Appendix A

Uniqueness of weak solutions for linear problems

The purpose of this appendix is to prove the following uniqueness result, which is a slight generalization to the case of variable coefficients of the uniqueness result of [8, Section 5] for (2.1).

Lemma A.1. *Let* $\Omega = (x_0, x_1) \times (z_b, z_t)$*, where* $x_0 < x_1$ *and* $z_b < 0 < z_t$*. Let* $\alpha \in C^2(\bar{\Omega})$ *such that* $\inf \alpha > 0$ *and* $\beta \in L^\infty(\Omega)$*. Assume that one of the two following conditions is satisfied:*

- *either* $\|\beta\|_\infty \ll 1$ *and* $\|\partial_z \alpha\|_\infty \ll 1$*,*
- *or* $|z_b|, z_t \leq z_0$*, for some small constant* z_0 *depending only on* α*.*

Let $g \in L^2_x H^{-1}_z$*,* $\delta_0 \in \mathcal{L}^2_z(\Sigma_0)$*,* $\delta_1 \in \mathcal{L}^2_z(\Sigma_1)$*. There exists at most one weak solution* $U \in L^2_x H^1_0$ *to*

$$\begin{cases} z\partial_x U + \beta \partial_z U - \partial_{zz}(\alpha U) = g, \\ U|_{\Sigma_0} = \delta_0, \\ U|_{\Sigma_1} = \delta_1, \\ U|_{z=z_t} = U|_{z=z_b} = 0. \end{cases} \tag{A.1}$$

The proof follows the arguments of Baouendi and Grisvard in [8], which concern the case of the model equation (2.1). For the reader's convenience, we recall the main steps of the proof here, and adapt them to the present (slightly different) context. The proof involves the spaces $\mathcal{B}$ defined in (1.25) and $\mathcal{A} := \mathcal{B} \cap H^1(\Omega)$.

Note that if $U \in L^2((x_0, x_1), H^1_0(z_b, z_t))$ is a weak solution to (A.1), then $U \in \mathcal{B}$. Indeed, it follows from the weak formulation that for any $V \in H^1_0(\Omega)$,

$$\langle z\partial_x U, V\rangle_{L^2(H^{-1}),L^2(H^1_0)} = -\int_\Omega \partial_z(\alpha U)\partial_z V - \int_\Omega \beta \partial_z U V + \langle g, V\rangle_{L^2(H^{-1}),L^2(H^1_0)}.$$

By density, this formula still holds for $V \in L^2_x(H^1_0)$, and therefore $z\partial_x U \in L^2_x(H^{-1}_z)$.

We then recall the following result from [8].

Lemma A.2. *The set* $\mathcal{A}$ *is dense in* $\mathcal{B}$*. Furthermore, there exists a constant* C *depending only on* Ω*, such that for* $i \in \{0, 1\}$*,*

$$\forall v \in \mathcal{A}, \quad \int_{z_b}^{z_t} |z|\, |v(x_i, y)|^2 \,\mathrm{d}y \leq C \|v\|^2_{\mathcal{B}}.$$

As a consequence, the applications

$$v \in \mathcal{A} \mapsto v_{|x=x_i} \in \mathcal{L}^2_z(z_b, z_t)$$

can be uniquely extended into continuous applications on $\mathcal{B}$*.*

As a consequence, Baouendi and Grisvard [8] obtain the following corollary.

Corollary A.3. *For all $u, v \in \mathcal{B}$,*

$$\langle z\partial_x u, v\rangle_{L^2(H^{-1}),L^2(H^1_0)} + \langle z\partial_x v, u\rangle_{L^2(H^{-1}),L^2(H^1_0)} = \int_{z_b}^{z_t} (zuv)|_{x=x_1} - \int_{z_b}^{z_t} (zuv)|_{x=x_0}.$$

Proof. Thanks to Lemma A.2, it suffices to prove the identity when $u, v \in \mathcal{A}$. In that case, the left-hand side is simply

$$\int_\Omega z\partial_x uv + zu\partial_x v = \int_\Omega \partial_x(zuv).$$

The result follows by integration. ∎

Proof of Lemma A.1. Let $U \in L^2_x(H^1_0)$ be a weak solution to (A.1) with $g = 0$ and $\delta_i = 0$. As mentioned above, $U \in \mathcal{B}$. According to Corollary A.3, for any $V \in \mathcal{B}$ such that $V = 0$ on $\partial\Omega \setminus (\Sigma_0 \cup \Sigma_1)$,

$$-\langle z\partial_x V, U\rangle_{L^2(H^{-1}),L^2(H^1_0)} + \int_\Omega (\beta\partial_z UV + \alpha_z U\partial_z V + \alpha\partial_z U\partial_z V) = 0.$$

Now, let $h \in C^\infty_c(\Omega)$ be arbitrary, and let $V \in L^2(H^1_0)$ be a weak solution to

$$\begin{cases} -z\partial_x V - \partial_z(\beta V) - \alpha\partial_{zz} V = h, \\ V_{|\partial\Omega\setminus(\Sigma_0\cup\Sigma_1)} = 0. \end{cases}$$

(The existence of weak solutions for this adjoint problem is proved in the same way as existence for the direct problem in Proposition 2.2 in the case $\|\beta\|_\infty \ll 1$, $\|\alpha_z\|_\infty \ll 1$, and Lemma 5.3 in the case $|z_b|$, z_t small).

Then $V \in \mathcal{B}$, and choosing U as a test function in the variational formulation for V, we obtain

$$\int_\Omega hU = 0.$$

Thus $U = 0$. Uniqueness of weak solutions to (A.1) follows. ∎