## Chapter 4

# A viscous Burgers equation

In this chapter, we consider the following nonlinear parabolic forward-backward system, which can be envisioned as a kind of stationary Burgers equation with transverse viscosity:

$$\begin{cases} u\partial_x u - \partial_{yy} u = f, \\ u_{|\Sigma_i} = \mathbb{u}_{|\Sigma_i} + \delta_i, \\ u_{|y=\pm 1} = \mathbb{u}_{|y=\pm 1}. \end{cases} \tag{4.1}$$

As detailed in the introduction, the perturbation $(f, \delta_0, \delta_1)$ is small and we look for solutions $u$ which are close to the shear flow $\mathbb{u}(x, y) := y$, which corresponds to $(f, \delta_0, \delta_1) = (0, 0, 0)$. Thanks to the nonlinear change of variables described in the Introduction and detailed in Section 4.1, the local well-posedness of (4.1) can be proved using the formalism of Section 3.5 (see Sections 4.2 and 4.3).

## 4.1 A nonlinear change of variables

As is classical for problems with free boundaries, we perform a change of variables which straightens the critical curve $\{u = 0\}$. Heuristically, we swap the roles of the vertical coordinate $y$ and the unknown $u$, the latter becoming the vertical coordinate, and the former the unknown of the new PDE. Keeping in mind that we are looking for perturbative solutions with $u$ close enough to $\mathbb{u}$ (in particular $\|u_y - 1\|_{L^\infty} \ll 1$), we change the vertical coordinate $y$ into $z$, defined as

$$z(x, y) := u(x, y).$$

The new unknown $Y(x, z)$ is defined by the implicit relation

$$u(x, Y(x, z)) = z. \tag{4.2}$$

In particular, thanks to the boundary conditions $u_{|y=\pm 1} = \mathbb{u}_{|y=\pm 1} = \pm 1$, one checks that the domain $(x, y) \in \Omega = [x_0, x_1] \times [-1, 1]$ is indeed mapped to $(x, z) \in \Omega$, and one still has $Y_{|z=\pm 1} = \pm 1$. Similarly, if $\delta_i(0) = 0$ and $\delta_i((-1)^i) = 0$, the inflow boundary regions $\Sigma_i$ are also left invariant by this change of variable.

**Remark 4.1.** More rigorously, given $u$ defined on $\Omega$ and close enough to $\mathbb{u}$ (for example in $H^1_x H^2_z$ topology), for each $x \in [x_0, x_1]$, the map $y \mapsto u(x, y)$ is a $C^1$ monotone increasing bijection from $[-1, 1]$ to itself, and the implicit definition (4.2) is equivalent to setting

$$Y(x, z) := (u(x, \cdot))^{-1}(z). \tag{4.3}$$

From (4.2), we successively derive the relations

$$\begin{aligned} \partial_y u(x, Y(x,z)) &= \frac{1}{\partial_z Y(x,z)}, \\ \partial_x u(x, Y(x,z)) &= -\partial_x Y(x,z)\partial_y u(x, Y(x,z)) = -\frac{\partial_x Y(x,z)}{\partial_z Y(x,z)}, \\ \partial_{yy} u(x, Y(x,z)) &= -\frac{\partial_{zz} Y(x,z)}{(\partial_z Y(x,z))^3}. \end{aligned} \tag{4.4}$$

These identities lead to the following PDE for $Y$:

$$z\partial_x Y - (\partial_z Y)^{-2}\partial_{zz} Y = -\partial_z Y f(x,Y). \tag{4.5}$$

Moreover, by (4.3), denoting by $(\cdot + \delta_i(\cdot))^{-1}$ the functional inverse of the function $z \mapsto z + \delta_i(z)$ and letting

$$\Upsilon[\delta_i](z) := z - (\cdot + \delta_i(\cdot))^{-1}(z), \tag{4.6}$$

we have $Y(x,z) = z - \Upsilon[\delta_i](z)$ for $(x,z) \in \Sigma_i$, where we used $\delta_i(0) = 0$ and $\delta_i((-1)^i) = 0$.

Therefore, we obtain the system

$$\begin{cases} z\partial_x Y - (\partial_z Y)^{-2}\partial_{zz} Y = -\partial_z Y f(x,Y), \\ Y_{|\Sigma_i} = z - \Upsilon[\delta_i](z), \\ Y_{|z=\pm 1} = \pm 1. \end{cases} \tag{4.7}$$

Eventually, to make the perturbative nature of this system explicit, we write $Y(x,z) = z - \widetilde{Y}(x,z)$, which leads to the system

$$\begin{cases} z\partial_x \widetilde{Y} - \partial_{zz} \widetilde{Y} = N_B(f, \widetilde{Y}), \\ \widetilde{Y}_{|\Sigma_i} = \Upsilon[\delta_i], \\ \widetilde{Y}_{|z=\pm 1} = 0, \end{cases} \tag{4.8}$$

where the nonlinearity is given by

$$N_B(f, \widetilde{Y}) := \frac{\partial_z \widetilde{Y}(2 - \partial_z \widetilde{Y})}{(1 - \partial_z \widetilde{Y})^2}\partial_{zz}\widetilde{Y} + (1 - \partial_z \widetilde{Y}) f(x, z - \widetilde{Y}). \tag{4.9}$$

We prove the well-posedness of (4.8) in Section 4.2 and use it to prove Theorem 3 in Section 4.3.

**Remark 4.2.** The initial PDE $u\partial_x u - \partial_{yy} u = f$ is quasilinear. After the change of variables described in this paragraph, we obtain system (4.7), which is still a quasilinear one (since the viscosity in front of $\partial_{zz}\widetilde{Y}$ depends on $\widetilde{Y}$). However, we know from

Chapter 2 that, for the linear problem $z\partial_x u - \partial_{zz} u = f$, there is no loss of derivative in the vertical direction. This key point allows us to apprehend (4.7) under the form (4.8), treating this nonlinearity perturbatively as the first term of $N_B$ in (4.9). The fact that there is no loss of vertical derivative explains why we will be able to prove in the following paragraph that the nonlinearity $N_B$ satisfies the mild estimates of Theorem 6. This would not have been possible in the initial form $u\partial_x u - \partial_{yy} u = f$, since the linear theory involves a loss of $\frac{1}{3}$ derivative in the horizontal direction.

## 4.2 Well-posedness in the new variables

We now prove the following well-posedness result with $Z^1(\Omega)$ regularity under two orthogonality conditions for system (4.8). Let

$$\mathscr{H}_B := \left\{(f, \delta_0, \delta_1) \in \mathscr{H}_K; \delta_i''(z)/z \in \mathcal{H}_z^1(\Sigma_i), \delta_i''((-1)^i) = 0\right\}, \tag{4.10}$$

$$\mathcal{X}_B := \left\{(f, \delta_0, \delta_1) \in \mathscr{H}_B; f \in H_x^1 H_z^2, f_{|\Sigma_i} = 0, \right.$$
$$\left. \delta_i \in H^5(\Sigma_i), \delta_i(0) = \delta_i''(0) = 0\right\}, \tag{4.11}$$

where we recall that the spaces $\mathscr{H}_K$ and $\mathcal{H}_z^1$ are defined in (2.13) and (1.22) respectively, with the norms

$$\|(f, \delta_0, \delta_1)\|_{\mathscr{H}_B} := \|(f, \delta_0, \delta_1)\|_{\mathscr{H}_K} + \|\delta_0''(z)/z\|_{\mathcal{H}_z^1} + \|\delta_1''(z)/z\|_{\mathcal{H}_z^1}, \tag{4.12}$$

$$\|(f, \delta_0, \delta_1)\|_{\mathcal{X}_B} := \|(f, \delta_0, \delta_1)\|_{\mathscr{H}_B} + \|f\|_{H_x^1 H_z^2} + \|\delta_0\|_{H^5} + \|\delta_1\|_{H^5}. \tag{4.13}$$

The restriction that $f_{|\Sigma_i} = 0$ lightens the exposition but could be partially relaxed. The space $\mathcal{X}_B$ of (4.11) is the same as the one defined in (1.3) in the introduction.

Our result on system (4.8) is the following proposition.

**Proposition 4.3.** *There exist $\eta > 0$ and a local Lipschitz submanifold $\mathcal{M}_B$ of $\mathcal{X}_B$ included in the ball of radius $\eta$, modeled on $\mathcal{X}_B \cap \ker(\overline{\ell^0}, \overline{\ell^1})$ (of codimension 2) and tangent to it at 0 such that, for every $(f, \delta_0, \delta_1) \in \mathcal{X}_B$ such that $\|(f, \delta_0, \delta_1)\|_{\mathcal{X}_B} \leq \eta$, (4.8) has a solution $\widetilde{Y} \in Z^1(\Omega)$ if and only if $(f, \delta_0, \delta_1) \in \mathcal{M}_B$. Such solutions are unique and satisfy $\|\widetilde{Y}\|_{Z^1} \lesssim \|(f, \delta_0, \delta_1)\|_{\mathcal{X}_B}$.*

*Proof.* Our strategy is to apply the same nonlinear argument as for our kinetic theory toy model (see Chapter 3). Before moving on to the formal proof using the abstract Theorem 6, let us give a heuristic overview of the corresponding concrete nonlinear scheme.

*Heuristic overview of the nonlinear scheme.* We follow the scheme described in Section 3.3. Let $(f, \delta_0, \delta_1) \in \mathcal{X}_B$ with $\|(f, \delta_0, \delta_1)\|_{\mathcal{X}_B} \leq \eta$ small enough to be chosen later on. We construct a sequence $\widetilde{Y}_n$ of $Z^1(\Omega)$ functions using Proposition 2.17, accommodating for the two orthogonality conditions at each step. We take $\widetilde{Y}_0 := 0$ and, for $n \in \mathbb{N}$, given $\widetilde{Y}_n \in Z^1(\Omega)$, we define $\widetilde{Y}_{n+1} \in Z^1(\Omega)$ as the solution to

$$\begin{cases} z\partial_x \widetilde{Y}_{n+1} - \partial_{zz}\widetilde{Y}_{n+1} = N_B(f, \widetilde{Y}_n) + \nu^0_{n+1} f^0 + \nu^1_{n+1} f^1, \\ (\widetilde{Y}_{n+1})_{|\Sigma_i} = \Upsilon[\delta_i] + \nu^0_{n+1}\delta^0_i + \nu^1_{n+1}\delta^1_i, \\ (\widetilde{Y}_{n+1})_{|z=\pm 1} = 0, \end{cases}$$

where the triplets $(f^k, \delta^k_0, \delta^k_1) \in \mathcal{X}_B$ for $k \in \{0,1\}$ are such that $\overline{\ell^j}(f^k, \delta^k_0, \delta^k_1) = \mathbf{1}_{j=k}$ and are constructed as in Corollary 2.19 and

$$\nu^j_{n+1} := -\overline{\ell^j}(N_B(f, \widetilde{Y}_n), \Upsilon[\delta_0], \Upsilon[\delta_1]).$$

This choice ensures that the two orthogonality conditions

$$\begin{aligned} \overline{\ell^j}\big(N_B(f, \widetilde{Y}_n) + \nu^0_{n+1} f^0 + \nu^1_{n+1} f^1, \\ \Upsilon[\delta_0] + \nu^0_{n+1}\delta^0_0 + \nu^1_{n+1}\delta^1_0, \\ \Upsilon[\delta_1] + \nu^0_{n+1}\delta^0_1 + \nu^1_{n+1}\delta^1_1\big) = 0 \end{aligned}$$

are satisfied. One checks that Proposition 2.17 can be applied, yielding $\widetilde{Y}_{n+1} \in Z^1(\Omega)$. One can then prove that $(\widetilde{Y}_n)_{n\in\mathbb{N}}$ is uniformly bounded by $C\eta$ and is a Cauchy sequence in $Z^1(\Omega)$.

*Proof using our abstract toolbox.* More precisely, this result follows from Theorem 6, applied with the following setting: $\mathcal{H}_B$ defined in (4.10) $\mathcal{X}_B$ defined in (4.11),

- the solution space

  $$\mathcal{Z}_B := \left\{u \in Z^1(\Omega);\, u_{|z=\pm 1} = 0,\, \partial_{zz}u(x_i, z)/z \in \mathcal{H}^1_z(\Sigma_i),\, \partial_{zz}u(x_i, (-1)^i) = 0\right\},$$

  with

  $$\|u\|_{\mathcal{Z}_B} := \|u\|_{Z^1} + \sum_{i\in\{0,1\}} \|\partial_{zz}u(x_i, z)/z\|_{\mathcal{H}^1_z(\Sigma_i)};$$

- $d = 2$, $\ell := (\overline{\ell^0}, \overline{\ell^1})_{|\mathcal{H}_B}$ defined in Definition 2.12, continuous on $\mathcal{H}_B$ by Lemma 2.13 and $\mathcal{H}_B \hookrightarrow \mathcal{H}_K$, satisfying $\ell(\mathcal{X}_B) = \mathbb{R}^2$ by Proposition 2.15, since $C^\infty_c(\Omega) \times \{0\} \times \{0\} \subset \mathcal{X}_B$;
- $L : \mathcal{Z}_B \to \mathcal{H}_B$ defined by $Lu := (z\partial_x u - \partial_{zz}u, u_{|\Sigma_0}, u_{|\Sigma_1})$, for which one easily checks that the assumption Item (i) of Theorem 6 is satisfied thanks to Proposition 2.17;
- $N : \mathcal{X}_B \times \mathcal{Z}_B \to \mathcal{H}_B$ defined by $N(\Xi, \widetilde{Y}) := (N_B(f, \widetilde{Y}), \Upsilon[\delta_0], \Upsilon[\delta_1])$. To prove that $N$ takes values in $\mathcal{H}_B \subset \mathcal{H}_K$, we must check that
  (a) $N_B(f, \widetilde{Y}) \in H^1_x L^2_z$: this follows from Lemmas 4.6 and 4.9 below;
  (b) $\Upsilon[\delta_i] \in \mathcal{H}^1_z(\Sigma_i)$ and $\partial_{zz}\Upsilon[\delta_i]/z \in \mathcal{H}^1_z(\Sigma_i)$: this follows essentially from the chain rule, and is proved in Lemma 4.15 below;

(c) $(N_B(f, \widetilde{Y})/z)_{|\Sigma_i} \in \mathcal{H}^1_z(\Sigma_i)$: this follows from Corollary 4.13 below;

(d) $N_B(f, \widetilde{Y})(x_i, (-1)^i) = 0$: this property follows from the fact that, for $\Xi \in \mathcal{X}_B$, $f_{|\Sigma_i} = 0$ and, for $\widetilde{Y} \in \mathcal{Z}_B$, $\partial_{zz}\widetilde{Y}(x_i, (-1)^i) = 0$;

(e) $\Upsilon[\delta_i]((-1)^i) = \partial_{zz}\Upsilon[\delta_i]((-1)^i) = 0$: this follows from the properties $\delta_i((-1)^i) = \delta_i''((-1)^i)$, see also the proof of Lemma 4.15 below.

Eventually, we claim that $N$ is strongly Fréchet-differentiable at $(0, 0)$ in the sense of Definition 3.14 with $\partial_u N(0, 0) = 0$ and $\partial_\Xi N(0, 0) = \mathrm{Id}$, which corresponds to the following estimate, as $\Xi, \Xi' \in \mathcal{X}_B$ and $\widetilde{Y}, \widetilde{Y}' \in \mathcal{Z}_B$ go to 0,

$$\begin{aligned}\|(N_B(f, \widetilde{Y}), \Upsilon[\delta_0], \Upsilon[\delta_1]) - (N_B(f', \widetilde{Y}'), \Upsilon[\delta_0'], \Upsilon[\delta_1']) - (\Xi - \Xi')\|_{\mathcal{H}_B} \\ = o\big(\|\Xi - \Xi'\|_{\mathcal{X}_B} + \|\widetilde{Y} - \widetilde{Y}'\|_{\mathcal{Z}_B}\big).\end{aligned} \tag{4.14}$$

This follows from Corollary 4.10 for the part involving $N_B$, and from Corollary 4.13 and Corollary 4.16 for the estimate of the boundary terms. ■

Note also that Item (iii) from Theorem 6 follows from a variant of Corollary 2.19 for the space $\mathcal{H}_B$, stepping on Proposition 2.15 and recalling that $\ell_N = \ell$.

The next sections are dedicated to the proof of Item (a) and Item (c) and of estimate (4.14) above. We will repeatedly use the following classical result.

**Lemma 4.4.** *The pointwise product is (bilinearly) continuous from $H^1_x H^1_z \times H^1_x L^2_z$ to $H^1_x L^2_z$.*

### 4.2.1 Forcing term

We first derive estimates for the main forcing term $(1 - \partial_z \widetilde{Y}) f(x, z - \widetilde{Y}(x, z))$ from (4.9). We start with an easy one-dimensional lemma.

**Lemma 4.5.** *For $\phi, \psi \in (H^2 \cap H^1_0)(-1, 1)$ small enough (so that the changes of variables $z \mapsto z - \phi(z)$ and $z \mapsto z - \psi(z)$ are well defined on $[-1, 1]$) and $f \in H^1(-1, 1)$, one has*

$$\|f(z - \phi(z))\|_{L^2} \lesssim \|f\|_{L^2}, \tag{4.15}$$

$$\|f(z - \phi(z)) - f(z - \psi(z))\|_{L^2} \lesssim \|\partial_z f\|_{L^2} \|\phi - \psi\|_{L^\infty}. \tag{4.16}$$

*Proof.* First, (4.15) is straightforward since the Jacobian of the change of variables $z \mapsto z - \phi(z)$ is bounded from below and from above for $\phi$ small enough in $(H^2 \cap H^1_0)(-1, 1)$.

Second, for $z \in [-1, 1]$, we write

$$f(z - \phi(z)) - f(z - \psi(z)) = (\psi(z) - \phi(z)) \int_0^1 \partial_z f(z - s\phi(z) - (1 - s)\psi(z))\, \mathrm{d}s.$$

Hence, by Cauchy–Schwarz,

$$\|f(z-\phi(z))-f(z-\psi(z))\|_{L^2}^2 \\ \lesssim \|\phi-\psi\|_{L^\infty}^2 \int_0^1 \|\partial_z f(z-(s\phi(z)+(1-s)\psi(z)))\|_{L^2}^2 \,\mathrm{d}s$$

so that (4.16) follows from (4.15) applied to $\partial_z f$ and $s\phi+(1-s)\psi$. ■

The next lemma states that the main forcing term belongs to $H_x^1 L_z^2$.

**Lemma 4.6.** *For $\phi,\psi \in H_x^1(H_z^2\cap H_0^1)$ small enough and $f\in H_x^1 H_z^2$, one has*

$$\|(1-\partial_z\phi)f(x,z-\phi)\|_{H_x^1L_z^2} \lesssim \|f\|_{H_x^1H_z^2},$$
$$\|f(x,z-\phi(x,z))-f(x,z-\psi(x,z))\|_{H_x^1L_z^2} \lesssim \|f\|_{H_x^1H_z^2}\|\phi-\psi\|_{H_x^1H_z^2}.$$

*Proof.* First, we observe that $\partial_z\phi,\partial_z\psi\in L^\infty$, and $\partial_z f\in L^\infty$. Since

$$\partial_x(f(x,z-\phi)) = \partial_x f(x,z-\phi) - \partial_x\phi\partial_z f(x,z-\phi)$$

we infer that $f(x,z-\phi)\in H_x^1L_z^2$. From there, we easily deduce the first estimate.

We then turn towards the second estimate. By the chain rule and the triangular inequality, one has

$$\begin{aligned}\|f(x,z-\phi)-f(x,z-\psi)\|_{H_x^1L_z^2} &\lesssim \|f(x,z-\phi)-f(x,z-\psi)\|_{L_x^2L_z^2}\\ &+\|\partial_x f(x,z-\phi)-\partial_x f(x,z-\psi)\|_{L_x^2L_z^2}\\ &+\|(\partial_z f(x,z-\phi)-\partial_z f(x,z-\psi))\phi_x\|_{L_x^2L_z^2}\\ &+\|\partial_z f(x,z-\psi)(\phi_x-\psi_x)\|_{L_x^2L_z^2}.\end{aligned}$$

By (4.16), the first two terms are bounded by $\|\partial_z f\|_{L^2}\|\phi-\psi\|_{L^\infty}$, $\|\partial_{xz} f\|_{L^2}\|\phi-\psi\|_{L^\infty}$.

For the third term, using (4.16),

$$\begin{aligned}&\|(\partial_z f(x,z-\phi)-\partial_z f(x,z-\psi))\phi_x\|_{L_x^2L_z^2}\\ &\quad\le \|\partial_z f(x,z-\phi)-\partial_z f(x,z-\psi)\|_{L_x^\infty L_z^2}\|\phi_x\|_{L_x^2L_z^\infty}\\ &\quad\lesssim \|\partial_{zz} f\|_{L_x^\infty L_z^2}\|\phi-\psi\|_{L^\infty}\|\phi_x\|_{L_x^2L_z^\infty}.\end{aligned}$$

For the fourth term, using (4.15),

$$\begin{aligned}\|\partial_z f(x,z-\psi)(\phi_x-\psi_x)\|_{L_x^2L_z^2} &\le \|\partial_z f(x,z-\psi)\|_{L_x^\infty L_z^2}\|\phi_x-\psi_x\|_{L_x^2L_z^\infty}\\ &\lesssim \|\partial_z f\|_{L_x^\infty L_z^2}\|\phi_x-\psi_x\|_{L_x^2L_z^\infty}.\end{aligned}$$

Gathering these inequalities concludes the proof using usual Sobolev embeddings. ■

We then prove the strong Fréchet-differentiability at (0, 0) of the main forcing term.

**Lemma 4.7.** *For $\phi_1, \phi_2 \in H^1_x(H^2_z \cap H^1_0)$ small enough and $f_1, f_2 \in H^1_x H^2_z$,*

$$\|(1 - \partial_z\phi_1) f_1(x, z - \phi_1) - (1 - \partial_z\phi_2) f_2(x, z - \phi_2) - (f_1 - f_2)\|_{H^1_x L^2_z}$$
$$\lesssim (\|f_1\| + \|f_2\| + \|\phi_1\| + \|\phi_2\|) \times (\|\phi_1 - \phi_2\| + \|f_1 - f_2\|),$$

*where $\|\cdot\|$ on the right-hand side denotes the $H^1_x H^2_z$ norm.*

*Proof.* First, we write

$$f_1(x, z - \phi_1) - f_2(x, z - \phi_2) - (f_1 - f_2)$$
$$= (f_1 - f_2)(x, z - \phi_1) - (f_1 - f_2)(x, z - 0) + f_2(x, z - \phi_1) - f_2(x, z - \phi_2).$$

Applying Lemma 4.6 to both lines, we have

$$\|f_1(x, z - \phi_1) - f_2(x, z - \phi_2) - (f_1 - f_2)\|_{H^1_x L^2_z}$$
$$\lesssim \|f_1 - f_2\|_{H^1_x H^2_z}\|\phi_1 - 0\|_{H^1_x H^2_z} + \|f_2\|_{H^1_x H^2_z}\|\phi_1 - \phi_2\|_{H^1_x H^2_z},$$

which allows to conclude the proof thanks to Lemma 4.4. ∎

### 4.2.2 Nonlinear viscous term

We derive estimates for the main nonlinear viscous term $\frac{\partial_z \widetilde{Y}(2-\partial_z \widetilde{Y})}{(1-\partial_z \widetilde{Y})^2}\partial_{zz}\widetilde{Y}$ from (4.9).

**Lemma 4.8.** *Let $g : \mathbb{R} \to \mathbb{R}$ be a $C^3$ function in a neighborhood of 0 with $g(0) = 0$. Then the map $G : H^1_x H^2_z \to H^1_x H^1_z$ given by $G(\phi) := g(\phi_z)$ is well defined and Lipschitz-continuous in a neighborhood of 0, and satisfies $G(0) = 0$.*

*Proof.* First, for $\phi, \psi \in H^1_x H^2_z$ small enough,

$$\|g(\phi_z) - g(\psi_z)\|_{L^\infty} \le \|g'\|_{L^\infty}\|\phi_z - \psi_z\|_{L^\infty} \lesssim \|\phi_z - \psi_z\|_{H^1_x H^1_z}.$$

Second, for $\phi \in H^1_x H^2_z$ small enough,

$$\partial_{xz}(g(\phi_z)) = g'(\phi_z)\phi_{xzz} + g''(\phi_z)\phi_{xz}\phi_{zz}.$$

Hence, for $\phi, \psi \in H^1_x H^2_z$ small enough, using that $g \in C^3$ and decomposing the difference, one obtains

$$\|\partial_{xz}(g(\phi_z) - g(\psi_z))\|_{L^2} \lesssim \|\phi - \psi\|_{H^1_x H^2_z},$$

which concludes the proof. ∎

The next lemma proves that the nonlinear viscous term belongs to $H^1_x L^2_z$.

**Lemma 4.9.** *For $\phi, \psi \in H^1_x H^2_z$ small enough, one has*

$$\left\| \frac{\partial_z \phi (2 - \partial_z \phi)}{(1 - \partial_z \phi)^2} \partial_{zz} \phi - \frac{\partial_z \psi (2 - \partial_z \psi)}{(1 - \partial_z \psi)^2} \partial_{zz} \psi \right\|_{H^1_x L^2_z} \lesssim (\|\phi\|_{H^1_x H^2_z} + \|\psi\|_{H^1_x H^2_z}) \|\phi - \psi\|_{H^1_x H^2_z}.$$

*Proof.* Since $\partial_{zz}$ is Lipschitz-continuous from $H^1_x H^2_z$ to $H^1_x L^2_z$, by Lemma 4.4, the result follows from the Lipschitz-continuity of $\phi \mapsto \partial_z \phi (2 - \partial_z \phi)(1 - \partial_z \phi)^{-2}$ from $H^1_x H^2_z$ to $H^1_x H^1_z$, which is a consequence of Lemma 4.8 with the relation $g(s) := s(2 - s)(1 - s)^{-2}$. ■

Gathering Lemma 4.7 (for the part involving $f$) and Lemma 4.9 (for the quadratic part involving $\widetilde{Y}$ only), we obtain the Fréchet-differentiability of $N_B$ at $(0, 0)$.

**Corollary 4.10.** *For $\Xi = (f, \delta_0, \delta_1)$, $\Xi' = (f', \delta_0', \delta_1') \in \mathcal{X}_B$ and $\widetilde{Y}, \widetilde{Y}' \in \mathcal{Z}_B$ small enough,*

$$\|N_B(f, \widetilde{Y}) - N_B(f', \widetilde{Y}') - (f - f')\|_{H^1_x L^2_z} = o(\|\Xi - \Xi'\|_{\mathcal{X}_B} + \|\widetilde{Y} - \widetilde{Y}'\|_{\mathcal{Z}_B}).$$

### 4.2.3 Boundary contribution of the nonlinearity

We now derive estimates concerning the $\mathcal{H}^1_z(\Sigma_i)$ contribution of $N_B(f, \widetilde{Y})$.

**Lemma 4.11.** *For $\psi \in \mathcal{H}^1_z(0, 1)$, one has $z\psi \in L^\infty(0, 1)$ with $\|z\psi\|_{L^\infty} \lesssim \|\psi\|_{\mathcal{H}^1_z}$.*

*Proof.* Let $\psi \in \mathcal{H}^1_z(0, 1)$. First, $\psi \in H^1(1/2, 1)$ and one has $|\psi(1)| \lesssim \|\psi\|_{\mathcal{H}^1_z}$. Thus, for $z_0 \in (0, 1)$,

$$|\psi(z_0)| \leq |\psi(1)| + \int_{z_0}^1 |\psi_z| \leq |\psi(1)| + |\ln z_0|^{\frac{1}{2}} \|z^{1/2} \psi_z\|_{L^2(0,1)} \lesssim |\ln z_0|^{\frac{1}{2}} \|\psi\|_{\mathcal{H}^1_z},$$

which proves that $|\ln z|^{-\frac{1}{2}} \psi \in L^\infty$, so that, in particular, $z\psi \in L^\infty$. ■

**Lemma 4.12.** *Let $g : \mathbb{R} \to \mathbb{R}$ be a $C^2$ function in a neighborhood of 0 with $g(0) = 0$. Let $\mathcal{E} := \{\psi \in L^2(0, 1); \psi_{zz}/z \in \mathcal{H}^1_z\}$ with the associated canonical norm. There exists $\eta > 0$ small enough such that, for $\phi, \psi \in \mathcal{E}$ with $\|\phi\|_{\mathcal{E}} \leq \eta$ and $\|\psi\|_{\mathcal{E}} \leq \eta$,*

$$\left\| \frac{g(\phi_z)\phi_{zz}}{z} - \frac{g(\psi_z)\psi_{zz}}{z} \right\|_{\mathcal{H}^1_z} \lesssim (\|\phi\|_{\mathcal{E}} + \|\psi\|_{\mathcal{E}}) \|\phi - \psi\|_{\mathcal{E}}.$$

*Proof.* First, $\mathcal{E} \hookrightarrow H^2 \hookrightarrow W^{1,\infty}$ and, thanks to Lemma 4.11, $\mathcal{E} \hookrightarrow W^{2,\infty}$. We write

$$\frac{g(\phi_z)\phi_{zz}}{z} - \frac{g(\psi_z)\psi_{zz}}{z} = (g(\phi_z) - g(\psi_z)) \frac{\phi_{zz}}{z} + g(\psi_z) \frac{\phi_{zz} - \psi_{zz}}{z}.$$

For the first term, we have

$$\left\| (g(\phi_z) - g(\psi_z)) \frac{\phi_{zz}}{z} \right\|_{\mathcal{H}^1_z} \lesssim \|\phi_z - \psi_z\|_{L^\infty} \|\phi_{zz}/z\|_{\mathcal{H}^1_z} + \|\phi_{zz}/z\|_{\mathcal{L}^2_z} \|\partial_z(g(\phi_z) - g(\psi_z))\|_{L^\infty},$$

where

$$\begin{aligned}\|\partial_z(g(\phi_z) - g(\psi_z))\|_{L^\infty} &\leq \|(g'(\phi_z) - g'(\psi_z))\phi_{zz}\|_{L^\infty} + \|g'(\phi_z)(\phi_{zz} - \psi_{zz})\|_{L^\infty} \\ &\lesssim \|\phi_z - \psi_z\|_{L^\infty} \|\phi_{zz}\|_{L^\infty} + \|\phi_{zz} - \psi_{zz}\|_{L^\infty}.\end{aligned}$$

For the second term,

$$\left\| g(\psi_z) \frac{\phi_{zz} - \psi_{zz}}{z} \right\|_{\mathcal{H}^1_z} \lesssim \|g(\psi_z)\|_{L^\infty} \|(\phi_{zz} - \psi_{zz})/z\|_{\mathcal{H}^1_z} + \|(\phi_{zz} - \psi_{zz})/z\|_{\mathcal{L}^2_z} \|g'(\psi_z)\psi_{zz}\|_{L^\infty}.$$

Hence, the claimed estimate follows from the embedding $\mathcal{E} \hookrightarrow W^{2,\infty}$. ∎

We infer that the boundary contribution of the nonlinearity is Fréchet-differentiable at $(0, 0)$.

**Corollary 4.13.** *For $\Xi = (f, \delta_0, \delta_1)$, $\Xi' = (f', \delta_0', \delta_1') \in \mathcal{X}_B$ and $\widetilde{Y}, \widetilde{Y}' \in \mathcal{Z}_B$ small enough, $z^{-1} N_B(f, \widetilde{Y})|_{\Sigma_i} \in \mathcal{H}^1_z(\Sigma_i)$ and $z^{-1} N_B(f', \widetilde{Y}')|_{\Sigma_i} \in \mathcal{H}^1_z(\Sigma_i)$ and*

$$\left\| \frac{N_B(f, \widetilde{Y}) - N_B(f', \widetilde{Y}')}{z} \right\|_{\mathcal{H}^1_z(\Sigma_i)} = o(\|\widetilde{Y} - \widetilde{Y}'\|_{\mathcal{Z}_B}).$$

*Proof.* Since $\Xi \in \mathcal{X}_B$, $f|_{\Sigma_i} = 0$. Thus

$$N_B(f, \widetilde{Y})|_{\Sigma_i} = g(\partial_z \widetilde{Y}|_{\Sigma_i}) \partial_{zz} \widetilde{Y}|_{\Sigma_i} \quad \text{with } g(a) = \frac{a(2-a)}{(1-a)^2}.$$

The result follows from Lemma 4.12, noting that, for $\widetilde{Y} \in \mathcal{Z}_B$, $\widetilde{Y}|_{\Sigma_i} \in \mathcal{E}$ of Lemma 4.12. ∎

### 4.2.4 Contribution of the inversion of the boundary data

We now move on to estimates concerning the Fréchet-differentiability of the map $\Upsilon$ of (4.6).

**Lemma 4.14.** *For $\phi \in H^2(0, 1)$ such that $\phi(0) = 0$,*

$$\|\phi(z)/z\|_{H^1} \lesssim \|\phi\|_{H^2}.$$

*Proof.* Writing a second-order Taylor expansion, one has

$$\phi(z) = z\phi'(0) + \int_0^z (z-s)\phi''(s)\,\mathrm{d}s.$$

Thus

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{\phi(z)}{z}\right) = \frac{1}{z^2}\int_0^z s\phi''(s)\,\mathrm{d}s,$$

from which the conclusion follows by the Hardy inequality of Lemma B.8. ∎

**Lemma 4.15.** *Consider the spaces*

$$\mathcal{E}_1 := \left\{\delta \in \mathcal{H}^1_z(0,1); \delta''(z)/z \in \mathcal{H}^1_z(0,1), \delta(0)=\delta(1)=\delta''(0)=\delta''(1)=0\right\}, \quad (4.17)$$
$$\mathcal{E}_2 := \left\{\delta \in H^5(0,1); \delta(0) = \delta(1) = \delta''(0) = \delta''(1) = 0\right\}. \quad (4.18)$$

*Then the map* $\Upsilon[\delta](z) := z - (\cdot + \delta(\cdot))^{-1}(z)$ *as in* (4.6) *is well defined for* $\delta$ *small enough and strongly Fréchet-differentiable at* 0 *from* $\mathcal{E}_2$ *to* $\mathcal{E}_1$. *More precisely, for* $\delta, \eta \in \mathcal{E}_2$ *small enough,*

$$\|\Upsilon[\delta] - \Upsilon[\eta] - (\delta - \eta)\|_{\mathcal{E}_1} \lesssim (\|\delta\|_{\mathcal{E}_2} + \|\eta\|_{\mathcal{E}_2})\|\delta - \eta\|_{\mathcal{E}_2}. \quad (4.19)$$

*Proof. Step 1. We first check that* $\Upsilon$ *is well defined.* Since $\mathcal{E}_2 \hookrightarrow W^{1,\infty}$, $\tilde{\delta} := \Upsilon[\delta]$ is well defined for $\delta \in \mathcal{E}_2$ small enough, and the boundary conditions $\delta(0) = \delta(1) = 0$ of $\mathcal{E}_2$ entail that $\tilde{\delta}(0) = \tilde{\delta}(1) = 0$.

Moreover, one has

$$\tilde{\delta}(z) = \delta(z - \tilde{\delta}(z)). \quad (4.20)$$

From this relation, we derive that

$$\tilde{\delta}'(z) = \frac{\delta'}{1+\delta'}(z - \tilde{\delta}(z)) \quad \text{and} \quad \tilde{\delta}''(z) = \frac{\delta''}{(1+\delta')^3}(z - \tilde{\delta}(z)) \quad (4.21)$$

which ensures that $\tilde{\delta}''(0) = \tilde{\delta}''(1) = 0$ since $\delta''(0) = \delta''(1) = 0$.

*Step 2. We prove the strong Fréchet-differentiability at* 0. To control the $\mathcal{E}_1$ norm, it suffices to control the $L^2$ norm and the $\mathcal{H}^1_z$ norm of the quotient $\partial_{zz}(\cdot)/z$. For $\delta, \eta \in \mathcal{E}_2$ by (4.20),

$$(\tilde{\delta} - \tilde{\eta})(z) = (\delta - \eta)(z - \tilde{\delta}) + (\eta(z - \tilde{\delta}) - \eta(z - \tilde{\eta})). \quad (4.22)$$

Hence

$$\|\tilde{\delta} - \tilde{\eta}\|_{L^\infty} \le \|\delta - \eta\|_{L^\infty} + \|\partial_z \eta\|_{L^\infty}\|\tilde{\delta} - \tilde{\eta}\|_{L^\infty}.$$

In particular, for $\eta$ small enough in $\mathcal{E}_2$,

$$\|\tilde{\delta} - \tilde{\eta}\|_{L^\infty} \le 2\|\delta - \eta\|_{L^\infty}.$$

Thus, applying estimate (4.16) to (4.22), we obtain

$$\begin{aligned}\|(\tilde{\delta}-\tilde{\eta})-(\delta-\eta)\|_{L^2} &\leq \|(\delta-\eta)(\cdot-\tilde{\delta})-(\delta-\eta)(\cdot)\|_{L^2}+\|\eta(\cdot-\tilde{\delta})-\eta(\cdot-\tilde{\eta})\|_{L^2}\\ &\lesssim \|\partial_z(\delta-\eta)\|_{L^2}\|\tilde{\delta}\|_{L^\infty}+\|\partial_z\eta\|_{L^2}\|\tilde{\delta}-\tilde{\eta}\|_{L^\infty}\\ &\lesssim \|\partial_z(\delta-\eta)\|_{L^2}\|\delta\|_{L^\infty}+\|\partial_z\eta\|_{L^2}\|\delta-\eta\|_{L^\infty}\\ &\lesssim (\|\delta\|_{H^1}+\|\eta\|_{H^1})\|\delta-\eta\|_{H^1}. \end{aligned} \tag{4.23}$$

We now move to the estimate of the $\mathcal{H}^1_z$ norm of the quotient $\partial_{zz}(\cdot)/z$. By Lemma 4.14 (which even yields an $H^1$ estimate, not only $\mathcal{H}^1_z$), since all our functions have null second derivative at 0, it suffices to obtain an $H^4$ estimate. Differentiating (4.21) twice, we obtain

$$\partial_z^4\tilde{\delta}(z)=\frac{\partial_z^4\delta}{(1+\partial_z\delta)^5}(z-\tilde{\delta}(z))+\text{lower order terms.}$$

Decomposing the difference in a similar manner as in (4.23) and applying (4.16), one can prove

$$\|(\tilde{\delta}-\tilde{\eta})-(\delta-\eta)\|_{H^4}=(\|\delta\|_{H^5}+\|\eta\|_{H^5})\|\delta-\eta\|_{H^5}.$$

Together with Lemma 4.14, this concludes the proof of (4.19). ∎

**Corollary 4.16.** *For* $\Xi=(f,\delta_0,\delta_1)$, $\Xi'=(f',\delta_0',\delta_1')\in\mathcal{X}_B$ *small enough,*

$$\|(0,\Upsilon[\delta_0],\Upsilon[\delta_1])-(0,\Upsilon[\delta_0'],\Upsilon[\delta_1'])-(0,\delta_0-\delta_0',\delta_1-\delta_1')\|_{\mathcal{H}_B}=o(\|\Xi-\Xi'\|_{\mathcal{X}_B}).$$

*Proof.* Recalling that, for $\Xi=(f,\delta_0,\delta_1)\in\mathcal{X}_B$, $f|_{\Sigma_i}=0$, this is a direct consequence of Lemma 4.15 and the definitions (4.10) and (4.11) of $\mathcal{H}_B$ and $\mathcal{X}_B$. ∎

## 4.3 Reverse change of variables

*Proofs of Theorem 3 and Proposition 1.1.* It only remains to prove that the change of variables of Section 4.1 is justified in both directions.

First, given $(f,\delta_0,\delta_1)\in\mathcal{M}_B$, let $\widetilde{Y}\in Z^1$ be the solution to (4.8) given by Proposition 4.3 and let $Y(x,z):=z-\widetilde{Y}(x,z)$ the associated solution to (4.5). By Proposition 4.3, $\|Y\|_{Z^1}\lesssim 1+\|(f,\delta_0,\delta_1)\|_{\mathcal{X}_B}$. By Lemma 1.15, $\|Y\|_{Q^1}\lesssim 1+\|(f,\delta_0,\delta_1)\|_{\mathcal{X}_B}$. Since $Y$ is a solution to (4.5), we have

$$\partial_{zz}Y=(\partial_z Y)^2(z\partial_x Y)-(\partial_z Y)f(x,Y(x,z)).$$

We check that the right-hand side is $L^2_xH^1_z$, from which we deduce that $\partial_z^3Y\in L^2$. Repeating this argument, we find that the right-hand side of the above equation is in fact $L^2_xH^2_z$ and that

$$\|\partial_z^4Y\|_{L^2}\lesssim\|Y\|_{Z^1}+\|f\|_{L^2_xH^2_y}.$$

Thus, $Y \in Q^1 \cap L^2_x H^4_z$ and

$$\|Y(x,z) - z\|_{Q^1} + \|Y(x,z) - z\|_{L^2_x H^4_z} \lesssim \|(f, \delta_0, \delta_1)\|_{\mathcal{X}_B}.$$

By Corollary B.6, (4.2) defines a $u \in Q^1 \cap L^2_x H^4_y$ such that

$$\|u(x,y) - y\|_{Q^1} + \|u(x,y) - y\|_{L^2_x H^4_y} \lesssim \|(f, \delta_0, \delta_1)\|_{\mathcal{X}_B}.$$

In particular, since $\partial_y u$ and $\partial_z Y$ are continuous functions on $\Omega$ with $\|\partial_y u - 1\|_{L^\infty} \ll 1$ and $\|\partial_z Y - 1\|_{L^\infty} \ll 1$, the computations of Section 4.1 hold. Thus, we have constructed a $u \in Q^1$ solution to (4.1). This proves the existence claim of Theorem 3.

Reciprocally, to prove the claim of Theorem 3 concerning the uniqueness of the solution to (4.1) and the one of Proposition 1.1 concerning the necessity of the nonlinear orthogonality conditions $(f, \delta_0, \delta_1) \in \mathcal{M}_B$, we must perform the reasoning in the other direction. Let $(f, \delta_0, \delta_1) \in \mathcal{X}_B$ small enough, and let $u \in Q^1$ be a solution to (1.8) such that $\|u\|_{Q^1} \ll 1$. Writing the PDE as

$$\partial_y^2 u = u \partial_x u - f,$$

we obtain that $u \in Q^1 \cap L^2_x H^4_y$. By Corollary B.6, (4.2) defines a function $Y \in Q^1 \cap L^2_x H^4_z$ such that

$$\|Y(x,z) - z\|_{Q^1} + \|Y(x,z) - z\|_{L^2_x H^4_z} \lesssim \|u(x,y) - y\|_{Q^1} + \|u(x,y) - y\|_{L^2_x H^4_z} \ll 1.$$

In particular, since $\partial_y u$ and $\partial_z Y$ are continuous functions on $\Omega$ with $\|\partial_y u - 1\|_{L^\infty} \ll 1$ and $\|\partial_z Y - 1\|_{L^\infty} \ll 1$, the computations of Section 4.1 hold. Thus, $Y$ is a solution to (4.5). Since $Y \in Q^1 \cap L^2_x H^4_z$, we have $Y \in H^1_x H^2_z$. From the equation (4.5), we recover that $z \partial_x (\partial_x Y) \in L^2$. Thus $Y \in Z^1(\Omega)$. Hence, the conclusions of Proposition 4.3 apply: $Y$ is unique and $(f, \delta_0, \delta_1) \in \mathcal{M}_B$. ∎