Chapter 3

A first nonlinear example in kinetic theory

In this chapter, we explain how the linear theory of Chapter 2 can be used in a simple nonlinear context. Before moving on to nonlinear examples from fluid mechanics in Chapters 4 and 5 (which involve additional difficulties), we encourage the reader to start by reading this section, where we set up the basics of our method to construct perturbative solutions to semilinear or quasilinear problems despite orthogonality conditions. In particular, we formulate a black-box abstract result in Section 3.5 which we will use in the sequel.

3.1 Description of the model and main result

As an example, we will show how one can build regular solutions to a stationary nonlinear system of Vlasov–Poisson–Fokker–Planck-type, set on a bounded interval. For the sake of readability, we will focus on the following system:

$$\begin{cases} z\partial_x u + E[u]\partial_z u - \partial_{zz} u = f, \\ u_{|\Sigma_i} = \delta_i, \\ u_{|z=\pm 1} = 0, \end{cases} \tag{3.1}$$

where $E[u]$ is an electric force deriving from a potential $V[u]$ satisfying a Poisson equation:

$$E = \partial_x V, \quad \text{where} \quad \begin{cases} \partial_{xx} V(x) = \int_{-1}^{1} u(x,z)\,\mathrm{d}z \quad \text{for } x \in (x_0, x_1), \\ \partial_x V_{|x=x_0} = 0. \end{cases} \tag{3.2}$$

In this toy model, the term $E[u]\partial_z u$ corresponds to a semilinear contribution, which is easily estimated since explicit integration of (3.2) and the Cauchy–Schwarz inequality yield

$$\|E[u]\|_{L^\infty(x_0,x_1)} \lesssim \|E[u]\|_{H^1(x_0,x_1)} \lesssim \|u\|_{L^2(\Omega)}. \tag{3.3}$$

Remark 3.1. Our toy kinetic model (3.1)–(3.2) departs from classical kinetic models such as the one studied in [32] in the following ways:

- As mentioned before, the variable z is more commonly denoted by v and represents the velocity of the particles. We keep the notation z by consistency with the remainder of the memoir.

- Usually, even if the position variable x lives in a bounded domain, the velocity variable z lives in $\mathbb{R}$ so that particles can take arbitrary speeds. Since our motivation is to understand what happens near the critical line $\{z = 0\}$ we focus here on the region $z \in [-1, 1]$. We expect that our techniques can be applied to the unbounded case to obtain similar results, provided that one works in the appropriate functional spaces to encode decay as $|z| \to \infty$.
- One could also enforce a non-zero Neumann boundary condition for the potential V at the left endpoint $x = x_0$, namely $\partial_x V_{|x=x_0} = g_0 \in \mathbb{R}$ as in [32]. This is a straightforward adaptation of the results presented below.

The goal of the next paragraphs is to prove the following counterparts of Propositions 2.5 and 2.17 concerning the linear model (2.1) for our nonlinear toy model. We will work with the following spaces of data triplets:

$$\mathscr{H}_{FP} := \left\{(f, \delta_0, \delta_1) \in \mathscr{H}_K; \delta_i'(z)/z \in \mathcal{H}_z^1(\Sigma_i) \text{ and } \delta_i'((-1)^i) = 0 \text{ for } i \in \{0, 1\}\right\} \tag{3.4}$$

with the norm

$$\|(f, \delta_0, \delta_1)\|_{\mathscr{H}_{FP}} := \|(f, \delta_0, \delta_1)\|_{\mathscr{H}_K} + \|\delta_0'(z)/z\|_{\mathcal{H}_z^1} + \|\delta_1'(z)/z\|_{\mathcal{H}_z^1}, \tag{3.5}$$

where we recall that the space $\mathcal{H}_z^1$ is defined in (1.22) and the space $\mathscr{H}_K$ in (2.13). We also define

$$\mathscr{H}_{FP,\mathrm{sg}}^{\perp} := \left\{(f, \delta_0, \delta_1) \in \mathscr{H}_{FP}; \overline{\ell^0}(f, \delta_0, \delta_1) = \overline{\ell^1}(f, \delta_0, \delta_1) = 0\right\}.$$

Theorem 5. *There exists a constant $\eta > 0$, and a Lipschitz submanifold $\mathcal{M}_{FP}$ of $\mathscr{H}_{FP}$ of codimension 2, containing 0 and included in the ball of radius η in $\mathscr{H}_{FP}$, modeled on $\mathscr{H}_{FP,\mathrm{sg}}^{\perp}$ and tangent to it at 0 (see Remark 3.12), such that the following statements hold:*

(1) *For all $(f, \delta_0, \delta_1) \in L^2(\Omega) \times \mathcal{H}_z^1(\Sigma_0) \times \mathcal{H}_z^1(\Sigma_1)$ with $\delta_0(1) = \delta_1(-1) = 0$ such that*

$$\|f\|_{L^2} + \|\delta_0\|_{\mathcal{H}_z^1} + \|\delta_1\|_{\mathcal{H}_z^1} \leq \eta, \tag{3.6}$$

system (3.1)–(3.2) has a solution $u \in Z^0(\Omega)$ satisfying

$$\|u\|_{Z^0} \lesssim \|f\|_{L^2} + \|\delta_0\|_{\mathcal{H}_z^1} + \|\delta_1\|_{\mathcal{H}_z^1} \tag{3.7}$$

and which is unique in a neighborhood of 0 in $Z^0(\Omega)$.

(2) *For all $(f, \delta_0, \delta_1) \in \mathscr{H}_{FP}$ such that*

$$\|(f, \delta_0, \delta_1)\|_{\mathscr{H}_{FP}} \leq \eta,$$

the locally unique solution $u \in Z^0(\Omega)$ to (3.1)–(3.2) enjoys $Z^1(\Omega)$ regularity if and only if $(f, \delta_0, \delta_1) \in \mathcal{M}_{FP}$, which corresponds to two nonlinear orthogonality conditions.

For such data, one has

$$\|u\|_{Z^1} \lesssim \|(f, \delta_0, \delta_1)\|_{\mathcal{H}_{FP}}. \tag{3.8}$$

Remark 3.2. The nonlinearity of the Vlasov–Poisson–Fokker–Planck system (3.1) is sufficiently mild to allow for a theory of weak solutions, leading to the first statement of Theorem 5. The Prandtl system in the vicinity of the recirculation zone enjoys the same feature, accounting for the first part of Theorem 4. However, the nonlinearity in the Burgers system (1.1) is stronger, and prevents us from proving the analogue of the first statement of the above theorem.

3.2 Well-posedness theory with low regularity

We prove in this section Item (1) of Theorem 5, which corresponds to the well-posedness theory at regularity Z^0, and is therefore a nonlinear counterpart of Proposition 2.5.

Lemma 3.3 (Existence of Z^0 solutions of (3.1)–(3.2)). *There exists $\eta > 0$ such that, for any $(f, \delta_0, \delta_1) \in L^2(\Omega) \times \mathcal{H}^1_z(\Sigma_0) \times \mathcal{H}^1_z(\Sigma_1)$ with $\delta_0(1) = \delta_1(-1) = 0$ satisfying (3.6), there exists a solution $u \in Z^0(\Omega)$ to (3.1)–(3.2) with (3.7).*

Proof. Let $(f, \delta_0, \delta_1) \in L^2(\Omega) \times \mathcal{H}^1_z(\Sigma_0) \times \mathcal{H}^1_z(\Sigma_1)$ with $\delta_0(1) = \delta_1(-1) = 0$ satisfying (3.6) for some $\eta > 0$ small enough to be chosen later.

- *Definition of the sequence.* We construct a sequence by setting $u_0 := 0$ and, for all $n \in \mathbb{N}$, we define $u_{n+1} \in Z^0(\Omega)$ by induction as the solution to

$$\begin{cases} z\partial_x u_{n+1} - \partial_{zz} u_{n+1} = f - E_n \partial_z u_n, \\ (u_{n+1})_{|\Sigma_i} = \delta_i, \\ (u_{n+1})_{|z=\pm 1} = 0, \end{cases}$$

  where $E_n := E[u_n]$. At each step, by (3.3), $E_n \in L^\infty(x_0, x_1)$. Hence, since $u_n \in Z^0(\Omega)$, $f - E_n \partial_z u_n \in L^2(\Omega)$, so the existence of $u_{n+1} \in Z^0(\Omega)$ follows from Proposition 2.5.

- *Uniform bound in Z^0.* Let us prove by induction that $\|u_n\|_{Z^0} \le 2C_P\eta$ for all $n \in \mathbb{N}$, where C_P is the constant in Pagani's estimate (2.7), provided that η is small enough. The statement is true for $n = 0$. For $n \ge 0$, by (3.3), $\|E_n\|_{L^\infty} \lesssim \|u_n\|_{L^2} \lesssim \eta$. As a consequence, it follows from Proposition 2.5 that

$$\|u_{n+1}\|_{Z^0} \le C_P(\|f\|_{L^2} + \|\delta_0\|_{\mathcal{H}^1_z} + \|\delta_1\|_{\mathcal{H}^1_z} + \|E_n\|_\infty \|\partial_z u_n\|_{L^2}) \le C_P\eta + C\eta^2,$$

  for some C depending only on Ω. Therefore, if $C\eta < C_P$, the bound propagates by induction.

- *Convergence.* Now, let $w_n := u_{n+1} - u_n$. Then, for $n \geq 1$, w_n is a solution to

$$\begin{cases} z\partial_x w_n - \partial_{zz} w_n = -(E_n - E_{n-1})\partial_z u_{n-1} - E_n \partial_z (u_n - u_{n-1}), \\ (w_n)_{|\Sigma_i} = 0, \\ (w_n)_{|z=\pm 1} = 0. \end{cases}$$

By (3.3), $\|E_n - E_{n-1}\|_{L^\infty} \lesssim \|u_n - u_{n-1}\|_{L^2}$ and $\|E_n\|_{L^\infty} \lesssim \|u_n\|_{L^2}$. Hence, by Proposition 2.5,

$$\|w_n\|_{Z^0} \lesssim \|(E_n - E_{n-1})\partial_z u_{n-1}\|_{L^2} + \|E_n \partial_z (u_n - u_{n-1})\|_{L^2} \lesssim \eta \|w_{n-1}\|_{Z^0}$$

and thus $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $Z^0(\Omega)$ provided that η is small enough. Passing to the limit as $n \to \infty$, we obtain a strong solution $u \in Z^0$ with $\|u\|_{Z^0} \leq 2C_P \eta$ to (3.1)–(3.2).

Eventually, the uniform bound propagated on the sequence also passes to the limit and implies (3.7). ■

Lemma 3.4 (Uniqueness of Z^0 solutions of (3.1)–(3.2)). *There exists $\eta > 0$ such that, for any $(f, \delta_0, \delta_1) \in L^2(\Omega) \times \mathcal{H}^1_z(\Sigma_0) \times \mathcal{H}^1_z(\Sigma_1)$, (3.1)–(3.2) has at most one solution $u \in Z^0(\Omega)$ such that $\|u\|_{Z^0} \leq \eta$.*

Proof. Let $(f, \delta_0, \delta_1) \in L^2(\Omega) \times \mathcal{H}^1_z(\Sigma_0) \times \mathcal{H}^1_z(\Sigma_1)$ and $u, u' \in Z^0(\Omega)$ be two solutions to (3.1)–(3.2). Then $w := u - u' \in Z^0(\Omega)$ is a solution to

$$\begin{cases} z\partial_x w - \partial_{zz} w = (E[u'] - E[u])\partial_z u' - E[u]\partial_z w, \\ w_{|\Sigma_i} = 0, \\ w_{|z=\pm 1} = 0. \end{cases} \tag{3.9}$$

Multiplying (3.9) by w, integrating by parts and using the boundary conditions and $\partial_z E[u] = 0$, we obtain

$$\int_\Omega (\partial_z w)^2 \leq \int_\Omega |(E[u'] - E[u])\partial_z u' w|.$$

By (3.3), $\|E[u'] - E[u]\|_{L^\infty} \lesssim \|w\|_{L^2}$. Hence, since $w_{|z=\pm 1} = 0$, Poincaré's inequality entails that

$$\|w\|_{L^2}^2 \lesssim \|\partial_z w\|_{L^2}^2 \lesssim \|E[u'] - E[u]\|_{L^\infty} \|\partial_z u'\|_{L^2} \|w\|_{L^2} \lesssim \|\partial_z u'\|_{L^2} \|w\|_{L^2}^2.$$

Hence, there exists $C_1 > 0$ (depending only on Ω) such that

$$\|w\|_{L^2}^2 \leq C_1 \|\partial_z u'\|_{L^2} \|w\|_{L^2}^2. \tag{3.10}$$

If $C_1 \|\partial_z u'\|_{L^2} < 1$, (3.10) implies $w = 0$, so uniqueness holds in the ball of radius $1/C_1$ of $Z^0(\Omega)$. ■

3.3 Nonlinear orthogonality conditions for higher regularity

We now prove Item (2) of Theorem 5, which corresponds to the well-posedness theory at regularity Z^1, under orthogonality conditions, and is therefore a nonlinear counterpart of Proposition 2.17.

Lemma 3.5. *There exist* $(f^k, \delta_0^k, \delta_1^k) \in \mathcal{H}_{FP}$ *for* $k \in \{0,1\}$ *such that*

$$\forall j,k \in \{0,1\}, \quad \overline{\ell^j}(f^k, \delta_0^k, \delta_1^k) = \mathbf{1}_{j=k}.$$

Proof. As Corollary 2.19, this follows from Proposition 2.15. ∎

Proposition 3.6. *There exist* $\eta > 0$ *and maps* $U_{FP} : B_\eta \to Z^1(\Omega)$ *and* $(\nu_{FP}^0, \nu_{FP}^1) : B_\eta \to \mathbb{R}^2$, *where* B_η *is the ball of radius* η *in* $\mathcal{H}_{FP}$ *such that, for any* $(f, \delta_0, \delta_1) \in B_\eta$, $u := U_{FP}(f, \delta_0, \delta_1) \in Z^1(\Omega)$ *and* $\nu^j := \nu_{FP}^j(f, \delta_0, \delta_1)$ *obey the equation*

$$\begin{cases} z\partial_x u + E[u]\partial_z u - \partial_{zz} u = f + \nu^0 f^0 + \nu^1 f^1, \\ u_{|\Sigma_i} = \delta_i + \nu^0 \delta_i^0 + \nu^1 \delta_i^1, \\ u_{|z=\pm 1} = 0, \end{cases} \tag{3.11}$$

where the triplets $(f^k, \delta_0^k, \delta_1^k)$ *for* $k \in \{0,1\}$ *are defined in Lemma 3.5. Furthermore,* u *and* ν *satisfy the estimate*

$$\|u\|_{Z^1} + |\nu^0| + |\nu^1| \lesssim \|(f, \delta_0, \delta_1)\|_{\mathcal{H}_{FP}} \tag{3.12}$$

and the orthogonality conditions

$$\nu^j = -\overline{\ell^j}(f - E[u]\partial_z u, \delta_0, \delta_1) \quad \textit{for } j \in \{0,1\}. \tag{3.13}$$

Proof. Let $(f, \delta_0, \delta_1) \in \mathcal{H}_{FP}$ with $\|(f, \delta_0, \delta_1)\|_{\mathcal{H}_{FP}} \leq \eta$ small enough to be chosen later on. We modify our iterative scheme to construct Z^1 solutions using Proposition 2.17 and accommodate for the two orthogonality conditions at each step.

- *Definition of the sequence.* More precisely, we take $u_0 := 0$ and, for $n \in \mathbb{N}$, given $u_n \in Z^1(\Omega)$ such that $u_n|_{\Sigma_i} \in \mathcal{H}_z^1(\Sigma_i)$, we define $u_{n+1} \in Z^1(\Omega)$ as the solution to

$$\begin{cases} z\partial_x u_{n+1} - \partial_{zz} u_{n+1} = f - E_n \partial_z u_n + \nu_{n+1}^0 f^0 + \nu_{n+1}^1 f^1, \\ (u_{n+1})_{|\Sigma_i} = \delta_i + \nu_{n+1}^0 \delta_i^0 + \nu_{n+1}^1 \delta_i^1, \\ (u_{n+1})_{|z=\pm 1} = 0, \end{cases} \tag{3.14}$$

  where $E_n := E[u_n]$, the triplets $(f^k, \delta_0^k, \delta_1^k)$ for $k \in \{0,1\}$ are defined in Lemma 3.5 and

$$\nu_{n+1}^j := -\overline{\ell^j}(f - E_n \partial_z u_n, \delta_0, \delta_1). \tag{3.15}$$

This choice ensures that the two orthogonality conditions

$$\overline{\ell^j}\big(f - E_n\partial_z u_n + \nu^0_{n+1} f^0 + \nu^1_{n+1} f^1, \delta_0 + \nu^0_{n+1}\delta^0_0 + \nu^1_{n+1}\delta^1_0,\\ \delta_1 + \nu^0_{n+1}\delta^0_1 + \nu^1_{n+1}\delta^1_1\big) = 0$$

are satisfied.

We now verify that the data of (3.14) satisfy the assumptions of Proposition 2.17. This mostly follows from the inclusion $\mathcal{H}_{FP} \subset \mathcal{H}_K$. It only remains to check that $(-E_n\partial_z u_n, 0, 0) \in \mathcal{H}_K$, i.e., that $-E_n\partial_z u_n \in H^1_x L^2_z$, $(-E_n\partial_z u_n/z)_{|\Sigma_i} \in \mathcal{H}^1_z(\Sigma_i)$ and $(-E_n\partial_z u_n/z)(x_i, (-1)^i) = 0$. The condition $\partial_z u_n(x_i, (-1)^i) = 0$ is guaranteed by the constraint $\delta_i'((-1)^i) = 0$ contained in definition (3.4) which also entails $(\delta_i^k)'((-1)^i) = 0$. We now estimate the norms of $E_n\partial_z u_n$. First, from the lateral boundary conditions, we derive that

$$\begin{aligned} &\|\partial_z u_{n+1}(x_i, z)/z\|_{\mathcal{H}^1_z(\Sigma_i)} \\ &\quad \le \|\delta_i'(z)/z\|_{\mathcal{H}^1_z(\Sigma_i)} + \sum_{k\in\{0,1\}} |\nu^k_{n+1}| \cdot \|(\delta^k_i)'(z)/z\|_{\mathcal{H}^1_z(\Sigma_i)} \\ &\quad \lesssim \|(f, \delta_0, \delta_1)\|_{\mathcal{H}_{FP}} + |\nu^0_{n+1}| + |\nu^1_{n+1}|. \end{aligned} \tag{3.16}$$

Since E_n does not depend on z, we obtain, using (3.3),

$$\|E_n(x_i)\partial_z u_n(x_i, z)/z\|_{\mathcal{H}^1_z(\Sigma_i)} \le \|E_n\|_{L^\infty}\|\partial_z u_n(x_i, z)/z\|_{\mathcal{H}^1_z(\Sigma_i)}, \tag{3.17}$$

and therefore $(-E_n\partial_z u_n/z)_{|\Sigma_i} \in \mathcal{H}^1_z(\Sigma_i)$. Moreover, using again (3.3),

$$\begin{aligned} \|E_n\partial_z u_n\|_{H^1_x L^2_z} &\lesssim \|E_n\|_{H^1_x}\|\partial_z u_n\|_{L^\infty_x L^2_z} + \|E_n\|_{L^\infty_x}\|\partial_z u_n\|_{H^1_x L^2_z} \\ &\lesssim \|u_n\|_{L^2}\|u_n\|_{Z^1}. \end{aligned} \tag{3.18}$$

By Proposition 2.17, we conclude that $u_{n+1} \in Z^1(\Omega)$.

- *Uniform bound in Z^1.* Let us prove by induction that there exists a constant $C_1 > 0$ such that, if η is small enough, then, for all $n \in \mathbb{N}$,

$$U_n := \|u_n\|_{Z^1} + \sum_{i\in\{0,1\}} \|\partial_z u_n(x_i, z)/z\|_{\mathcal{H}^1_z(\Sigma_i)} \le 2C_1\eta.$$

This holds for $n = 0$. For $n \in \mathbb{N}$, it follows from (2.16) that

$$\begin{aligned} \|u_{n+1}\|_{Z^1} \lesssim \|(f, \delta_0, \delta_1)\|_{\mathcal{H}_K} + |\nu_{n+1}| + \|E_n\partial_z u_n\|_{H^1_x L^2_z} \\ + \sum_{i\in\{0,1\}} \|(E_n\partial_z u_n)|_{\Sigma_i}/z\|_{\mathcal{H}^1_z(\Sigma_i)}. \end{aligned}$$

We obtain from (3.15) and Lemma 2.13 that

$$|\nu^j_{n+1}| \lesssim \|(f, \delta_0, \delta_1)\|_{\mathcal{H}_K} + \|E_n\partial_z u_n\|_{H^1_x L^2_z} + \sum_{i\in\{0,1\}} \|(E_n\partial_z u_n)|_{\Sigma_i}/z\|_{\mathcal{H}^1_z(\Sigma_i)}.$$

Using (3.16), (3.17), (3.18), we infer that there exists $C_1 > 0$ (depending only on Ω) such that

$$U_{n+1} \leq C_1(\|(f, \delta_0, \delta_1)\|_{\mathcal{H}_{FP}} + U_n^2).$$

Thus, if $\eta \leq 1/(4C_1^2)$, then the bound $U_n \leq 2C_1\eta$ propagates by induction.

- *Convergence.* As in the low regularity case, we let $w_n := u_{n+1} - u_n$. Then, for $n \geq 1$, w_n is now a solution to:

$$\begin{cases} z\partial_x w_n - \partial_{zz} w_n = -(E_n - E_{n-1})\partial_z u_{n-1} - E_n \partial_z (u_n - u_{n-1}) \\ \qquad\qquad + (\nu_{n+1}^0 - \nu_n^0) f^0 + (\nu_{n+1}^1 - \nu_n^1) f^1, \\ (w_n)_{|\Sigma_i} = (\nu_{n+1}^0 - \nu_n^0)\delta_i^0 + (\nu_{n+1}^1 - \nu_n^1)\delta_i^1, \\ (w_n)_{|z=\pm 1} = 0. \end{cases}$$

Using the same type of proof as above, we derive from (3.15) and Definition 2.12 that

$$|\nu_{n+1}^j - \nu_n^j| \lesssim \eta \|w_{n-1}\|_{Z^1}.$$

Therefore, estimate (2.16) of Proposition 2.17 entails that

$$\|w_n\|_{Z^1} \lesssim \eta \|w_{n-1}\|_{Z^1}.$$

Thus $(u_n)_{n\in\mathbb{N}}$ and $(\nu_n^j)_{n\in\mathbb{N}}$ are Cauchy sequences. Passing to the limit, we deduce that there exist $u \in Z^1(\Omega)$ and $(\nu^0, \nu^1) \in \mathbb{R}^2$ satisfying (3.11), (3.12) and (3.13).

∎

Definition 3.7. For $\eta > 0$ small enough, we define $\mathcal{M}_{FP}$ as

$$\mathcal{M}_{FP} := \left\{(f, \delta_0, \delta_1) \in \mathcal{H}_{FP};\, \|(f, \delta_0, \delta_1)\|_{\mathcal{H}_{FP}} \leq \eta \text{ and } {}_{FP}(f, \delta_0, \delta_1) = (0, 0)\right\}. \tag{3.19}$$

By definition, for any $(f, \delta_0, \delta_1) \in \mathcal{M}_{FP}$, there exists a solution $u \in Z^1(\Omega)$ to (3.1) (since (3.11) is satisfied with $\nu^0 = \nu^1 = 0$), which satisfies (3.8) thanks to (3.12).

Proposition 3.8. *There exists $\eta > 0$ such that, for any $(f, \delta_0, \delta_1) \in \mathcal{H}_{FP}$ and $u \in Z^1(\Omega)$ solution to (3.1)–(3.2) satisfying $\|(f, \delta_0, \delta_1)\|_{\mathcal{H}_{FP}} \leq \eta$ and $\|u\|_{Z^1} \leq \eta$, one has $(f, \delta_0, \delta_1) \in \mathcal{M}_{FP}$.*

Proof. Let $(f, \delta_0, \delta_1) \in \mathcal{H}_{FP}$ and $u \in Z^1(\Omega)$ be a solution to (3.1)–(3.2). Assume that $\|u\|_{Z^1} \leq \eta$ and $\|(f, \delta_0, \delta_1)\|_{\mathcal{H}_{FP}} \leq \eta$ for some $\eta > 0$ small enough to be chosen later.

Since $u \in Z^1(\Omega)$, one has $-E[u]\partial_z u \in H_x^1 L_z^2$. Thus, viewing (3.1) as a linear equation with source term $f - E[u]\partial_z u$, Proposition 2.17 implies that

$$\overline{\ell^j}(f - E[u]\partial_z u, \delta_0, \delta_1) = 0.$$

Now, let $(\tilde{u}, \nu^0, \nu^1) \in Z^1(\Omega) \times \mathbb{R}^2$ be the solution to (3.11) constructed from (f, δ_0, δ_1) in Proposition 3.6. By (3.13),

$$\nu^j = -\overline{\ell^j}(f - E[\tilde{u}]\partial_z \tilde{u}, \delta_0, \delta_1).$$

Combining both equalities leads to

$$|\nu^0| + |\nu^1| \lesssim (\|u\|_{Z^1} + \|\tilde{u}\|_{Z^1})\|u - \tilde{u}\|_{Z^1}.$$

Therefore, writing the system satisfied by $w := u - \tilde{u}$ and applying estimate (2.16) of Proposition 2.10 leads to

$$\|w\|_{Z^1} \lesssim \eta \|w\|_{Z^1}.$$

If $\eta > 0$ is small enough, this implies that $w = 0$, so $\nu^0 = \nu^1 = 0$, and $(f, \delta_0, \delta_1) \in \mathcal{M}_{FP}$. ■

Remark 3.9 (An alternative approach). Another potential proof of Item 2 of Theorem 5 could be the following. Consider the map

$$\nu^j_{FP} : (f, \delta_0, \delta_1) \in \mathcal{H}_{FP} \mapsto -\overline{\ell^j}(f, \delta_0, \delta_1) + \overline{\ell^j}(E[u]\partial_z u, 0, 0) \in \mathbb{R}^2,$$

where $u \in Z^0$ is the unique solution to system (3.1)–(3.2), provided by Lemma 3.3.

Since $u \in Z^0$, $E[u]\partial_z u \in H^{1/3}_x L^2_z$, and therefore ν^j_{FP} is well defined thanks to Remark 2.33. We then set $\mathcal{M}_{FP} := \{(f, \delta_0, \delta_1) \in \mathcal{H}_{FP}; \nu^j_{FP}(f, \delta_0, \delta_1) = 0\}$. Then for all $(f, \delta_0, \delta_1) \in \mathcal{H}_{FP}$, u is a solution to an equation of the type $z\partial_x - \partial_{zz} u = g$, where the right-hand side g belongs to $H^{1/3}_x L^2_z$ and satisfies orthogonality conditions. It follows from the interpolation result Proposition 2.36 and Lemma 1.15 that $u \in Z^{1/3} \hookrightarrow H^1_x L^2_z \cap H^{1/3}_x H^2_z$, and therefore $E[u]\partial_z u \in H^{2/3}_x L^2_z$. Bootstrapping twice the same argument, we eventually infer that $u \in Z^1$.

However, this argument is based on the existence of Z^0 solutions of the nonlinear problem without any orthogonality condition. For the Burgers equation, the nonlinearity is too strong for such a theory of weak solutions to be available. Therefore, in order to unify the presentation, we have chosen to present a different proof, based on a modification of the iterative scheme.

3.4 Regularity and tangent space of the manifold

We now give another description of the set $\mathcal{M}_{FP}$ defined in (3.19), which we use to prove that it is indeed a Lipschitz submanifold of $\mathcal{H}_{FP}$ of codimension 2, modeled on $\mathcal{H}^{\perp}_{FP,\text{sg}}$, and we describe its tangent space at the origin. Throughout this paragraph, we denote by $\Xi = (f, \delta_0, \delta_1)$ an element of $\mathcal{H}_{FP}$.

We recall that there exist $\Xi^0, \Xi^1 \in \mathcal{H}_{FP}$ such that $\overline{\ell^j}(\Xi^k) = \mathbf{1}_{j=k}$ (see Lemma 3.5), and such that $\mathcal{H}_{FP,\mathrm{sg}}^{\perp} := \ker \overline{\ell^0} \cap \ker \overline{\ell^1} \cap \mathcal{H}_{FP} = (\mathbb{R}\Xi^0 + \mathbb{R}\Xi^1)^{\perp}$. For every $\Xi \in \mathcal{H}_{FP}$, one has the decomposition

$$\Xi = \Xi^{\perp} + \langle \Xi; \Xi^0 \rangle_{\mathcal{H}_{FP}} \Xi^0 + \langle \Xi; \Xi^1 \rangle_{\mathcal{H}_{FP}} \Xi^1, \tag{3.20}$$

where $\Xi^{\perp} \in \mathcal{H}_{FP,\mathrm{sg}}^{\perp}$ and the linear maps $\Xi \mapsto \Xi^{\perp}$ and $\Xi \mapsto \langle \Xi^k; \Xi \rangle_{\mathcal{H}_{FP}}$ are continuous.

Lemma 3.10. *For $\eta > 0$ small enough, the set $\mathcal{M}_{FP}$ defined in (3.19) is equal to*

$$\widetilde{M}_{FP} := \left\{ \widetilde{\Xi} \in \mathcal{H}_{FP}; \|\widetilde{\Xi}\|_{\mathcal{H}_{FP}} \leq \eta \text{ and } \langle \widetilde{\Xi}; \Xi^j \rangle = \nu_{FP}^j(\widetilde{\Xi}^{\perp}) \text{ for } j \in \{0,1\} \right\}. \tag{3.21}$$

Proof. We proceed by double inclusion.

Let $\widetilde{\Xi} \in \widetilde{M}_{FP}$. Consider the solution $(u, \nu^0, \nu^1) \in Z^1(\Omega) \times \mathbb{R}^2$ constructed for the data $\widetilde{\Xi}^{\perp}$ in Proposition 3.6. Then $u \in Z^1(\Omega)$ is a solution to (3.1)–(3.2) with data $\widetilde{\Xi}^{\perp} + \nu_{FP}^0(\widetilde{\Xi}^{\perp})\Xi^0 + \nu_{FP}^1(\widetilde{\Xi}^{\perp})\Xi^1$. Since $\widetilde{\Xi} \in \widetilde{M}_{FP}$, we infer from (3.20) that $u \in Z^1(\Omega)$ is actually a solution with data $\widetilde{\Xi}$. Thus, Proposition 3.8 implies that $\widetilde{\Xi} \in \mathcal{M}_{FP}$.

Let $\Xi \in \mathcal{M}_{FP}$. We introduce

$$\widetilde{\Xi} := \Xi^{\perp} + \nu_{FP}^0(\Xi^{\perp})\Xi^0 + \nu_{FP}^1(\Xi^{\perp})\Xi^1,$$

which can be thought of as a good projection of Ξ on $\widetilde{M}_{FP}$ since $\widetilde{\Xi}^{\perp} = \Xi^{\perp}$ and $\widetilde{\Xi} \in \widetilde{M}_{FP}$. Let $u, \tilde{u} \in Z^1(\Omega)$ denote the solutions constructed in Proposition 3.6 from Ξ and $\Xi^{\perp}$. For $k \in \{0,1\}$, we also introduce the coefficients $\mu^k := \nu_{FP}^k(\Xi^{\perp}) - \langle \Xi; \Xi^k \rangle_{\mathcal{H}_{FP}}$, which characterize how far Ξ is from $\widetilde{M}_{FP}$. Then $w := \tilde{u} - u$ belongs to $Z^1(\Omega)$ with

$$\|w\|_{Z^1} \lesssim \eta$$

and is a solution to

$$\begin{cases} z\partial_x w - \partial_{zz} w = E[u]\partial_z u - E[\tilde{u}]\partial_z \tilde{u} + \mu^0 f^0 + \mu^1 f^1, \\ w_{|\Sigma_i} = \mu^0 \delta_i^0 + \mu^1 \delta_i^1, \\ w_{|z=\pm 1} = 0. \end{cases} \tag{3.22}$$

By Proposition 2.17, since $w \in Z^1(\Omega)$, the following orthogonality conditions are satisfied for $j = 0, 1$:

$$\begin{aligned} 0 &= \overline{\ell^j}(E[u]\partial_z u - E[\tilde{u}]\partial_z \tilde{u} + \mu^0 f^0 + \mu^1 f^1, \mu^0\delta_0^0 + \mu^1\delta_0^1, \mu^0\delta_1^0 + \mu^1\delta_1^1) \\ &= \overline{\ell^j}(E[u]\partial_z u - E[\tilde{u}]\partial_z \tilde{u}, 0, 0) + \mu^j. \end{aligned} \tag{3.23}$$

Moreover, since

$$\|E[u]\partial_z u - E[\tilde{u}]\partial_z \tilde{u}\|_{H_x^1 L_z^2} \lesssim \eta \|w\|_{Z^1},$$

we infer from (3.23) that

$$|\mu^j| \lesssim \eta \|w\|_{Z^1}.$$

Applying estimate (1.7) to (3.22), we obtain

$$\|w\|_{Z^1} \lesssim \eta \|w\|_{Z^1} + |\mu^0| + |\mu^1| \lesssim \eta \|w\|_{Z^1}.$$

For $\eta > 0$ small enough, this entails that $w = 0$ and $\mu^0 = \mu^1 = 0$, so that $\Xi \in \widetilde{\mathcal{M}}_{FP}$. ∎

Lemma 3.11. *The maps U_{FP} and ν_{FP} of Proposition 3.6 are Lipschitz-continuous.*

Proof. Taking two triplets $\Xi, \Xi' \in \mathcal{H}_{FP}$, one can consider the constructed sequences $u_n, u'_n \in Z^1(\Omega)$ and $\nu_n, \nu'_n \in \mathbb{R}^2$ from (3.14). Then, for $n \geq 1$, $w_n := u_n - u'_n$ is the solution to

$$\begin{cases} z\partial_x w_n - \partial_{zz} w_n = (f - f') - E[w_{n-1}]\partial_z u_{n-1} - E[u'_{n-1}]\partial_z w_{n-1} \\ \qquad\qquad\qquad + (\nu_n^0 - \nu_n'^0) f^0 + (\nu_n^1 - \nu_n'^1) f^1, \\ (w_n)_{|\Sigma_i} = (\delta_i - \delta'_i) + (\nu_n^0 - \nu_n'^0)\delta_i^0 + (\nu_n^1 - \nu_n'^1)\delta_i^1, \\ (w_n)_{|z=\pm 1} = 0, \end{cases}$$

where, from (3.15) and Definition 2.12,

$$|\nu_n - \nu'_n| \lesssim \|\Xi - \Xi'\|_{\mathcal{H}_{FP}} + \eta \|w_{n-1}\|_{Z^1}.$$

Thus, we obtain from Proposition 2.17 that

$$\|w_n\|_{Z^1} \lesssim \|\Xi - \Xi'\|_{\mathcal{H}_{FP}} + \eta \|w_{n-1}\|_{Z^1}.$$

For η small enough, we obtain at the limit that

$$\|u - u'\|_{Z^1} + |\nu - \nu'| \lesssim \|\Xi - \Xi'\|_{\mathcal{H}_{FP}},$$

which concludes the proof. ∎

Remark 3.12. Since we only proved Lipschitz regularity for the map ν_{FP}, (3.19) (and equivalently (3.21)) *a priori* only defines a Lipschitz manifold. Hence, it is difficult to define tangent spaces to $\mathcal{M}_{FP}$. Nevertheless, one can say that $\mathcal{H}^{\perp}_{FP,\mathrm{sg}}$ is tangent to $\mathcal{M}_{FP}$ at 0 in the following weak senses:

- For $\Xi \in \mathcal{M}_{FP}$, $\mathrm{d}(\Xi, \mathcal{H}^{\perp}_{FP,\mathrm{sg}}) \lesssim \|\Xi\|^2_{\mathcal{H}_{FP}}$.
- For every $\Xi^{\perp} \in \mathcal{H}^{\perp}_{FP,\mathrm{sg}}$, for $\varepsilon \in \mathbb{R}$ small enough, $\mathrm{d}(\varepsilon\Xi^{\perp}, \mathcal{M}_{FP}) \lesssim \varepsilon^2$.

Both facts are straightforward consequences of the equivalent definitions (3.19) and (3.21) and of the estimate

$$|\nu^j(\Xi) + \overline{\ell^j}(\Xi)| \lesssim \|\Xi\|^2_{\mathcal{H}_{FP}}$$

which follows from (3.13) and (3.12).

Remark 3.13. It is likely that similar techniques can be used to prove that $\mathcal{M}_{FP}$ has in fact more regularity (say C^1 for example) and characterize its tangent spaces in a neighborhood of the origin by computing the orthogonality conditions associated with the linearized problems around small enough solutions $u \in Z^1(\Omega)$, but this is not our focus here.

3.5 A general formalization

The construction used in Section 3.3 can be seen as a particular case (see Remark 3.16) of a more general approach to construct solutions to semilinear or quasilinear equations in the presence of orthogonality conditions, in a perturbative regime. We give here a statement in an abstract framework which we will use in the following sections for the Burgers and Prandtl systems.

Our abstract result is related with general results for semilinear problems associated with Fredholm operators with negative index, such as the ones of [60, Chapter 11, Section 4.2.3]. However, the approach in this reference consists in modifying parameters in the nonlinearity to ensure the orthogonality conditions, while we focus on constructing a submanifold of data for which the nonlinear problem has a regular solution.

We intend to construct solutions to problems of the form $Lu = N(\Xi, u)$, where $u \in \mathcal{Z}$ (the space of solutions), $\Xi \in \mathcal{X}$ (the space of data for the nonlinear problem), $N : \mathcal{X} \times \mathcal{Z} \to \mathcal{H}$ is the nonlinearity, with values in $\mathcal{H}$ (the space of source terms $\Theta \in \mathcal{H}$ for the linear problem $Lu = \Theta$).

To avoid investigating the C^1 dependency of the solutions to our nonlinear systems on the data, we use a version of the implicit function theorem for functions which are not C^1 but only "strongly Fréchet-differentiable at a point". We refer the reader to [57, Chapter 25].

Definition 3.14. Let E, F be Banach spaces, $f : E \to F$ and $x^* \in E$. We say that f is *strongly Fréchet-differentiable* at x^* when there exists a continuous linear map $L : E \to F$ such that for all x_1, x_2 with $x_1, x_2 \to x^*$

$$\|f(x_1) - f(x_2) - L(x_1 - x_2)\|_F = o(\|x_1 - x_2\|_E).$$

The following implicit function theorem is proved in [57, Paragraph 25.13].

Lemma 3.15. *Let E_1, E_2, F be Banach spaces and $f : E_1 \times E_2 \to F$ such that $f(0,0) = 0$. Assume f is strongly Fréchet-differentiable at $(0,0)$ and that $\partial_2 f(0,0) : E_2 \to F$ is a linear isomorphism. Then there exists a Lipschitz-continuous map $g : E_1 \to E_2$ defined in a neighborhood of $0 \in E_1$ such that, for every (x, y) in a neighborhood of $(0,0) \in E_1 \times E_2$, $f(x, y) = 0$ if and only if $y = g(x)$. Moreover, g is strongly Fréchet-differentiable at 0 and $Dg(0) = -(\partial_2 f(0,0))^{-1}\partial_1 f(0,0)$.*

Our general result is the following theorem.

Theorem 6. *Let $\mathcal{H}, \mathcal{X}, \mathcal{Z}$ be Banach spaces and $d \in \mathbb{N}$. Let $\ell : \mathcal{H} \to \mathbb{R}^d$ and $L : \mathcal{Z} \to \mathcal{H}$ be continuous linear maps. Let N be a (nonlinear) map from $\mathcal{X} \times \mathcal{Z}$ to $\mathcal{H}$ such that $N(0,0) = 0$. Assume that*

(i) *for all $\Theta \in \mathcal{H}$, the equation $Lu = \Theta$ has a unique solution $u \in \mathcal{Z}$ if and only if $\Theta \in \ker \ell$, which moreover satisfies $\|u\|_{\mathcal{Z}} \lesssim \|\Theta\|_{\mathcal{H}}$;*

(ii) *N is strongly Fréchet-differentiable at $(0,0)$ and $\partial_u N(0,0) = 0$, i.e., there exists a continuous linear map $\partial_\Xi N(0,0) : \mathcal{X} \to \mathcal{H}$ such that, as $\Xi, \Xi' \in \mathcal{X}$ and $u, u' \in \mathcal{Z}$ go to 0,*

$$\begin{aligned} &\|N(\Xi,u) - N(\Xi',u') - (\partial_\Xi N(0,0))(\Xi - \Xi')\|_{\mathcal{H}} \\ &\qquad = o\big(\|\Xi - \Xi'\|_{\mathcal{X}} + \|u - u'\|_{\mathcal{Z}}\big); \end{aligned} \tag{3.24}$$

(iii) *$\ell_N := \ell \circ \partial_\Xi N(0,0)$ is onto from $\mathcal{X}$ to $\mathbb{R}^d$.*

Then there exists a local Lipschitz submanifold $\mathcal{M}$ of $\mathcal{X}$, modeled on $\ker \ell_N$ (of codimension d) and tangent to it at 0, such that, for any $\Xi \in \mathcal{X}$ small enough, the equation $Lu = N(\Xi, u)$ has a solution $u \in \mathcal{Z}$ if and only if $\Xi \in \mathcal{M}$. Such a solution satisfies $\|u\|_{\mathcal{Z}} \lesssim \|\Xi\|_{\mathcal{X}}$ and is unique.

Proof. Using Item (iii), we fix $\Xi^1, \dots, \Xi^d \in \mathcal{X}$ such that $\ell_N^j(\Xi^k) = \mathbf{1}_{j=k}$, and we set $\Theta^k := \partial_\Xi N(0,0)\Xi^k$. We could then mimic the iterative scheme of Section 3.3 by defining sequences $u_n \in \mathcal{Z}$ and $v_n \in \mathbb{R}^d$ such that

$$Lu_{n+1} = N\left(\Xi + \sum_k v_{n+1}^k \Xi^k, u_n\right).$$

Instead, we provide a shorter proof directly relying on the bundled result Lemma 3.15.

Let $\Xi \mapsto \Xi^\perp$ be the linear continuous projection from $\mathcal{X}$ to $\ker \ell_N$ parallel to the space span $(\Xi^1, \dots, \Xi^d)$, i.e., $\Xi^\perp = \Xi - \sum_{j=1}^d \ell_N^j(\Xi)\Xi^j$. Let $f : \ker \ell_N \times (\mathcal{Z} \times \mathbb{R}^d) \to \mathcal{H}$ defined by

$$f(\Xi^\perp, (u,a)) := Lu - N(\Xi^\perp + a_1 \Xi^1 + \dots + a_d \Xi^d, u).$$

By Item (ii) and continuity of L on $\mathcal{Z}$, f is strongly Fréchet-differentiable at $(0,0)$. Moreover, $\partial_2 f(0,0) : (u,a) \mapsto Lu - a_1\Theta^1 - \dots - a_d\Theta^d$ is a linear isomorphism from $\mathcal{Z} \times \mathbb{R}^d$ to $\mathcal{H}$ by Item (i) and continuity of ℓ on $\mathcal{H}$. Indeed, given $h \in \mathcal{H}$, setting $a^h := -\ell(h)$ and $u^h \in \mathcal{Z}$ the solution to $Lu^h = h + a_1^h\Theta^1 + \dots + a_d^h\Theta^d$, one has $\partial_2 f(0,0)(u^h, a^h) = h$ and $\|u_h\|_{\mathcal{Z}} \lesssim \|h\|_{\mathcal{H}}$, $|a^h| \lesssim \|h\|_{\mathcal{H}}$.

Hence, the implicit function theorem stated in Lemma 3.15 yields the existence of Lipschitz-continuous functions $(U, \mu) : \ker \ell_N \to \mathcal{Z} \times \mathbb{R}^d$ such that, for every $\Xi^\perp \in \ker \ell_N$, $u \in \mathcal{Z}$ and $a \in \mathbb{R}^d$ small enough,

$$Lu = N(\Xi^\perp + a_1 \Xi^1 + \dots + a_d \Xi^d, u)$$

if and only if $a = \mu(\Xi^\perp)$ and $u = U(\Xi^\perp)$. From there, we infer that for all $\Xi \in \mathcal{X}$ and $u \in \mathcal{Z}$ small enough, $Lu = N(\Xi, u)$ if and only if $\ell_N(\Xi) = \mu(\Xi^\perp)$ and $u = U(\Xi^\perp)$. Thus the conclusions of the theorem hold provided that we set

$$\mathcal{M} := \left\{\Xi \in \mathcal{X}; \|\Xi\|_{\mathcal{X}} \leq \eta \text{ and } \ell_N(\Xi) = \mu(\Xi^\perp)\right\}. \tag{3.25}$$

Indeed, (3.25) corresponds to the graph characterization of a local Lipschitz submanifold of $\mathcal{X}$ containing 0 and modeled on $\ker \ell_N$; therefore of codimension d by Item (iii). Eventually, μ is strongly Fréchet-differentiable at 0 and, since $Dg(0) = -(\partial_2 f(0,0))^{-1}\partial_1 f(0,0)$ with the notation of Lemma 3.15, we obtain that $D\mu(0) = -\ell \circ \partial_\Xi N(0,0) = -\ell_N$ so $D\mu(0) = 0$ on $\ker \ell_N$, which justifies the claim that $\mathcal{M}$ is tangent to $\ker \ell_N$ at 0. ∎

Remark 3.16. Item 2 of Theorem 5 can be recovered as a particular case of Theorem 6 with the following setting:

- $\mathcal{X} = \mathcal{H} = \mathcal{H}_{FP}$ defined in (3.4);
- the solution space

  $$\mathcal{Z} := \left\{u \in Z^1(\Omega); u_{|z=\pm 1} = 0, (\partial_z u(x_i, z))/z \in \mathcal{H}^1_z(\Sigma_i), \partial_z u(x_i, (-1)^i) = 0\right\},$$

  with

  $$\|u\|_{\mathcal{Z}} := \|u\|_{Z^1} + \sum_{i\in\{0,1\}} \|(\partial_z u(x_i, z))/z\|_{\mathcal{H}^1_z};$$

- $L : \mathcal{Z} \to \mathcal{X}$ defined by $Lu := (z\partial_x u - \partial_{zz} u, u_{|\Sigma_0}, u_{|\Sigma_1})$, for which one easily checks that the assumption Item (i) of Theorem 6 is satisfied thanks to Proposition 2.17;
- $N : \mathcal{X} \times \mathcal{Z} \to \mathcal{H}$ defined by $N(\Xi, u) := (f, \delta_0, \delta_1) - (E[u]\partial_z u, 0, 0)$. In particular, one has $\partial_\Xi N(0,0) = \mathrm{Id}$. To check that N takes values in $\mathcal{H} = \mathcal{H}_{FP} \subset \mathcal{H}_K$, we must check that $E[u](x_i)\partial_z u(x_i, (-1)^i) = 0$, which follows from the fact that, for $u \in \mathcal{Z}$, $\partial_z u(x_i, (-1)^i) = 0$. We now check that N satisfies Item (ii) of Theorem 6.

  First, for $u, u' \in Z^1(\Omega)$, by (3.3),

  $$\begin{aligned}\|E[u]\partial_z u - E[u']\partial_z u'\|_{H^1_x L^2_z} &\leq \|E[u-u']\partial_z u\|_{H^1_x L^2_z} + \|E[u']\partial_z(u-u')\|_{H^1_x L^2_z}\\ &\lesssim \|u-u'\|_{L^2}\|\partial_z u\|_{H^1_x L^2_z} + \|u'\|_{L^2}\|\partial_z(u-u')\|_{H^1_x L^2_z}\\ &\lesssim (\|u\|_{Z^1} + \|u'\|_{Z^1})\|u-u'\|_{Z^1}.\end{aligned}$$

  Second, one similarly checks that

  $$\|(E[u]\partial_z u - E[u']\partial_z u')(x_i, z)/z\|_{\mathcal{H}^1_z(\Sigma_i)} \lesssim (\|u\|_{\mathcal{Z}} + \|u'\|_{\mathcal{Z}})\|u-u'\|_{\mathcal{Z}}.$$

Hence, we conclude that

$$\begin{aligned}\|(E[u]\partial_z u - E[u']\partial_z u', 0, 0)\|_{\mathcal{H}_{FP}} &= \|(E[u]\partial_z u - E[u']\partial_z u', 0, 0)\|_{\mathcal{H}_K} \\ &\lesssim (\|u\|_{\mathcal{Z}} + \|u'\|_{\mathcal{Z}})\|u - u'\|_{\mathcal{Z}}\end{aligned}$$

so that estimate (3.24) is satisfied.

- $d = 2$, $\ell := (\overline{\ell^0}, \overline{\ell^1})_{|\mathcal{H}_{FP}}$ defined in Definition 2.12, which is continuous on $\mathcal{H}_{FP}$ thanks to Lemma 2.13 and the embedding $\mathcal{H}_{FP} \hookrightarrow \mathcal{H}_K$, satisfying $\ell_N(\mathcal{X}) = \ell(\mathcal{X}) = \mathbb{R}^2$ by Lemma 3.5 and $\partial_\Xi N(0,0) = \mathrm{Id}$.