \partial_z(\partial_x v) = \int_\Omega f(\partial_x v) + \int_{\Sigma_0} z\delta_0(\partial_x v) - \int_{\Sigma_1} z\delta_1(\partial_x v).$$

The $H^1_x H^1_z$ regularity of u legitimates integrations by parts in x on the left-hand side. Thus

$$\left[-\int_{-1}^1 zu\partial_x v\right]_{x_0}^{x_1} + \int_\Omega z(\partial_x u)\partial_x v + \left[\int_{-1}^1 \partial_z u \partial_z v\right]_{x_0}^{x_1} - \int_\Omega \partial_z(\partial_x u)\partial_z v \\ = \left[\int_{-1}^1 fv\right]_{x_0}^{x_1} - \int_\Omega f_x v + \int_{\Sigma_0} z\delta_0(\partial_x v) - \int_{\Sigma_1} z\delta_1(\partial_x v),$$

which, after taking the boundary conditions into account, integrating by parts in z in the boundary terms $\int_{-1}^1 \partial_z u \partial_z v$ and recalling (2.8) yields

$$-\int_\Omega z(\partial_x u)\partial_x v + \int_\Omega \partial_z(\partial_x u)\partial_z v = \int_\Omega f_x v + \int_{\Sigma_0} z\Delta_0 v - \int_{\Sigma_1} z\Delta_1 v.$$

Since $\mathcal{V}$ is dense in the set of test functions for Definition 2.1, this proves that $\partial_x u$ is the weak solution to (2.11). ∎

We start by defining "dual profiles" which are necessary to state our orthogonality conditions.

Lemma 2.7 (Dual profiles). *We define* $\overline{\Phi^0}$, $\overline{\Phi^1} \in Z^0(\Omega_\pm)$ *as the unique solutions to*

$$\begin{cases} -z\partial_x \overline{\Phi^j} - \partial_{zz}\overline{\Phi^j} = 0 \quad \text{in } \Omega_\pm, \\ \left[\overline{\Phi^j}\right]_{|z=0} = \mathbf{1}_{j=1}, \\ \left[\partial_z \overline{\Phi^j}\right]_{|z=0} = -\mathbf{1}_{j=0}, \\ \overline{\Phi^j}_{|\partial\Omega\setminus(\Sigma_0\cup\Sigma_1)} = 0. \end{cases} \tag{2.12}$$

Proof. Uniqueness is straightforward. Given $j \in \{0,1\}$ and two solutions to (2.12), let ϕ denote their difference. Then $\phi \in Z^0(\Omega_\pm)$ and both ϕ and $\partial_z \phi$ are continuous across the line $\{z=0\}$. Hence, $\phi \in Z^0(\Omega)$ and ϕ is the solution to a problem of the form (2.1) (with reversed tangential direction). So $\phi = 0$ since weak solutions to such problems are unique in Z^0.

We prove the existence of $\overline{\Phi^0}$. We define $\overline{\Phi^0}(x,z) := -z\mathbf{1}_{z>0}\zeta(z) + \Psi^0(x,z)$, where we choose $\zeta \in C_c^\infty(\mathbb{R})$ such that $\zeta \equiv 1$ in a neighborhood of $z=0$ and $\operatorname{supp}\zeta \subset (-1/2,1/2)$, and where $\Psi^0 \in L^2((x_0,x_1);H_0^1(-1,1))$ is the unique weak solution to

$$\begin{cases} -z\partial_x\Psi^0 - \partial_{zz}\Psi^0 = -2\mathbf{1}_{z>0}\zeta'(z) - z\mathbf{1}_{z>0}\zeta''(z) & \text{in } \Omega, \\ \Psi^0(x_0,z) = 0 & \text{for } z \in (-1,0), \\ \Psi^0(x_1,z) = z\zeta(z) & \text{for } z \in (0,1), \\ \Psi^0_{|z=\pm1} = 0. \end{cases}$$

By Proposition 2.5, $\Psi^0 \in Z^0(\Omega)$. Hence $\partial_{zz}\overline{\Phi^0} \in L^2(\Omega_\pm)$ and $z\partial_x\overline{\Phi^0} \in L^2(\Omega_\pm)$.

The construction of the profile $\overline{\Phi^1}$ is similar and is left to the reader. For example, one can decompose the profile $\overline{\Phi^1}$ as $\overline{\Phi^1}(x,z) = \mathbf{1}_{z>0}\zeta(z) + \Psi^1(x,z)$, where, similarly, $\Psi^1 \in Z^0(\Omega)$. ■

Remark 2.8. The jump conditions in (2.12) prevent the dual profiles from enjoying vertical regularity across the line $\{z=0\}$. More subtly, even inside each half-domain, neither the $\overline{\Phi^j}$ nor their lifted version the Ψ^j enjoy tangential regularity. Indeed, formally, $\partial_x\overline{\Phi^j}$ and $\partial_x\Psi^j$ satisfy systems of the form (2.1) (with reversed tangential direction) with zero source term and zero boundary data. Hence, if they were sufficiently regular, they would be zero by the uniqueness results of Appendix A, and so would $\overline{\Phi^j}$ and Ψ^j by integration, contradicting (2.12). We will see in Corollary 2.32 that these dual profiles indeed do contain an explicit singular part localized near the endpoints $(x_i,0)$.

We now turn to the main result of this section, which gives a necessary and sufficient condition for the solutions to enjoy the mentioned tangential regularity. Strangely, we could not find a proof of Proposition 2.10 in the literature, although some works mention orthogonality conditions (see [22, equation (4.2)] or [55]). Hence we provide here a full proof. This strategy will be extended in Section 5.2 to equations with smooth variable coefficients (see Proposition 5.5). We prove further that these orthogonality conditions are not empty.

We will work with the following space of data triplets:

$$\begin{aligned} \mathcal{H}_K := \big\{ &(f,\delta_0,\delta_1) \in H_x^1L_z^2 \times \mathcal{H}_z^1(\Sigma_0) \times \mathcal{H}_z^1(\Sigma_1); (\Delta_0,\Delta_1) \in \mathcal{H}_z^1(\Sigma_0)\times\mathcal{H}_z^1(\Sigma_1) \\ &\text{and } \delta_0(1) = \delta_1(-1) = \Delta_0(1) = \Delta_1(-1) = 0 \big\}, \end{aligned} \tag{2.13}$$

where Δ_i is defined in (2.8), with the associated norm

$$\|(f,\delta_0,\delta_1)\|_{\mathcal{H}_K} := \|f\|_{H^1_x L^2_z} + \sum_{i\in\{0,1\}} \|\delta_i\|_{\mathcal{H}^1_z} + \|\Delta_i\|_{\mathcal{H}^1_z}. \tag{2.14}$$

Lemma 2.9. *If $(f,\delta_0,\delta_1)\in\mathcal{H}_K$, then $\delta_i\in H^2(\Sigma_i)$ and $\|\delta_i\|_{H^2}\lesssim\|(f,\delta_0,\delta_1)\|_{\mathcal{H}_K}$.*

Proof. For $i\in\{0,1\}$, recalling (2.8), one has

$$\|\delta_i''\|_{L^2(\Sigma_i)} \le \|\delta_i''+f(x_i,\cdot)\|_{L^2(\Sigma_i)} + \|f(x_i,\cdot)\|_{L^2(\Sigma_i)} \lesssim \|\Delta_i\|_{\mathcal{L}^2_z} + \|f\|_{H^1_x L^2_z}.$$

Moreover, proceeding for example as in the proof of Lemma B.7, one checks that $\|\delta_i\|_{L^2}\lesssim\|\delta_i\|_{\mathcal{L}^2_z}+\|\delta_i''\|_{L^2}$. ■

Proposition 2.10. *Let $(f,\delta_0,\delta_1)\in\mathcal{H}_K$. The unique weak solution u to (2.1) belongs to $H^1_x H^1_z$ if and only if, for $j=0$ and $j=1$,*

$$\int_\Omega \partial_x f\,\overline{\Phi^j} + \int_{\Sigma_0} z\Delta_0\overline{\Phi^j} - \int_{\Sigma_1} z\Delta_1\overline{\Phi^j} = \partial_z^j\delta_1(0) - \partial_z^j\delta_0(0), \tag{2.15}$$

where $\overline{\Phi^0}$ and $\overline{\Phi^1}$ are defined in Lemma 2.7.

Furthermore, in this case, u actually belongs to $Z^1(\Omega)$ and the following estimate holds:

$$\|u\|_{Z^1}\lesssim\|(f,\delta_0,\delta_1)\|_{\mathcal{H}_K}. \tag{2.16}$$

Proof. Step 1. We exhibit possible discontinuities. Let us consider the unique solution $u\in Z^0(\Omega)$ to (2.1). Following the strategy sketched by Goldstein and Mazumdar [25, Theorem 4.2] (see Remark 2.11 for further comments), we introduce the unique strong solution $w\in Z^0(\Omega)$ to (2.11), so that w is a good candidate for $\partial_x u$. The idea is then to introduce the function u_1 defined by

$$u_1(x,z) := \begin{cases} \delta_0(z) + \int_{x_0}^{x} w(x',z)\,\mathrm{d}x' & \text{in } \Omega_+, \\ \delta_1(z) - \int_{x}^{x_1} w(x',z)\,\mathrm{d}x' & \text{in } \Omega_- \end{cases} \tag{2.17}$$

so that $\partial_x u_1 = w$ almost everywhere. Furthermore, it can be easily proved that, in $\mathcal{D}'(\Omega_\pm)$,

$$z\partial_x u_1 - \partial_{zz}u_1 = f.$$

However, this does not entail that u_1 is a solution to this equation in the whole domain. Indeed, u_1 and $\partial_z u_1$ may have discontinuities across the line $\{z=0\}$. One checks that u_1 and $\partial_z u_1$ are continuous across $z=0$ if and only if

$$\int_{x_0}^{x_1} w(x,0)\,\mathrm{d}x = \delta_1(0)-\delta_0(0), \quad \int_{x_0}^{x_1} w_z(x,0)\,\mathrm{d}x = \partial_z\delta_1(0)-\partial_z\delta_0(0). \tag{2.18}$$

The two integrals are well defined since w_z and w_{zz} belong to $L^2(\Omega)$.

Step 2. We compute the horizontal mean value of w and w_z using the dual profiles. Let $\phi \in Z^0(\Omega_\pm)$ such that $\phi_{|\partial\Omega\setminus(\Sigma_0\cap\Sigma_1)} = 0$. Since $w \in Z^0(\Omega)$, it satisfies (2.11) almost everywhere, so that we can multiply the equation by ϕ and integrate over Ω_+. Hence,

$$\int_{\Omega_+} f_x\phi = \int_{\Omega_+} (z\partial_x w - \partial_{zz} w)\phi,$$

where, on the one hand,

$$\int_{\Omega_+} z(\partial_x w)\phi = \int_{\Sigma_1} z\Delta_1\phi - \int_{\Omega_+} zw\partial_x\phi$$

and on the other hand,

$$-\int_{\Omega_+} (\partial_{zz} w)\phi = \int_{x_0}^{x_1} (\partial_z w\phi - w\partial_z\phi)(x, 0^+)\,\mathrm{d}x - \int_{\Omega_+} w\partial_{zz}\phi.$$

Thus, performing the same computation on Ω_- and summing both contributions yields

$$\begin{aligned}\int_{x_0}^{x_1} &(\partial_z w[\phi]_{|z=0} - w[\partial_z\phi]_{|z=0})(x, 0)\,\mathrm{d}x \\ &= \int_\Omega f_x\phi + \int_{\Sigma_0} z\Delta_0\phi - \int_{\Sigma_1} z\Delta_1\phi + \sum_\pm \int_{\Omega_\pm} w(z\partial_x\phi + \partial_{zz}\phi).\end{aligned}$$

Hence, for $j \in \{0, 1\}$,

$$\int_{x_0}^{x_1} \partial_z^j w(x, 0)\,\mathrm{d}x = \int_\Omega f_x\overline{\Phi^j} + \int_{\Sigma_0} z\Delta_0\overline{\Phi^j} - \int_{\Sigma_1} z\Delta_1\overline{\Phi^j},$$

where the dual profiles $\overline{\Phi^0}$ and $\overline{\Phi^1}$ are defined in Lemma 2.7.

Step 3. Conclusion. Assume that the orthogonality conditions (2.15) are satisfied for $j = 0$ and $j = 1$. Then (2.18) holds, and as a consequence, $[u_1]_{|z=0} = [\partial_z u_1]_{|z=0} = 0$. Thus $u_1 \in L^2((x_0, x_1); H_0^1(-1, 1))$ is a weak solution to (2.1). We infer from the uniqueness of weak solutions that $u = u_1$, and therefore $\partial_x u = w \in Z^0$. Hence $u \in H^1((x_0, x_1); H_0^1(-1, 1))$. Estimate (2.16) follows from (2.7) and (2.10).

Conversely, if u is a solution to (2.1) with $H^1((x_0, x_1); H_0^1(-1, 1))$ regularity, then $\partial_x u$ is a weak solution to (2.11) (see the proof of Proposition 2.6) and u is given in terms of $\partial_x u$ by (2.17) almost everywhere. Thus $[u_1]_{|z=0} = [\partial_z u_1]_{|z=0} = 0$. Hence $\int_{x_0}^{x_1} u_x(x, 0)\,\mathrm{d}x = \delta_1(0) - \delta_0(0)$ and $\int_{x_0}^{x_1} u_{xz}(x, 0)\,\mathrm{d}x = \partial_z\delta_1(0) - \partial_z\delta_0(0)$, and thus the orthogonality conditions (2.15) are satisfied. ∎

Remark 2.11. Oddly, in [25, Theorem 4.2], Goldstein and Mazumdar do not mention the orthogonality conditions (2.15). They merely state that, "since $\partial_{zz} u_1 = z\partial_x u_1 - f$

in $\mathcal{D}'(\Omega_\pm)$, since $zu_1, f \in C^0([x_0,x_1];L^2(-1,1))$, consequently $z\partial_x u_1 - \partial_{zz}u_1 = f$ in $L^2(\Omega)$". However, these orthogonality conditions are non-empty, as we show below (see Proposition 2.15).

Definition 2.12. In the sequel, we denote by $\overline{\ell^j}$ the linear forms associated with the orthogonality conditions (2.15) for the linear shear flow problem, i.e., for $(f,\delta_0,\delta_1) \in \mathcal{H}_K$, we set

$$\overline{\ell^j}(f,\delta_0,\delta_1) := \partial_z^j \delta_0(0) - \partial_z^j \delta_1(0) + \int_\Omega \partial_x f\, \overline{\Phi^j} + \int_{\Sigma_0} z\Delta_0 \overline{\Phi^j} - \int_{\Sigma_1} z\Delta_1 \overline{\Phi^j}.$$

Lemma 2.13. *The linear forms $\overline{\ell^j}$ for $j \in \{0,1\}$ are continuous over $\mathcal{H}_K$.*

Proof. First, by Lemma 2.9, for $(f,\delta_0,\delta_1) \in \mathcal{H}_K$, $\delta_i \in H^2(\Sigma_i)$ so that $\delta_i(0)$ and $\delta_i'(0)$ depend continuously on $(f,\delta_0,\delta_1) \in \mathcal{H}_K$. Second, by Lemma 2.7, $\overline{\Phi^j} \in Z^0(\Omega_\pm)$ so, in particular $\overline{\Phi^j} \in L^2(\Omega)$. Hence $f \mapsto \int_\Omega \partial_x f\, \overline{\Phi^j}$ is continuous on $H^1_x L^2_z$. Eventually, by Lemma 1.10, $\overline{\Phi^j}(x_i,\cdot) \in \mathcal{H}^1_z(\Sigma_i) \hookrightarrow \mathcal{L}^2_z(\Sigma_i)$, so

$$(f,\delta_0,\delta_1) \mapsto \int_{\Sigma_i} z\Delta_i(z)\overline{\Phi^j}(x_i,z)\,\mathrm{d}z$$

is continuous on $\mathcal{H}_K$. ∎

Remark 2.14. Although this continuity result will be sufficient for most of our purpose, the linear forms $\overline{\ell^j}$ are in fact continuous for weaker topologies than the one of $\mathcal{H}_K$. In particular, one does not need $f \in H^1_x L^2_z$ (see Remark 2.33).

We now prove that the orthogonality conditions (2.15) are non-empty and independent.

Proposition 2.15 (Independence of the orthogonality conditions). *The linear forms $\overline{\ell^0}$ and $\overline{\ell^1}$ are linearly independent over $C^\infty_c(\Omega) \times \{0\} \times \{0\} \subset \mathcal{H}_K$.*

Proof. Proceeding by contradiction, let $(c_0,c_1) \in \mathbb{R}^2$ such that, for every $f \in C^\infty_c(\Omega)$, there holds $c_0\overline{\ell^0}(f,0,0) + c_1\overline{\ell^1}(f,0,0) = 0$. Define $\Phi^c := c_0\overline{\Phi^0} + c_1\overline{\Phi^1}$. Then, for every $f \in C^\infty_c(\Omega)$, it satisfies $\int_\Omega \partial_x f\,\Phi^c = 0$. Hence $\partial_x \Phi^c = 0$ in $\mathcal{D}'(\Omega_+)$. Since $\Phi^c(x_1,z) = 0$ for $z \in (0,1)$ and $\Phi^c \in Z^0(\Omega_+)$, this implies that $\Phi^c = 0$ in Ω_+ (since Z^0 functions have traces in the usual sense, see Lemma 1.10). The same holds in Ω_-. Hence $[\Phi^c]_{|z=0} = [\partial_z \Phi^c]_{|z=0} = 0$, which implies $c_0 = c_1 = 0$. ∎

Remark 2.16. Proposition 2.15 of course implies that $\overline{\ell^0}$ and $\overline{\ell^1}$ are linearly independent on $\mathcal{H}_K$. Although Proposition 2.15 gives a prominent role to the source term f, we will actually also prove that $\overline{\ell^0}$ and $\overline{\ell^1}$ are linearly independent on $\{0\} \times C^\infty_c(\Sigma_0) \times C^\infty_c(\Sigma_1) \subset \mathcal{H}$. This property relies on the structure of the dual profiles

$\overline{\Phi^j}$ near the points $(x_i, 0)$, and will be proved at the end of this section (see Proposition 2.34).

Eventually, gathering all of the above results, definitions and notations, we have proved the following well-posedness result for the linear problem.

Proposition 2.17. *Let $\mathcal{H}_K$ be the vector space defined in (2.13). There exists a vector subspace $\mathcal{H}_{K,\mathrm{sg}}^{\perp} = (\ker \overline{\ell^0}) \cap (\ker \overline{\ell^1})$ of codimension 2 in $\mathcal{H}_K$ such that, for each $(f, \delta_0, \delta_1) \in \mathcal{H}_K$, there exists a solution $u \in Z^1(\Omega)$ to (2.1) if and only if $(f, \delta_0, \delta_1) \in \mathcal{H}_{K,\mathrm{sg}}^{\perp}$. Such a solution is unique and satisfies estimate (2.16).*

Theorem 1 of the introduction is a rough restatement of the above Proposition 2.17 allowing us to avoid introducing more notations and functional spaces at this early stage.

Proof of Theorem 1. One easily checks from (1.3) and (2.13) that $\mathcal{X}_B \hookrightarrow \mathcal{H}_K$. Moreover, setting $\mathcal{X}_{B,\mathrm{sg}}^{\perp} := \mathcal{X}_B \cap (\ker \overline{\ell^0}) \cap (\ker \overline{\ell^1})$, one obtains that $\mathcal{X}_{B,\mathrm{sg}}^{\perp}$ is of codimension 2 in $\mathcal{X}_B$ from Proposition 2.15 or Proposition 2.34. Hence, given $(f, \delta_0, \delta_1) \in \mathcal{X}_{B,\mathrm{sg}}^{\perp}$, Proposition 2.17 gives a solution $u \in Z^1$. By Lemma 1.15, $Z^1 \hookrightarrow Q^1$ (defined in (1.5)), so $u \in Q^1$. Reciprocally, given a solution $u \in Q^1$ corresponding to some data triple $(f, \delta_0, \delta_1) \in \mathcal{X}_B$, one has $u \in H_x^1 H_z^2$ (see (1.5)), so Proposition 2.10 applies and one has $(f, \delta_0, \delta_1) \in \mathcal{X}_{B,\mathrm{sg}}^{\perp}$. ∎

Similarly, going further, it can be easily checked that the control of k derivatives in x requires the cancellation of $2k$ independent conditions. Although controlling a single x-derivative will be sufficient in the sequel to obtain our nonlinear result, we establish here this short higher-regularity statement as an illustration. More precisely, we have the following result.

Lemma 2.18. *Let $k \geq 1$. Let $f \in C^\infty(\overline{\Omega})$, $\delta_i \in C^\infty(\overline{\Sigma_i})$. Define recursively Δ_i^n for $0 \leq n \leq k$ and $z \in \Sigma_i$ by*

$$\Delta_i^0(z) := \delta_i(z), \tag{2.19}$$

$$\Delta_i^n(z) := \frac{1}{z}\big(\partial_x^{n-1} f(x_i, z) + \partial_{zz} \Delta_i^{n-1}(z)\big). \tag{2.20}$$

Assume that the following compatibility conditions are satisfied:

$$\forall n \in \{0, \dots, k\}, \quad \Delta_0^n(1) = \Delta_1^n(-1) = 0.$$

Assume furthermore that for all $n \in \{0, \dots, k\}$, $\Delta_i^n \in \mathcal{H}_z^1(\Sigma_i)$.

Let u be the unique solution to (1.6). Then $u \in H_x^k H_z^2$ if and only if the following orthogonality conditions are satisfied:

$$\overline{\ell^j}(\partial_x^n f, \Delta_0^n, \Delta_1^n) = 0 \quad \forall n \in \{0, \dots, k-1\},\ j \in \{0, 1\}.$$

Furthermore, these $2k$ orthogonality conditions are linearly independent.

Proof. First, notice that $\partial_x^n u$ satisfies formally

$$\begin{cases} (z\partial_x - \partial_{zz})\partial_x^n u = \partial_x^n f & \text{in } \Omega, \\ \partial_x^n u_{|z=\pm 1} = 0, \\ \partial_x^n u_{|\Sigma_i} = \Delta_i^n. \end{cases}$$

The first part of the statement follows easily from Proposition 2.10 and Proposition 2.6 and from an induction argument.

Let us now check the independence of the orthogonality conditions. We extend the methodology used in the proof of Proposition 2.15. Assume that there exist $c_n^j \in \mathbb{R}$, $0 \le n \le k-1$, $j = 0, 1$ such that for all (f, δ_0, δ_1) satisfying the assumptions of the lemma,

$$\sum_{j=0,1} \sum_{n=0}^{k-1} c_n^j \overline{\ell^j}(\partial_x^n f, \Delta_0^n, \Delta_1^n) = 0.$$

In particular, for any $f \in C_c^\infty(\Omega)$,

$$\sum_{j=0,1} \sum_{n=0}^{k-1} c_n^j \overline{\ell^j}(\partial_x^n f, 0, 0) = 0,$$

i.e.,

$$\sum_{n=0}^{k-1} \int_\Omega \partial_x^n f \left(\sum_{j=0,1} c_n^j \overline{\Phi^j} \right) = 0.$$

This means that

$$\sum_{j=0,1} \sum_{n=0}^{k-1} (-1)^n c_n^j \partial_x^n \overline{\Phi^j} = 0$$

in the sense of distributions. Since $[\partial_x^n \overline{\Phi^j}]_{|z=0} = [\partial_x^n \partial_z \overline{\Phi^j}]_{|z=0} = 0$ for $n \ge 1$, we infer that

$$\left[c_0^0 \overline{\Phi^0} + c_0^1 \overline{\Phi^1}\right]_{|z=0} = \left[\partial_z (c_0^0 \overline{\Phi^0} + c_0^1 \overline{\Phi^1})\right]_{|z=0} = 0.$$

Once again, using the jump conditions on $\overline{\Phi^j}$, we deduce that $c_0^j = 0$, and thus

$$\partial_x \left(\sum_{j=0,1} \sum_{n=1}^{k-1} (-1)^n c_n^j \partial_x^{n-1} \overline{\Phi^j} \right) = 0.$$

It follows that

$$\sum_{j=0,1} \sum_{n=1}^{k-1} (-1)^n c_n^j \partial_x^{n-1} \overline{\Phi^j} = p(z)$$

for some function p. Note that by parabolic regularity, the profiles $\overline{\Phi^j}$ (and therefore the function p) are smooth away from the line $\{z = 0\}$. Taking the trace of the above identity on $\{x_0\} \times (-1, 0) \cup \{x_1\} \times (0, 1)$, we find that $p = 0$. Arguing by induction, we infer eventually that $c_n^j = 0$ for all $0 \leq n \leq k-1$, $j = 0, 1$. ∎

Corollary 2.19 (Biorthogonal basis). *There exist* $\Xi^k = (f^k, \delta_0^k, \delta_1^k) \in \mathcal{H}_K$ *for* $k \in \{0, 1\}$ *such that, for every* $j, k \in \{0, 1\}$,

$$\overline{\ell^j}(\Xi^k) = \overline{\ell^j}(f^k, \delta_0^k, \delta_1^k) = \mathbf{1}_{j=k}$$

and such that, within $\mathcal{H}_K$,

$$(\mathbb{R}\,\Xi^0 + \mathbb{R}\,\Xi^1)^{\perp} = \ker \overline{\ell^0} \cap \ker \overline{\ell^1} \tag{2.21}$$

is a vector subspace of codimension 2.

Proof. Since $\overline{\ell^0}$ and $\overline{\ell^1}$ are continuous linear forms on $\mathcal{H}_K$, by the Riesz representation theorem, they can be written as scalar products with two given triplets, say $\overline{\Xi^0}, \overline{\Xi^1} \in \mathcal{H}_K$ which are linearly independent thanks to Proposition 2.15. Then one looks for $\Xi^k = (f^k, \delta_0^k, \delta_1^k)$ as $a_k \overline{\Xi^0} + b_k \overline{\Xi^1}$, where $a_k, b_k \in \mathbb{R}^2$ are such that $a_k \langle \overline{\Xi^j}; \overline{\Xi^0} \rangle + b_k \langle \overline{\Xi^j}; \overline{\Xi^1} \rangle = \mathbf{1}_{j=k}$. These systems can be solved since $\overline{\Xi^0}$ and $\overline{\Xi^1}$ are free. This proves the equality (2.21). The independence of the linear forms guarantees that (2.21) is of codimension 2 in $\mathcal{H}_K$. ∎

2.4 Singular radial solutions in the half-plane and profile decomposition

In this section, we give a full description of the singularities that appear when the orthogonality conditions are not satisfied. We start by constructing singular solutions to the homogeneous equation set in the half-plane, using separation of variables in polar-like coordinates. We then localize these solutions near the critical points $(x_i, 0)$ to obtain the decomposition result of Theorem 2.

Our approach is similar to the one developed by Grisvard in [28, Section 4.4] for elliptic problems in polygonal domains (see in particular the singular profiles of equation (4.4.3.7) and the decomposition result of Theorem 4.4.3.7 therein). The main difference is that we cannot use usual polar coordinates and that the construction of the elementary singular profiles is much more technical than, for instance, the classical solution of the form $r^{\frac{1}{2}} \sin(\theta/2)$ which is involved in the resolution of Dirichlet–Neumann junctions as in the elliptic problem (1.12) mentioned in the introduction.

2.4.1 Construction of singular solutions in the half-plane

In this paragraph, we look for elementary singular radial solutions to the following problem without source-term in the half-plane:

$$\begin{cases} z\partial_x u - \partial_{zz} u = 0 & x \geq 0, z \in \mathbb{R}, \\ u(0,z) = 0 & z > 0. \end{cases} \tag{2.22}$$

Remark 2.20. In [23], Fleming considered the related problem of finding a "fairly explicit formula" for solutions to $z\partial_x u - \partial_{zz} u = 0$ in a strip $(0,1) \times \mathbb{R}$, with prescribed boundary data at $x = 0, z > 0$ and $x = 1, z < 0$. His proof involves Whittaker functions, which are related to the confluent hypergeometric functions we use below.

In [27], Gor'kov computes a representation formula for solutions to (2.22) with a non-zero source term and boundary data, and proves uniqueness of such solutions, under a growth assumption of the form $|u(0,z)| \lesssim |z|^\sigma$ for $0 \leq \sigma < \frac{1}{2}$ on the line $\{x = 0\}$, for which he claims that uniqueness holds. The threshold $\sigma = \frac{1}{2}$ is precisely the scaling (at which uniqueness indeed breaks) of the first fundamental singular solution v_0 which we construct below.

Our setting is a little different from the works mentioned above, as we look for (non-zero) solutions to the homogeneous equation. Similar computations were also performed in [30, 31], albeit with different boundary conditions, and therefore with a different exponent for r, and a different asymptotic behavior for the profile Λ in Proposition 2.21. However, we were not able to find the specific expression of the profiles from Proposition 2.21 in previous works.

Near the point $(0,0)$ which is expected to be singular, balancing the terms $z\partial_x$ and ∂_{zz} leads to the natural scaling $z \sim x^{\frac{1}{3}}$. Thus, we introduce the following polar-like coordinates $(r,t) \in [0,+\infty) \times \mathbb{R}$:

$$r := (z^2 + x^{\frac{2}{3}})^{\frac{1}{2}} \quad \text{and} \quad t := zx^{-\frac{1}{3}}. \tag{2.23}$$

The reverse change of coordinates is given by

$$x = \frac{r^3}{(1+t^2)^{\frac{3}{2}}} \quad \text{and} \quad z = \frac{rt}{(1+t^2)^{\frac{1}{2}}}. \tag{2.24}$$

Since it will be convenient to switch from cartesian coordinates (x,z) to the polar-like coordinates (r,t), we compute the Jacobian

$$J(r,t) = \begin{pmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial z} \\ \frac{\partial t}{\partial x} & \frac{\partial t}{\partial z} \end{pmatrix} = \begin{pmatrix} \frac{1}{3x^{\frac{1}{3}} r} & \frac{z}{r} \\ -\frac{t}{3x} & \frac{1}{x^{\frac{1}{3}}} \end{pmatrix} = \begin{pmatrix} \frac{(1+t^2)^{\frac{1}{2}}}{3r^2} & \frac{t}{(1+t^2)^{\frac{1}{2}}} \\ -\frac{t(1+t^2)^{\frac{3}{2}}}{3r^3} & \frac{(1+t^2)^{\frac{1}{2}}}{r} \end{pmatrix}, \tag{2.25}$$

where we have used the equalities (2.24). In particular,

$$\det J(r,t) = \frac{(1+t^2)^2}{3r^3}, \tag{2.26}$$

which we will use to compute integrals using the (r,t) variables.

By (2.25), for any C^1 function φ,

$$\partial_x \varphi = \frac{(1+t^2)^{\frac{1}{2}}}{3r^2}\partial_r\varphi - \frac{t(1+t^2)^{\frac{3}{2}}}{3r^3}\partial_t\varphi, \tag{2.27}$$

$$\partial_z \varphi = \frac{t}{(1+t^2)^{\frac{1}{2}}}\partial_r\varphi + \frac{(1+t^2)^{\frac{1}{2}}}{r}\partial_t\varphi. \tag{2.28}$$

In particular, if $u(r,t) = r^\lambda \Lambda(t)$,

$$z\partial_x u = \frac{r^{\lambda-2}}{3}\big[\lambda t\Lambda(t) - t^2(1+t^2)\partial_t\Lambda(t)\big] \tag{2.29}$$

and

$$\begin{aligned}\partial_{zz}u &= \left[\frac{t}{(1+t^2)^{\frac{1}{2}}}\partial_r + \frac{(1+t^2)^{\frac{1}{2}}}{r}\partial_t\right]\left(r^{\lambda-1}\left(\frac{\lambda t\Lambda(t)}{(1+t^2)^{\frac{1}{2}}} + (1+t^2)^{\frac{1}{2}}\partial_t\Lambda(t)\right)\right)\\ &= r^{\lambda-2}\Bigg[(\lambda-1)\left(\frac{\lambda t^2}{1+t^2}\Lambda(t) + t\partial_t\Lambda(t)\right)\\ &\qquad + (1+t^2)^{\frac{1}{2}}\partial_t\left(\frac{\lambda t}{(1+t^2)^{\frac{1}{2}}}\Lambda(t) + (1+t^2)^{\frac{1}{2}}\partial_t\Lambda(t)\right)\Bigg].\end{aligned} \tag{2.30}$$

We are now ready to construct solutions to (2.22) using these coordinates.

Proposition 2.21. *For every $k \in \mathbb{Z}$, equation (2.22) has a solution of the form*

$$v_k := r^{\frac{1}{2}+3k}\Lambda_k(t)$$

with the variables (r,t) of (2.23) and $\Lambda_k \in C^\infty(\mathbb{R};\mathbb{R})$ is a smooth bounded function satisfying $\Lambda_k(-\infty) = 1$ and $\Lambda_k(+\infty) = 0$. The profile Λ_0 is presented in Figure 1.2.

Proof. By separation of variables, we look for a solution to (2.22) under the form $u := r^\lambda\Lambda(t)$, where $\lambda \in \mathbb{R}$ and $\Lambda : \mathbb{R} \to \mathbb{R}$ is a smooth function. The boundary condition $u(0,z) = 0$ for $z > 0$ translates to $\Lambda(+\infty) = 0$. From (2.29) and (2.30) above, one checks that such a u satisfies $z\partial_x u - \partial_{zz}u = 0$ if and only if

$$\partial_t^2\Lambda(t) + \left(\frac{t^2}{3} + \frac{2\lambda t}{1+t^2}\right)\partial_t\Lambda(t) + \lambda\left(-\frac{1}{3}\frac{t}{1+t^2} + \frac{1+(\lambda-1)t^2}{(1+t^2)^2}\right)\Lambda(t) = 0. \tag{2.31}$$

To absorb the $(1+t^2)$ factors, we let $H(t) := (1+t^2)^{\frac{\lambda}{2}}\Lambda(t)$. Then, Λ satisfies (2.31) if and only if H is a solution to

$$\partial_t^2 H(t) + \frac{t^2}{3}\partial_t H(t) - \frac{\lambda t}{3}H(t) = 0. \tag{2.32}$$

Moreover, for $t \neq 0$, using the change of variable $\zeta := -t^3/9$, and looking for $H(t) =: W(-t^3/9)$, we obtain that H solves (2.32) on $\mathbb{R}\setminus\{0\}$ if and only if W is a solution to

$$\zeta\partial_\zeta^2 W(\zeta) + \left(\frac{2}{3} - \zeta\right)\partial_\zeta W(\zeta) - \left(-\frac{\lambda}{3}\right)W(\zeta) = 0, \tag{2.33}$$

which corresponds to Kummer's equation, with $a = -\frac{\lambda}{3}$ and $b = \frac{2}{3}$. It is known (see [50, Section 13.2]) that (2.33) has a unique solution behaving like ζ^{-a} as $\zeta \to \infty$. This (complex valued) solution is usually denoted by $U(a,b,\zeta)$ and called *confluent hypergeometric function of the second kind*, or *Tricomi's function*. In general, U has a branch point at $\zeta = 0$. More precisely, the asymptotic ζ^{-a} holds in the region $|\arg\zeta| < \frac{3\pi}{2}$ and the principal branch of $U(a,b,\zeta)$ corresponds to the principal value of ζ^{-a}. Moreover, when b is not an integer, which is our case, one has (see [50, equation (13.2.42)]),

$$U(a,b,\zeta) = \frac{\Gamma(1-b)}{\Gamma(a-b+1)}M(a,b,\zeta) + \frac{\Gamma(b-1)}{\Gamma(a)}\zeta^{1-b}M(a-b+1,2-b,\zeta), \tag{2.34}$$

where M is the *confluent hypergeometric function of the first kind* or *Kummer's function*,

$$M(a,b,\zeta) := \sum_{n\in\mathbb{N}} \frac{(a)_n}{(b)_n}\frac{\zeta^n}{n!},$$

where $(a)_n$ and $(b)_n$ denote the rising factorial. In particular, M is an entire function of ζ. From (2.34), we see that the singularity in Tricomi's function U stems from the fractional power $\zeta^{1-b} = \zeta^{\frac{1}{3}}$. When $\zeta = -\rho$ (for $\rho > 0$), $\zeta^{\frac{1}{3}} = e^{\frac{i\pi}{3}}\rho^{\frac{1}{3}}$.

We therefore set

$$W(\zeta) := \Re\left\{e^{\frac{i\pi}{3}}U\left(-\frac{\lambda}{3}, \frac{2}{3}, \zeta\right)\right\}. \tag{2.35}$$

By linearity, W is still a solution to (2.33). Moreover, by [50, equation (13.7.3)], as $\zeta \to \infty$,

$$W(\zeta) = \Re\left\{e^{\frac{i\pi}{3}}\zeta^{-a}\left(1 + O\left(\frac{1}{|\zeta|}\right)\right)\right\}. \tag{2.36}$$

In particular, when $\lambda = \frac{1}{2} + 3k$ for $k \in \mathbb{Z}$ (and only in this situation), as $\rho \to +\infty$,

$$W(-\rho) = O(\rho^{-a-1}),$$

because $\Re\{e^{i\pi/3}e^{-ia\pi}\rho^{-a}\} = \Re\{(-1)^k e^{i\pi/3}e^{i\pi/6}\rho^{-a}\} = (-1)^k\rho^{-a}\Re\{i\} = 0$. Defining $H(t) := W(-t^3/9)$ for W as in (2.35) and recalling that $\Lambda(t) = (1+t^2)^{-\lambda/2}H(t)$ implies that $\Lambda(+\infty) = 0$. Indeed, as $t \to +\infty$,

$$\Lambda(t) = (1+t^2)^{-\frac{\lambda}{2}}O(t^{3(\frac{\lambda}{3}-1)}) = O(t^{-3}). \tag{2.37}$$

Moreover, from (2.36), we obtain that Λ is bounded as $t \to -\infty$. Indeed, as $t \to -\infty$,

$$\begin{aligned}\Lambda(t) &= (1+t^2)^{-\frac{\lambda}{2}}\Re\left\{e^{\frac{i\pi}{3}}\left(-\frac{t^3}{9}\right)^{-a}\left(1+O\left(\frac{1}{|t|^3}\right)\right)\right\}\\ &= \frac{1}{2}9^{-\frac{1}{6}-k}(1+t^2)^{-\frac{\lambda}{2}}|t|^{-3a}(1+O(|t|^{-3})) = \frac{1}{2}9^{-\frac{1}{6}-k} + O(|t|^{-2}).\end{aligned} \tag{2.38}$$

Eventually, let us check that H is an entire function of t, which will entail that Λ is smooth. First, note that $M(-\lambda/3, 2/3, -t^3/9)$ and $M(-(\lambda-1)/3, 4/3, -t^3/9)$ are real valued and entire functions of t. Additionally,

$$\begin{aligned}\Re\left\{e^{i\pi/3}\left(-\frac{t^3}{9}\right)^{1/3}\right\} &= 9^{-1/3}\times\begin{cases}\frac{|t|}{2} & \text{if } t<0,\\ t\Re(e^{2i\pi/3}) & \text{if } t>0\end{cases}\\ &= -\frac{1}{2}\frac{t}{9^{1/3}}.\end{aligned}$$

Using (2.35) and (2.34), we obtain

$$\begin{aligned}H(t) = {}&\frac{1}{2}\frac{\Gamma(1-b)}{\Gamma(a-b+1)}M\left(a,b,-\frac{t^3}{9}\right)\\ &-\frac{1}{2}\frac{t}{9^{1/3}}\frac{\Gamma(b-1)}{\Gamma(a)}M\left(a-b+1,2-b,-\frac{t^3}{9}\right),\end{aligned} \tag{2.39}$$

so that H is entire because M is. This entails that H solves (2.32) even across $t = 0$. Moreover, (2.37) and (2.38) imply that Λ is bounded on $\mathbb{R}$. Eventually, using (2.38), we can define Λ_k as $2\cdot 9^{\frac{1}{6}+k}\Lambda$, which ensures that $\Lambda_k(-\infty) = 1$. For this normalization, one deduces from (2.39) that

$$\Lambda_k(0) = 9^{\frac{1}{6}+k}\frac{\Gamma(1/3)}{\Gamma(1/6-k)}, \tag{2.40}$$

which will be used below. ■

If u is a solution to (2.22), then, formally, $\partial_x u$ too (the operator $z\partial_x - \partial_{zz}$ commutes with ∂_x, and the boundary condition at $x = 0$ and $z > 0$ is satisfied thanks to the equation). This property entails that the solutions $v_k = r^{\frac{1}{2}+3k}\Lambda_k(t)$ are related by a recurrence relation on the profiles Λ_k.

Lemma 2.22 (Recurrence relations). *Let $k \in \mathbb{Z}$ and $c_k := \frac{1}{4} - 9k^2$. One has*

$$\partial_x v_k = c_k v_{k-1}. \tag{2.41}$$

Moreover, for every $t \in \mathbb{R}$,

$$c_k \Lambda_{k-1}(t) = \frac{(1+t^2)^{\frac{1}{2}}}{3}\left(\left(\frac{1}{2}+3k\right)\Lambda_k(t) - t(1+t^2)\Lambda_k'(t)\right), \tag{2.42}$$

or, equivalently,

$$\Lambda_k'(t) = \frac{1}{t(1+t^2)}\left(\left(\frac{1}{2}+3k\right)\Lambda_k(t) - \frac{3c_k\Lambda_{k-1}(t)}{(1+t^2)^{\frac{1}{2}}}\right). \tag{2.43}$$

Proof. By (2.27), one has $\partial_x v_k = r^{\frac{1}{2}+3(k-1)} H_k(t)$, where $H_k(t)$ is the right-hand side of (2.42). Thus $\partial_x v_k$ is a solution to (2.22) of the form studied in Proposition 2.21. Since the proof of Proposition 2.21 proceeds by equivalence, v_{k-1} is the only solution of the form $r^{\frac{1}{2}+3(k-1)}$. This entails that $H_k(t)$ is proportional to $\Lambda_{k-1}(t)$ and the constant can be identified by comparing the values at 0 using (2.40), yielding (2.42), (2.41) and (2.43) (which are all equivalent) with $c_k = \frac{1}{4} - 9k^2$.

Actually, these identities are linked with recurrence relations on Tricomi's function U. Let us give another proof of (2.43) using this approach. By the proof of Proposition 2.21,

$$\Lambda_k(t) = 2\cdot 9^{\frac{1}{6}+k}(1+t^2)^{-\frac{1}{4}-\frac{3}{2}k}\cdot\Re\left\{e^{\frac{i\pi}{3}}U\left(-\frac{1}{6}-k,\frac{2}{3},-\frac{t^3}{9}\right)\right\}.$$

First, using the relation $\partial_\zeta U(a-1,b,\zeta) = (1-a)U(a,b+1,\zeta)$ (see [50, equation (13.3.22)]),

$$\begin{aligned}\Lambda_k'(t) = -\left(\frac{1}{2}+3k\right)\frac{t}{1+t^2}\Lambda_k(t) + 2\cdot 9^{\frac{1}{6}+k}(1+t^2)^{-\frac{1}{4}-\frac{3}{2}k}\\ \cdot\frac{3}{t}\left(k+\frac{1}{6}\right)\Re\left\{e^{\frac{i\pi}{3}}\left(-\frac{t^3}{9}\right)U\left(-\frac{1}{6}-k+1,\frac{5}{3},-\frac{t^3}{9}\right)\right\}.\end{aligned}$$

Eventually, (2.43) follows from $(b-a)U(a,b,\zeta) + U(a-1,b,\zeta) - \zeta U(a,b+1,\zeta) = 0$ (see [50, equation (13.3.10)]). ∎

Remark 2.23. We will see below that v_0 is linked with two solutions to (2.1) which have Z^0 regularity, but do not belong to $H_x^1 H_z^1$. Similarly, for each $k \geq 0$, v_k is linked with solutions u such that $\partial_x^k u \in Z^0(\Omega)$ but $u \notin H_x^{k+1} H_z^1$. Conversely, for $k = -1$, one could expect to be able to construct a very weak solution u based on v_{-1} which would entail that uniqueness fails for solutions with less than $L_x^2 H_z^1$ regularity.

Lemma 2.22 entails the following decay estimates, which will be useful in the sequel.

Lemma 2.24. *For every $k \in \mathbb{Z}$, there exists $C_k > 0$ such that, for every $t \in \mathbb{R}$,*

$$|\Lambda_k(t)| + |t^3 \Lambda_k'(t)| + |t^4 \Lambda_k''(t)| + |t^5 \Lambda_k'''(t)| \leq C_k.$$

Proof. For all $k \in \mathbb{Z}$, the bound $|\Lambda_k(t)| \leq C_k$ is already contained in Proposition 2.21 which claims that Λ_k is bounded. Since Λ_{k-1} and Λ_k are uniformly bounded over $\mathbb{R}$, we deduce from (2.43) that $t^3 \Lambda_k'(t)$ is also bounded on $\mathbb{R}$. Eventually, differentiating (2.43) with respect to t leads to a uniform bound for $|t^4 \Lambda_k''(t)|$ and $|t^5 \Lambda_k'''(t)|$ over $\mathbb{R}$. ■

Moreover, the recurrence relations of Lemma 2.22 also imply that the solutions v_k to (2.22) are smooth, up to the boundary $\{x = 0\}$, except at the origin $(0, 0)$.

Lemma 2.25. *For every $k \in \mathbb{Z}$, $v_k \in C^\infty(P_*)$, where $P_* := ([0, +\infty) \times \mathbb{R}) \setminus \{(0, 0)\}$.*

Proof. The smoothness inside the half-plane $\{x > 0\}$ follows directly from Proposition 2.21 since $\Lambda_k \in C^\infty(\mathbb{R})$ and the function $r \mapsto r^{\frac{1}{2}+3k}$ as well as the change of coordinates of (2.23) are smooth inside this domain.

By Proposition 2.21, since Λ_k is continuous on $\mathbb{R}$ and has limits at $t = \pm\infty$, we obtain that $v_k = r^{\frac{1}{2}+3k} \Lambda_k(t)$ is continuous up to the boundary $\{x = 0\}$, except at the origin: $v_k \in C^0(P_*)$.

We now turn to the continuity of derivatives. Using (2.28), we obtain

$$\partial_z v_k = r^{-\frac{1}{2}+3k} \left[\left(\frac{1}{2} + 3k \right) \frac{t}{(1+t^2)^{\frac{1}{2}}} \Lambda_k(t) + (1+t^2)^{\frac{1}{2}} \Lambda_k'(t) \right].$$

Since Λ_k has limits at $t = \pm\infty$ and since, by Lemma 2.24, $t^3 \Lambda_k'(t) = O(1)$, we obtain that $\partial_z v_k$ has limits at $t = \pm\infty$. Hence $\partial_z v_k \in C^0(P_*)$.

Eventually, the $C^\infty(P_*)$ regularity follows from an induction argument. Indeed, by (2.41), $\partial_x v_k = c_k v_{k-1}$, so $\partial_x v_k \in C^0(P_*)$ because $v_{k-1} \in C^0(P_*)$. And, similarly, in the vertical direction, using (2.22), $\partial_{zz} v_k = z \partial_x v_k = z c_k v_{k-1}$ so $\partial_{zz} v_k \in C^0(P_*)$. Iterating the argument concludes the proof. ■

2.4.2 Localization and decomposition

We now introduce singular profiles $\bar{u}^i_{\text{sing}}$, for $i = 0, 1$, localized in the vicinity of $(x_i, 0)$ and based on the singular profiles of the previous paragraph. Let $\chi_i \in C^\infty(\bar{\Omega})$ be a cut-off function such that $\chi_i \equiv 1$ in a neighborhood of $(x_i, 0)$, and $\operatorname{supp} \chi \subset B((x_i, 0), \bar{R})$ for some $\bar{R} < \min(1, x_1 - x_0)/2$. These localized profiles are the ones involved in the main decomposition result of Theorem 2.

Definition 2.26. For $i \in \{0, 1\}$, let

$$\bar{u}^i_{\text{sing}}(x, z) := r_i^{\frac{1}{2}} \Lambda_0(t_i) \chi_i(x, z),$$

where Λ_0 is constructed in Proposition 2.21 and

$$r_i := (z^2 + |x - x_i|^{\frac{2}{3}})^{\frac{1}{2}} \quad \text{and} \quad t_i := (-1)^i z|x - x_i|^{-\frac{1}{3}}.$$

Lemma 2.27. *For $i \in \{0, 1\}$, there exists $\overline{f_i} \in C^\infty(\overline{\Omega})$, with $\overline{f_i} \equiv 0$ in neighborhoods of $(x_i, 0)$ and $\{z = \pm 1\}$, such that $\bar{u}^i_{\text{sing}}$ is the unique solution with $Z^0(\Omega)$ regularity to*

$$\begin{cases} z\partial_x \bar{u}^i_{\text{sing}} - \partial_{zz}\bar{u}^i_{\text{sing}} = \overline{f_i}, \\ \bar{u}^i_{\text{sing}|\Sigma_0\cup\Sigma_1} = 0, \\ \bar{u}^i_{\text{sing}|z=\pm 1} = 0. \end{cases} \tag{2.44}$$

Moreover, $\bar{u}^i_{\text{sing}} \in C^\infty(\overline{\Omega} \setminus \{(x_i, 0)\})$ but $\bar{u}^i_{\text{sing}} \notin H^1_x H^1_z$.

Proof. By symmetry, we only prove the statement for $\overline{f_0}$ and $\bar{u}^0_{\text{sing}}$. In order to alleviate the notation, we drop the index 0 in r_0, t_0 and χ_0. We introduce positive numbers $0 < r_- < r_+$ such that $\chi \equiv 1$ for $r \le r_-$ and $\chi \equiv 0$ for $r \ge r_+$. In particular, all derivatives of χ are smooth, bounded, and supported in $\mathbf{1}_{r_-<r<r_+}$.

Straightforward computations lead to (2.44), provided that one defines

$$\overline{f_0} := r^{\frac{1}{2}}\Lambda_0(t)(z\partial_x\chi - \partial_{zz}\chi) - 2\partial_z(r^{\frac{1}{2}}\Lambda_0(t))\partial_z\chi = v_0(z\partial_x\chi - \partial_{zz}\chi) - 2\partial_z v_0 \partial_z\chi. \tag{2.45}$$

Since the derivatives of χ are supported away from the point $(x_0, 0)$, the $C^\infty(\overline{\Omega})$ regularity of $\overline{f_0}$ follows directly from the smoothness of v_0 away from the origin proved in Lemma 2.25. Since $\bar{u}^0_{\text{sing}}(x, z) = v_0(x, z)\chi(x, z)$, the $C^\infty(\overline{\Omega} \setminus \{(x_i, 0)\})$ regularity of $\bar{u}^0_{\text{sing}}$ follows from Lemma 2.25.

Therefore, to prove the lemma, there remains to prove that $\bar{u}^0_{\text{sing}}$, $\partial_{zz}\bar{u}^0_{\text{sing}}$ and $z\partial_x\bar{u}^0_{\text{sing}}$ are in $L^2(\Omega)$ but $\partial_x\partial_z\bar{u}^0_{\text{sing}} \notin L^2(\Omega)$. We will use the change of coordinates from cartesian to polar-like ones (see (2.23)), whose Jacobian is given by equation (2.26), so that, for $\varphi : \Omega \to \mathbb{R}$,

$$\|\varphi\|^2_{L^2(\Omega)} = \int_0^\infty \int_{\mathbb{R}} \frac{3r^3}{(1+t^2)^2}\varphi(r, t)^2 \,\mathrm{d}t\,\mathrm{d}r.$$

In particular, we have the following integrability criterion. Assume that φ is of the form $r^\mu H(t)\psi$, where $H(t) = O_{t\to\pm\infty}(|t|)$ and $\operatorname{supp}\psi \subset \mathbf{1}_{r<r_+}$. If $\mu > -2$ or $\operatorname{supp}\psi \subset \mathbf{1}_{r_-<r}$, then $\varphi \in L^2(\Omega)$.

Step 1. Preliminary estimates. Let ψ such that $\operatorname{supp}\psi \subset \mathbf{1}_{r<r_+}$. By the previous integrability criterion, since $\Lambda_0(t) = O(1)$, $r^{\frac{1}{2}}\Lambda_0(t)\psi \in L^2(\Omega)$. Using (2.28), we obtain

$$\partial_z(r^{\frac{1}{2}}\Lambda_0(t)) = r^{-\frac{1}{2}}\left[\frac{t}{2(1+t^2)^{\frac{1}{2}}}\Lambda_0(t) + (1+t^2)^{\frac{1}{2}}\Lambda_0'(t)\right].$$

By Lemma 2.24, $|t|\Lambda_0'(t) = O(|t|^{-2})$. Thus, $\partial_z(r^{\frac{1}{2}}\Lambda_0(t))\psi \in L^2(\Omega)$. Using (2.28) again,

$$\partial_{zz}(r^{\frac{1}{2}}\Lambda_0(t)) = -\frac{1}{2}r^{-\frac{3}{2}}\frac{t}{(1+t^2)^{\frac{1}{2}}}\left[\frac{t}{2(1+t^2)^{\frac{1}{2}}}\Lambda_0(t) + (1+t^2)^{\frac{1}{2}}\Lambda_0'(t)\right] + r^{-\frac{3}{2}}(1+t^2)^{\frac{1}{2}}\partial_t\left[\frac{t}{2(1+t^2)^{\frac{1}{2}}}\Lambda_0(t) + (1+t^2)^{\frac{1}{2}}\Lambda_0'(t)\right].$$

Using (2.27), we obtain

$$\partial_x(r^{\frac{1}{2}}\Lambda_0(t)) = r^{-\frac{5}{2}}\left[\frac{(1+t^2)^{\frac{1}{2}}}{6}\Lambda_0(t) - \frac{t(1+t^2)^{\frac{3}{2}}}{3}\Lambda_0'(t)\right].$$

By Lemma 2.24, $\Lambda_0'(t) = O(|t|^{-3})$. Hence, $|t|\Lambda_0(t) = O(|t|)$ and $|t|^4\Lambda_0'(t) = O(|t|)$ so, assuming that supp $\psi \subset \mathbf{1}_{r_-<r<r_+}$, one concludes that $\partial_x(r^{\frac{1}{2}}\Lambda_0(t))\psi \in L^2(\Omega)$.

Eventually, using (2.28), we obtain

$$\partial_{xz}(r^{\frac{1}{2}}\Lambda_0(t)) = r^{-\frac{7}{2}}\left(-\frac{t}{4}\Lambda_0(t) - \frac{1}{6}\Lambda_0'(t)(1+t^2)(1+3t^2) - \frac{t}{3}(1+t^2)^2\Lambda_0''(t)\right). \tag{2.46}$$

Step 2. Z^0 estimates on $\bar{u}^0_{\text{sing}}$. By Step 1, $\bar{u}^0_{\text{sing}}$ and $\partial_{zz}\bar{u}^0_{\text{sing}}$ belong to $L^2(\Omega)$. Since $z\partial_x\bar{u}^0_{\text{sing}} = \overline{f_0} + \partial_{zz}\bar{u}^0_{\text{sing}}$ and $\overline{f_0} \in L^2(\Omega)$, we infer that $z\partial_x\bar{u}^0_{\text{sing}} \in L^2(\Omega)$. Hence $\bar{u}^0_{\text{sing}} \in Z^0(\Omega)$.

Step 3. Lack of $H^1_x H^1_z$ estimate for $\bar{u}^0_{\text{sing}}$. Recalling (2.46),

$$\partial_x\partial_z\bar{u}^0_{\text{sing}} = r^{-\frac{7}{2}}h(t)\chi + \partial_x(r^{\frac{1}{2}}\Lambda_0(t))\partial_z\chi + \partial_z(r^{\frac{1}{2}}\Lambda_0(t)\partial_x\chi), \tag{2.47}$$

where, by (2.46), the function h is given by

$$h(t) = -\frac{t}{4}\Lambda_0(t) - \frac{1}{6}\Lambda_0'(t)(1+t^2)(1+3t^2) - \frac{t}{3}(1+t^2)^2\Lambda_0''(t).$$

Using (2.43) together with the relation $\Lambda_k(-\infty) = 9^{-\frac{1}{6}-k}/2$, we find that as $t \to -\infty$

$$\Lambda_0(t) = a + \frac{b}{t^2} + \frac{c}{t^3} + O(t^{-4}), \quad \Lambda_0'(t) = -\frac{2b}{t^3} - \frac{3c}{t^4} + O(t^{-5}),$$
$$\Lambda_0''(t) = \frac{6b}{t^4} + \frac{12c}{t^5} + O(t^{-6}),$$

where the coefficients a, b, c are defined by $a = \Lambda_0(-\infty)$, $-2b = a/2$, $-3c = 3c_0\Lambda_{-1}(-\infty)$. We infer that as $t \to -\infty$,

$$h(t) = \frac{3c}{2} - \frac{12c}{3} + O(t^{-1}) \sim \frac{5}{2}c_0\Lambda_{-1}(-\infty) \neq 0.$$

Hence $h \neq 0$.

The last two terms on the right-hand side of (2.47) belong to $L^2(\Omega)$ according to the previous computations. Since $h \neq 0$, the L^2 norm of the first term is bounded from below by

$$c \int_0^{r_-} r^{-7} r^3 \, \mathrm{d}r = +\infty$$

and thus $\partial_x \partial_z \bar{u}^0_{\mathrm{sing}} \notin L^2(\Omega)$. ■

Actually, we have the following regularity on the profiles $\bar{u}^i_{\mathrm{sing}}$, which is slightly better than Z^0.

Lemma 2.28. *For all $\sigma < \frac{1}{2}$, $\bar{u}^i_{\mathrm{sing}} \in H_x^{\frac{2+\sigma}{3}} L^2_z \cap L^2_x H_z^{2+\sigma} \hookrightarrow H_x^{\frac{2+\sigma}{3}} L^2_z \cap H_x^{\frac{\sigma}{3}} H^2_z$ and this is optimal. More precisely, $\bar{u}^i_{\mathrm{sing}} \notin H_x^{\frac{5}{6}} L^2_z \cap H_x^{\frac{1}{6}} H^2_z$.*

Proof. The proof follows from an easy scaling argument. We start with the z derivative and focus on $\bar{u}^0_{\mathrm{sing}}$. Dropping the index 0 in r_0 and t_0 as in the previous proof, we have, using (2.23) and Definition 2.26, and setting $\chi(x,z) := \chi_0(x_0 + x, z)$,

$$\bar{u}^0_{\mathrm{sing}}(x_0 + x, z) = x^{\frac{1}{6}} \varphi\left(\frac{z}{x^{\frac{1}{3}}}\right) \chi(x,z),$$

where $\varphi(t) = (1+t^2)^{\frac{1}{4}} \Lambda_0(t)$. Therefore,

$$\partial_z^2 \bar{u}^0_{\mathrm{sing}}(x_0 + x, z) = x^{-\frac{1}{2}} \varphi''\left(\frac{z}{x^{\frac{1}{3}}}\right) \chi + 2x^{-\frac{1}{6}} \varphi'\left(\frac{z}{x^{\frac{1}{3}}}\right) \chi_z + x^{\frac{1}{6}} \varphi\left(\frac{z}{x^{\frac{1}{3}}}\right) \chi_{zz}.$$

We focus on the regularity of the first term, which is the most singular. We have, for any $\sigma > 0$,

$$\left\| x^{-\frac{1}{2}} \varphi''\left(\frac{z}{x^{\frac{1}{3}}}\right) \chi(x,z) \right\|^2_{L^2_x H^\sigma_z} \leq \int \frac{1}{x} \frac{1}{|z-z'|^{1+2\sigma}} \left(\varphi''\left(\frac{z}{x^{\frac{1}{3}}}\right) - \varphi''\left(\frac{z'}{x^{\frac{1}{3}}}\right) \right)^2 \mathrm{d}x \, \mathrm{d}z \, \mathrm{d}z'.$$

Changing variables in the above integral, we get

$$\left\| x^{-\frac{1}{2}} \varphi''\left(\frac{z}{x^{\frac{1}{3}}}\right) \chi(x,z) \right\|^2_{L^2_x H^\sigma_z} \lesssim \|\varphi''\|^2_{H^\sigma(\mathbb{R})} \int_0^{x_1 - x_0} |x|^{-\frac{2}{3} - \frac{2\sigma}{3}} \, \mathrm{d}x.$$

The integral on the right-hand side is finite if and only if $\sigma < \frac{1}{2}$. Moreover,

$$\|\varphi''\|^2_{H^\sigma(\mathbb{R})} \leq \|\varphi''\|^2_{H^1(\mathbb{R})}.$$

From the definition of φ and the decay bounds of Lemma 2.24, we infer that $\varphi'' \in H^1(\mathbb{R})$. This shows that $\bar{u}^i_{\mathrm{sing}} \in L^2_x H_z^{2+\sigma}$ for $\sigma < \frac{1}{2}$.

The bound in $H_x^{\frac{2+\sigma}{3}} L_z^2$ is obtained similarly and left to the reader.

Conversely, if one had $\bar{u}^i_{\text{sing}} \in H_x^{\frac{5}{6}} L_z^2 \cap H_x^{\frac{1}{6}} H_z^2$, by the fractional trace theorem [42, equation (4.7), Chapter 1], one would have $\bar{u}^i_{\text{sing}} \in C_z^0(H_x^{2/3})$. In particular, $\bar{u}^i_{\text{sing}}(\cdot, 0) \in H^{2/3}(x_0, x_1)$. But, in a neighborhood of $x = 0$, $\bar{u}^i_{\text{sing}}(x_0 + x, 0) = \Lambda_0(0)x^{\frac{1}{6}}$ with $\Lambda_0(0) \neq 0$. One checks that $x \mapsto x^{\frac{1}{6}} \in H^s(0,1)$ if and only if $s < 2/3$, which completes the proof. ■

Eventually, we introduce the following 2×2 nonsingular matrix which translates the fact that $\bar{u}^0_{\text{sing}}$ and $\bar{u}^1_{\text{sing}}$ are indeed independent elementary solutions related with the non-satisfaction of the orthogonality constraints associated with $\overline{\ell^0}$ and $\overline{\ell^1}$. We will use this reference matrix multiple times in the sequel for perturbations of this shear flow situation.

Lemma 2.29. *Let $\overline{f_0}$, $\overline{f_1}$ as in Lemma 2.27 and $\overline{\Phi^0}$, $\overline{\Phi^1}$ as in Lemma 2.7. The matrix*

$$\bar{M} := \left(\int_\Omega \partial_x \overline{f_j} \, \overline{\Phi^i} \right)_{0 \le i,j \le 1} \in \mathcal{M}_2(\mathbb{R})$$

is invertible.

Proof. Let $c \in \mathbb{R}^2$ such that $\bar{M} c = 0$. Then, for $j = 0, 1$,

$$\int_\Omega \partial_x (c_0 \overline{f_0} + c_1 \overline{f_1}) \overline{\Phi^j} = 0.$$

Thus, the source term for the function $c_0 \bar{u}^0_{\text{sing}} + c_1 \bar{u}^1_{\text{sing}}$ satisfies the orthogonality conditions (2.15) (note that in this case, the boundary data are null). It then follows from Proposition 2.10 that $c_0 \bar{u}^0_{\text{sing}} + c_1 \bar{u}^1_{\text{sing}} \in H^1_x H^1_z$. Localizing in the vicinity of $(x_i, 0)$, we infer that $c_i \bar{u}^i_{\text{sing}} \in H^1_x H^1_z$, which, since $\bar{u}^i_{\text{sing}} \notin H^1_x H^1_z$ (by Lemma 2.27), implies that $c_i = 0$. Therefore, $c = 0$ and $\bar{M}$ is invertible. ■

Corollary 2.30 (Decomposition into singular profiles). *Let $(f, \delta_0, \delta_1) \in \mathcal{H}_K$ and $u \in Z^0(\Omega)$ be the unique solution to (2.1). Then there exists two real constants c_0, c_1 and a function $u_{\text{reg}} \in Z^1(\Omega)$, as defined in (1.24), such that*

$$u = c_0 \bar{u}^0_{\text{sing}} + c_1 \bar{u}^1_{\text{sing}} + u_{\text{reg}}.$$

Proof. We recall the definition of the matrix $\bar{M}$ from Lemma 2.29. Since $\bar{M}$ is invertible, we may define $c = (c_0, c_1)$ such that

$$\bar{M} c = \begin{pmatrix} \overline{\ell^0}(f, \delta_0, \delta_1) \\ \overline{\ell^1}(f, \delta_0, \delta_1) \end{pmatrix}. \tag{2.48}$$

Let $\overline{f_0}$ and $\overline{f_1}$ as in Lemma 2.27. By construction, the triplet $(f - c_0\overline{f_0} - c_1\overline{f_1}, \delta_0, \delta_1)$ satisfies the orthogonality conditions from Proposition 2.10. It follows that the solution u_{reg} to

$$\begin{cases} z\partial_x u_{\text{reg}} - \partial_{zz} u_{\text{reg}} = f - c_0\overline{f_0} - c_1\overline{f_1} & \text{in } \Omega, \\ u_{\text{reg}|\Sigma_i} = \delta_i, \\ u_{\text{reg}|z=\pm 1} = 0 \end{cases}$$

satisfies $u_{\text{reg}} \in H^1_x(H^1_z)$. Thus, estimate (2.10) of Proposition 2.6 ensures that $\partial_x u_{\text{reg}} \in Z^0(\Omega)$, i.e., $u_{\text{reg}} \in Z^1$. Now, u and $u_{\text{reg}} + c_0\bar{u}^0_{\text{sing}} + c_1\bar{u}^1_{\text{sing}}$ both belong to $Z^0(\Omega)$ and satisfy the system (2.1). By the uniqueness result of Proposition 2.2, the result follows. ∎

Theorem 2 follows easily from Corollary 2.30. Indeed, one easily checks from (1.3) and (2.13) that $\mathcal{X}_B \hookrightarrow \mathcal{H}_K$. Moreover, by Proposition 1.7, $Z^0 \hookrightarrow H^{2/3}_x L^2_z \cap L^2_x H^2_z$ and, by Lemma 1.15, $Z^1 \hookrightarrow Q^1$ (defined in (1.5)). The rest of the conclusions on $\bar{u}^i_{\text{sing}}$ are derived in Lemma 2.27.

Remark 2.31. The constants c_0, c_1 from Corollary 2.30 depend (linearly) on u, but do not depend on the choice of the truncation functions χ_i. Indeed, if χ'_0, χ'_1 is another truncation, associated with constants c'_0, c'_1, then applying Corollary 2.30 twice yields

$$c_0\bar{u}^0_{\text{sing}} + c_1\bar{u}^1_{\text{sing}} - c'_0(\bar{u}^0_{\text{sing}})' - c'_1(\bar{u}^1_{\text{sing}})' \in Z^1.$$

Therefore, in a small neighborhood $V_i = \chi_i^{-1}(\{1\}) \cap (\chi'_i)^{-1}(\{1\})$ of $(x_i, 0)$, we obtain

$$(c_i - c'_i) r_i^{\frac{1}{2}} \Lambda_0(t_i) \in H^1_x H^1_z(V_i),$$

and therefore $c_i = c'_i$.

As already claimed in Remark 2.8, we can also prove a related decomposition result for the dual profiles $\overline{\Phi^j}$ defined in Lemma 2.7. Here, the decomposition *always* involves a singular part.

Corollary 2.32. *Let* $(c_0, c_1) \in \mathbb{R}^2 \setminus \{0\}$. *There exists* $(d_0, d_1) \in \mathbb{R}^2 \setminus \{0\}$ *and* $\Phi_{\text{reg}} \in Z^1$, *as defined in* (1.24), *such that*

$$c_0\overline{\Phi^0} + c_1\overline{\Phi^1} = (-c_0 z + c_1)\mathbf{1}_{z>0}\zeta(z) + d_0\bar{u}^0_{\text{sing}}(x, -z) + d_1\bar{u}^1_{\text{sing}}(x, -z) + \Phi_{\text{reg}}, \tag{2.49}$$

where ζ *is a smooth cut-off function, equal to* 1 *near* $z = 0$ *and compactly supported in* $(-1, 1)$.

Proof. Using the same decomposition as in Lemma 2.7, set

$$\Psi^c := c_0\overline{\Phi^0} + c_1\overline{\Phi^1} - (-c_0 z + c_1)\mathbf{1}_{z>0}\zeta(z).$$

Then $\widetilde{\Psi^c}(x,z) := \Psi^c(x,-z)$ is the solution to

$$\begin{cases} z\partial_x \widetilde{\Psi^c} - \partial_{zz}\widetilde{\Psi^c} = g_c & \text{in } \Omega, \\ \Psi^c(x_0,z) = 0 & \text{for } z \in (0,1), \\ \Psi^c(x_1,z) = (-c_0 z - c_1)\zeta(-z) & \text{for } z \in (-1,0), \\ \Psi^c_{|z=\pm 1} = 0, \end{cases}$$

where $g_c = (c_1\zeta''(-z) - 2c_0\zeta'(-z) + c_0 z\zeta''(-z))\mathbf{1}_{z<0}$. Thus, (2.49) follows from Corollary 2.30, applied with $f = g_c \in C^\infty(\overline{\Omega})$, $\delta_0 = \Delta_0 = 0$ and $\delta_1(z) = (-c_0 z - c_1)\zeta(-z)$ and $\Delta_1 = 0$.

It remains to prove that $(d_0, d_1) \neq (0,0)$. By Proposition 2.10, $\widetilde{\Psi^c} \in H^1_x H^1_z$ if and only if $\overline{\ell^j}(g_c, 0, \delta_1) = 0$ for $j = 0, 1$. By Definition 2.12, since $\partial_x g_c = 0$ and $\Delta_0 = \Delta_1 = 0$,

$$\overline{\ell^j}(g_c, 0, \delta_1) = 0 \iff \partial_z^j \delta_0(0) - \partial_z^j \delta_1(0) = 0.$$

Since $\delta_0 = 0$ and $\delta_1(0) = -c_1$ and $\delta_1'(0) = -c_0$, $(d_0, d_1) = (0,0)$ if and only if $\widetilde{\Psi^c} \in H^1_x H^1_z$, if and only if $(c_0, c_1) = 0$. ∎

Remark 2.33. Using Corollary 2.32 and the regularity result Lemma 2.28 on $\bar{u}^i_{\text{sing}}$, we see that $f \mapsto \overline{\ell^j}(f, 0, 0) = \int_\Omega \partial_x f \overline{\Phi^j}$ is not only continuous on $H^1_x L^2_z$ but also on $H^\sigma_x L^2_z$ for every $\sigma > \frac{1}{6}$. We will encounter a related threshold of tangential regularity in Proposition 2.36.

Using the decomposition of the dual profiles, we can show that the orthogonality conditions are also independent when considering only variations of the inflow boundary data.

Proposition 2.34. *The linear forms $\overline{\ell^0}$ and $\overline{\ell^1}$ are independent on $\{0\} \times C^\infty_c(\Sigma_0) \times C^\infty_c(\Sigma_1)$.*

Proof. By contradiction, let $(c_0, c_1) \in \mathbb{R}^2 \setminus \{0\}$ such that, for every $\delta_0 \in C^\infty_c(\Sigma_0)$ and $\delta_1 \in C^\infty_c(\Sigma_1)$,

$$c_0 \overline{\ell^0}(0, \delta_0, \delta_1) + c_1 \overline{\ell^1}(0, \delta_0, \delta_1) = 0.$$

Let $(d_0, d_1) \in \mathbb{R}^2 \setminus \{0\}$, ζ and $\Phi_{\text{reg}} \in Z^1$ be given by Corollary 2.32. By symmetry, assume that $d_0 \neq 0$. By Definition 2.12, for every $\delta_0 \in C^\infty_c(\Sigma_0)$, letting $\Delta_0(z) := \delta_0''(z)/z$,

$$\begin{aligned} 0 &= c_0 \overline{\ell^0}(0, \delta_0, 0) + c_1 \overline{\ell^1}(0, \delta_0, 0) \\ &= \int_{\Sigma_0} z\Delta_0 \big[(-c_0 z + c_1)\zeta(z) + d_0 \bar{u}^0_{\text{sing}}(x_0, -z) + \Phi_{\text{reg}}(x_0, z)\big] \\ &= \int_{\Sigma_0} \delta_0'' \big[(-c_0 z + c_1)\zeta(z) + d_0 \bar{u}^0_{\text{sing}}(x_0, -z) + \Phi_{\text{reg}}(x_0, z)\big]. \end{aligned}$$

For $\bar{z} > 0$ small enough, on $(0, \bar{z})$, $\zeta = 1$ and $\bar{u}^0_{\text{sing}}(x_0, -z) = z^{\frac{1}{2}} \Lambda_0(-\infty) = z^{\frac{1}{2}}$ (see Definition 2.26 and Proposition 2.21). Since $Z^1 \hookrightarrow H^1_x H^2_z$, $\Phi_{\text{reg}|\Sigma_0} \in H^2(\Sigma_0)$.

If $\operatorname{supp} \delta_0 \subset (0, \bar{z})$, integrating by parts yields

$$0 = d_0 \int_0^1 \left[-\frac{1}{4} z^{-\frac{3}{2}} + \varphi(z) \right] \delta_0(z),$$

where $\varphi(z) := \partial_{zz} \Phi_{\text{reg}}(x_0, z) \in L^2(\Sigma_0)$. Since $z \mapsto z^{-\frac{3}{2}}$ does not belong to $L^2(0, \bar{z})$ but φ does, one easily deduces that there exists $\delta_0 \in C^\infty_c((0, \bar{z}))$ such that the right-hand side is non-zero, reaching a contradiction. ∎

Let us conclude this section with an easy consequence of the decomposition result from Corollary 2.30, which will be used in Section 5.2.

Corollary 2.35 (Single orthogonality condition for localized solutions). *There exists a couple* $(a_0, a_1) \in \mathbb{R}^2 \setminus \{(0, 0)\}$ *such that the following result holds.*

Let $(f, \delta_0, \delta_1) \in \mathcal{H}_K$ *and let* $u \in Z^0(\Omega)$ *be the unique solution to* (2.1). *Assume that there exists* $0 < r < \min(x_1 - x_0, 1)$ *such that* $\operatorname{supp} u \subset B((x_1, 0), r)^c$. *Then* $u \in Z^1(\Omega)$ *if and only if*

$$\left(a_0 \overline{\ell^0} + a_1 \overline{\ell^1}\right)(f, \delta_0, \delta_1) = 0.$$

Proof. Let us choose the cut-off function χ_1 from Definition 2.26 such that $\operatorname{supp} u \cap \operatorname{supp} \chi_1 = \emptyset$. According to Corollary 2.30, there exists $(c_0, c_1) \in \mathbb{R}^2$ and $u_{\text{reg}} \in Z^1(\Omega)$ such that $u = c_0 \bar{u}^0_{\text{sing}} + c_1 \bar{u}^1_{\text{sing}} + u_{\text{reg}}$. Multiplying this identity by χ_1, we infer that $c_1 \bar{u}^1_{\text{sing}} \chi_1 = -u_{\text{reg}} \chi_1 \in Z^1(\Omega)$. Lemma 2.27 then entails that $c_1 = 0$.

Therefore $u \in Z^1(\Omega)$ if and only if $c_0 = 0$. We then recall (2.48), and we denote by (a_0, a_1) the two coefficients in the first line of $\bar{M}^{-1}$. The result follows. ∎

2.5 Interpolation and fractional regularity

For further purposes, we will also need some fractional regularity results. Their proof relies on interpolation arguments, and therefore on the explicit expressions of the singular profiles. Due to a subtle technical difficulty, the proof of these results are postponed to Chapter 6.

Proposition 2.36. *Let* $\sigma \in (0, 1) \setminus \{1/6, 1/2\}$. *Let* $f \in H^\sigma_x L^2_z$, $\delta_0 \in H^2(\Sigma_0)$, $\delta_1 \in H^2(\Sigma_1)$ *such that* $\delta_0(1) = \delta_1(-1) = 0$.

- *If* $\sigma > 1/6$, *assume that* $\overline{\ell^0}(f, \delta_0, \delta_1) = \overline{\ell^1}(f, \delta_0, \delta_1) = 0$.
- *If* $\sigma > 1/2$, *assume also that* $\Delta_i \in \mathcal{H}^1_z(\Sigma_i)$ *and* $\Delta_1(-1) = \Delta_0(1) = 0$ *(recall* (2.8)*).*

The unique strong solution $u \in Z^0(\Omega)$ to (2.1) is in $Z^\sigma(\Omega) := [Z^0(\Omega), Z^1(\Omega)]_\sigma$, with

$$\|u\|_{Z^\sigma} \lesssim \|f\|_{H^\sigma_x L^2_z} + \|\delta_0\|_{H^2} + \|\delta_1\|_{H^2} + \mathbf{1}_{\sigma>1/2}\big(\|\Delta_0\|_{\mathcal{H}^1_z} + \|\Delta_1\|_{\mathcal{H}^1_z}\big). \quad (2.50)$$

Remark 2.37. The case $\sigma = 1/6$ is not covered in the above result. This critical level of regularity corresponds to the maximal continuity of the orthogonality conditions. Such critical levels are excluded from the abstract interpolation results on which we rely (see Lemma 6.2). In this case, one would expect a similar result to hold, but with a supplementary norm on the data, in the spirit of [5, 6]. The case $\sigma = 1/2$ is also excluded, but it would be possible to include it provided one introduces an appropriate additional norm.

Remark 2.38. The regularity assumptions on the δ_i's are not optimal and could be weakened.

We also obtain the following analogue of Corollary 2.35 in fractional regularity.

Corollary 2.39 (Single orthogonality condition for localized solutions in fractional regularity). *Let $(a_0, a_1) \in \mathbb{R}^2 \setminus \{(0,0)\}$ be the couple from Corollary 2.35. Let $\sigma \in (1/6, 1) \setminus \{1/2\}$. Let $f \in H^\sigma_x L^2_z$, $\delta_0 \in H^2(\Sigma_0)$ such that $\delta_0(1) = 0$. For $\sigma > 1/2$, assume also that $\Delta_0 \in \mathcal{H}^1_z(\Sigma_0)$ and $\Delta_0(1) = 0$. Let $u \in Z^0(\Omega)$ be the unique solution to (2.1) associated with $(f, \delta_0, 0)$.*

Assume that there exists $0 < r < \min(x_1 - x_0, 1)$ such that $\operatorname{supp} u \subset B((x_1, 0), r)^c$. Then $u \in Z^\sigma(\Omega)$ if and only if

$$\big(a_0\overline{\ell^0} + a_1\overline{\ell^1}\big)(f, \delta_0, 0) = 0,$$

and in this case

$$\|u\|_{Z^\sigma} \lesssim \|f\|_{H^\sigma_x L^2_z} + \|\delta_0\|_{H^2(\Sigma_0)} + \mathbf{1}_{\sigma>1/2}\|\Delta_0\|_{\mathcal{H}^1_z(\Sigma_0)}.$$

Proof. The proof follows the same structure as the Z^1 case.

Using Proposition 2.36, we first prove an analogue of the decomposition result Corollary 2.30 for source terms $f \in H^\sigma_x L^2_z$ with $\sigma \in (1/6, 1)$, where the conclusion is that $u_{\text{reg}} \in Z^\sigma$.

The conclusion then stems from the fact that $\bar{u}^0_{\text{sing}} \notin Z^\sigma$. Indeed, by Lemma 1.15, for $\sigma \geq 1/6$, $Z^\sigma \hookrightarrow H^{\frac{5}{6}}_x L^2_z \cap H^{\frac{1}{6}}_x H^2_z$. But, from Lemma 2.28, $\bar{u}^0_{\text{sing}} \notin H^{\frac{5}{6}}_x L^2_z \cap H^{\frac{1}{6}}_x H^2_z$. ∎