Chapter 6

Interpolation estimate for the linear shear flow problem

This chapter is devoted to the proof of Proposition 2.36, which is used in particular in the construction of weak solutions for the Prandtl system (see Proposition 5.5). The idea is to interpolate between the Z^0 estimate from Proposition 2.5, and the Z^1 estimate from Proposition 2.10. However, because of the orthogonality conditions, justifying that the interpolation space for the source terms is the expected one turns out to be quite complicated.

We introduce the following spaces for the source terms:

$$\mathcal{Y}_0 := \{f \in L^2(\Omega)\}, \tag{6.1}$$

$$\mathcal{Y}_1 := \left\{f \in H^1_x L^2_y;\ f_{|\Sigma_0 \cup \Sigma_1} = 0\right\} \tag{6.2}$$

endowed with their usual norms and

$$\mathcal{Y}_1^{\bar{\ell}} := \left\{f \in \mathcal{Y}_1; \overline{\ell^0}(f,0,0) = \overline{\ell^1}(f,0,0) = 0\right\}, \tag{6.3}$$

endowed with the norm of $\mathcal{Y}_1$, where $\overline{\ell^0}$ and $\overline{\ell^1}$ are defined in Definition 2.12. Since $\overline{\ell^0}$ and $\overline{\ell^1}$ are continuous for the $H^1_x L^2_y$ norm, $\mathcal{Y}_1^{\bar{\ell}}$ is a closed subspace of $\mathcal{Y}_1$.

We wish to interpolate between $\mathcal{Y}_0$ and $\mathcal{Y}_1^{\bar{\ell}}$. Using classical interpolation theory, one can determine $\mathcal{Y}_\sigma := [\mathcal{Y}_0, \mathcal{Y}_1]_\sigma$ quite easily (see Lemma 6.5 below). Nevertheless, there is a difficulty in the determination of the space $[\mathcal{Y}_0, \mathcal{Y}_1^{\bar{\ell}}]_\sigma$. This corresponds to the well-known problem of "subspace interpolation", for which we give a short survey in Section 6.1.

The proof of Proposition 2.36 relies on a careful analysis of the dual profiles $\overline{\Phi^j}$, and in particular on a decomposition of the latter into an explicit singular part and a regular part. This decomposition allows us to have quantitative upper and lower bounds on the functions $\tau \mapsto I(\tau, \overline{\ell^j})$, which play a paramount role in interpolation theory (see [43] and Section 6.1.2 below).

The organization of this section is as follows. We start by introducing the theory of subspace interpolation, and associated notations in Section 6.1. We then turn towards the proof of Proposition 2.36 in Section 6.2, illustrating how the general theory can be applied for our problem, thanks to the knowledge of the singular profiles of Section 2.4.

6.1 A primer on subspace interpolation

Using interpolation theory in a context where constraints are enforced on the data comes with a specific difficulty, known as "subspace interpolation". In this section, we give a short introduction and set up notations and a lemma that will be used in the next sections.

6.1.1 An introduction to subspace interpolation

Let us start by a short introduction to the topic of subspace interpolation and the associated difficulty. This difficulty is *not* linked with the difference between complex and real interpolation methods. Indeed, it occurs even in the case of "quadratic" interpolation between separable Hilbert spaces, for which all methods construct the same interpolation spaces (see [14, Remark 3.6] and [13, Section 3.3, Item (4)] based on the initial geometric argument of [46]).

Setting of the problem. Let $\mathcal{Y}_0$ and $\mathcal{Y}_1$ denote two Banach spaces with a dense continuous embedding $\mathcal{Y}_1 \hookrightarrow \mathcal{Y}_0$. Let $\mathcal{Y}_\sigma := [\mathcal{Y}_0, \mathcal{Y}_1]_\sigma$, for $\sigma \in (0,1)$, say for the complex method to fix ideas. Let ℓ be a continuous linear form on $\mathcal{Y}_1$, which is however unbounded on $\mathcal{Y}_0$, and define its kernel $\mathcal{Y}_1^\ell := \{f \in \mathcal{Y}_1; \ell(f) = 0\}$, which is a closed subspace of $\mathcal{Y}_1$. The question of "subspace interpolation" consists in determining the relation between $\mathcal{Y}_\sigma$ and $[\mathcal{Y}_0, \mathcal{Y}_1^\ell]_\sigma$. This question of course admits a straightforward generalization to the case of a finite number of orthogonality conditions.

Generally, one checks that the closure of $[\mathcal{Y}_0, \mathcal{Y}_1^\ell]_\sigma$ in $\mathcal{Y}_\sigma$ is either a subspace of codimension 1, when ℓ is continuous on $\mathcal{Y}_\sigma$, or the whole of $\mathcal{Y}_\sigma$, when ℓ is unbounded on $\mathcal{Y}_\sigma$. In the former case, there is no guarantee that $[\mathcal{Y}_0, \mathcal{Y}_1^\ell]_\sigma$ itself is closed in $\mathcal{Y}_\sigma$ (or, equivalently, that the associated norms are equivalent on $[\mathcal{Y}_0, \mathcal{Y}_1^\ell]_\sigma$). The first systematic occurrence of this question seems to date back to [45, Problem 18.5, Chapter 1], which claims that a major difficulty to use interpolation theory is that "*l'interpolé de sous-espaces fermés n'est pas nécessairement un sous-espace fermé dans l'interpolé*" (the interpolation space between closed subspaces is not necessarily a closed subspace in the interpolation space), and asks for sufficient conditions for $[\mathcal{Y}_0, \mathcal{Y}_1^\ell]_\sigma$ to be closed in $\mathcal{Y}_\sigma$.

Remark 6.1. When ℓ is continuous for the topology of $\mathcal{Y}_0$, there is no difficulty. Indeed, one checks that, for every $\sigma \in (0,1)$, $[\mathcal{Y}_0, \mathcal{Y}_1^\ell]_\sigma = \{f \in \mathcal{Y}_\sigma; \ell(f) = 0\}$, endowed with the topology of $\mathcal{Y}_\sigma$, for which ℓ is continuous (see, e.g., the related result [45, Theorem 13.3, Chapter 1]).

Some examples. The best known and most simple example of such a phenomenon, introduced in [45, Theorem 11.7, Chapter 1] concerns the construction of the space $H_{00}^{1/2}(0,1) = [L^2(0,1), H_0^1(0,1)]_{1/2}$. The space $H_{00}^{1/2}(0,1)$ is not closed in $H^{1/2}(0,1)$ and the associated norm involves a non-equivalent "additional term".

In [62], using real interpolation between L^1 and L^∞, Wallstén constructed examples illustrating that this pathological behavior is not limited to exceptional values of the interpolation parameter, since there exist constraints for which it occurs for every $\sigma \in (0, 1)$.

Short survey of known results. Precising earlier results of Löfström [43,44], Ivanov and Kalton proved in [33] that, in the general case, there exist two thresholds $0 \le \sigma_0 \le \sigma_1 \le 1$ such that

- when $0 < \sigma < \sigma_0$, $[\mathcal{Y}_0, \mathcal{Y}_1^\ell]_\sigma = \mathcal{Y}_\sigma$, with equivalent norms,
- when $\sigma_0 \le \sigma \le \sigma_1$, the norm on $[\mathcal{Y}_0, \mathcal{Y}_1^\ell]_\sigma$ is not equivalent to the one on $\mathcal{Y}_\sigma$,
- when $\sigma_1 < \sigma < 1$, $[\mathcal{Y}_0, \mathcal{Y}_1^\ell]_\sigma$ is a closed subspace of codimension 1 in $\mathcal{Y}_\sigma$.

In the first case, ℓ is unbounded on $\mathcal{Y}_\sigma$ (the constraint does not make sense). In the second and third cases, ℓ admits a continuous extension to $\mathcal{Y}_\sigma$ and the closure of $[\mathcal{Y}_0, \mathcal{Y}_1^\ell]_\sigma$ in $\mathcal{Y}_\sigma$ is of codimension 1.

This classification has generalizations to the case of multiple constraints (see [2]), potentially involving multiple pathological intervals, associated with each constraint.

In the difficult regime $\sigma_0 \le \sigma \le \sigma_1$, more precise results [5,6] allow the computation of the "additional norm" stemming from the presence of the constraints.

The recent work [63] considers a kind of dual problem, by computing interpolation spaces between $\mathcal{Y}_0$ and $\mathcal{Y}_1 \oplus \mathbb{R}\omega$, where ω is a singular function of $\mathcal{Y}_0 \setminus \mathcal{Y}_1$, whose singularity is expressed in polar coordinates. In this work, $\sigma_0 = \sigma_1$. This is also our case below, and our dual profiles also involve singular parts which are expressed in radial-like coordinates, as constructed in Section 2.4.

6.1.2 A variant of a criterion due to Löfström

To prove Proposition 2.36, we will rely on an abstract interpolation result proved by Löfström in [43]. Let $\mathcal{Y}_0$ and $\mathcal{Y}_1$ denote two Hilbert spaces with a dense continuous embedding $\mathcal{Y}_1 \hookrightarrow \mathcal{Y}_0$.

For $f \in \mathcal{Y}_1$ and $\tau \in (0, 1)$, let

$$\|f\|_\tau^2 := \|f\|_{\mathcal{Y}_0}^2 + \tau^2 \|f\|_{\mathcal{Y}_1}^2. \tag{6.4}$$

This notation stems from [33], while [43] uses instead $\max(\|f\|_{\mathcal{Y}_0}, \tau\|f\|_{\mathcal{Y}_1})$. Since $\|f\|_\tau/\sqrt{2} \le \max(\|f\|_{\mathcal{Y}_0}, \tau\|f\|_{\mathcal{Y}_1}) \le \|f\|_\tau$, both quantities can be used equivalently.

Given a linear form ℓ on $\mathcal{Y}_1$, one defines, for $\tau \in (0, 1)$,

$$I(\tau, \ell) := \sup_{f \in \mathcal{Y}_1 \setminus \{0\}} \frac{\ell(f)}{\|f\|_\tau}. \tag{6.5}$$

As $\tau \to 0$, upper bounds on $I(\tau, \ell)$ are linked with the boundedness of ℓ on intermediate spaces between $\mathcal{Y}_0$ and $\mathcal{Y}_1$, while lower bounds on $I(\tau, \ell)$ are linked with the non-degeneracy of ℓ on these spaces. In particular, one has the following result, which is a reformulation of [43, Theorem 2] in the particular case of two constraints having the same "order".

Lemma 6.2. *Let $\mathcal{Y}_0$ and $\mathcal{Y}_1$ denote two Hilbert spaces with a dense continuous embedding $\mathcal{Y}_1 \hookrightarrow \mathcal{Y}_0$. Let ℓ^0, ℓ^1 be two linear forms on $\mathcal{Y}_1$. Assume that there exists $C_\pm > 0$ and $\bar{\sigma} \in (0,1)$ such that, for every $(c_0, c_1) \in \mathbb{S}^1$ and every $\tau \in (0,1)$,*

$$C_- \tau^{-\bar{\sigma}} \le I(\tau, c_0 \ell^0 + c_1 \ell^1) \le C_+ \tau^{-\bar{\sigma}}. \tag{6.6}$$

As in Section 6.1.1, let $\mathcal{Y}_1^\ell := \{f \in \mathcal{Y}_1; \ell^0(f) = \ell^1(f) = 0\}$ and, for $\sigma \in (0,1)$, $\mathcal{Y}_\sigma := [\mathcal{Y}_0, \mathcal{Y}_1]_\sigma$, for the complex interpolation method. Then,

- *for every $\sigma \in (0, \bar{\sigma})$, $[\mathcal{Y}_0, \mathcal{Y}_1^\ell]_\sigma = \mathcal{Y}_\sigma$, with equivalent norms,*
- *for every $\sigma \in (\bar{\sigma}, 1)$, the linear forms ℓ^0 and ℓ^1 have continuous extensions to $\mathcal{Y}_\sigma$ and $[\mathcal{Y}_0, \mathcal{Y}_1^\ell]_\sigma = \{f \in \mathcal{Y}_\sigma; \ell^0(f) = \ell^1(f) = 0\}$, endowed with the norm of $\mathcal{Y}_\sigma$.*

Remark 6.3. Lemma 6.2 does not say anything on $[\mathcal{Y}_0, \mathcal{Y}_1^\ell]_\sigma$ for the critical value $\sigma = \bar{\sigma}$. In fact, with the notations of [33] mentioned above, one has $\sigma_0 = \sigma_1 = \bar{\sigma}$, so the norm of $[\mathcal{Y}_0, \mathcal{Y}_1^\ell]_{\bar{\sigma}}$ is not equivalent to the norm of $\mathcal{Y}_{\bar{\sigma}}$.

Remark 6.4. In assumption (6.6), it is important to consider arbitrary linear combinations of the two linear forms ℓ^0 and ℓ^1. It would not be sufficient to assume (6.6) with $(c_0, c_1) = (1, 0)$ and $(c_0, c_1) = (0, 1)$. Indeed, the lower bound of this condition ensures that the two linear forms remain sufficiently independent on the intermediate spaces. We state here a formulation giving a symmetrical role to ℓ^0 and ℓ^1, whereas [43] uses a hierarchical formulation. We prove below that our formulation indeed implies Löfström's one.

Proof of Lemma 6.2. This is an application of [43, Theorem 2]. By (6.6) applied with $(c_0, c_1) = (1, 0)$ and $(c_0, c_1) = (0, 1)$, both ℓ^0 and ℓ^1 have "order" $\bar{\sigma}$ in Löfström's vocabulary. Therefore, there only remains to check that they form a "strongly independent basis", i.e., that there exists $C > 0$ such that, for every $\tau \in (0,1)$,

$$I(\tau, \ell^1) \le C I_0(\tau, \ell^1), \tag{6.7}$$

where

$$I_0(\tau, \ell^1) := \sup \left\{ \frac{\ell^1(f)}{\|f\|_\tau}; f \in \mathcal{Y}_1 \setminus \{0\} \text{ and } \ell^0(f) = 0 \right\}. \tag{6.8}$$

Let $\tau \in (0,1)$. Denote by $\langle \cdot, \cdot \rangle_\tau$ the scalar product associated with the norm $\|\cdot\|_\tau$ on $\mathcal{Y}_1$. By the Riesz representation theorem, there exists $g_\tau^0, g_\tau^1 \in \mathcal{Y}_1$ such that $\ell^j = \langle g_\tau^j, \cdot \rangle_\tau$. In particular, $I(\tau, \ell^j) = \|g_\tau^j\|_\tau$. Moreover, by (6.8), $I_0(\tau, \ell^1)$ is the supremum of ℓ^1 on the intersection of $\ker \ell^0$ with the unit ball in $\mathcal{Y}_1$ for the norm $\|\cdot\|_\tau$. Thus, a natural candidate to bound $I_0(\tau, \ell^1)$ from below is the orthogonal projection of $g_\tau^1/\|g_\tau^1\|_\tau$ on $\ker \ell^0$, namely,

$$f_\tau^1 := \frac{g_\tau^1}{\|g_\tau^1\|_\tau} - R_\tau \frac{g_\tau^0}{\|g_\tau^0\|_\tau}, \quad \text{where } R_\tau := \left\langle \frac{g_\tau^1}{\|g_\tau^1\|_\tau}, \frac{g_\tau^0}{\|g_\tau^0\|_\tau} \right\rangle_\tau .$$

In particular, $\|f_\tau^1\|_\tau = (1 - R_\tau^2)^{\frac{1}{2}}$ and $\ell^0(f_\tau^1) = \langle g_\tau^0, f_\tau^1 \rangle = 0$. Thus

$$I_0(\tau, \ell^1) \geq \frac{\langle f_\tau^1, g_\tau^1 \rangle_\tau}{\|f_\tau^1\|_\tau} = (1 - R_\tau^2)^{\frac{1}{2}} \|g_\tau^1\|_\tau = (1 - R_\tau^2)^{\frac{1}{2}} I(\tau, \ell^1). \tag{6.9}$$

Thus, to prove (6.7), it is sufficient to prove that the ratio R_τ^2 is bounded away from 1. By (6.6), for every $(c_0, c_1) \in \mathbb{S}^1$,

$$C_- \tau^{-\bar{\sigma}} \leq \|c_0 g_\tau^0 + c_1 g_\tau^1\|_\tau \leq C_+ \tau^{-\bar{\sigma}}. \tag{6.10}$$

In particular,

$$C_- \tau^{-\bar{\sigma}} \leq \|g_\tau^j\|_\tau \leq C_+ \tau^{-\bar{\sigma}}. \tag{6.11}$$

By homogeneity, from (6.10), for every $(c_0, c_1) \in \mathbb{R}^2$,

$$C_-^2 \tau^{-2\bar{\sigma}} (c_0^2 + c_1^2) \leq c_0^2 \|g_\tau^0\|_\tau^2 + c_1^2 \|g_\tau^1\|_\tau^2 + 2 c_0 c_1 \langle g_\tau^0, g_\tau^1 \rangle_\tau \leq C_+^2 \tau^{-2\bar{\sigma}} (c_0^2 + c_1^2).$$

Substituting $c_j \leftarrow c_j / \|g_\tau^j\|_\tau$ and using (6.11) leads to the fact that, for every $(c_0, c_1) \in \mathbb{R}^2$,

$$\rho^2 (c_0^2 + c_1^2) \leq c_0^2 + c_1^2 + 2 R_\tau c_0 c_1 \leq \rho^{-2} (c_0^2 + c_1^2),$$

where $\rho := C_-/C_+$. In particular, using $(c_0, c_1) = (1, 1)$ and $(1, -1)$ yields $\rho^2 \leq 1 + R_\tau$ and $\rho^2 \leq 1 - R_\tau$. Hence, (6.9) proves that

$$I_0(\tau, \ell^1) \geq \rho^2 I(\tau, \ell^1),$$

which implies (6.7) with $C = \rho^{-2}$. So ℓ^0 and ℓ^1 form a "strongly independent basis" and Lemma 6.2 follows from [43, Theorem 2]. ∎

6.2 Interpolated theory in the case of the linear shear flow

In this section, we consider the problem (2.1) at the linear shear flow, with vanishing boundary data. We proved in Section 2.2 (see Proposition 2.5) that, when $f \in L_x^2 L_z^2$, the solutions to this problem have Z^0 regularity, and in Section 2.3 (see Proposition 2.10) that they have Z^1 regularity when $f \in H_x^1 L_z^2$ and the two orthogonality conditions (2.15) are satisfied. Here, we establish an interpolated theory for the problem (2.1) with source terms $f \in H_x^\sigma L_z^2$, $\sigma \in (0, 1)$, see Proposition 2.36. This interpolated theory involves the difficulty exposed in Section 6.1. We define $\mathcal{Y}_0$, $\mathcal{Y}_1$ and $\mathcal{Y}_1^{\bar{\ell}}$ by (6.1), (6.2) and (6.3) respectively, endowed with their usual norms. For $\sigma \in (0, 1)$, let $\mathcal{Y}_\sigma := [\mathcal{Y}_0, \mathcal{Y}_1]_\sigma$. The identification of the space $\mathcal{Y}_\sigma$ is classical and provided by the following lemma.

Lemma 6.5. *Let $\sigma \in (0,1)$. Let $\mathcal{Y}_\sigma := [\mathcal{Y}_0, \mathcal{Y}_1]_\sigma$, for the complex interpolation method.*

- *When $\sigma \in (0, 1/2)$, $\mathcal{Y}_\sigma = H^\sigma_x L^2_y$.*
- *When $\sigma = 1/2$, recalling that $\Omega_\pm = \Omega \cap \{\pm y > 0\}$*

$$\|f\|^2_{\mathcal{Y}_{1/2}} \approx \|f\|^2_{H^{1/2}_x L^2_y} + \int_{\Omega_+} \frac{f^2(x,y)}{|x - x_0|}\,\mathrm{d}x\,\mathrm{d}y + \int_{\Omega_-} \frac{f^2(x,y)}{|x - x_1|}\,\mathrm{d}x\,\mathrm{d}y.$$

- *When $\sigma \in (1/2, 1)$, $\mathcal{Y}_\sigma = \{f \in H^\sigma_x L^2_y;\ f_{|\Sigma_0 \cup \Sigma_1 = 0}\}$, with the usual norm.*

Proof. This follows from classical interpolation theory for intersections (see [45, Theorem 13.1 and equation (13.4), Chapter 1]), and from (one-sided versions of) the equality $H^{1/2}_{00}(x_0, x_1) = [H^1_0(x_0, x_1), L^2(x_0, x_1)]_{\frac{1}{2}}$ (see also [45, Theorem 11.7, Chapter 1]). ■

In order to extend the theory of Chapter 2 to fractional tangential regularity, we start by identifying the spaces $[\mathcal{Y}_0, \mathcal{Y}^{\bar\ell}_1]_\sigma$. More precisely, we prove the following characterization.

Lemma 6.6. *Let $\mathcal{Y}_0$, $\mathcal{Y}_1$ and $\mathcal{Y}^{\bar\ell}_1$ be given by (6.1), (6.2) and (6.3) respectively. Then,*

- *For every $\sigma \in (0, 1/6)$, $[\mathcal{Y}_0, \mathcal{Y}^{\bar\ell}_1]_\sigma = \mathcal{Y}_\sigma$ with equivalent norms.*
- *For every $\sigma \in (1/6, 1)$, the linear forms $\overline{\ell^0}$ and $\overline{\ell^1}$ admit continuous extensions to $\mathcal{Y}_\sigma$ and*

$$[\mathcal{Y}_0, \mathcal{Y}^{\bar\ell}_1]_\sigma = \left\{ f \in \mathcal{Y}_\sigma; \overline{\ell^0}(f, 0, 0) = \overline{\ell^1}(f, 0, 0) = 0 \right\},$$

endowed with the norm of $\mathcal{Y}_\sigma$.

Remark 6.7. The threshold at $1/6$ is consistent with the observation of Remark 2.33 that the maps $\overline{\ell^j}(\cdot, 0, 0)$ are bounded on $H^\sigma_x L^2_z$ for every $\sigma > 1/6$.

For $\tau \in (0,1)$, we use the notations of the previous paragraph, in particular the norm $\|\cdot\|_\tau$ of (6.4) and the function $I(\tau, \cdot)$ of (6.5), with $\mathcal{Y}_0$ and $\mathcal{Y}_1$ defined as above.

To derive the estimates required to apply Lemma 6.2, two strategies would be possible. Both rely on the explicit knowledge of the singular radial solutions constructed in Section 2.4, which are involved in the orthogonality conditions. First, one could impose periodic boundary conditions on f, compute a 2D Fourier-series representation of (an extension by parity of) the singular profiles and estimate the functions I working in the Fourier space. Such a frequency-domain approach is carried out in [5], assuming some appropriate asymptotic decay of the Fourier transform of the profile defining the orthogonality condition. We choose a second strategy, which stays in the spatial domain, and involves estimates using cut-off functions whose space-scale are linked with the parameter τ. This strategy is related to the one used in [63] and inspired by the links between the K functional of real interpolation theory and the notions of modulus of continuity and modulus of smoothness of functions (see, e.g., [37]).

To prove Lemma 6.6, we intend to apply Lemma 6.2. Hence, we need to bound from below and from above the functions $I(\tau, \overline{\ell^j})$. By Definition 2.12, $\overline{\ell^j}(f, 0, 0) = \int_\Omega \partial_x f \overline{\Phi^j}$. As highlighted in Corollary 2.32, the profiles $\overline{\Phi^j}$ can be decomposed as the sum of a singular radial part, an x-independent part, and a regular part. The singular radial part is the one that will be dominating the behavior of the orthogonality conditions. Thus, we start by two lemmas concerning estimates from above and from below for integrals of the form $\int_\Omega (\partial_x f)\bar{u}^i_{\mathrm{sing}}$, before moving to the general case.

Lemma 6.8. *Let $h \in H^1_x L^2_z$ such that $h = 0$ on $\Sigma_0 \cup \Sigma_1$. Then, for $\tau \in (0, 1)$,*

$$\left| \int_\Omega (\partial_x h(x, z))\bar{u}^i_{\mathrm{sing}}(x, -z)\, \mathrm{d}x\, \mathrm{d}z \right| \lesssim \tau^{-1/6}(\|h\|_{L^2} + \tau \|\partial_x h\|_{L^2}),$$

where $\bar{u}^i_{\mathrm{sing}}$ is defined in Definition 2.26.

Proof. By symmetry, it is sufficient to prove the result with $i = 0$, which we assume from now on, and we drop the indexes $i = 0$ on r_i and t_i involved in Definition 2.26. We also let $\chi(x, z) := \chi_i(x, -z)$ of Definition 2.26 and $\Lambda(t) := \Lambda_0(-t)$, where Λ_0 is defined in Proposition 2.21. With these notations

$$\bar{u}^0_{\mathrm{sing}}(x, -z) = r^{\frac{1}{2}}\Lambda(t)\chi(x, z). \tag{6.12}$$

In particular, since $\Lambda_0(+\infty) = 0$, $\bar{u}^0_{\mathrm{sing}}(x_0, -z) = 0$ for $z \in (-1, 0)$. We split the integral to be estimated depending on whether $r \le \tau^\alpha$ or $r \ge \tau^\alpha$, where $\alpha > 0$ is to be chosen later. Let $\eta \in C^\infty(\mathbb{R}; [0, 1])$ such that $\eta(s) \equiv 1$ for $s \le 1$ and $\eta(s) \equiv 0$ for $s \ge 2$.

Step 1. Estimate in the region: $r \le \tau^\alpha$. By Cauchy–Schwarz,

$$\left| \int_\Omega \partial_x h \cdot r^{\frac{1}{2}}\Lambda(t)\chi \cdot \eta(r/\tau^\alpha) \right| \le \|\chi\|_\infty \|\Lambda\|_\infty \|\partial_x h\|_{L^2} \left(\int_\Omega r\eta^2(r/\tau^\alpha) \right)^{\frac{1}{2}}.$$

Using the polar-like change of coordinates of (2.23) and (2.26), one has

$$\int_\Omega r\eta^2(r/\tau^\alpha) = \int_0^\infty \int_{\mathbb{R}} \frac{3r^3}{(1+t^2)^2} r\eta^2(r/\tau^\alpha)\, \mathrm{d}r\, \mathrm{d}t \lesssim (\tau^\alpha)^5.$$

Hence, in this region,

$$\left| \int_\Omega \partial_x h \cdot r^{\frac{1}{2}}\Lambda(t)\chi \cdot \eta(r/\tau^\alpha) \right| \lesssim (\tau^\alpha)^{5/2}\|\partial_x h\|_{L^2}.$$

Step 2. Estimate in the region: $r \ge \tau^\alpha$. We intend to integrate by parts in x. At $x = x_1$, $\bar{u}^0_{\mathrm{sing}}(x, -z) = 0$ for $z \in (-1, 1)$ because $\chi = 0$. At $x = x_0$, when $z > 0$, $h = 0$ by assumption, and, when $z < 0$, $\bar{u}^0_{\text{sing}}(x, -z) = 0$ as recalled above. Hence, there is no boundary term and

$$\int_\Omega \partial_x h \cdot r^{\frac{1}{2}} \Lambda(t) \chi \cdot (1 - \eta(r/\tau^\alpha)) = -\int_\Omega h \partial_x \big(\chi \cdot r^{\frac{1}{2}} \Lambda(t)(1 - \eta(r/\tau^\alpha))\big).$$

First, one easily bounds

$$\left| \int_\Omega h \partial_x \chi \cdot r^{\frac{1}{2}} \Lambda(t)(1 - \eta(r/\tau^\alpha)) \right| \le \|h\|_{L^2} \|\Lambda\|_\infty \|\partial_x \chi\|_{L^2} \max_\Omega r^{\frac{1}{2}} \lesssim \|h\|_{L^2}.$$

For the second term, when ∂_x hits on the function expressed in (r, t) coordinates, we use the derivative formula (2.27):

$$\begin{aligned} \int_\Omega h \chi \partial_x \big(r^{\frac{1}{2}} \Lambda(t)(1 - \eta(r/\tau^\alpha))\big) &= \int_\Omega h \chi \frac{(1+t^2)^{\frac{1}{2}}}{3r^2} \Lambda(t) \partial_r \big(r^{\frac{1}{2}}(1 - \eta(r/\tau^\alpha))\big) \\ &\quad - \int_\Omega h \chi \frac{t(1+t^2)^{\frac{3}{2}}}{3r^3} r^{\frac{1}{2}} (1 - \eta(r/\tau^\alpha)) \partial_t \Lambda(t). \end{aligned}$$

We bound both terms using the Cauchy–Schwarz inequality and the polar-like change of coordinates (2.23) with Jacobian determinant (2.26). In particular, on the one hand,

$$\begin{aligned} &\int_\Omega \frac{1+t^2}{9r^4} \Lambda^2(t) \big(\partial_r (r^{\frac{1}{2}}(1 - \eta(r/\tau^\alpha)))\big)^2 \\ &\quad = \int_0^\infty \int_{\mathbb{R}} \frac{3r^3}{(1+t^2)^2} \frac{1+t^2}{9r^4} \Lambda^2 \big(\partial_r (r^{\frac{1}{2}}(1 - \eta(r/\tau^\alpha)))\big)^2 \, \mathrm{d}t \, \mathrm{d}r \\ &\quad \lesssim \int_0^\infty \big(r^{-1}(1 - \eta(r/\tau^\alpha))^2 + r(\eta'(r/\tau^\alpha))^2/(\tau^\alpha)^2\big) \frac{\mathrm{d}r}{r} \\ &\quad = (\tau^\alpha)^{-1} \int_1^\infty \big(s^{-1}(1 - \eta(s))^2 + s(\eta'(s))^2\big) \frac{\mathrm{d}s}{s} \lesssim (\tau^\alpha)^{-1}. \end{aligned}$$

On the other hand,

$$\begin{aligned} &\int_\Omega \frac{t^2(1+t^2)^3}{9r^6} r(1 - \eta(r/\tau^\alpha))^2 (\partial_t \Lambda(t))^2 \\ &\quad = \int_0^\infty \int_{\mathbb{R}} \frac{3r^3}{(1+t^2)^2} \frac{t^2(1+t^2)^3}{9r^6} r(1 - \eta(r/\tau^\alpha))^2 (\partial_t \Lambda(t))^2 \, \mathrm{d}t \, \mathrm{d}r \\ &\quad \lesssim \left(\int_{\mathbb{R}} t^2(1+t^2)(\partial_t \Lambda(t))^2 \, \mathrm{d}t \right) \int_0^\infty \frac{1}{3r^2} (1 - \eta(r/\tau^\alpha)) \, \mathrm{d}r \lesssim (\tau^\alpha)^{-1} \end{aligned}$$

by the integrability property $t^3 \partial_t \Lambda(t) = O(1)$ of Lemma 2.24.

Thus, gathering the estimates in this region proves that

$$\left| \int_\Omega \partial_x h \cdot r^{\frac{1}{2}} \Lambda(t) \chi \cdot (1 - \eta(r/\tau^\alpha)) \right| \lesssim (\tau^\alpha)^{-\frac{1}{2}} \|h\|_{L^2}.$$

Gathering the estimates in both regions and choosing $\alpha = 1/3$ concludes the proof. ∎

Lemma 6.9. *There exists a family* $(h_\tau^i)_{\tau\in(0,1)}$ *of non-zero, smooth, compactly supported functions on* Ω *such that, as* $\tau \to 0$,

$$\left| \int_\Omega (\partial_x h_\tau^i(x,z))\bar{u}_{\mathrm{sing}}^i(x,-z)\,\mathrm{d}x\,\mathrm{d}z \right| \gtrsim \tau^{-1/6}(\|h_\tau^i\|_{L^2} + \tau\|\partial_x h_\tau^i\|_{L^2}),$$

and $\int_\Omega \partial_x h_\tau^j \bar{u}_{\mathrm{sing}}^i = 0$ *for* $j \neq i$, *where* $\bar{u}_{\mathrm{sing}}^i$ *is defined in Definition* 2.26.

Proof. As in the previous lemma, by symmetry, it is sufficient to prove the result with $i = 0$, which we assume from now on, and we drop the indexes $i = 0$ on r_i and t_i involved in Definition 2.26. We also let $\chi(x,z) := \chi_i(x,-z)$ of Definition 2.26 and $\Lambda(t) := \Lambda_0(-t)$, where Λ_0 is defined in Proposition 2.21. With these notations, one has (6.12).

Let $\alpha > 0$. Let $H \in C_c^\infty(\mathbb{R};[-1,1])$ and $\eta \in C_c^\infty(\mathbb{R};[-1,1])$ such that $\operatorname{supp}\eta \subset (1/2, 3/2)$. For $\tau \in (0,1)$, we define

$$h_\tau := \eta(r/\tau^\alpha)H(t).$$

By the support properties of H and η, one checks that h_τ is both smooth and compactly supported in Ω. Moreover, it is non-zero if $H \neq 0$ and $\eta \neq 0$.

Let $\tau > 0$ be sufficiently small such that the support of h_τ is included in the region where $\chi \equiv 1$. Note that with this choice, we also have $\int_\Omega \partial_x h_\tau \bar{u}_{\mathrm{sing}}^1 = 0$. Then, using the formula (2.27) for ∂_x and the determinant (2.26),

$$\begin{aligned}
&\int_\Omega \partial_x h_\tau \bar{u}_{\mathrm{sing}}(x,-z)\,\mathrm{d}x\,\mathrm{d}z \\
&= \int_0^\infty \int_{\mathbb{R}} r^{\frac{1}{2}}\Lambda(t)\Big(\frac{(1+t^2)^{\frac{1}{2}}}{3r^2}(\tau^\alpha)^{-1}\eta'(r/\tau^\alpha)H(t) \\
&\qquad\qquad - \frac{t(1+t^2)^{\frac{3}{2}}}{3r^3}\eta(r/\tau^\alpha)H'(t)\Big)\frac{3r^3}{(1+t^2)^2}\,\mathrm{d}t\,\mathrm{d}r \\
&= \int_0^\infty \int_{\mathbb{R}} \frac{r^{\frac{1}{2}}\Lambda(t)}{(1+t^2)^{\frac{3}{2}}}\big(r(\tau^\alpha)^{-1}\eta'(r/\tau^\alpha)H(t) - t(1+t^2)\eta(r/\tau^\alpha)H'(t)\big)\,\mathrm{d}t\,\mathrm{d}r \\
&= (\tau^\alpha)^{3/2}\Big(\int_{\mathbb{R}} \frac{H(t)\Lambda(t)}{(1+t^2)^{\frac{3}{2}}}\,\mathrm{d}t \int_0^\infty s^{\frac{3}{2}}\eta'(s)\,\mathrm{d}s - \int_{\mathbb{R}} \frac{tH'(t)\Lambda(t)}{(1+t^2)^{\frac{1}{2}}}\,\mathrm{d}t \int_0^\infty s^{\frac{1}{2}}\eta(s)\,\mathrm{d}s\Big).
\end{aligned}$$

Note that since $\eta \in C_c^\infty((0,+\infty))$,

$$\int_0^\infty s^{\frac{3}{2}}\eta'(s)\,\mathrm{d}s = -\frac{3}{2}\int_0^\infty s^{\frac{1}{2}}\eta(s)\,\mathrm{d}s.$$

We claim that we may choose η and H such that

$$\int_0^\infty s^{\frac{1}{2}}\eta(s)\,\mathrm{d}s = \frac{3}{2}\int_{\mathbb{R}} \frac{H(t)\Lambda(t)}{(1+t^2)^{\frac{3}{2}}}\,\mathrm{d}t + \int_{\mathbb{R}} \frac{tH'(t)\Lambda(t)}{(1+t^2)^{\frac{1}{2}}}\,\mathrm{d}t = 1.$$

The claim for η is obvious. As for H, we assume that supp $H \subset (0, +\infty)$ and we write the sum of integrals as

$$\int_{\mathbb{R}} H(t)\left[\frac{3}{2}\frac{\Lambda(t)}{(1+t^2)^{\frac{3}{2}}} - \frac{d}{dt}\left(\frac{t\Lambda(t)}{(1+t^2)^{\frac{1}{2}}}\right)\right] \mathrm{d}t$$
$$= -\int_0^\infty H(t)\frac{d}{dt}\left(\frac{t\Lambda(t)}{(1+t^2)^{1/2}}t^{-3/2}(1+t^2)^{3/4}\right)t^{3/2}(1+t^2)^{-3/4}\,\mathrm{d}t.$$

Since $\Lambda(t) \neq Ct^{1/2}(1+t^2)^{-1/4}$ on $\mathbb{R}_+$, the claim for H follows.

The above choice of η and H ensures that

$$\int_\Omega \partial_x h_\tau \cdot r^{\frac{1}{2}}\Lambda(t)\chi = -(\tau^\alpha)^{3/2}.$$

Using once again the formula (2.27) for ∂_x and the change of coordinates of the Jacobian (2.26), one obtains that $\|h_\tau\|_{L^2} \lesssim (\tau^\alpha)^2$ and $\|\partial_x h_\tau\|_{L^2} \lesssim 1/\tau^\alpha$. Similarly, using (2.28) to compute $\partial_z^3 h_\tau$ and the same technique, $\|\partial_z^3 h_\tau\|_{L^2} \lesssim 1/\tau^\alpha$. Thus, choosing $\alpha = 1/3$ leads to

$$\|h_\tau\|_{L^2} + \tau\|\partial_x h_\tau\|_{L^2} + \tau\|\partial_z^3 h_\tau\|_{L^2} \lesssim (\tau^\alpha)^2,$$

which concludes the proof. ∎

We are now ready to prove Lemma 6.6.

Proof of Lemma 6.6. This is an application of Lemma 6.2 with $\bar\sigma = 1/6$. Therefore, we need to find constants $C_\pm > 0$ such that, for every $\tau \in (0,1)$ and $(c_0, c_1) \in \mathbb{S}^1$,

$$C_-\tau^{-1/6} \le I(\tau, c_0\overline{\ell^0} + c_1\overline{\ell^1}) \le C_+\tau^{-1/6}.$$

Let $(c_0, c_1) \in \mathbb{S}^1$ and $f \in \mathcal{Y}_1$. By Definition 2.12,

$$\overline{\ell^j}(f, 0, 0) = \int_\Omega \partial_x f\,\overline{\Phi^j},$$

where $\overline{\Phi^j}$ is the solution to (2.12).

By Corollary 2.32, there exists $(d_0, d_1) \in \mathbb{R}^2 \setminus \{0\}$ such that

$$c_0\overline{\ell^0}(f,0,0) + c_1\overline{\ell^1}(f,0,0) = \int_\Omega \partial_x f(d_0\bar u^0_{\text{sing}}(x,-z) + d_1\bar u^1_{\text{sing}}(x,-z))$$
$$+ \int_\Omega \partial_x f\left(\Phi_{\text{reg}} + (c_1 - zc_0)\chi(z)\mathbf{1}_{z>0}\right), \quad (6.13)$$

where $\Phi_{\text{reg}} \in Z^1$. By linearity, d_0, d_1 and Φ_{reg} are uniformly bounded for $(c_0, c_1) \in \mathbb{S}^1$. The first term corresponds to the one studied in Lemmas 6.8 and 6.9. We want to integrate by parts in the second term. Since $f \in \mathcal{Y}_1$, $f_{|\Sigma_0\cup\Sigma_1} = 0$. At $x = x_0$

and $z \in (-1,0)$, $\overline{\Phi^j} = 0$ by (2.12). Moreover, $\bar{u}^0_{\text{sing}}(x_0,-z) = 0$ because $\Lambda_0(+\infty) = 0$ and $\bar{u}^1_{\text{sing}}(x_0,-z) = 0$ because $\bar{u}^1_{\text{sing}}$ is compactly supported near $(x_1,0)$. Hence, $\Phi_{\text{reg}}(x_0,z) + (c_1 - zc_0)\chi(z)\mathbf{1}_{z>0} \equiv 0$ on $(-1,0)$. The same conclusion holds at $x = x_1$ and $z \in (0,1)$. Thus, we can integrate by parts with no boundary term and the second term is estimated as

$$\left| \int_\Omega \partial_x f(\Phi_{\text{reg}} + (c_1 - zc_0)\chi(z)\mathbf{1}_{z>0}) \right| \leq \|f\|_{L^2} \|\partial_x \Phi_{\text{reg}}\|_{L^2}. \tag{6.14}$$

Step 1. Bound from above. For $\tau \in (0,1)$, using Lemma 6.8 and (6.14),

$$|c_0\overline{\ell^0}(f,0,0) + c_1\overline{\ell^1}(f,0,0)| \lesssim \tau^{-1/6}(\|f\|_{L^2} + \tau\|f\|_{\mathcal{Y}_1}).$$

Step 2. Bound from below. For $\tau \in (0,1)$, let us assume that $f_\tau := h_\tau$, where h_τ is constructed in Lemma 6.9, which ensures that f_τ is compactly supported in Ω so satisfies $(f_\tau)_{|\Sigma_0 \cup \Sigma_1} = 0$. Substituting in (6.13) and integrating by parts yields

$$c_0\overline{\ell^0}(f_\tau,0,0) + c_1\overline{\ell^1}(f_\tau,0,0) = -\int_\Omega h_\tau \partial_x \Phi_{\text{reg}} + \sum_{i\in\{0,1\}} d_i \int_\Omega (\partial_x h_\tau)\bar{u}^i_{\text{sing}}(x,-z).$$

By Corollary 2.32 and linearity, $\min(|d_0|,|d_1|)$ is uniformly bounded from below. We choose h_τ as either h^0_τ or h^1_τ of Lemma 6.9 accordingly. Thus, by Lemma 6.9, as $\tau \to 0$,

$$\begin{aligned}
|c_0\overline{\ell^0}(f_\tau,0,0)+c_1\overline{\ell^1}(f_\tau,0,0)| &\gtrsim \tau^{-1/6}(\|h_\tau\|_{L^2}+\tau\|h_\tau\|_{\mathcal{Y}_1}) - C\|h_\tau\|_{L^2}\|\partial_x\Phi_{\text{reg}}\|_{L^2} \\
&\gtrsim \tau^{-1/6}(\|h_\tau\|_{L^2} + \tau\|h_\tau\|_{\mathcal{Y}_1}) \\
&= \tau^{-1/6}(\|f_\tau\|_{L^2} + \tau\|f_\tau\|_{\mathcal{Y}_1})
\end{aligned}$$

for $\tau > 0$ sufficiently small. This concludes the proof. ∎

To conclude this section, we turn towards the proof of Proposition 2.36.

Proof of Proposition 2.36. *Step 1. Case* $\delta_0 = \delta_1 = 0$ *and* $\mathbf{1}_{\sigma>1/2} f_{|\Sigma_i} = 0$. By Proposition 2.5, for every $f \in L^2(\Omega)$, there exists a unique solution $u \in Z^0(\Omega)$ to (2.1) with $\delta_0 = \delta_1 = 0$ and $\|u\|_{Z^0} \lesssim \|f\|_{L^2}$. By Proposition 2.10 and Proposition 2.6, for every $f \in H^1_x L^2_z$ such that $f_{|\Sigma_0\cup\Sigma_1} = 0$ (so that $\Delta_0 = \Delta_1 = 0$) and $\overline{\ell^0}(f,0,0) = \overline{\ell^1}(f,0,0) = 0$, this solution satisfies $u \in Z^1(\Omega)$ with $\|u\|_{Z_1} \lesssim \|f\|_{H^1_x L^2_z}$. Hence, by interpolation, the mapping $f \mapsto u$ is bounded from $[\mathcal{Y}_0, \mathcal{Y}^{\bar{\ell}}_1]_\sigma$ to $Z^\sigma(\Omega)$. Moreover, by Lemma 6.5 and Lemma 6.6, when $\sigma \in (0,1) \setminus \{1/6, 1/2\}$, $[\mathcal{Y}_0, \mathcal{Y}^{\bar{\ell}}_1]_\sigma = H^\sigma_x L^2_z$ (with null boundary conditions on $\Sigma_0 \cup \Sigma_1$ when $\sigma > 1/2$, and null linear forms constraints when $\sigma > 1/6$). This proves estimate (2.50) in the case of vanishing boundary data.

Step 2. Arbitrary boundary data. When δ_0 and δ_1 are arbitrary, we extend them to $(-1,1)$ in such a way that the extension belongs to $H_0^2(-1,1)$. We then lift the boundary data by setting $u_l(x,z) = \chi(x-x_0)\delta_0 + \chi(x-x_1)\delta_1$, with $\chi \in C_c^\infty(\mathbb{R})$, supported in $B(0,(x_1-x_0)/2)$, and equal to 1 in a neighborhood of zero. This introduces a source term $f_l = z\partial_x u_l - \partial_{zz} u_l \in H_x^1 L_z^2$ in the equation, whose trace on Σ_i is $-\delta_i''$, so that the trace of $f - f_l$ on Σ_i is $z\Delta_i$. When $\sigma < 1/2$, we immediately obtain the desired result thanks to the previous step.

For $\sigma > 1/2$, we first note that, since $u, u_l \in Z^1(\Omega)$, by Proposition 2.10,

$$\overline{\ell^j}(f - f_l, 0, 0) = 0.$$

We further decompose $f - f_l$ into $f - f_l = z\Delta_0\chi(x-x_0) + z\Delta_1\chi(x-x_1) + g_l$, where $g_l \in H_x^\sigma L_z^2$ is such that $g_{l|\Sigma_0\cup\Sigma_1} = 0$. Using Proposition 2.15 we construct $h_l \in C_c^\infty(\Omega)$ such that

$$\|h_l\|_{H_x^1 L_z^2} \lesssim \|\Delta_0\|_{\mathcal{H}^1(\Sigma_0)} + \|\Delta_1\|_{\mathcal{H}^1(\Sigma_1)}$$

and

$$\overline{\ell^j}(z\Delta_0\chi(x-x_0) + z\Delta_1\chi(x-x_1) + h_l, 0, 0) = \overline{\ell^j}(g_l - h_l, 0, 0) = 0.$$

We then apply the result of the first step to the system with source term $g_l - h_l$ (which vanishes on $\Sigma_0 \cup \Sigma_1$) and homogeneous boundary data, and the result of Proposition 2.10 to the system with source term $z\Delta_0\chi(x-x_0) + z\Delta_1\chi(x-x_1) + h_l$ and homogeneous boundary data, using the conditions $\Delta_0(1) = \Delta_1(-1) = 0$. This concludes the proof. ∎