Chapter 5

The Prandtl system in the recirculation zone

Let us now continue our analysis of nonlinear parabolic forward-backward systems by considering the Prandtl equation in the vicinity of a recirculating flow $(\mathbb{u}_P, \mathbb{v}_P)$, revisiting the results of Iyer and Masmoudi from [34, 35]. Throughout this chapter, the index t stands for 'top' and the index b for 'bottom'. We refer to Section 1.1.4 of the introduction for the assumptions on $(\mathbb{u}_P, \mathbb{v}_P)$.

We consider the system

$$\begin{cases} uu_x + vu_y - u_{yy} = -\partial_x p + f & \text{in } \Omega_P, \\ u_x + v_y = 0 & \text{in } \Omega_P, \end{cases} \tag{5.1}$$

where the pressure gradient $\partial_x p$ is the one associated with $(\mathbb{u}_P, \mathbb{v}_P)$, and where we recall that the domain Ω_P is defined by

$$\Omega_P := \{(x, y) \in (x_0, x_1) \times \mathbb{R}_+; \gamma_b(x) < y < \gamma_t(x)\}. \tag{5.2}$$

This system is endowed with the boundary conditions (1.9), (1.10), (1.11), which we now recall for the reader's convenience:

$$\begin{cases} u|_{y=\gamma_b} = z_b,\ \partial_y u|_{y=\gamma_b} = \partial_y \mathbb{u}_P|_{y=\overline{\gamma_b}} + \delta_b,\ v|_{y=\gamma_b} = \mathbb{v}_P|_{y=\overline{\gamma_b}} + v_b & \text{(bottom BC)}, \\ u|_{y=\gamma_t} = z_t,\ \partial_y u|_{y=\gamma_t} = \partial_y \mathbb{u}_P|_{y=\overline{\gamma_t}} + \delta_t & \text{(top BC)}, \\ u|_{\Sigma_i^P} = \mathbb{u}_P|_{\Sigma_i^P} + \delta_i & \text{(lateral BC)}. \end{cases} \tag{5.3}$$

We recall that the lines $\{y = \gamma_j(x)\}$ for $j \in \{t, b\}$, which are level sets of the function u, are free boundaries which are expected to lie in the vicinity of the level sets $\{y = \overline{\gamma_j}(x)\}$ of the function $\mathbb{u}_P$. We refer to the introduction for further comments on these boundary conditions. See Figure 5.1 for a sketch of the geometry of the domain.

The source term f in (5.1) is a small regular perturbation of the pressure term. From the physical point of view, it is relevant to consider perturbations which depend only on x, since the right-hand side in the Prandtl system is the trace of the pressure gradient of some outer Euler flow on the boundary. However, the analysis is essentially unchanged if we allow f to depend on the vertical variable y, and therefore in the following f will be a smooth function depending on both x and y, for the sake of generality.

Our analysis in this section follows the one from Chapter 4. We first perform in Section 5.1 a nonlinear change of variables in order to straighten the free boundary $\{(x, y); u(x, y) = 0\}$. The whole analysis then takes place in these new variables. One remarkable point lies in the fact that the linear problem associated with the Prandtl

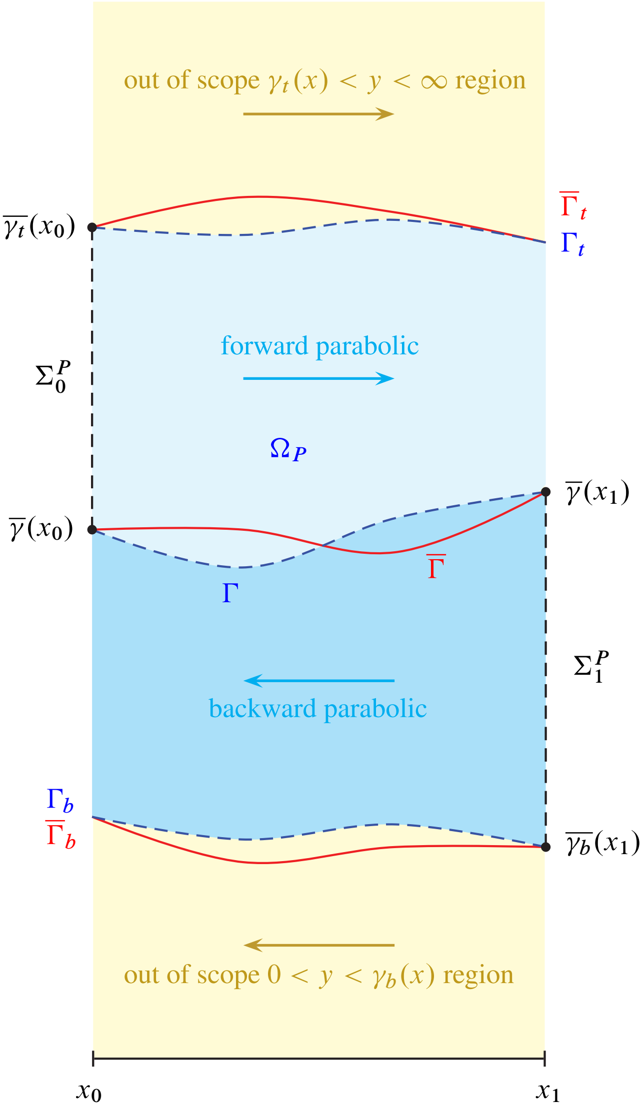

Figure 5.1. Fluid domain Ω_P defined in (5.2) with free top and bottom boundaries Γ_t and Γ_b, and fixed inflow boundaries Σ_0^P and Σ_1^P.

system is similar to, but slightly different from the one for the Burgers equation. In fact, the linear problem associated with the *vorticity* studied in Section 5.2 has the same structure as (1.6). Retrieving the velocity from the vorticity in Section 5.3 gives rise to an additional orthogonality condition. Moreover, since the vorticity plays the same role as the function u from Chapter 4, it turns out that the Prandtl system is actually *more regular* than the Burgers equation (1.1): indeed, there is a gain of one vertical derivative (corresponding to a vertical integration of the velocity) between Burgers and Prandtl. This will allow us to construct solutions with a minimal requirement of regularity, and just one orthogonality condition. We construct solutions to the nonlinear problem in the new variables in Section 5.4, and conclude the proof of Theorem 4 in Section 5.5.

We recall that we focus here on the behavior of the system in the vicinity of the curve $\{u = 0\}$. When studying the system in the whole infinite strip $(x_0, x_1) \times \mathbb{R}_+$, special care must be taken to "glue together" the different zones. As explained in [35], information flows from bottom to top. The analysis of the system in the vicinity of the lower boundary and for large values of y requires specific tools, which go beyond the scope of the present memoir. We refer the interested reader to [34,35] for the study of the Prandtl system in the whole domain, and for a description of the difficulties associated with the interplay between the different zones. We also present in Section 5.6 a potential strategy to construct a solution to the Prandtl system in the whole infinite strip, stepping on the analysis of the present memoir. In particular, we explain why the analysis of the system in an infinite vertical domain may call for an assumption on the horizontal size of the domain $x_1 - x_0$: in [35], the well-posedness of the system holds when $|x_1 - x_0|$ is either small, or outside a countable set (corresponding to the zeros of an analytic function). No such assumption is required when the Prandtl system is studied in the recirculation zone only, see Theorem 4 or Proposition 5.2 below. Let us also recall that our purpose here is merely to present, in a unified framework, different forward-backward problems. Therefore we will put an emphasis on the specific features associated with the Prandtl system in the recirculation zone Ω_P, and on the similarities and differences with the Burgers-type system (1.1) studied in Chapter 4.

5.1 Nonlinear change of variables

At this stage, we assume that a smooth solution to (5.1) exists in order to write the equation in a form that is more amenable to mathematical analysis. We will come back on the justification of the computations below in Section 5.5.

As in Section 4.1, we change variables by setting $(x, z) = (x, u(x, y))$, where u is the unknown tangential velocity. This maps the unknown domain $\Omega_P = \{\gamma_b(x) < y < \gamma_t(x)\}$ depending on the solution u (since the lines γ_b and γ_t are defined by $u(x, \gamma_j(x)) = z_j$ for $j \in \{b, t\}$) to the fixed rectangular domain $(x_0, x_1) \times (z_b, z_t)$.

We denote by $(x, Y(x,z))$ the diffeomorphism such that $u(x, Y(x,z)) = z$. As a consequence, we have the same relations (4.4) between the derivatives of u and Y as for the Burgers case. The top and bottom boundary conditions become $Y(x, z_j) = \gamma_j(x)$ for $j \in \{b, t\}$.

Furthermore, integrating the divergence-free condition and using (1.9),

$$\begin{aligned} v(x, Y(x,z)) &= v|_{\Gamma_b} - \int_{\gamma_b(x)}^{Y(x,z)} \partial_x u(x, y')\, \mathrm{d}y' \\ &= \mathbb{v}_{\mathrm{P}}|_{\overline{\Gamma_b}} + v_b + \int_{z_b}^{z} \frac{\partial_x Y(x,z')}{\partial_z Y(x,z')} \partial_z Y(x,z')\, \mathrm{d}z' \\ &= \mathbb{v}_{\mathrm{P}}|_{\overline{\Gamma_b}} + v_b + \int_{z_b}^{z} \partial_x Y(x,z')\, \mathrm{d}z'. \end{aligned}$$

Replacing this expression and (4.4) into (5.1) and evaluating the equation at $y = Y(x,z)$, we find that

$$-\frac{1}{\partial_z Y}\left[z\partial_x Y - \int_{z_b}^{z} \partial_x Y - \mathbb{v}_{\mathrm{P}}|_{y=\overline{\gamma_b}} - v_b \right] + \frac{1}{(\partial_z Y)^3}\partial_z^2 Y = -\partial_x p + f(x, Y(x,z)).$$

Let us now denote by $\mathbb{Y}_{\mathrm{P}}$ the function such that $\mathbb{u}_{\mathrm{P}}(x, \mathbb{Y}_{\mathrm{P}}(x,z)) = z$. Following the same computations as above, this function satisfies

$$-\frac{1}{\partial_z \mathbb{Y}_{\mathrm{P}}}\left[z\partial_x \mathbb{Y}_{\mathrm{P}} - \int_{z_b}^{z} \partial_x \mathbb{Y}_{\mathrm{P}} - \mathbb{v}_{\mathrm{P}}|_{\overline{\Gamma_b}} \right] + \frac{1}{(\partial_z \mathbb{Y}_{\mathrm{P}})^3}\partial_z^2 \mathbb{Y}_{\mathrm{P}} = -\partial_x p.$$

Let $\widetilde{Y} := \mathbb{Y}_{\mathrm{P}} - Y$. Then

$$\begin{aligned} \frac{1}{(\partial_z \mathbb{Y}_{\mathrm{P}})^2}\partial_z^2 \mathbb{Y}_{\mathrm{P}} - \frac{1}{(\partial_z Y)^2}\partial_z^2 Y &= \partial_z\left(\frac{1}{\partial_z Y} - \frac{1}{\partial_z \mathbb{Y}_{\mathrm{P}}} \right) \\ &= \partial_z\left(\frac{\partial_z \widetilde{Y}}{(\partial_z \mathbb{Y}_{\mathrm{P}})^2} \right) + \partial_z\left(\frac{(\partial_z \widetilde{Y})^2}{(\partial_z \mathbb{Y}_{\mathrm{P}})^2 \partial_z Y} \right). \end{aligned}$$

We obtain eventually the following very simple equation:

$$z\partial_x \widetilde{Y} - \int_{z_b}^{z} \partial_x \widetilde{Y}(x,z')\, \mathrm{d}z' - \partial_x p \partial_z \widetilde{Y} - \partial_z\left(\frac{\partial_z \widetilde{Y}}{(\partial_z \mathbb{Y}_{\mathrm{P}})^2} \right) = g(x,z), \tag{5.4}$$

where

$$g(x,z) := f(x, Y(x,z))\partial_z(\mathbb{Y}_{\mathrm{P}} - \widetilde{Y}) - v_b(x) + \partial_z\left(\frac{(\partial_z \widetilde{Y})^2}{(\partial_z \mathbb{Y}_{\mathrm{P}})^2 \partial_z Y} \right). \tag{5.5}$$

The top and bottom boundary conditions (1.10) and (1.9) become, for $j \in \{t, b\}$,

$$\begin{aligned} \partial_z \widetilde{Y}(x, z_j) &= \partial_z \mathbb{Y}_{\mathrm{P}}(x, z_j) - \partial_z Y(x, z_j) \\ &= \frac{1}{\partial_y \mathbb{u}_{\mathrm{P}}(x, \overline{\gamma_j}(x))} - \frac{1}{\partial_y \mathbb{u}_{\mathrm{P}}(x, \overline{\gamma_j}(x)) + \delta_j(x)} =: \Upsilon_{\mathrm{P}}^j[\delta_j](x). \end{aligned} \tag{5.6}$$

The unknown function γ_j can be retrieved from $\widetilde{Y}$ by

$$\gamma_j(x) = Y(x, z_j) = \mathbb{Y}_{\mathrm{P}}(x, z_j) - \widetilde{Y}(x, z_j) = \overline{\gamma_j}(x) - \widetilde{Y}(x, z_j).$$

We still denote by Σ_0 and Σ_1 the lateral boundaries, i.e., $\Sigma_0 = \{x_0\} \times (0, z_t)$, $\Sigma_1 = \{x_1\} \times (z_b, 0)$. In order to simplify the definition of the functional spaces for the lateral boundary data, we assume $\delta_0(\gamma_{\mathrm{P}}(x_0)) = \delta_1(\gamma_{\mathrm{P}}(x_1)) = \delta_0(\overline{\gamma_t}(x_0)) = \delta_1(\overline{\gamma_b}(x_1)) = 0$. The lateral boundary conditions (1.11) are then given by the implicit equation

$$z = \mathbb{u}_{\mathrm{P}}(x_i, Y(x_i, z)) + \delta_i(Y(x_i, z)) \quad \text{on } \Sigma_i,$$

which becomes, after noticing that $\mathbb{u}_{\mathrm{P}}(x_i, \cdot) + \delta_i$ is strictly increasing on Σ_i^P and therefore bijective from Σ_i^P to Σ_i,

$$\widetilde{Y}(x_i, z) = \mathbb{Y}_{\mathrm{P}}(z) - (\mathbb{u}_{\mathrm{P}}(x_i, \cdot) + \delta_i)^{-1}(z) =: \Upsilon_{\mathrm{P}}^i[\delta_i] \quad \text{on } \Sigma_i. \tag{5.7}$$

For further purposes, we note that the function $\Upsilon_{\mathrm{P}}^j[\delta_j]$ (resp. $\Upsilon_{\mathrm{P}}^i[\delta_i]$) has the same regularity and size as δ_j (resp. δ_i).

Remark 5.1. When $\mathbb{u}_{\mathrm{P}}(x, y) = y$ (linear shear flow), (5.4) simply becomes, at main order

$$z\partial_x \widetilde{Y} + \widetilde{V} - \partial_z^2 \widetilde{Y} = g,$$

where

$$\widetilde{V} = -\int_{z_b}^{z} \partial_x \widetilde{Y}.$$

Differentiating this equation with respect to z, and setting $W := \partial_z \widetilde{Y}$ (W is the vorticity in our new variables) we find

$$z\partial_x W - \partial_z^2 W = \partial_z g.$$

Therefore, when we consider the Prandtl equation in the vicinity of the linear shear flow, the equation for the vorticity in the new variables is (1.6). We retrieve here the following fact, which was already identified by Iyer and Masmoudi in [35]: the Prandtl system in vorticity form is very close to (1.6). This will also be central in our analysis below.

Let us now state our main result on system (5.4). Since we will state two results within different regularity frameworks, we will work with two different functional spaces for the data. Note that since the boundaries γ_b, γ_t are free, we allow the function f to be defined on a domain that is possibly larger, in the vertical direction, than the reference domain $\{(x, y) \in (x_0, x_1) \times (0, +\infty),\ \overline{\gamma_b}(x) < y < \overline{\gamma_t}(x)\}$. Hence, in order to simplify the statements, we assume that f is defined in the whole infinite strip $(x_0, x_1) \times (0, +\infty)$.

In the low regularity setting, we choose an index $\sigma \in (0, 1/6)$. Our function space will be

$$\begin{aligned}\mathcal{X}^\sigma := \big\{(f, \delta_0, \delta_1, \delta_t, \delta_b, v_b) \mid\ & f \in H^\sigma_x H^2_z, \\ & f \in L^4_x H^3_z \cap H_x^{\frac12+\sigma} H^1_z \cap L^\infty_x W^{2,\infty}_z, (x-x_0)(x-x_1)\partial_x\partial_z f \in L^2, \\ & \delta_i \in H^4(\Sigma_i^P), \delta_t, \delta_b \in H^2(x_0, x_1), v_b \in H^\sigma(x_0, x_1), \\ & \delta_0(\gamma_{\mathrm{P}}(x_0)) = \delta_1(\gamma_{\mathrm{P}}(x_1)) = \delta_0(\overline{\gamma_t}(x_0)) = \delta_1(\overline{\gamma_b}(x_1)) = 0, \\ & \Upsilon^t_{\mathrm{P}}[\delta_t](x_0) = \partial_z \Upsilon^0_{\mathrm{P}}[\delta_0](z_t), \Upsilon^b_{\mathrm{P}}[\delta_b](x_1) = \partial_z \Upsilon^1_{\mathrm{P}}[\delta_1](z_b)\big\},\end{aligned} \tag{5.8}$$

which we endow with its canonical norm.

In the high regularity setting, our function space will be

$$\begin{aligned}\mathcal{X}^1 := \big\{(f, \delta_0, \delta_1, \delta_t, \delta_b, v_b) \mid\ & f \in H^1_x H^3_z \text{ with } f|_{\Sigma_i^P} = 0, \\ & \delta_i \in H^6(\Sigma_i^P), \delta_t, \delta_b \in H^2(x_0, x_1), v_b \in H^1(x_0, x_1), \\ & \delta_0(\overline{\gamma_t}(x_0)) = \delta_1(\overline{\gamma_b}(x_1)) = \partial_z^k \delta_i(\gamma_{\mathrm{P}}(x_i)) = 0\ \forall k \in \{0, \dots, 3\}, \\ & \Upsilon^t_{\mathrm{P}}[\delta_t](x_0) = \partial_z \Upsilon^0_{\mathrm{P}}[\delta_0](z_t), \Upsilon^b_{\mathrm{P}}[\delta_b](x_1) = \partial_z \Upsilon^1_{\mathrm{P}}[\delta_1](z_b), \\ & \Delta_0(z_t) = \partial_x \Upsilon^t_{\mathrm{P}}[\delta_t](x_0), \Delta_1(z_b) = \partial_x \Upsilon^b_{\mathrm{P}}[\delta_b](x_1)\big\},\end{aligned} \tag{5.9}$$

where

$$\Delta_i(z) := \frac{1}{z}\partial_z^2\left[\frac{\partial_z \Upsilon^i_{\mathrm{P}}[\delta_i]}{\partial_z \mathbb{Y}_{\mathrm{P}}(x_i, z)(\partial_z \mathbb{Y}_{\mathrm{P}}(x_i, \cdot) - \partial_z \Upsilon^i_{\mathrm{P}}[\delta_i])} + \partial_x p(x_i)\Upsilon^i_{\mathrm{P}}[\delta_i]\right].$$

Once again, we endow $\mathcal{X}^1$ with its canonical norm. The assumptions on f, δ_0 and δ_1 could be relaxed slightly: in particular, it is not compulsory to assume that δ_0 and δ_1 vanish up to order three near $z = 0$, or that f vanishes on the lateral boundary. However, this simplifies the formulation of some compatibility conditions.

Our result is the following proposition.

Proposition 5.2. *Let* $(\mathbb{u}_{\mathrm{P}}, \mathbb{v}_{\mathrm{P}})$ *be a smooth solution to* (5.1) *on* $(x_0, x_1) \times (0, +\infty)$ *such that* $\partial_y \mathbb{u}_{\mathrm{P}} > 0$ *on* $\{\overline{\gamma_b}(x) \le y \le \overline{\gamma_t}(x),\ x \in [x_0, x_1]\}$. *Let* $\sigma \in (0, 1/6)$. *There exist* $\eta > 0$ *and* $z_0 > 0$ *such that if* $|z_b|, z_t \le z_0$, *the following result holds.*

- *There exists a manifold* $\mathcal{M}_\sigma \subset \mathcal{X}^\sigma$, *of codimension 1 within the ball of radius* η *in* $\mathcal{X}^\sigma$, *such that* (5.4), (5.7), (5.6) *have a solution in* $H_x^{\frac23+\sigma} H^1_z \cap H^\sigma_x H^3_z$ *if and only if* $(f, \delta_0, \delta_1, \delta_b, \delta_t, v_b) \in \mathcal{M}_\sigma$.
  This solution, if it exists, is unique.
- *There exists a manifold* $\mathcal{M}_1 \subset \mathcal{X}^1$, *of codimension 3 within the ball of radius* η *in* $\mathcal{X}^1$, *such that* (5.4), (5.7), (5.6) *has a solution in* $H^{5/3}_x H^1_z \cap H^1_x H^3_z$ *if and only if* $(f, \delta_0, \delta_1, \delta_b, \delta_t, v_b) \in \mathcal{M}_1$.

The proof of Proposition 5.2 is similar to the one of Theorem 3. We construct a solution to (5.4) thanks to an iterative scheme (or equivalently, thanks to the abstract Theorem 6), relying on several important observations.

- First, the left-hand side of (5.4) depends *linearly* on $\widetilde{Y}$, and the right-hand side depends smoothly on $\widetilde{Y}$. This nice feature stems directly from our change of variables. Note also that our choice of boundary conditions (1.9), (1.10), which are slightly unusual when we formulate them on the unknown function u, are in fact designed so that they become classical boundary conditions in the variable $\widetilde{Y}$. Indeed, the top and bottom boundaries in the z variable are now fixed (and flat), and the boundary condition for $\widetilde{Y}$ on these boundaries is merely a Neumann condition (so a Dirichlet condition for the vorticity $\partial_z \widetilde{Y}$).
- Second, as mentioned above, the vorticity $\partial_z \widetilde{Y}$ satisfies an equation with a very nice structure. More precisely, setting

$$\alpha(x,z) := \frac{1}{(\partial_z \mathbb{Y}_\mathrm{P}(x,z))^2} > 0,$$
$$\beta(x) := -\partial_x p,$$

and differentiating (5.4) with respect to z, we find that $W := \partial_z \widetilde{Y}$ is a solution to

$$\begin{cases} z\partial_x W + \beta \partial_z W - \partial_z^2(\alpha W) = \partial_z g & \text{in } (x_0,x_1)\times(z_b,z_t), \\ W|_{\Sigma_i} = \partial_z \Upsilon_\mathrm{P}^i[\delta_i] & \text{for } i \in \{0,1\}, \\ W|_{z=z_j} = \Upsilon_\mathrm{P}^j[\delta_j] & \text{for } j \in \{t,b\}. \end{cases} \tag{5.10}$$

The coefficients α and β are smooth and depend only on the underlying flow $(\mathbb{u}_\mathrm{P}, \mathbb{v}_\mathrm{P})$. Furthermore, $\inf \alpha > 0$ in $(x_0,x_1)\times(z_b,z_t)$ by assumption. Hence the structure of system (5.10) is very similar to the one of (1.6), albeit with variable coefficients. The smallness condition on z_b and z_t ensures that we have nice *a priori* estimates for (5.10) (see Lemma 5.3 below).
- Eventually, we observe that, from (5.5),

$$\partial_z g = \partial_y f(x,Y)(\partial_z \mathbb{Y}_\mathrm{P} - W)^2 + f(x,Y)(\partial_z^2 \mathbb{Y}_\mathrm{P} - \partial_z W)$$
$$+ \partial_z^2 \left(\frac{W^2}{(\partial_z \mathbb{Y}_\mathrm{P})^2(\partial_z \mathbb{Y}_\mathrm{P} - W)} \right).$$

In order to design a convergent iterative scheme for (5.10), it is necessary to work in a function space controlling the L^∞ norm of W (for example to ensure that the denominator does not vanish, or that the application $W \mapsto \partial_z^2(W^2) \in L^2$ is Lipschitz continuous). Having $W \in Z^0$ is not sufficient as we barely miss the embedding in L^∞ (see Remark 1.12). However, the function space $W \in H_x^{\frac{2}{3}+\sigma} L_z^2 \cap H_x^\sigma H_z^2$, with σ strictly positive and small, will be suitable for our purposes. This is in sharp contrast with the nonlinear scheme for the Burgers system, for which we also needed that $\partial_z \widetilde{Y} = W \in L^\infty$ but for which the function $\widetilde{Y}$ (rather than $\partial_z \widetilde{Y} = W$) was a solution to (1.6). Therefore, having $W \in L^\infty$ required $\widetilde{Y} \in H_x^{\frac{2}{3}+\sigma} L_z^2 \cap H_x^\sigma H_z^2$ for some $\sigma > 1/3$. Such a regularity requires two orthogonality conditions (see Proposition 2.36). This gain of one derivative in the vertical variable (corresponding to a gain of $1/3$ of derivative in the horizontal variable) allows us to get rid of two of the orthogonality conditions, leading to the first statement of Proposition 5.2.

5.2 The linearized vorticity equation

This section is devoted to the analysis of system (5.10), for a given source term $\partial_z g \in L^2(\Omega)$. Adapting and stepping on the analysis of Chapter 2, we prove the existence and uniqueness of solutions in $Z^0(\Omega)$. We also exhibit necessary and sufficient conditions for higher regularity.

For the sake of simplicity, within this section, Ω denotes the rectangle $(x_0, x_1) \times (z_b, z_t)$, which is a slight abuse of notation since $(z_b, z_t) \neq (-1, 1)$. We still denote by $\Sigma_0 = \{x_0\} \times (0, z_t)$ and $\Sigma_1 = \{x_1\} \times (z_b, 0)$ the lateral boundaries.

Lemma 5.3 (Well-posedness of the linear vorticity equation). *Let $\alpha \in C^2(\overline{\Omega})$ and $\beta \in L^\infty(x_0, x_1)$. Assume that there exists $\lambda > 0$ such that*

$$\forall (x, z) \in \Omega, \quad \frac{1}{\lambda} \leq \alpha(x, z) \leq \lambda. \tag{5.11}$$

There exists $z_0 > 0$, depending only on α, such that if $|z_b|, z_t \leq z_0$, the following result holds.

Let $h \in L^2(\Omega)$, $w_t, w_b \in H^{3/4}(x_0, x_1)$, and $w_i \in \mathcal{H}_z^1(\Sigma_i)$. Assume that the compatibility conditions $w_t(x_0) = w_0(z_t)$, $w_b(x_1) = w_1(z_b)$ are satisfied.

Consider the system

$$\begin{cases} z\partial_x W + \beta \partial_z W - \partial_z^2(\alpha W) = h & \text{in } \Omega, \\ W|_{\Sigma_i} = w_i & \text{for } i \in \{0, 1\}, \\ W|_{z=z_j} = w_j & \text{for } j \in \{t, b\}. \end{cases} \tag{5.12}$$

Then (5.12) has a unique solution $W \in Z^0(\Omega)$, which moreover satisfies

$$\|W\|_{Z^0} \leq C\big(\|h\|_{L^2} + \|w_t\|_{H^{3/4}} + \|w_b\|_{H^{3/4}} + \|w_0\|_{\mathcal{H}_z^1(\Sigma_0)} + \|w_1\|_{\mathcal{H}_z^1(\Sigma_1)}\big),$$

where the constant C depends only on λ, $\|\beta\|_\infty$ and $\|\partial_z \alpha\|_\infty$.

Proof. According to [53, Theorem 2.1] it is sufficient to prove the result when $w_t = w_b = w_0 = w_1 = 0$, since one could lift these boundary conditions for the given regularity.

In this case, we note that since $\partial_z \beta = 0$, we have the $L^2_x H^1_z$ energy estimate

$$\int_\Omega \alpha(\partial_z W)^2 \le \|h\|_{L^2}\|W\|_{L^2} + \|\partial_z\alpha\|_{L^\infty}\|W\|_{L^2}\|\partial_z W\|_{L^2}.$$

If $|z_b|, z_t \le z_0$, then $\|W\|_{L^2(\Omega)} \le z_0\|\partial_z W\|_{L^2(\Omega)}$. Thus, if $z_0 \le 1/(2\lambda\|\partial_z\alpha\|_\infty)$, we obtain $\|W\|_{L^2_x H^1_z} \lesssim \|h\|_{L^2}$. From there, following the same arguments as in Proposition 2.2, we infer that there exists a solution $W \in \mathcal{B}$ to (5.12) satisfying $\|W\|_{\mathcal{B}} \lesssim \|h\|_{L^2}$. The uniqueness of this solution is proved in Appendix A. Eventually, we see $W \in \mathcal{B}$ as the solution to

$$z\partial_x W - \partial_z(\alpha\partial_z W) = h - \beta\partial_z W + \partial_z(\partial_z\alpha W),$$

where the right-hand side belongs to $L^2(\Omega)$ since $\beta \in L^\infty$ and $\alpha \in C^2(\overline{\Omega})$. Since $\alpha \in C^2(\overline{\Omega})$, applying Pagani's result [53, Theorem 5.1] to the operator $z\partial_x - \partial_z(\alpha\partial_z\cdot)$ which is in conservative form, we obtain that $W \in Z^0$ and $\|W\|_{Z^0} \lesssim \|h\|_{L^2} + \|W\|_{\mathcal{B}}$. ∎

We now rely on the analysis of Chapter 2 in order to identify two necessary and sufficient orthogonality conditions for higher regularity. Let us first remark that the only potential singular points are $(x_0, 0)$ and $(x_1, 0)$. Indeed, we recall that $z\partial_x W \in L^2(\Omega)$, and therefore $W \in H^1_x L^2_z(\{|z| \ge z_0\})$ for all $z_0 > 0$. Regularity away from the lateral boundaries is ensured by the following lemma.

Lemma 5.4. *Let $\alpha \in C^3(\overline{\Omega})$ satisfying (5.11) and $\beta \in C^1([x_0, x_1])$. There exists $z_0 > 0$, depending only on α, such that if $|z_b|, z_t \le z_0$, the following result holds.*

Let $h \in L^2(\Omega)$ such that $(x - x_0)(x - x_1)\partial_x h \in L^2$. Let $w_t, w_b \in H^2(x_0, x_1)$ and $w_i \in H^2(\Sigma_i)$ such that the compatibility conditions $w_t(x_0) = w_0(z_t)$, $w_b(x_1) = w_1(z_b)$ are satisfied.

Let $W \in Z^0$ be the unique solution to (5.12). Then $(x - x_0)(x - x_1)\partial_x W \in Z^0$.

The proof is postponed to Appendix C, in order not to burden this section. We are now ready to state our orthogonality conditions for system (5.12). To that end, for $\alpha \in C^4(\overline{\Omega})$, $\beta \in C^1([x_0, x_1])$, $\sigma \in (0, 1]$, we introduce the space

$$\begin{aligned}\mathcal{H}^\sigma_{\alpha,\beta} := \Big\{&(h, w_0, w_1, w_t, w_b) \in H^\sigma_x L^2_z \times H^2(\Sigma_0) \times H^2(\Sigma_1) \times H^2(x_0, x_1)^2,\\ &(x - x_0)(x - x_1)\partial_x h \in L^2, w_t(x_0) = w_0(z_t), w_b(x_1) = w_1(z_b),\\ &\text{and } \Delta_i \in \mathcal{H}^1_z(\Sigma_i) \text{ if } \sigma > 1/2,\\ &\text{and } \Delta_0(z_t) = \partial_x w_t(x_0), \Delta_1(z_b) = \partial_x w_b(x_1) \text{ if } \sigma > 1/2,\\ &\text{where } \Delta_i := \frac{1}{z}(h(x_i, \cdot) + \partial_z^2(\alpha(x_i, \cdot)w_i) - \beta(x_i)\partial_z w_i)\Big\}.\end{aligned} \tag{5.13}$$

We now state a proposition extending the results of Chapter 2 to equations with smooth variable coefficients.

Proposition 5.5. *Let $\alpha \in C^4(\overline{\Omega})$ satisfying (5.11) and $\beta \in C^1([x_0, x_1])$. There exist two linear forms $\widehat{\ell^0}$, $\widehat{\ell^1}$, continuous on $\mathcal{H}^\sigma_{\alpha,\beta}$ for all $\sigma \in (1/6, 1]$, such that the following result holds.*

- *Let $\sigma \in (0, 1/6)$, and let $(h, w_0, w_1, w_t, w_b) \in \mathcal{H}^\sigma_{\alpha,\beta}$. Let $W \in Z^0$ be the unique solution to (5.12).*
  Then $W \in Z^\sigma = [Z^0, Z^1]_\sigma \hookrightarrow H_x^{\frac{2}{3}+\sigma} L^2_z \cap H^\sigma_x H^2_z \hookrightarrow L^\infty(\Omega)$, and
  $$\|W\|_{Z^\sigma} \lesssim \|h\|_{H^\sigma_x L^2_z} + \|(x-x_0)(x-x_1)\partial_x h\|_{L^2} + \sum_{j\in\{b,t\}} \|w_j\|_{H^2(x_0,x_1)} + \sum_{i\in\{0,1\}} \|w_i\|_{H^2(\Sigma_i)}.$$
- *Let $\sigma \in (1/6, 1] \setminus \{1/2\}$, and let $(h, w_0, w_1, w_t, w_b) \in \mathcal{H}^\sigma_{\alpha,\beta}$. Let $W \in Z^0$ be the unique solution to (5.12). Then $W \in Z^\sigma$ if and only if*
  $$\widehat{\ell^0}(h, w_0, w_1, w_t, w_b) = \widehat{\ell^1}(h, w_t, w_b, w_0, w_1) = 0,$$
  and in that case
  $$\|W\|_{Z^\sigma} \lesssim \|h\|_{H^\sigma_x L^2_z} + \|(x-x_0)(x-x_1)\partial_x h\|_{L^2} + \sum_{j\in\{b,t\}} \|w_j\|_{H^2(x_0,x_1)} + \sum_{i\in\{0,1\}} \|w_i\|_{H^2(\Sigma_i)} + \|\Delta_i\|_{\mathcal{H}^{\frac{1}{2}}_z(\Sigma_i)}.$$

Proof. We start with the first statement, and we take $\sigma \in (0, 1/6)$ fixed.

Step 1. Lifting the top and bottom boundary conditions. In order to use the theory from Chapter 2, which is stated with homogeneous Dirichlet boundary conditions at the top and bottom, we first lift the latter. We change W into $W - \rho(z - z_t)w_t - \rho(z - z_b)w_b$, where $\rho \in C^\infty_c(\mathbb{R})$ is such that $\rho \equiv 1$ in a neighborhood of zero, and $\operatorname{supp} \rho \subset (-r, r)$ for some $r < \min(|z_b|, z_t)/2$. This changes the source term h into

$$h - \sum_{j\in\{t,b\}} (z\partial_x w_j \rho(z - z_j) + \beta w_j \rho'(z - z_j) - w_j \partial^2_z(\alpha\rho(z - z_j))),$$

which belongs to $H^\sigma_x L^2_z$, and also changes the boundary condition w_0 (resp. w_1) into $w_0 - w_0(z_t)\eta(z - z_t)$ (resp. $w_1 - w_1(z_b)\eta(z - z_b)$), which belongs to $H^2(\Sigma_0)$ (resp. $H^2(\Sigma_1)$). With a slight abuse of notation, we still denote by W the unknown function, and by $(h, w_0, w_1, 0, 0)$ the data. Note that this operation does not affect the compatibility conditions in the corners.

Step 2. Localization in the vicinity of the singular points. We then localize horizontally the solution in the vicinity of x_0 and x_1. We only treat the localization in the vicinity of x_0 since the other boundary is identical. Let $\chi_0 \in C_c^\infty(\mathbb{R})$ be such that $\chi_0 \equiv 1$ in a neighborhood of x_0, and $\operatorname{supp} \chi_0 \subset B(x_0, r)$ for some small $0 < r < (x_1 - x_0)/2$. Then $W_0 := W\chi_0(x)$ is a solution to

$$z\partial_x W_0 + \beta\partial_z W_0 - \partial_z^2(\alpha W_0) = h\chi_0 + zW\partial_x\chi_0. \tag{5.14}$$

Since $W \in Z^0$, $W \in H_x^{2/3} L_z^2$, so the right-hand side belongs to $H_x^\sigma L_z^2$. We then localize the coefficient α. Let $\alpha_0(z) := \alpha(x_0, z)$. Then

$$\begin{aligned} z\partial_x W_0 - \partial_z^2(\alpha_0(z)W_0) = h\chi_0 + zW\partial_x\chi_0 - \beta\partial_z W\chi_0 \\ - \partial_z^2((\alpha_0 - \alpha)W_0). \end{aligned} \tag{5.15}$$

On the support of χ_0, there exists a constant C such that $|\alpha_0 - \alpha| \le C|x - x_0|$ and $(\alpha - \alpha_0)/(x - x_0)$ is a C^3 function of (x, z). According to Lemma 5.4, $(\alpha_0 - \alpha)\partial_z^2 W_0 \in H_x^1 L_z^2$. Hence, the right-hand side of (5.15) belongs to $H_x^\sigma L_z^2$. Note furthermore that W_0 vanishes on $\{z = z_t\}$ and $\{z = z_b\}$ thanks to the first step.

Step 3. Vertical change of variables to work with constant coefficients. In order to use the theory from Chapter 2, we now change the vertical coordinate so that the equation in the new variables is formulated thanks to the Kolmogorov operator. More precisely, we set $W_0(x, z) = \omega_0(x, \zeta)$, where ζ is a function of z such that $\zeta(0) = 0$. We have

$$\partial_z^2(\alpha_0 W_0) = \alpha_0(\zeta')^2\partial_\zeta^2\omega_0 + (\alpha_0\zeta'' + 2\partial_z\alpha_0\zeta')\partial_\zeta\omega_0 + (\partial_z^2\alpha_0)\omega_0.$$

We first choose the function ζ so that $\zeta(0) = 0$ and

$$\frac{z}{\alpha_0(z)(\zeta'(z))^2} = \zeta, \quad \text{i.e.} \quad \zeta'(z) = \sqrt{\frac{z}{\alpha_0(z)\zeta(z)}}.$$

Explicit resolution for $z > 0$ yields (with a similar formula for $z < 0$):

$$\zeta(z) = \left(\frac{3}{2}\int_0^z \sqrt{\frac{t}{\alpha_0(t)}}\, dt\right)^{2/3}. \tag{5.16}$$

It can be easily checked that the function ζ thus defined has the same regularity as α_0 on (z_b, z_t) and that $C^{-1} \le \zeta' \le C$ for some positive constant C. Moreover, $(\alpha_0(0))^{\frac{1}{3}}\zeta(z) \sim z$ as $z \to 0$.

The function ω_0 then solves

$$\zeta\partial_x\omega_0 - \partial_\zeta^2\omega_0 = s_0(x, \zeta), \quad \text{where } s_0 \in H_x^\sigma L_z^2. \tag{5.17}$$

Furthermore, ω_0 is supported in the vicinity of $(x_0, 0)$. We denote by μ_0 the lateral boundary condition on Σ_0 in the new vertical variable i.e. $\mu_0(\zeta(z)) = \chi_0(x_0, z)w_0(z)$. Note that μ_0 and w_0 enjoy the same regularity, so that $\mu_0 \in H^2(\Sigma_0)$.

Step 4. Small fractional regularity. We now consider (5.17), whose right-hand side belongs to $H^\sigma_x L^2_z$. The equation is endowed with homogeneous data on $\{z = z_t\} \cup \{z = z_b\} \cup \Sigma_1$, and with H^2 data on Σ_0 satisfying a compatibility condition at (x_0, z_t). Using Proposition 2.36 and Lemma 1.15, we infer $\omega_0 \in Z^\sigma \hookrightarrow H_x^{\frac{2}{3}+\sigma} L^2_\zeta \cap H^\sigma_x H^2_\zeta$, and thus W_0 enjoys the same regularity. Performing a similar change of variables near $(x_1, 0)$, we deduce that $W \in Z^\sigma$. This completes the proof of the first statement from Proposition 5.5.

Step 5. Identification of the orthogonality conditions. Let us now assume that $\sigma \in (1/6, 1/3] \sum$ and $h \in H^\sigma_x L^2_z$. The right-hand side of (5.17) now belongs to $H^\sigma_x L^2_z$. Furthermore, in a neighborhood of $\zeta = 0$,

$$
\begin{aligned}
s_0(x_0, \zeta) = \frac{1}{\alpha_0(z)(\zeta'(z))^2}\big[&h(x_0, z) - \beta(x_0) w_0'(z) \\
&+ (\alpha_0\zeta'' + 2\alpha_0'\zeta')(z)\partial_\zeta \mu_0(\zeta) + \alpha_0''(z)\mu_0(\zeta)\big],
\end{aligned}
$$

where the primes always denote derivatives with respect to z. Using this equality together with the identity $\partial_\zeta \mu_0(\zeta) = w_0'(z)/\zeta'(z)$, we find, after some tedious but straightforward computations, and for ζ in a neighborhood of zero,

$$
\partial_\zeta^2 \mu_0(\zeta) + s_0(x_0, \zeta) = \frac{\zeta(z)}{z} \Delta_0(z).
$$

Hence $(\partial_\zeta^2 \mu_0 + s_0(x_0, \cdot))/\zeta \in \mathcal{H}^1_\zeta(\Sigma_0)$. Note also that the compatibility conditions in the corners are satisfied. We then apply Corollary 2.39 to (5.17) whose right-hand side is in $H^\sigma_x L^2_\zeta$. We infer that if

$$
(a_0 \overline{\ell^0} + a_1 \overline{\ell^1})(s_0, \mu_0, 0) = 0,
$$

then $\omega_0 \in Z^\sigma \hookrightarrow H_x^{\frac{2}{3}+\sigma} L^2_\zeta \cap H^\sigma_x H^2_\zeta$ by Lemma 1.15. Similarly, $\omega_1 \in Z^\sigma$, so $W \in Z^\sigma$.

For $\sigma \geq 1/3$ and $\sigma \neq 1/2$, we use a bootstrap argument. Going back to (5.14), we now know that the right-hand side is in $H_x^{\min(\sigma, 2/3)} L^2_z$, so that we can apply Corollary 2.39 to (5.17) whose right-hand side belongs to $H_x^{\min(\sigma,2/3)} L^2_\zeta$. This implies that $W \in Z^{\min(\sigma,2/3)}$. We then repeat this procedure one last time if $\sigma \geq 2/3$.

Setting

$$
\widehat{\ell^0}(h, w_0, w_1, w_t, w_b) = (a_0 \overline{\ell^0} + a_1 \overline{\ell^1})(s_0, \mu_0, 0), \tag{5.18}
$$

and defining in a similar fashion the linear form $\widehat{\ell^1}$ associated with the regularity in the vicinity of $(x_1, 0)$, we obtain the desired result.

Eventually, it follows from the definition of $\widehat{\ell^0}$ in (5.18) and from Remark 2.33 that the linear forms $\widehat{\ell^j}$ are continuous on $\mathcal{H}^\sigma_{\alpha,\beta}$ for all $\sigma > 1/6$. ■

Lemma 5.6. *The two linear forms $\widehat{\ell_0}, \widehat{\ell_1} : \mathcal{H}^1_{\alpha,\beta} \to \mathbb{R}$ defined in Proposition 5.5 are independent. Furthermore, there exist $g^0, g^1 \in C_c^\infty(\Omega)$ such that*

$$\widehat{\ell^j}(g^i, 0, 0, 0, 0) = \delta_{i,j} \quad \forall i, j \in \{0, 1\}.$$

Proof. We begin with the following remark. Following the notations of the proof of Proposition 5.5 above, we set $\alpha_i(z) := \alpha(x_i, z)$. With the same change of variables as in Step 3 of the proof (see (5.16)), we define

$$U^0(x, z) = \bar{u}^0_{\text{sing}}(x, \zeta_0).$$

Then

$$z\partial_x U^0 + \beta\partial_z U^0 - \partial_z^2(\alpha_0 U^0) = \alpha_0(\zeta_0')^2 \overline{f^0} + \lambda_0 \bar{u}^0_{\text{sing}}(x, \zeta_0) + \gamma_0 \partial_{\zeta_0} \bar{u}^0_{\text{sing}}(x, \zeta_0)$$

for some smooth functions λ_0, γ_0 depending on β and α. The right-hand side therefore belongs to $H^{1/3}_x L^2_z \cap H^1_x L^2_z((x - x_0)^2(x - x_1)^2)$. Furthermore, U^0 vanishes on $\Sigma_0 \cup \Sigma_1 \cup \{z = z_b\} \cup \{z = z_t\}$.

Of course, we may perform the same procedure around $(x_1, 0)$, and we define a function $U^1(x, z)$, localized in a neighborhood of $(x_1, 0)$ and with the same regularity as $\bar{u}^1_{\text{sing}}$, such that

$$z\partial_x U^1 + \beta\partial_z U^1 - \partial_z^2(\alpha_1 U^1) \in H^{1/3}_x L^2_z.$$

Note that U^0 and U^1 vanish on $\Sigma_0 \cup \Sigma_1 \cup \{z = z_b\} \cup \{z = z_t\}$. Now, for $i = 0, 1$, let

$$h^i := z\partial_x U^i + \beta\partial_z U^i - \partial_z^2(\alpha U^i).$$

By construction, h_i and U_i are localized in the vicinity of $(x_i, 0)$, and $h^i \in H^{1/3}_x L^2_z$, $(x - x_0)(x - x_1)\partial_x h_i \in L^2$. Furthermore, $h_i|_{\Sigma_0 \cup \Sigma_1} = 0$. As a consequence, recalling the definition of $\widehat{\ell^0}$ and $\widehat{\ell^1}$ (see (5.18) together with Corollary 2.39), we infer that

$$\widehat{\ell^0}(h_1, 0, 0, 0, 0) = \widehat{\ell^1}(h_0, 0, 0, 0, 0) = 0.$$

Now, assume that $c_0\widehat{\ell^0} + c_1\widehat{\ell^1} = 0$ for some $(c_0, c_1) \in \mathbb{R}^2$. We deduce from the above equalities that

$$\widehat{\ell^0}(c_0 h_0 + c_1 h_1, 0, 0, 0, 0) = c_0\widehat{\ell^0}(h_0, 0, 0, 0, 0) = (c_0\widehat{\ell^0} + c_1\widehat{\ell^1})(h_0, 0, 0, 0, 0) = 0,$$

and similarly $\widehat{\ell^1}(c_0 h_0 + c_1 h_1, 0, 0, 0, 0) = 0$. Using Proposition 5.5, we infer that $c_0 U^0 + c_1 U^1 \in Z^{1/3} \hookrightarrow H^1_x L^2_z \cap H^{1/3}_x H^2_z$. Since U^i has the regularity of $\bar{u}^i_{\text{sing}}$ and is localized in the vicinity of $(x_i, 0)$, it follows from Lemma 2.28 that $c_0 = c_1 = 0$.

Note that the above argument also ensures that $\widehat{\ell^i}(h^i, 0, 0, 0, 0) \neq 0$. Hence, up to a multiplication by a constant, we may always assume that $\widehat{\ell^j}(h^i, 0, 0, 0, 0) = \delta_{i,j}$.

Let us now take, for $\varepsilon > 0$ small, $h^i_\varepsilon \in C^\infty_c(\Omega)$ such that $\|h^i_\varepsilon - h^i\|_{H^{1/3}_x L^2_z} \leq \varepsilon$ and $\|(x-x_0)(x-x_1)\partial_x(h^i - h^i_\varepsilon)\|_{L^2} \leq \varepsilon$. Then, since the linear forms $\widehat{\ell^j}$ are continuous on $\mathcal{H}^{1/3}_{\alpha,\beta}$, we obtain $|\widehat{\ell^j}(h^i_\varepsilon, 0, 0, 0, 0) - \delta_{i,j}| \lesssim \varepsilon$. As a consequence, there exists $a^0_\varepsilon, a^1_\varepsilon, b^0_\varepsilon, b^1_\varepsilon$ such that

$$\widehat{\ell^0}(a^0_\varepsilon h^0_\varepsilon + a^1_\varepsilon h^1_\varepsilon, 0, 0, 0, 0) = \widehat{\ell^1}(b^0_\varepsilon h^0_\varepsilon + b^1_\varepsilon h^1_\varepsilon, 0, 0, 0, 0) = 1,$$
$$\widehat{\ell^0}(b^0_\varepsilon h^0_\varepsilon + b^1_\varepsilon h^1_\varepsilon, 0, 0, 0, 0) = \widehat{\ell^1}(a^0_\varepsilon h^0_\varepsilon + a^1_\varepsilon h^1_\varepsilon, 0, 0, 0, 0) = 0,$$

and $|a^0_\varepsilon - 1|, |b^1_\varepsilon - 1| \lesssim \varepsilon$, $|a^1_\varepsilon|, |b^0_\varepsilon| \lesssim \varepsilon$. The result follows, taking $g^0 = a^0_\varepsilon h^0_\varepsilon + a^1_\varepsilon h^1_\varepsilon$ and $g^1 = b^0_\varepsilon h^0_\varepsilon + b^1_\varepsilon h^1_\varepsilon$. ∎

5.3 Reconstructing the velocity from the vorticity

Let $(g, \tilde{\delta}_0, \tilde{\delta}_1) \in L^2_x H^1_z \times H^3(\Sigma_0) \times H^3(\Sigma_1)$ and $w_t, w_b \in H^2(x_0, x_1)$. Assume that $\partial_z\widetilde{\delta_1}(z_b) = w_b(x_1)$ and $\partial_z\widetilde{\delta_0}(z_t) = w_t(x_0)$. According to Lemma 5.3, there exists a unique solution $W \in Z^0$ to (5.12) with $h = \partial_z g$ and $w_i = \partial_z\tilde{\delta}_i$. The purpose of this section is to construct a solution to the system

$$\begin{cases} z\partial_x\widetilde{Y} - \int_{z_b}^z \partial_x\widetilde{Y} + \beta\partial_z\widetilde{Y} - \partial_z(\alpha\partial_z\widetilde{Y}) = g & \text{in } \Omega, \\ \widetilde{Y}|_{\Sigma_i} = \tilde{\delta}_i & \text{for } i \in \{0,1\}, \\ \partial_z\widetilde{Y}|_{z=z_j} = w_j & \text{for } j \in \{t,b\}. \end{cases} \tag{5.19}$$

We therefore set, for $(x,z) \in \Omega$,

$$\widetilde{Y}(x,z) := \widetilde{\gamma_b}(x) + \int_{z_b}^z W(x,z')\,\mathrm{d}z', \tag{5.20}$$

where the function $\widetilde{\gamma_b}$ solves the differential equation

$$\begin{aligned} z_b\partial_x\widetilde{\gamma_b} + (\beta - \partial_z\alpha(\cdot,z_b))w_b - \alpha(x,z_b)\partial_z W(x,z_b) = g(x,z_b), \\ \widetilde{\gamma_b}(x_1) = \widetilde{\delta_1}(z_b). \end{aligned} \tag{5.21}$$

Since $W \in Z^0$, the trace $\partial_z W(\cdot, z_b)$ belongs to $H^{1/4}(x_0,x_1)$ by Lemma 1.11. Thus $\widetilde{\gamma_b} \in H^1(x_0,x_1)$, and from (5.20) we infer that $\widetilde{Y} \in H^{2/3}_x H^1_z \cap L^2_x H^3_z \subset C^0(\overline{\Omega})$. Furthermore, $\widetilde{Y} \in H^1_x H^1_z(z_b, z_b/2)$.

By construction, we have, in the sense of distributions on Ω,

$$\partial_z\left[z\partial_x\widetilde{Y} - \int_{z_b}^z \partial_x\widetilde{Y}(x,z')\,\mathrm{d}z' + \beta\partial_z\widetilde{Y} - \partial_z(\alpha\partial_z\widetilde{Y}) - g\right] = 0,$$

and therefore there exists a function G depending only on x such that

$$z\partial_x \widetilde{Y} - \int_{z_b}^{z} \partial_x \widetilde{Y}(x,z')\,\mathrm{d}z' + \beta\partial_z \widetilde{Y} - \alpha\partial_z^2 \widetilde{Y} = g(x,z) + G(x).$$

The choice of the function $\widetilde{\gamma_b}$ (see (5.21)) then ensures that $G \equiv 0$. By definition of $\widetilde{Y}$, we have $\partial_z \widetilde{Y}|_{z=z_j} = W|_{z=z_j} = w_j$ for $j \in \{t,b\}$.

Let us now investigate the lateral boundary conditions. On Σ_i, we have

$$\partial_z \widetilde{Y}(x_i,z) = W(x_i,z) = \partial_z \tilde{\delta}_i(z).$$

Hence, in order to ensure that $\widetilde{Y}|_{\Sigma_i} = \tilde{\delta}_i$, it suffices to check that $\widetilde{Y}(x_i,z_i) = \tilde{\delta}_i(z_i)$ for some $(x_i,z_i) \in \overline{\Sigma_i}$. From there, we treat separately (and differently) the two boundaries Σ_0 and Σ_1.

- On Σ_1, we note that $\widetilde{Y}(x_1,z_b) = \tilde{\delta}_1(z_b)$ by definition of $\widetilde{\gamma_b}$. Therefore $\widetilde{Y}|_{\Sigma_1} = \tilde{\delta}_1$.
- On Σ_0, the situation is different, since $\widetilde{Y}(x_0,0) \neq \tilde{\delta}_0(0)$ *a priori*. Indeed,

$$\begin{aligned}\widetilde{Y}(x_0,0) &= \widetilde{\gamma_b}(x_0) + \int_{z_b}^{0} W(x_0,z')\,\mathrm{d}z' \\ &= \frac{1}{z_b}\Big(- \int_{x_0}^{x_1} g(\cdot,z_b) + (\partial_z\alpha(\cdot,z_b) - \beta)w_b + \alpha(\cdot,z_b)\partial_z W(\cdot,z_b)\Big) \\ &\quad + \int_{z_b}^{0} W(x_0,z')\,\mathrm{d}z' + \tilde{\delta}_1(z_b).\end{aligned}$$

The right-hand side of the above equality is a linear form in $(g,\tilde{\delta}_0,\tilde{\delta}_1,w_t,w_b)$, which leads to the following definition.

Definition 5.7 (Additional linear form for the solvability of Prandtl). Let $(g,\tilde{\delta}_0,\tilde{\delta}_1) \in L_x^2 H_z^1 \times H^3(\Sigma_0) \times H^3(\Sigma_1)$, $w_t, w_b \in H^2(x_0,x_1)$ such that $w_t(x_0) = \partial_z\tilde{\delta}_0(z_t)$, $w_b(x_1) = \partial_z\tilde{\delta}_1(z_b)$. Let $W \in Z^0$ be the unique solution to (5.12) with $h = \partial_z g$ and $w_i = \partial_z \tilde{\delta}_i$.

The linear form ℓ^2 is defined by

$$\begin{aligned}&\ell^2(g,\widetilde{\delta_0},\widetilde{\delta_1},w_t,w_b) \\ &\quad := \int_{z_b}^{0} W(x_0,z')\,\mathrm{d}z' + \tilde{\delta}_1(z_b) - \tilde{\delta}_0(0) \\ &\qquad - \frac{1}{z_b}\int_{x_0}^{x_1} (g(x,z_b) + (\partial_z\alpha(x,z_b) - \beta(x))w_b(x) + \alpha(x,z_b)\partial_z W(x,z_b))\,\mathrm{d}x.\end{aligned}$$

The above computations lead to the following result.

Lemma 5.8. *Let* $(g,\tilde{\delta}_0,\tilde{\delta}_1) \in L_x^2 H_z^1 \times H^3(\Sigma_0) \times H^3(\Sigma_1)$, $w_t, w_b \in H^2(x_0,x_1)$ *such that* $w_t(x_0) = \partial_z\tilde{\delta}_0(z_t)$, $w_b(x_1) = \partial_z\tilde{\delta}_1(z_b)$.

Then system (5.19) *has a solution* $\widetilde{Y} \in H^{2/3}_x H^1_z \cap L^2_x H^3_z$ *if and only if*

$$\ell^2\big(g, \widetilde{\delta_0}, \widetilde{\delta_1}, w_t, w_b\big) = 0.$$

This solution is given by (5.20)*, and satisfies the estimate*

$$\begin{aligned}\|\widetilde{Y}\|_{H^{2/3}_x H^1_z} + \|\widetilde{Y}\|_{L^2_x H^3_z} + \|\partial_x \widetilde{Y}\|_{L^2_x H^1_z(\{z<z_b/2\})}\\ \lesssim \|g\|_{L^2_x H^1_z} + \|\widetilde{\delta}_i\|_{H^3(\Sigma_i)} + \|w_j\|_{H^2(x_0,x_1)},\end{aligned}$$

where we implicitly sum over $i \in \{0, 1\}$ *and* $j \in \{t, b\}$ *on the right-hand side.*

Proof. First, assume that (5.19) has a solution $\widetilde{Y} \in H^{2/3}_x H^1_z \cap L^2_x H^3_z$. Then, $W = \partial_z \widetilde{Y}$ is an $L^2_x H^1_z$ solution to (5.12) with $h = \partial_z g$ and $w_i = \partial_z \widetilde{\delta}_i$. By uniqueness arguments such as in Lemma A.1, it is equal to the unique Z^0 solution to (5.12) constructed in Lemma 5.3. Furthermore, for $z \neq 0$,

$$z^2 \partial_z \left(\frac{\int_{z_b}^z \partial_x \widetilde{Y}}{z} \right) = g + \partial_z(\alpha \partial_z \widetilde{Y}) - \beta \partial_z \widetilde{Y} \in L^2_x H^1_z.$$

It follows that $\partial_z \big(\frac{\int_{z_b}^z \partial_x \widetilde{Y}}{z}\big) \in L^2_x H^1_z(\{z < z_b/2\})$, and thus $\partial_x \widetilde{Y} \in L^2_x H^1_z(\{z < z_b/2\})$. In particular, $\partial_x \widetilde{Y}|_{z=z_b} \in L^2(x_0, x_1)$.

Taking the trace of (5.19) at $z = z_b$, we infer that

$$z_b \partial_x \widetilde{Y}|_{z=z_b} + (\beta - \partial_z \alpha(x, z_b)) w_b - \alpha \partial_z W(x, z_b) = g(x, z_b)$$

and

$$\widetilde{Y}(x_1, z_b) = \widetilde{\delta_1}(z_b).$$

Therefore $\widetilde{Y}|_{z=z_b} = \widetilde{\gamma_b}$, where $\widetilde{\gamma_b}$ is defined by (5.21). Since $\widetilde{Y}(x_0, 0) = \widetilde{\delta_0}(0)$, we then deduce that

$$\widetilde{Y}(x_0, z_b) + \int_{z_b}^0 W(x_0, z)\, \mathrm{d}z = \widetilde{\delta_0}(0),$$

which is precisely the condition $\ell^2(g, \widetilde{\delta_0}, \widetilde{\delta_1}, w_t, w_b) = 0$.

Conversely, the above computations ensure that if $\ell^2(g, \widetilde{\delta_0}, \widetilde{\delta_1}, w_t, w_b) = 0$, the function defined by (5.20) is a solution to (5.19). ∎

Assume that $\ell^2(g, \widetilde{\delta_0}, \widetilde{\delta_1}, w_t, w_b) = 0$, and let $\widetilde{Y} \in H^{2/3}_x H^1_z \cap L^2_x H^3_z$ be the unique solution to (5.19). For further purposes, we define the function $\widetilde{\gamma_t}$ by

$$\widetilde{\gamma_t}(x) := \widetilde{Y}(x, z_t).$$

Since $\widetilde{Y} \in C^0(\overline{\Omega})$, we have

$$\widetilde{\gamma_t}(x_0) = \widetilde{\delta_0}(z_t).$$

Remark 5.9. As we already mentioned, the nonlocal term $v\partial_y u$ in the Prandtl equation (which becomes $-\int_{z_b}^{z} \partial_x Y$ in our new variables) creates a flow of information upwards, therefore inducing an asymmetry between z and $-z$. Because of the forward-backward nature of the equation, this results in an asymmetry between the lateral boundaries Σ_0 and Σ_1. We deal with this issue by introducing an additional orthogonality condition.

Note that this feature is also present, in a slightly different fashion, in the work of Iyer and Masmoudi [34, 35]. In their work, the left extremity of the curve $\{u = 0\}$ is left as a free parameter, and boundary data on the vorticity are enforced.

In order to simplify the future discussion, it will be useful to modify slightly the definition of the linear forms ℓ^i for $i \in \{0, 1\}$, so that they are defined on the same space as the linear form ℓ^2.

Definition 5.10. We denote by ℓ^i for $i \in \{0, 1\}$ the linear forms defined by

$$\ell^i(g, \widetilde{\delta_0}, \widetilde{\delta_1}, w_t, w_b) := \widehat{\ell^i}(\partial_z g, \partial_z \widetilde{\delta_0}, \partial_z \widetilde{\delta_1}, w_t, w_b).$$

Remark 5.11. In spite of their similar appearance, the purpose of the orthogonality conditions $\ell^0(g, \widetilde{\delta_0}, \widetilde{\delta_1}, w_t, w_b) = \ell^1(g, \widetilde{\delta_0}, \widetilde{\delta_1}, w_t, w_b) = 0$ on the one hand, and $\ell^2(g, \widetilde{\delta_0}, \widetilde{\delta_1}, w_t, w_b) = 0$ on the other hand is quite different. The former are necessary and sufficient conditions for the existence of smooth solutions to the vorticity equation (5.12), while the latter is a necessary and sufficient condition for the solvability of system (5.19) at a lower level of regularity, corresponding to Z^0 solutions of the vorticity equation (5.12). In other words, the condition $\ell^2 = 0$ is a necessary and sufficient condition to reconstruct $\widetilde{Y}$ from the vorticity.

Lemma 5.12. *The linear forms ℓ^0, ℓ^1, ℓ^2 are independent on $C^\infty(\bar{\Omega}) \times C_c^\infty(\Sigma_0) \times C_c^\infty(\Sigma_1) \times C_c^\infty(x_0, x_1)^2$. There exist Ξ^0, Ξ^1, Ξ^2 such that, for $i, j \in \{0, 1, 2\}$,*

$$\ell^i(\Xi_j) = \delta_{i,j}, \quad \Xi^j \in C^\infty(\bar{\Omega}) \times C_c^\infty(\Sigma_0) \times C_c^\infty(\Sigma_1) \times C_c^\infty(x_0, x_1)^2.$$

One may choose $\Xi^j = (f^j, 0, 0, 0, 0)$, with $f^j \in C^\infty(\bar{\Omega})$ such that $f^j|_{\Sigma_0 \cup \Sigma_1} = 0$.

Proof. Assume that there exists $(c_0, c_1, c_2) \in \mathbb{R}^3$ such that

$$c_0 \ell^0 + c_1 \ell^1 + c_2 \ell^2 = 0.$$

Let $W \in C_c^\infty(\bar{\Omega})$ such that $\operatorname{supp} W \subset [x_0, x_0 + \delta] \times [-\delta, -\delta/2]$ for some small $\delta > 0$ such that $\delta < (x_1 - x_0)/2$ and $\delta < |z_b|/2$. We further assume that $\int_{z_b}^{0} W(x_0, z)\,\mathrm{d}z = 1$ and $\int_{z_b}^{0} z\partial_x W(x_0, z)\,\mathrm{d}z = 0$. We set $w_t = w_b = 0$, $\widetilde{\delta_0} = \widetilde{\delta_1} = 0$, and

$$f^2(x, z) := \int_{z_b}^{z} (z'\partial_x W(x, z') + \beta(x)\partial_z W(x, z') - \partial_z^2(\alpha(x, z')W(x, z')))\,\mathrm{d}z'.$$

Then by definition, W is a solution to (5.12) with $h = \partial_z f^2$, and with homogeneous boundary data. Note also that $f^2(x_0, 0) = \int_{z_b}^0 z\partial_x W(x_0, z)\,dz = 0$. Therefore $f^2|_{\Sigma_0 \cup \Sigma_1} = 0$. The compatibility conditions from Proposition 5.5 are satisfied. Since W is smooth, according to Proposition 5.5,

$$\widehat{\ell^0}(\partial_z f^2, 0, 0, 0, 0) = \widehat{\ell^1}(\partial_z f^2, 0, 0, 0, 0) = 0.$$

Hence

$$c_2 \ell^2(f^2, 0, 0, 0, 0) = 0.$$

Now, by definition of ℓ^2 and f^2, since W and f^2 are identically zero for $z \le -\delta$,

$$\ell^2(f^2, 0, 0, 0, 0) = \int_{z_b}^0 W = 1.$$

We infer that $c_2 = 0$. The result then follows from Lemma 5.6, taking $f^i = \int_0^z g^i$ for $i = 0, 1$. ■

Gathering the results of Proposition 5.5 and Lemma 5.8, we obtain the following statement.

Corollary 5.13. *Let $\alpha \in C^4(\bar{\Omega})$ satisfying (5.11) and $\beta \in C^1(x_0, x_1)$.*

- *Let $\sigma \in (0, 1/6)$, and let $g \in H^\sigma_x H^1_z$ such that $(\partial_z g, \partial_z \widetilde{\delta_0}, \partial_z \widetilde{\delta_1}, w_t, w_b) \in \mathcal{H}^\sigma_{\alpha,\beta}$ defined in (5.13).*

  Then (5.19) has a solution $\widetilde{Y} \in H_x^{\frac{2}{3}+\sigma} H^1_z \cap H^\sigma_x H^3_z$ if and only if $\ell^2(g, \widetilde{\delta_0}, \widetilde{\delta_1}, w_t, w_b) = 0$, *and this solution, if it exists, is unique and satisfies the estimate*

$$\begin{aligned}&\|\widetilde{Y}\|_{H_x^{\frac{2}{3}+\sigma} H^1_z \cap H^\sigma_x H^3_z} + \|(x - x_0)(x - x_1)\partial_x \partial_z^3 \widetilde{Y}\|_{L^2} \\ &\quad + \|\partial_x \partial_z \widetilde{Y}\|_{L^2((x_0,x_1)\times(z_b, z_b/2))} \\ &\quad \lesssim \|g\|_{H^\sigma_x H^1_z} + \|(x - x_0)(x - x_1)\partial_x \partial_z g\|_{L^2} + \|w_j\|_{H^2_x} + \|\widetilde{\delta_i}\|_{H^3(\Sigma_i)}.\end{aligned}$$

- *Let $g \in H^1_x H^1_z$ and assume that $(\partial_z g, \partial_z \widetilde{\delta_0}, \partial_z \widetilde{\delta_1}, w_t, w_b) \in \mathcal{H}^1_{\alpha,\beta}$ defined in (5.13).*

  Assume that $\ell^2(g, \widetilde{\delta_0}, \widetilde{\delta_1}, w_t, w_b) = 0$, and let $\widetilde{Y} \in H_x^{\frac{2}{3}} H^1_z \cap L^2_x H^3_z$ be the unique solution to (5.19).

  Then $\widetilde{Y} \in H_x^{5/3} H^1_z \cap H^1_x H^3_z$ if and only if $\ell^j(g, \widetilde{\delta_0}, \widetilde{\delta_1}, w_t, w_b) = 0$ *for $j \in \{0, 1\}$, and in this case $\widetilde{Y}$ satisfies the estimate*

$$\|\widetilde{Y}\|_{H_x^{5/3} H^1_z} + \|\widetilde{Y}\|_{H^1_x H^3_z} \lesssim \|g\|_{H^1_x H^1_z} + \|(\partial_z g, \partial_z \widetilde{\delta_0}, \partial_z \widetilde{\delta_1}, w_t, w_b)\|_{\mathcal{H}^1_{\alpha,\beta}}.$$

Remark 5.14. The regularity assumptions on g in the first (resp. second) statement of the above corollary can be relaxed into $\partial_z g \in H^\sigma_x L^2_z$, $(x - x_0)(x - x_1)\partial_x \partial_z g \in L^2$ and $g|_{z=z_b} \in L^2(x_0, x_1)$ (resp. $\partial_z g \in H^1_x L^2_z$ and $g|_{z=z_b} \in H^{2/3}(x_0, x_1)$), but we have kept the above assumptions for the sake of simplicity.

5.4 Local nonlinear well-posedness in the new variables

We are now ready to prove Proposition 5.2. The spirit of the proof is very similar to the one of Chapter 4. In order to avoid repetition, we do not write the iterative scheme, and we rather apply Theorem 6 directly. We will work with two different settings:

(1) *Low regularity setting:* for $\sigma \in (0, 1/6)$ fixed, we take

$$\begin{aligned}
\mathcal{Z}^\sigma =& \big\{Y \in H_x^{\frac{2}{3}+\sigma} H_z^1 \cap H_x^\sigma H_z^3, (x-x_0)(x-x_1)\partial_x\partial_z^3 Y \in L^2(\Omega),\\
&\partial_x\partial_z Y \in L^2((x_0,x_1)\times(z_b,z_b/2)),\\
&Y|_{\Sigma_i} \in H^3(\Sigma_i), \partial_z Y|_{z=z_j} \in H^2(x_0,x_1)\big\},\\
\mathcal{H}^\sigma =& \big\{(f,\tilde{\delta}_0,\tilde{\delta}_1,w_t,w_b) \in H_x^\sigma H_z^1 \times H^3(\Sigma_0)\times H^3(\Sigma_1)\times (H^2(x_0,x_1))^2,\\
&(x-x_0)(x-x_1)\partial_x\partial_z f \in L^2,\\
&w_t(x_0) = \partial_z\widetilde{\delta_0}(z_t), w_t(x_0) = \partial_z\widetilde{\delta_0}(z_t)\big\},
\end{aligned}$$

and our space of data is the space $\mathcal{X}^\sigma$ defined in (5.8). Furthermore, in the low regularity setting, $d = 1$ and the linear form ℓ coincides with the linear form ℓ^2 defined in Definition 5.7.

(2) *High regularity setting:* we take

$$\begin{aligned}
\mathcal{Z}^1 =& \big\{Y \in H_x^{\frac{5}{3}} H_z^1 \cap H_x^1 H_z^3, Y|_{\Sigma_i} \in H^5(\Sigma_i), \partial_z Y|_{z=z_j} \in H^2(x_0,x_1)\big\},\\
\mathcal{H}^1 =& \big\{(f,\widetilde{\delta}_0,\widetilde{\delta}_1,w_t,w_b) \in H_x^1 H_z^1 \times H^5(\Sigma_0)\times H^5(\Sigma_1)\times(H^2(x_0,x_1))^2,\\
&\partial_z^k\widetilde{\delta}_i(0) = 0\ \forall k \in \{0,\dots,3\}, z^{-1}\partial_z f(x_i,z) \in \mathcal{H}_z^1(\Sigma_i),\\
&w_t(x_0) = \partial_z\widetilde{\delta_0}(z_t), w_t(x_0) = \partial_z\widetilde{\delta_0}(z_t),\\
&\Delta_0(z_t) = \partial_x w_t(x_0), \Delta_1(z_b) = \partial_x w_b[\delta_b](x_1)\big\},
\end{aligned}$$

where

$$\Delta_i(z) = \frac{1}{z}\partial_z\big[f(x_i,z) + \partial_z(\alpha(x_i,z)\partial_z\tilde{\delta}_i) - \beta(x_i)\partial_z\tilde{\delta}_i\big].$$

Note that

$$(f,\widetilde{\delta}_0,\widetilde{\delta}_1,w_t,w_b) \in \mathcal{H}^1 \Rightarrow (\partial_z f,\partial_z\widetilde{\delta}_0,\partial_z\widetilde{\delta}_1,w_t,w_b) \in \mathcal{H}^1_{\alpha,\beta},$$

where the space $\mathcal{H}^\sigma_{\alpha,\beta}$ for $\sigma \in (0,1]$ is defined in (5.13). Our space of data is the space $\mathcal{X}^1$ defined in (5.9). In the high regularity setting, we take $d = 3$ and $\ell = (\ell^0,\ell^1,\ell^2)$ defined in Definitions 5.10 and 5.7.

Remark 5.15. As in the previous sections, in $\mathcal{X}^1$, we could also consider source terms f which do not vanish on Σ_i^P, up to additional technical complications.

In both settings, the linear operator L_P is defined as

$$L_P\widetilde{Y} := \left(z\partial_x\widetilde{Y} - \int_{z_b}^{z}\partial_x\widetilde{Y} + \beta\partial_z\widetilde{Y} - \partial_z(\alpha\partial_z\widetilde{Y}), \widetilde{Y}|_{\Sigma_0}, \widetilde{Y}|_{\Sigma_1}, \partial_z\widetilde{Y}|_{z=z_t}, \partial_z\widetilde{Y}|_{z=z_b}\right),$$

and the nonlinearity N is defined as

$$N(\Xi,\widetilde{Y}) := (N_P(\Xi,\widetilde{Y}), \Upsilon_{\mathrm{P}}^0[\delta_0], \Upsilon_{\mathrm{P}}^1[\delta_1], \Upsilon_{\mathrm{P}}^t[\delta_t], \Upsilon_{\mathrm{P}}^b[\delta_b]),$$
$$N_P(\Xi,\widetilde{Y}) := f(x,\mathbb{Y}_{\mathrm{P}} - \widetilde{Y})\partial_z(\mathbb{Y}_{\mathrm{P}} - \widetilde{Y}) - v_b + \partial_z\left(\frac{(\partial_z\widetilde{Y})^2}{(\partial_z\mathbb{Y}_{\mathrm{P}})^2(\partial_z\mathbb{Y}_{\mathrm{P}} - \partial_z\widetilde{Y})}\right),$$

where the operators Υ_{P}^i, Υ_{P}^j for $i \in \{0,1\}$, $j \in \{t,b\}$ are defined in (5.7) and (5.6) respectively.

Let us now check that the assumptions of Theorem 6 are satisfied in the two settings. The continuity of L_P from $\mathcal{Z}^\sigma$ to $\mathcal{H}^\sigma$ for $\sigma \in (0,1/6) \cup \{1\}$ is a consequence of the definition of the spaces $\mathcal{Z}^\sigma$. Item (i) follows from Corollary 5.13. Furthermore,

$$N(\Xi,0) = \big(f(x,\mathbb{Y}_{\mathrm{P}})\partial_z\mathbb{Y}_{\mathrm{P}} - v_b, \Upsilon_{\mathrm{P}}^0[\delta_0], \Upsilon_{\mathrm{P}}^1[\delta_1], \Upsilon_{\mathrm{P}}^t[\delta_t], \Upsilon_{\mathrm{P}}^b[\delta_b]\big).$$

Hence it is easily checked that $N(\cdot,0)$ is differentiable at $\Xi = 0$, and its (partial) differential is given by

$$\begin{aligned}
&\partial_\Xi N(0,0)(\Xi)\\
&= \big(f(x,\mathbb{Y}_{\mathrm{P}})\partial_z\mathbb{Y}_{\mathrm{P}} - v_b, \partial_z\mathbb{Y}_{\mathrm{P}}(x_0,z)\delta_0(\mathbb{Y}_{\mathrm{P}}(x_0,z)),\\
&\qquad \partial_z\mathbb{Y}_{\mathrm{P}}(x_1,z)\delta_1(\mathbb{Y}_{\mathrm{P}}(x_1,z)), (\partial_z\mathbb{Y}_{\mathrm{P}}(x,z_t))^2\delta_t(x), (\partial_z\mathbb{Y}_{\mathrm{P}}(x,z_b))^2\delta_b(x)\big).
\end{aligned}$$

As a consequence, $N_P(\Xi,\widetilde{Y}) - N_P(\Xi',\widetilde{Y}') - \partial_\Xi N_P(0,0)(\Xi - \Xi')$ is

$$\begin{aligned}
&\partial_z\mathbb{Y}_{\mathrm{P}}\big[\big(f(\cdot,\mathbb{Y}_{\mathrm{P}} - \widetilde{Y}) - f(\cdot,\mathbb{Y}_{\mathrm{P}})\big) - \big(f'(\cdot,\mathbb{Y}_{\mathrm{P}} - \widetilde{Y}') - f'(\cdot,\mathbb{Y}_{\mathrm{P}})\big)\big]\\
&\quad - \partial_z\widetilde{Y}(f(\cdot,\mathbb{Y}_{\mathrm{P}} - \widetilde{Y}) - f'(\cdot,\mathbb{Y}_{\mathrm{P}} - \widetilde{Y}')) - \partial_z(\widetilde{Y} - \widetilde{Y}')f'(\cdot,\mathbb{Y}_{\mathrm{P}} - \widetilde{Y}')\\
&\quad + \partial_z\left(\frac{(\partial_z\widetilde{Y})^2}{(\partial_z\mathbb{Y}_{\mathrm{P}})^2(\partial_z\mathbb{Y}_{\mathrm{P}} - \partial_z\widetilde{Y})}\right) - \partial_z\left(\frac{(\partial_z\widetilde{Y}')^2}{(\partial_z\mathbb{Y}_{\mathrm{P}})^2(\partial_z\mathbb{Y}_{\mathrm{P}} - \partial_z\widetilde{Y}')}\right).
\end{aligned} \tag{5.22}$$

We therefore turn towards the verification of Items (ii) and (iii) from Theorem 6.

Verification of Item (ii) in the low regularity setting. For $\sigma \in (0,1/6)$, let $\Xi, \Xi' \in \mathcal{X}^\sigma$ and $\widetilde{Y}, \widetilde{Y}' \in \mathcal{Z}^\sigma$ small enough. We need to estimate (5.22) in $H_x^\sigma H_z^1 \cap H_x^1 H_z^1((x-x_0)^2(x-x_1)^2)$.

In order not to burden the proof, we only estimate some of the norms above, and leave the other estimates to the reader. We focus for instance on

$$\big\|\partial_z\mathbb{Y}_{\mathrm{P}}\partial_z\big[\big(f(\cdot,\mathbb{Y}_{\mathrm{P}} - \widetilde{Y}) - f(\cdot,\mathbb{Y}_{\mathrm{P}})\big) - \big(f'(\cdot,\mathbb{Y}_{\mathrm{P}} - \widetilde{Y}') - f'(\cdot,\mathbb{Y}_{\mathrm{P}})\big)\big]\big\|_{H_x^\sigma L_z^2}.$$

Using Lemma B.3 in the Appendix, we bound this term by

$$\begin{aligned}
&\|(\partial_z \mathbb{Y}_{\mathrm{P}})^2\|_{L^\infty_z(H^{\frac12+\sigma}_x)} \big(\|\partial_y (f-f')(x,\mathbb{Y}_{\mathrm{P}}-\widetilde{Y}) - \partial_y (f-f')(x,\mathbb{Y}_{\mathrm{P}})\|_{H^\sigma_x L^2_z} \\
&\qquad\qquad + \|\partial_y f'(x,\mathbb{Y}_{\mathrm{P}}-\widetilde{Y}) - \partial_y f'(x,\mathbb{Y}_{\mathrm{P}}-\widetilde{Y}')\|_{H^\sigma_x L^2_z}\big) \\
&+ \|\partial_z \mathbb{Y}_{\mathrm{P}} \partial_z \widetilde{Y}\|_{L^\infty_z(H^{\frac12+\sigma}_x)} \|\partial_y (f-f')(x,\mathbb{Y}_{\mathrm{P}}-\widetilde{Y})\|_{H^\sigma_x L^2_z} \\
&+ \|\partial_z \mathbb{Y}_{\mathrm{P}} \partial_z \widetilde{Y}\|_{L^\infty_z(H^{\frac12+\sigma}_x)} \|\partial_y f'(x,\mathbb{Y}_{\mathrm{P}}-\widetilde{Y}) - \partial_y f'(x,\mathbb{Y}_{\mathrm{P}}-\widetilde{Y}')\|_{H^\sigma_x L^2_z} \\
&+ \|\partial_z \mathbb{Y}_{\mathrm{P}} (\partial_z \widetilde{Y} - \partial_z \widetilde{Y}')\|_{L^\infty_z(H^{\frac12+\sigma}_x)} \|\partial_y f'(x,\mathbb{Y}_{\mathrm{P}}-\widetilde{Y}')\|_{H^\sigma_x L^2_z}.
\end{aligned}$$

Using the fractional trace theorem [42, equation (4.7), Chapter 1], $\mathcal{Z}^\sigma \hookrightarrow C^1_z(H^{\frac12+\sigma}_x)$. Furthermore, since $L^\infty_z(H^{\frac12+\sigma}_x)$ is an algebra,

$$\begin{aligned}
\|\partial_z \mathbb{Y}_{\mathrm{P}} \partial_z \widetilde{Y}\|_{L^\infty_z(H^{\frac12+\sigma}_x)} &\lesssim \|\widetilde{Y}\|_{\mathcal{Z}^\sigma}, \\
\|\partial_z \mathbb{Y}_{\mathrm{P}} (\partial_z \widetilde{Y} - \partial_z \widetilde{Y}')\|_{L^\infty_z(H^{\frac12+\sigma}_x)} &\lesssim \|\widetilde{Y} - \widetilde{Y}'\|_{\mathcal{Z}^\sigma}.
\end{aligned}$$

There remains to estimate the norms involving f and f'. Using Lemma B.4 in the Appendix, we infer that

$$\begin{aligned}
&\|\partial_y (f-f')(x,\mathbb{Y}_{\mathrm{P}}-\widetilde{Y}) - \partial_y (f-f')(x,\mathbb{Y}_{\mathrm{P}})\|_{H^\sigma_x L^2_z} \\
&= \left\| \widetilde{Y} \int_0^1 \partial_y^2 (f-f')(x,\mathbb{Y}_{\mathrm{P}} - \tau \widetilde{Y})\, \mathrm{d}\tau \right\|_{H^\sigma_x L^2_z} \\
&\lesssim \|\widetilde{Y}\|_{H^{2/3}_x H^1_z} \big(\|\partial_y^2 (f-f')\|_{H^\sigma_x L^2_z} + \|\partial_y^3 (f-f')\|_{L^4_x L^2_y}\big) \lesssim \|\widetilde{Y}\|_{\mathcal{Z}^\sigma} \|\Xi - \Xi'\|_{\mathcal{X}^\sigma}.
\end{aligned}$$

In a similar fashion,

$$\begin{aligned}
\|\partial_y f'(x,\mathbb{Y}_{\mathrm{P}}-\widetilde{Y}')\|_{H^\sigma_x L^2_z} &\lesssim \|\Xi'\|_{\mathcal{X}^\sigma}, \\
\|\partial_y f'(x,\mathbb{Y}_{\mathrm{P}}-\widetilde{Y}) - \partial_y f'(x,\mathbb{Y}_{\mathrm{P}}-\widetilde{Y}')\|_{H^\sigma_x L^2_z} &\lesssim \|\widetilde{Y} - \widetilde{Y}'\|_{\mathcal{Z}^\sigma} \|\Xi'\|_{\mathcal{X}^\sigma}.
\end{aligned}$$

The other terms are evaluated in a similar way. For instance, using again Lemma B.3 and the embedding $\mathcal{Z}^\sigma \hookrightarrow C^1_z(H^{\frac12+\sigma}_x)$,

$$\begin{aligned}
&\left\| \partial_z^3 (\widetilde{Y} - \widetilde{Y}') \frac{\partial_z \widetilde{Y}}{(\partial_z \mathbb{Y}_{\mathrm{P}})^2 \partial_z (\mathbb{Y}_{\mathrm{P}} - \widetilde{Y})} \right\|_{H^\sigma_x L^2_z} \\
&\qquad \lesssim \|\partial_z^3 (\widetilde{Y} - \widetilde{Y}')\|_{H^\sigma_x L^2_z} \left\| \frac{\partial_z \widetilde{Y}}{(\partial_z \mathbb{Y}_{\mathrm{P}})^2 \partial_z (\mathbb{Y}_{\mathrm{P}} - \widetilde{Y})} \right\|_{L^\infty_z H^{\frac12+\sigma}_x} \\
&\qquad \lesssim \|\widetilde{Y} - \widetilde{Y}'\|_{H^\sigma_x H^3_z} \|\partial_z \widetilde{Y}\|_{L^\infty_z H^{\frac12+\sigma}_x} \\
&\qquad \lesssim \|\widetilde{Y} - \widetilde{Y}'\|_{\mathcal{Z}^\sigma} \|\widetilde{Y}\|_{\mathcal{Z}^\sigma},
\end{aligned}$$

and, using once again Lemma B.4,

$$\begin{aligned}&\left\|\partial_z^2\widetilde{Y}(f(x,\mathbb{Y}_{\mathrm{P}}-\widetilde{Y})-f(x,\mathbb{Y}_{\mathrm{P}}-\widetilde{Y}'))\right\|_{H_x^\sigma L_z^2}\\&\quad\lesssim\|\partial_z^2\widetilde{Y}\|_{H_x^\sigma H_z^1}\left\|f(x,\mathbb{Y}_{\mathrm{P}}-\widetilde{Y})-f(x,\mathbb{Y}_{\mathrm{P}}-\widetilde{Y}')\right\|_{H_x^{\frac12+\sigma}L_z^2}\\&\quad\lesssim\|\widetilde{Y}\|_{H_x^\sigma H_z^3}\|\widetilde{Y}-\widetilde{Y}'\|_{H_x^{\frac12+\sigma}H_z^1}\big(\|\partial_y f\|_{H_x^{\frac12+\sigma}L_z^2}+\|\partial_y^2 f\|_{L^\infty}\big).\end{aligned}$$

The estimate on the $H_x^1H_z^1((x-x_0)^2(x-x_1)^2)$ norm follows from similar arguments and is left to the reader.

We then turn towards the estimation of the boundary terms.

- For $i\in\{0,1\}$ and $\delta_i,\eta_i\in H^4(\Sigma_i^P)$, we obtain that

  $$\left\|\Upsilon_{\mathrm{P}}^i[\delta_i]-\Upsilon_{\mathrm{P}}^i[\eta_i]-\partial_z\mathbb{Y}_{\mathrm{P}}(x_i,z)(\delta_i-\eta_i)(\mathbb{Y}_{\mathrm{P}}(x_i,z))\right\|_{H^3(\Sigma_i)}=o(\|\delta_i-\eta_i\|_{H^4(\Sigma_i^P)}).$$

  The proof is similar to the one of Lemma 4.15 for the Burgers case, although slightly less technical because we only need a standard Sobolev estimate here, and slightly more technical because the reference flow is now $\mathbb{u}_{\mathrm{P}}$ instead of the linear shear flow.

- For $j\in\{t,b\}$ and $\delta_j,\eta_j\in H^2(x_0,x_1)$, we obtain that

  $$\left\|\Upsilon_{\mathrm{P}}^j[\delta_j]-\Upsilon_{\mathrm{P}}^j[\eta_j]-(\partial_z\mathbb{Y}_{\mathrm{P}}(x,z_j))^2(\delta_j-\eta_j)\right\|_{H_x^2}=o(\|\delta_j-\eta_j\|_{H_x^2}).$$

  The proof is immediate because the maps Υ_{P}^j defined in (5.6) are in fact of the form $\Upsilon_{\mathrm{P}}^j[\delta_j](x)=h_j(x,\delta_j(x))$, where $h_j:(x_0,x_1)\times\mathbb{R}\to\mathbb{R}$ is a smooth function with $h_j(\cdot,0)=0$.

Eventually, we conclude that

$$\|N(\Xi,\widetilde{Y})-N(\Xi',\widetilde{Y}')-\partial_\Xi N(0,0)(\Xi-\Xi')\|_{\mathcal{H}^\sigma}=o(\|\Xi-\Xi'\|_{\mathcal{X}^\sigma}+\|\widetilde{Y}-\widetilde{Y}'\|_{\mathcal{Z}^\sigma}).$$

Verification of Item (ii) in the high regularity setting. The estimates in this case are similar to the low regularity setting and left to the reader. They are actually slightly easier since $H^1(x_0,x_1)$ is an algebra, and close to the ones performed for the Burgers system.

The only new estimate bears on the boundary term. More precisely, taking two data tuples $\Xi=(f,\delta_0,\delta_1,\delta_t,\delta_b,v_b)$ and $\Xi'=(f',\eta_0,\eta_1,\eta_t,\eta_b,v_b')$, we need to bound in $\mathcal{H}_z^1(\Sigma_i)$ the quantity

$$z^{-1}\big[\partial_z(N_P(\Xi,\widetilde{Y})-N_P(\Xi',\widetilde{Y}')-\partial_\Xi N_P(0,0)(\Xi-\Xi'))|_{\Sigma_i}\big].$$

We recall that $f|_{\Sigma_i^P}=f'|_{\Sigma_i^P}=0$, so that the terms stemming from f and f' in N_P vanish on the boundary. We therefore consider

$$\left\|z^{-1}\partial_z^2\left[\frac{(\partial_z\Upsilon_{\mathrm{P}}^i[\delta_i])^2}{(\partial_z\mathbb{Y}_{\mathrm{P}})^2(\partial_z\mathbb{Y}_{\mathrm{P}}-\partial_z\Upsilon_{\mathrm{P}}^i[\delta_i])}-\frac{(\partial_z\Upsilon_{\mathrm{P}}^i[\eta_i])^2}{(\partial_z\mathbb{Y}_{\mathrm{P}})^2(\partial_z\mathbb{Y}_{\mathrm{P}}-\partial_z\Upsilon_{\mathrm{P}}^i[\eta_i])}\right]\right\|_{\mathcal{H}_z^1}. \tag{5.23}$$

Since $\partial_z^k \delta_i(0) = 0$ for $k \in \{0, \dots, 3\}$, we have $\partial_z^k \Upsilon_{\mathrm{P}}^i[\delta_i](z=0) = 0$. According to Lemma 4.14,

$$
\begin{aligned}
(5.23) &\lesssim \left\| \frac{(\partial_z \Upsilon_{\mathrm{P}}^i[\delta_i])^2}{(\partial_z \mathbb{Y}_{\mathrm{P}})^2(\partial_z \mathbb{Y}_{\mathrm{P}} - \partial_z \Upsilon_{\mathrm{P}}^i[\delta_i])} - \frac{(\partial_z \Upsilon_{\mathrm{P}}^i[\eta_i])^2}{(\partial_z \mathbb{Y}_{\mathrm{P}})^2(\partial_z \mathbb{Y}_{\mathrm{P}} - \partial_z \Upsilon_{\mathrm{P}}^i[\eta_i])} \right\|_{H^4(\Sigma_i)} \\
&\lesssim (\|\delta_i\|_{H^6(\Sigma_i)} + \|\eta_i\|_{H^6(\Sigma_i)}) \|\eta_i - \delta_i\|_{H^6(\Sigma_i)}.
\end{aligned}
$$

We obtain eventually

$$
\|N(\Xi, \widetilde{Y}) - N(\Xi', \widetilde{Y}') - \partial_\Xi N(0,0)(\Xi - \Xi')\|_{\mathcal{H}^1} = o(\|\Xi - \Xi'\|_{\mathcal{X}^1} + \|\widetilde{Y} - \widetilde{Y}'\|_{\mathcal{Z}^1}).
$$

Verification of Item (iii) in the low regularity setting. We just need to check that the application $\ell^2 \circ \partial_\Xi N(0,0)$ is not identically zero. This is actually trivial: take $\Xi = (0, 0, 0, 0, 0, v_b)$. Then the solution to the vorticity equation is zero, and recalling Definition 5.7,

$$
\ell^2 \circ \partial_\Xi N(0,0) = -\frac{1}{z_b} \int_{x_0}^{x_1} v_b(x)\, \mathrm{d}x.
$$

Therefore it suffices to choose v_b such that the above integral is non-zero.

Verification of Item (iii) in the high regularity setting. Using Lemma 5.12, we take $\Theta^j = (f^j, 0, 0, 0, 0)$ such that $\ell^i(\Theta^j) = \delta_{i,j}$ for $0 \le i, j \le 2$, with $f^i \in C^\infty(\overline{\Omega})$ such that $f|_{\Sigma_0 \cup \Sigma_1} = 0$. We then set $\Xi^j := (g^j, 0, 0, 0, 0, 0)$, where

$$
g^j(x, y) := \partial_y \mathbb{u}_{\mathrm{P}}(x, y) f^j(x, \mathbb{u}_{\mathrm{P}}(x, y)).
$$

Then, by design, $\partial_\Xi N(0,0)(\Xi^j) = \Theta^j$, so that $\ell^j \circ \partial_\Xi N(0,0)(\Xi^j) = \delta_{i,j}$. Furthermore, $\Xi^j \in \mathcal{X}^1$. The result follows.

Conclusion. We have checked the assumptions of Theorem 6 both in the low regularity case $\sigma \in (0, 1/6)$ and in the high regularity case $\sigma = 1$. Proposition 5.2 is now a straightforward consequence of our abstract framework.

5.5 Well-posedness of the Prandtl system

We conclude this section with the proof of Theorem 4, which follows from Proposition 5.2.

High regularity case. The proof of Theorem 4 in the high regularity case corresponding to $(f, \delta_0, \delta_1, \delta_t, \delta_b, v_b) \in \mathcal{M}_1$ is very similar to the proof for Burgers carried out in Section 4.3. We leave it to the reader. As in the Burgers case, one uses the equations satisfied by u and $\widetilde{Y}$ to check that they actually also enjoy $L_x^2 H_y^5$ regularity and one can prove a lemma similar to Corollary B.6 to prove that the formula $u(x, Y(x, z)) = z$ allows to transfer such a regularity back and forth.

Low regularity case. We focus on the case when $(f, \delta_0, \delta_1, \delta_t, \delta_b, v_b) \in \mathcal{M}_\sigma$ with $\sigma \in (0, 1/6)$, and we consider the unique solution $\widetilde{Y} \in \mathcal{Z}^\sigma$ of (5.4). Let $\Omega_P = \{(x, y) \in (x_0, x_1) \times \mathbb{R},\ \gamma_b(x) < y < \gamma_t(x)\}$, where $\gamma_j(x) = Y(x, z_j)$. For almost every $x \in (x_0, x_1)$, $z \mapsto \mathbb{Y}_\mathrm{P}(x, z) + \widetilde{Y}(x, z)$ is an H^3 diffeomorphism. We note that there exists a constant $\lambda > 0$ such that $\lambda^{-1} \leq \partial_z Y \leq \lambda$ in Ω. Let us define the reverse change of variables u such that $u(x, (\mathbb{Y}_\mathrm{P} + \widetilde{Y})(x, z)) = z$. Classical results ensure that for a.e. x, $u(x, \cdot) \in H^3_y$. Furthermore, differentiating the formula (4.4), we obtain

$$\partial_y^3 u(x, Y(x, z)) = -\frac{\partial_z^3 Y(x, z)}{(\partial_z Y(x, z))^4} + 3\frac{(\partial_z^2 Y(x, z))^2}{(\partial_z Y(x, z))^5},$$

which ensures that $\partial_y^3 u \in L^2(\Omega_P)$. Since

$$\partial_y u(x, y) = \frac{1}{\partial_z Y(x, u(x, y))},$$

we also infer that $\partial_y u \in L^\infty$ and $\lambda^{-1} \leq \partial_y u \leq \lambda$ for some $\lambda > 0$.

Additionally, since $(x - x_0)(x - x_1)Y \in H^1_x H^3_z$, we also infer that $(x - x_0)(x - x_1)\partial_y^k u(x, Y(x, z)) \in H^1_x L^2_y(\Omega_P)$ for $0 \leq k \leq 3$. From there, we deduce that $(x - x_0)(x - x_1)\partial_x \partial_y^k u \in L^2(\Omega_P)$ for $0 \leq k \leq 3$. Furthermore, since $z\partial_x \partial_z Y \in L^2$, we also deduce that $u\partial_x \partial_y u \in L^2$. Tracing back the computations at the beginning of Section 5.1, and noticing that $u \in H^1_x H^3_y(\omega)$ for all $\omega \Subset \Omega_P$ as well as in the vicinity of $\overline{\Gamma_b}$, we infer that u is a weak solution to the Prandtl system (5.1). This proves the existence of a solution to (5.1) and (5.3). In order to prove the continuity of u, we observe that for all $(x, y), (x', y') \in \Omega_P$, setting $z = u(x, y)$,

$$\begin{aligned}
|u(x, y) - u(x', y')| &\leq |u(x, y) - u(x', y)| + \|\partial_y u\|_\infty |y - y'| \\
&\leq |z - u(x', Y(x, z))| + \|\partial_y u\|_\infty |y - y'| \\
&\leq |u(x', Y(x', z)) - u(x', Y(x, z))| + \|\partial_y u\|_\infty |y - y'| \\
&\leq \|\partial_y u\|_\infty (|Y(x', z) - Y(x, z)| + |y - y'|).
\end{aligned}$$

Since $Y \in H^{\frac{2}{3}}_x H^1_z \hookrightarrow C^\alpha$ for some $\alpha > 0$, we infer that u is Hölder continuous.

Let us now prove the uniqueness of this solution within the regularity class

$$u \in L^2_x H^3_y(\Omega_P), \partial_y u \in L^\infty, (x - x_0)(x - x_1)u \in H^1_x H^3_y(\Omega_P), u\partial_x \partial_y u \in L^2(\Omega_P),$$

and assuming that u is close to $\mathbb{u}_\mathrm{P}$ in the associated norm. Note that this implies in particular that $\partial_y u$ is bounded pointwise from above and below. The associated function Y is such that $Y \in L^2_x H^3_z$, $\partial_z Y \in L^\infty$, $(x - x_0)(x - x_1)Y \in H^1_x H^3_y$, and $z\partial_x \partial_z Y \in L^2$. In particular, $\partial_z Y \in Z^0$. This regularity is sufficient to justify the computations of Section 5.1, and thus $\widetilde{Y} = Y - \mathbb{Y}_\mathrm{P}$ is a solution to (5.4) in the sense of distributions. It follows that $\partial_z \widetilde{Y}$ is a solution to (5.10), and $\partial_z Y$ is bounded pointwise from above and below by positive constants. From there, we deduce that $\partial_z Y \in \mathcal{Z}^\sigma$. Applying the first statement of Proposition 5.2, we deduce that $(f, \delta_0, \delta_1, \delta_b, \delta_t, v_b) \in \mathcal{M}_\sigma$.

Now, let u_1, u_2 be two solutions of (5.1) within the above regularity class, corresponding to solutions $\widetilde{Y}_1, \widetilde{Y}_2$ of (5.4). Let

$$g_i := f(x, Y_i)\partial_z Y_i - v_b + \partial_z\left(\frac{(\partial_z \widetilde{Y}_i)^2}{(\partial_z \mathbb{Y}_{\mathrm{P}})^2 \partial_z Y_i}\right).$$

Then $W := \partial_z(\widetilde{Y}_1 - \widetilde{Y}_2) \in Z^0$ is a solution to (5.12) with homogeneous boundary data and with a source term $h = \partial_z g_1 - \partial_z g_2$. Therefore, multiplying the equation by W and integrating by parts, we obtain

$$\int \alpha |\partial_z W|^2 \le C(\|g_1 - g_2\|_{L^2}^2 + \|W\|_{L^2}^2),$$

where the constant C depends only on the underlying flow $\mathbb{u}_{\mathrm{P}}$. As in the proof of Lemma 5.3, for $|z_b|, z_t \le z_0$, we infer that

$$\|W\|_{L^2_x H^1_z} \lesssim \|g_1 - g_2\|_{L^2}.$$

From there, using equation (5.12), we obtain

$$\|W\|_{\mathcal{B}} \lesssim \|g_1 - g_2\|_{L^2}.$$

Using the formula for g_i above, we deduce that

$$\begin{aligned}\|g_1 - g_2\|_{L^2} \lesssim\ & \|\partial_y f\|_\infty \|Y_1 - Y_2\|_{L^2} \|\partial_z Y_1\|_\infty + \|f\|_\infty \|\partial_z (Y_1 - Y_2)\|_{L^2} \\ & + \|\partial_z \widetilde{Y}_1\|_\infty \|\partial_z^2 (Y_1 - Y_2)\|_{L^2} \\ & + \|\partial_z (Y_1 - Y_2)\|_{L^\infty_z(L^3_x)} \|\partial_z^2 \widetilde{Y}_2\|_{L^2_z(L^6_x)}.\end{aligned}$$

Setting

$$\eta := \|\partial_z \widetilde{Y}_1\|_\infty + \|\partial_z \widetilde{Y}_2\|_{Z^0} + \|f\|_{L^\infty_x W^{1,\infty}_y},$$

and using the embeddings $Z^0 \hookrightarrow L^2_z H^{1/3}_x \hookrightarrow L^2_z(L^6_x)$, $\mathcal{B} \hookrightarrow C^0_z([z_b, z_t]; H^{1/6}_x) \hookrightarrow L^\infty_z(L^3_x)$ (see Lemma 1.14), we infer

$$\|g_1 - g_2\|_{L^2} \lesssim \eta \|W\|_{\mathcal{B}}.$$

Hence we obtain $\|W\|_{\mathcal{B}} \lesssim \eta \|W\|_{\mathcal{B}}$, and provided η is small enough, $W = 0$.

Remark 5.16. Note that in the case $\sigma \in (0, 1/6)$, we are not able to transfer completely the fractional horizontal regularity from Y to u. Indeed, one can easily check from the formulas in (4.4) that $\partial_y^k u(x, Y(x,z)) \in H_x^{\frac{3-k}{3}+\sigma} L_z^2 \cap H_x^\sigma H_z^{3-k}$ for $k \in \{1,\dots,3\}$. Then, one may try to get some regularity on u by computing

$$\|\partial_y u\|^2_{H_x^{\frac{2}{3}+\sigma} L_y^2} = \|\partial_y u\|^2_{L^2} + \int_{x_0}^{x_1}\int_{x_0}^{x_1}\int_{\mathbb{R}} \mathbf{1}_{(x,y)\in\Omega_P}\mathbf{1}_{(x',y)\in\Omega_P} \frac{|\partial_y u(x,y) - \partial_y u(x',y)|^2}{|x-x'|^{\frac{7}{3}+2\sigma}}\, dx\, dx'\, dy.$$

It is quite natural to change variables in the second integral on the right-hand side by setting $y = Y(x,z)$, the associated Jacobian being bounded from above and below, and to split the resulting integral into

$$\int_{x_0}^{x_1}\int_{x_0}^{x_1}\int_{z_b}^{z_t} \frac{|\partial_y u(x, Y(x,z)) - \partial_y u(x', Y(x',z))|^2}{|x-x'|^{\frac{7}{3}+2\sigma}}\, dx\, dx'\, dz + \int_{x_0}^{x_1}\int_{x_0}^{x_1}\int_{z_b}^{z_t} \frac{|\partial_y u(x', Y(x',z)) - \partial_y u(x', Y(x,z))|^2}{|x-x'|^{\frac{7}{3}+2\sigma}}\, dx\, dx'\, dz.$$

The first integral above is bounded by $\|\partial_y u(x, Y(x,z))\|^2_{H_x^{2/3+\sigma} L_z^2}$. As for the second integral, if $\partial_y u$ were Lipschitz-continuous with respect to y (or even Hölder continuous with some suitable exponent), we would bound this integral by $\|Y\|^2_{H_x^{2/3+\sigma} L_z^2}$. But unfortunately, this Lipschitz regularity does not hold in general. However, thanks to the regularity result far from the lateral boundaries from Lemma 5.4, we have sufficient regularity on u to ensure uniqueness.

5.6 Potential strategy in a whole infinite strip

In this section, we sketch a potential strategy to solve the Prandtl equation (5.1) in the whole infinite strip $(x_0, x_1) \times (0, +\infty)$, based on the previous analysis. To that end, we first propose a scheme to solve a system with a modified source term (and without any orthogonality condition). Once the solvability of this modified system is understood, the solvability of the original system follows for data within a finite codimensional manifold.

We start from a smooth solution $(\mathbb{u}_P, \mathbb{v}_P)$ to (5.1) such that $\mathbb{u}_P(x, \gamma_P(x)) = 0$ for some smooth function γ_P, and $\mathbb{u}_P(x,y) < 0$ (resp. $\mathbb{u}_P(x,y) > 0$) for $y \in (0, \gamma_P(x))$ (resp. for $y > \gamma_P(x)$). We also have the boundary conditions $\mathbb{u}_P|_{y=0} = \mathbb{v}_P|_{y=0} = 0$, and $\mathbb{u}_P(x,y) \to u_\infty(x)$ as $y \to \infty$, where $u_\infty u_\infty' = -\partial_x p$. As before, we fix two small numbers $z_b < 0 < z_t$, such that there exist smooth lines $\{y = \overline{\gamma_j}(x)\}$ with $\mathbb{u}_P(x, \overline{\gamma_j}(x)) = z_j$. We consider perturbations

$$\Theta := (\delta_0, \delta_1, f) \in H^k(0,+\infty) \times H^k(0,+\infty) \times H^k((x_0,x_1)\times(0,+\infty))$$

for some sufficiently large k, and for simplicity, we also assume that δ_i vanishes at $\gamma_{\mathrm{P}}(x_i)$. We then define an application $\mathcal{A} : (\Theta; \gamma_b, \delta_t) \mapsto (\gamma_b', \delta_t')$ in the following way.

(1) We solve the Prandtl system in the domain $\{(x,y) \in (x_0, x_1) \times (0, +\infty),\ y < \gamma_b(x)\}$ in the vicinity of the flow $(\mathbb{u}_{\mathrm{P}}, \mathbb{v}_{\mathrm{P}})$, with source term $-\partial_x p + f$ and boundary data

$$\begin{aligned} u|_{x=x_1} &= \mathbb{u}_{\mathrm{P}}|_{x=x_1} + \delta_1, \\ u|_{y=0} = v|_{y=0} &= 0, \\ u|_{y=\gamma_b(x)} &= z_b. \end{aligned}$$

In (the interior of) this domain, $\mathbb{u}_{\mathrm{P}} < 0$, and therefore the system is backward parabolic. Hence we expect that it is solvable (see [49]). A possible way to solve it could be to introduce the "von Mise-type good unknown" from [35]. Assuming that the above system is solvable, we set $v_b := v|_{y=\gamma_b} - \mathbb{v}_{\mathrm{P}}|_{y=\gamma_b}$, and $\delta_b := \partial_y u|_{y=\gamma_b} - \partial_y \mathbb{u}_{\mathrm{P}}|_{y=\gamma_b}$. Note that there are typically compatibility conditions which are necessary to ensure the existence of smooth solutions of this system. We leave this issue aside in the present discussion. The compatibility conditions are automatically ensured if f is supported in $(x_0 + \delta_x, x_1 - \delta_x)$ for $|\delta_x| \ll 1$, and if δ_0 (resp. δ_1) is compactly supported in $(\gamma_P(x_0), \overline{\gamma_t}(x_0))$ (resp. $\overline{\gamma_b}(x_1), \gamma_P(x_1))$.

(2) We then consider the Prandtl system in the recirculating zone. More precisely, using the analysis of the previous sections, we construct a solution to

$$\begin{aligned} u u_x + v u_y - \partial_{yy} u &= -\partial_x p + f \\ &\qquad - (\nu^0 f^0 + \nu^1 f^1 + \nu^2 f^2)(x, u(x,y)) \partial_y u(x,y), \\ u_x + v_y &= 0, \end{aligned}$$

together with the boundary conditions (1.9), (1.10), (1.11), in which the bottom data v_b, δ_b are provided by the first step. Note that the new free boundary $\{y = \gamma_b'(x) := Y(x, z_b)\}$, with the notations of the previous sections, is different from the boundary $\{y = \gamma_b(x)\}$ *a priori*. The coefficients (ν^0, ν^1, ν^2) are Lipschitz functions of the data (f, δ_0, δ_1) and ensure that the associated solution u belongs to $H^{5/3}_x H^1_y \cap H^1_x H^3_y$. Note that the structure of the right-hand side is designed so that the equation in the variables (x,z) is

$$\begin{aligned} z \partial_x Y - \int_{z_b}^{z} \partial_x Y - \frac{1}{(\partial_z Y)^2} \partial_z^2 Y &= (\partial_x p - f(x, Y)) \partial_z Y + \mathbb{v}_{\mathrm{P}}|_{y=\overline{\gamma_b}} \\ &\qquad + v_b + \nu^0 f^0 + \nu^1 f^1 + \nu^2 f^2. \end{aligned}$$

Let $\mathcal{V}(f, \delta_0, \delta_1; \gamma_b, \delta_t)$ denote the quantity $v|_{y=\gamma_t(x)}$, where $\gamma_t(x) = Y(x, z_t)$. The boundary $\{y = \gamma_t(x)\}$ will be the lower boundary of the upper domain considered in the next step, but is not a variable of the implicit function argument.

(3) Eventually, we solve the Prandtl system in $\{(x, y) \in (x_0, x_1) \times (0, +\infty),\ y > \gamma_t(x)\}$ in the vicinity of the flow $(\mathbb{u}_P, \mathbb{v}_P)$, with source term $-\partial_x p + f$ and boundary data

$$\begin{aligned} u|_{x=x_0} &= \mathbb{u}_P|_{x=x_0} + \delta_0, \\ u|_{y=\gamma_t(x)} &= z_t, \\ v|_{y=\gamma_t(x)} &= \mathcal{V}(f, \delta_0, \delta_1; \gamma_b, \delta_t), \\ \lim_{y\to\infty} u(x, y) &= u_\infty(x). \end{aligned}$$

This system is now forward parabolic. It can be solved with the tools of [49]. Note that $\inf \mathbb{u}_P > 0$ in the upper domain, so that the system is in fact non-degenerate after a suitable change of variables. We then define

$$\delta_t' = \partial_y u|_{y=\gamma_t(x)}.$$

Eventually, we set $\mathcal{A}(\Theta; \gamma_b, \delta_t) = (\gamma_b', \delta_t')$. The first question which needs to be solved is the following:

For every $\Theta \in H^k(0, +\infty) \times H^k(0, +\infty) \times H^k((x_0, x_1) \times (0, +\infty))$ *such that* $\|\Theta\| \leq \delta$,
find (γ_b, δ_t) *such that* $\mathcal{A}(\Theta; \gamma_b, \delta_t) = (\gamma_b, \delta_t)$.

For $\Theta = 0$, by definition of the application $\mathcal{A}$, one has $\mathcal{A}(0; \overline{\gamma_b}, \partial_y \mathbb{u}_P|_{y=\overline{\gamma_t}}) = (\overline{\gamma_b}, \partial_y \mathbb{u}_P|_{y=\overline{\gamma_t}})$. Hence a possible strategy could be to apply an implicit function theorem, in the spirit of [15] or Lemma 3.15. This requires to prove the invertibility of the function $d_{(\gamma_b,\delta_t)}\mathcal{A}(0; \overline{\gamma_b}, \partial_y \mathbb{u}_P|_{y=\overline{\gamma_t}}) - \mathrm{Id}$. In turn, this requires to prove the well-posedness of a linearized type Prandtl system (or of three coupled linearized Prandtl systems) in the infinite strip $(x_0, x_1) \times (0, +\infty)$. Such a result may typically involve restrictions on the size of the domain, as the following toy example demonstrates. Let $a \in L^\infty((x_0, x_1) \times \mathbb{R})$. Consider the forward-backward system

$$\begin{cases} z\partial_x u - \partial_{zz} u - au = 0 & \text{in } (x_0, x_1) \times \mathbb{R}, \\ u(x_0, z) = 0 & \text{for } z > 0, \\ u(x_1, z) = 0 & \text{for } z < 0. \end{cases}$$

Let us assume that there exists a solution with high enough decay for $|z| \gg 1$; our purpose is to prove that such a solution is identically zero. To that end, we multiply the above system by $u\rho$, where $\rho(x, z) := \exp(-(x - x_0)z/(x_1 - x_0))$, and perform integrations by parts. We obtain

$$\begin{aligned} &\frac{1}{2(x_1 - x_0)} \int_{x_0}^{x_1} \int_{\mathbb{R}} z^2 u^2 \rho \, \mathrm{d}x \, \mathrm{d}z + \int_{x_0}^{x_1} \int_{\mathbb{R}} (\partial_z u)^2 \rho \, \mathrm{d}x \, \mathrm{d}z \\ &\quad = -\int_{x_0}^{x_1} \int_{\mathbb{R}} a u^2 \rho \, \mathrm{d}x \, \mathrm{d}z + \frac{1}{2} \int_{x_0}^{x_1} \int_{\mathbb{R}} \left(\frac{x - x_0}{x_1 - x_0} \right)^2 u^2 \rho \, \mathrm{d}x \, \mathrm{d}z. \end{aligned}$$

For $|z| \geq 2(\|a\|_\infty + 1)^{1/2}(x_1 - x_0)^{1/2}$, the two terms on the right-hand side can be absorbed on the left-hand side. On the other hand, for $|z| \leq 2(\|a\|_\infty + 1)^{1/2}(x_1 - x_0)^{1/2}$ and $|x_1 - x_0| \leq 1$, the weight $\exp\left(-\frac{x-x_0}{x_1-x_0}z\right)$ is bounded from above and below. We then use the inequality

$$\|\phi\|_{L^2_z} \lesssim \|z\phi\|_{L^2_z} + \|\partial_z \phi\|_{L^2_z}$$

for any $\phi \in H^1(\mathbb{R})$ such that $z\phi \in L^2(\mathbb{R})$. The proof of the inequality follows from arguments similar to the ones of Lemma B.7 and is left to the reader. We infer that for $x_1 - x_0$ small enough, the only decaying solution to the above system is $\phi \equiv 0$. For $x_1 - x_0$ large, the situation is not so clear. These considerations could be seen as a toy example of why Iyer and Masmoudi in [35] need to exclude a "resonant set" of lengths $x_1 - x_0$ for which non-trivial solutions of a system similar to the one above may exist.