Appendix B

Proofs of functional analysis results

B.1 An abstract existence principle

As Fichera in [22], we use the following abstract existence principle (see [19, Theorem 1]), which allows skipping a viscous regularization scheme.

Lemma B.1. *Let $\mathcal{H}_1$, $\mathcal{H}_2$ and $\mathcal{H}$ be three Hilbert spaces. Let $F_i \in \mathcal{L}(\mathcal{H}_i; \mathcal{H})$ for $i \in \{1, 2\}$. Then the following statements are equivalent.*

- *It holds that* range $F_1 \subset$ range F_2.
- *There exists a constant $C > 0$ such that*

$$\forall h \in \mathcal{H}', \quad \|F_1^* h\|_{\mathcal{H}_1'} \le C \|F_2^* h\|_{\mathcal{H}_2'}. \tag{B.1}$$

- *There exists $G \in \mathcal{L}(\mathcal{H}_1; \mathcal{H}_2)$ such that $F_1 = F_2 G$.*

Moreover, when these hold, there exists a unique $G \in \mathcal{L}(\mathcal{H}_1; \mathcal{H}_2)$ such that $\ker G = \ker F_1$, range $G \subset (\text{range } F_2^*)^\perp$ and $\|G\| = \inf\{C > 0; (\text{B.1}) \text{ holds}\}$.

Indeed, this yields the following weak Lax–Migram result, where the linear right-hand side is assumed to be continuous for the weaker norm.

Lemma B.2. *Let $\mathcal{U}$ and $\mathcal{V}$ be two Hilbert spaces with $\mathcal{V}$ continuously embedded in $\mathcal{U}$. Let a be a continuous bilinear form on $\mathcal{U} \times \mathcal{V}$ and b be a continuous linear form on $\mathcal{U}$. Assume that there exists a constant $c > 0$ such that, for every $v \in \mathcal{V}$,*

$$a(v, v) \ge c \|v\|_{\mathcal{U}}^2.$$

Then, there exists $u \in \mathcal{U}$ such that $\|u\|_{\mathcal{U}} \le \frac{1}{c} \|b\|_{\mathcal{L}(\mathcal{U})}$ and, for every $v \in \mathcal{V}$, $a(u, v) = b(v)$.

Proof. Set $\mathcal{H} := \mathcal{L}(\mathcal{V})$, $\mathcal{H}_1 := \mathcal{L}(\mathcal{U})$, $F_1 := \text{Id}$ (from $\mathcal{L}(\mathcal{U})$ to $\mathcal{L}(\mathcal{V})$), $\mathcal{H}_2 := \mathcal{U}$ and $F_2 : \mathcal{U} \to \mathcal{L}(\mathcal{V})$ defined by $F_2 u := a(u, \cdot)$. Then $F_1^* = \text{Id}$ (from $\mathcal{V}$ to $\mathcal{U}$) and $F_2^* v = a(\cdot, v)$. Moreover,

$$\|F_2^* v\|_{\mathcal{L}(\mathcal{U})} \ge |a(v, v)| / \|v\|_{\mathcal{U}} \ge c \|v\|_{\mathcal{U}} = c \|F_1^* v\|_{\mathcal{U}}.$$

So (B.1) holds with $C = 1/c$ and Lemma B.1 yields the existence of $G \in \mathcal{L}(\mathcal{L}(\mathcal{U}); \mathcal{U})$ such that $F_1 = F_2 G$ and $\|G\| \le \frac{1}{c}$. The conclusions follow by setting $u := Gb$. ∎

B.2 Product and composition rules in Sobolev spaces

Lemma B.3 (Pointwise multiplication). *Pointwise multiplication of two functions is a continuous bilinear map*

- *from $H^{3/2}(-1, 1) \times H^{3/2}(-1, 1)$ to $H^{3/2}(-1, 1)$,*
- *from $H^{1/2}(x_0, x_1) \times H^s(x_0, x_1)$ to $H^{1/2}(x_0, x_1)$ for any $s > 1/2$,*
- *from $H^{1/2}(x_0, x_1) \times H^s(x_0, x_1)$ to $H^{s'}(x_0, x_1)$ for any $s' < \min(s, 1/2)$,*
- *from $H^s(x_0, x_1) \times H^{s'}(x_0, x_1)$ to $H^{s'}(x_0, x_1)$ for any $s > 1/2$, $s \geq s'$.*

Proof. These are particular cases of [10, Theorem 7.4]. ∎

Lemma B.4 (Composition of H^σ functions). *Let $\sigma \in (0, 1/6)$, and let $\Omega_y = (x_0, x_1) \times \mathbb{R}$, $\Omega_z = (x_0, x_1) \times (z_b, z_t)$. For $f \in H^\sigma_x H^1_y \cap L^4_x H^1_y(\Omega_y)$ and $Y \in H^{\frac{2}{3}+\sigma}_x H^1_z(\Omega_z)$, such that $\lambda \leq \partial_z Y \leq \lambda^{-1}$ for some positive constant λ,*

$$\| f(x, Y(x, z)) \|_{H^\sigma_x L^2_z(\Omega_z)} \lesssim C_{\|Y\|} (\| f \|_{H^\sigma_x L^2_y} + \| f \|_{L^4_x H^1_y})$$

and

$$\| f(x, Y(x, z)) \|_{L^\infty((z_b, z_t), H^\sigma(x_0, x_1))} \lesssim C_{\|Y\|} (\| f \|_{H^\sigma_x H^1_z} + \| f \|_{L^4_x H^1_y}).$$

In a similar way, if $f \in H^{\frac{1}{2}+\sigma}_x L^2_y \cap L^\infty_x W^{1,\infty}_y(\Omega_y)$,

$$\| f(x, Y(x, z)) \|_{H^{\frac{1}{2}+\sigma}_x L^2_z(\Omega_z)} \lesssim C_{\|Y\|} (\| f \|_{H^{\frac{1}{2}+\sigma}_x L^2_z} + \| f \|_{L^\infty_x W^{1,\infty}_z}).$$

Proof. Using the classical definition of fractional Sobolev spaces, for all $z \in (z_b, z_t)$,

$$\| f(\cdot, Y(\cdot, z)) \|^2_{H^\sigma(x_0, x_1)} = \| f(\cdot, Y(\cdot, z)) \|^2_{L^2(x_0, x_1)}$$
$$+ \int_{x_0}^{x_1} \int_{x_0}^{x_1} \frac{|f(x, Y(x, z)) - f(x', Y(x', z))|^2}{|x - x'|^{1+2\sigma}} \, dx \, dx'.$$

We start with the $H^\sigma_x L^2_z$ estimate. Integrating with respect to z, the norm of the first term is bounded by the square of the L^2 norm of f after a change of variable with bounded Jacobian. We then decompose the second integral into

$$\int_{x_0}^{x_1} \int_{x_0}^{x_1} \frac{|f(x, Y(x, z)) - f(x', Y(x, z))|^2}{|x - x'|^{1+2\sigma}} \, dx \, dx'$$
$$+ \int_{x_0}^{x_1} \int_{x_0}^{x_1} \frac{|f(x', Y(x, z)) - f(x', Y(x', z))|^2}{|x - x'|^{1+2\sigma}} \, dx \, dx'.$$

Once again, the first integral is bounded by $\| f \|^2_{H^\sigma_x L^2_z}$ after vertical integration. As for the second one, using the embedding $H^1(-1, 1) \hookrightarrow C^{1/2}$, we have

$$|f(x', Y(x, z)) - f(x', Y(x', z))|^2 \lesssim \| \partial_y f(x', \cdot) \|^2_{L^2_y} |Y(x, z) - Y(x', z)|.$$

Since $Y \in H_x^{\frac{2}{3}+\sigma} H_z^1$,

$$\int_{x_0}^{x_1} \int_{x_0}^{x_1} \|\partial_y f(x',\cdot)\|_{L_y^2}^2 \frac{|Y(x,z) - Y(x',z)|}{|x - x'|^{1+2\sigma}} \mathrm{d}x\mathrm{d}x'$$

$$\lesssim \|Y(\cdot,z)\|_{H_x^{\frac{2}{3}+\sigma}} \left(\int_{x_0}^{x_1} \int_{x_0}^{x_1} \|\partial_y f(x',\cdot)\|_{L_y^2}^4 |x - x'|^{\frac{1}{3}-3\sigma} \, \mathrm{d}x \, \mathrm{d}x' \right)^{1/2}$$

$$\lesssim \|Y\|_{H_x^{\frac{2}{3}+\sigma} H_z^1} \|f\|_{L_x^4 H_y^1}^2.$$

The first estimate follows. The other ones go along the same lines and are left to the reader. ∎

Lemma B.5 (Composition with a Q^1 function). *Let $\phi \in Q^1(\Omega)$ such that $\phi(x, \pm 1) = \pm 1$. Assume that there exists $m > 0$ such that $\partial_z \phi(x,z) \in [m^{-1}, m]$. Let $\sigma \in [0,1]$. There exists $C(m, \sigma)$ such that, for any $g \in H_x^\sigma L_y^2 \cap L_x^2(W_y^{\sigma,4})$,*

$$\|g(x, \phi(x,z))\|_{H_x^\sigma L_z^2} \leq C \left(\|g\|_{H_x^\sigma L_z^2} + (1 + \|\phi\|_{Q^1}^\sigma) \|g\|_{L_x^2(W_y^{\sigma,4})} \right). \tag{B.2}$$

Proof. Throughout the proof, we set $G(x,z) := g(x, \phi(x,z))$. First, note that, since the Jacobian of the change of variable $z \mapsto \phi(x,z)$ is bounded from below, for any $p, q \in [1, \infty]$,

$$\|G\|_{L_x^p L_z^q} \leq m^{\frac{1}{q}} \|g\|_{L_x^p L_z^q}. \tag{B.3}$$

In particular, $\|G\|_{L^2} \leq m^{\frac{1}{2}} \|g\|_{L^2}$. Furthermore, for $\sigma = 1$,

$$\partial_x G(x,z) = \partial_x g(x, \phi(x,z)) + \partial_x \phi(x,z) \partial_y g(x, \phi(x,z)).$$

Hence,

$$\|\partial_x G\|_{L^2} \leq \|(\partial_x g) \circ \phi\|_{L^2} + \|\partial_x \phi\|_{L_x^\infty L_z^4} \|(\partial_y g) \circ \phi\|_{L_x^2 L_z^4}. \tag{B.4}$$

By the "fractional trace theorem" [45, equation (4.7), Chapter 1],

$$\|\partial_x \phi\|_{L_x^\infty H_z^{1/2}} \lesssim \|\partial_x \phi\|_{H_x^{2/3} L_z^2} + \|\partial_x \phi\|_{L_x^2 H_z^2} \lesssim \|\phi\|_{Q^1}.$$

Hence, we obtain from (B.3) and (B.4) that

$$\|\partial_x G\|_{L^2} \lesssim \|\partial_x g\|_{L^2} + \|\phi\|_{Q^1} \|g\|_{L_x^2(W_y^{1,4})}.$$

Now, note that the application $g \mapsto G$ is linear. By interpolation, we obtain, for any $\sigma \in (0,1)$,

$$\|G\|_{H_x^\sigma L_z^2} \lesssim \|g\|_{H_x^\sigma L_z^2} + (1 + \|\phi\|_{Q^1}^\sigma) \|g\|_{L_x^2(W_y^{\sigma,4})},$$

which concludes the proof. ∎

Corollary B.6. *Let $\phi \in Q^1 \cap L_x^2 H_z^4$ such that $\phi(x, \pm 1) = \pm 1$ and $\|\phi - z\|_{Q^1} +$*
$\|\phi - z\|_{L_x^2 H_z^4} \ll 1$. Let $\psi(x, y)$ be such that $\psi(x, \phi(x, z)) = z$ for all $(x, z) \in \Omega$.
Then $\psi \in Q^1 \cap L_x^2 H_y^4$ and

$$\|\psi - y\|_{Q^1} + \|\psi - y\|_{L_x^2 H_y^4} \lesssim \|\phi - z\|_{Q^1} + \|\phi - z\|_{L_x^2 H_z^4}.$$

Proof. In this statement and this proof, we use the variable y as second argument for
ψ, z as second argument for ϕ. First, observe that $\partial_y \psi(x, y) = 1/(\partial_z \phi(x, \psi(x, y)))$,
so that $\|\partial_y \psi - 1\|_{L^\infty} \ll \|\partial_z \phi - 1\|_{L^\infty}$. In particular, the associated changes of ver-
tical variables are defined and bounded so that estimates such as (B.3) hold and will
be used abundantly.

Step 1. Vertical regularity of ψ. By the "fractional trace theorem" [45, equation (4.7),
Chapter 1], for $\phi \in Q^1 \cap L_x^2 H_z^4$, $\phi \in H_x^1 H_z^2 \cap L_x^2 H_z^4 \hookrightarrow C_x^0(H_z^3)$. In particular,
$\partial_z^2 \phi \in L^\infty$. Differentiating the definition $\psi(x, \phi(x, z)) = z$, we obtain the following
relations and estimates. First, we already said that $\partial_y \psi \in L^\infty$. Second, $\partial_y^2 \psi \in L^2$
since

$$-(\partial_z \phi)^2 \partial_y^2 \psi \circ \phi = \underbrace{(\partial_y \psi \circ \phi)}_{L^\infty} \underbrace{(\partial_{zz} \phi)}_{L^\infty}.$$

Third, $\partial_y^3 \psi \in L^2$ since

$$-(\partial_z \phi)^3 \partial_y^3 \psi \circ \phi = 3 \underbrace{\partial_y^2 \psi \circ \phi}_{L^2} \underbrace{\partial_z \phi \partial_z^2 \phi}_{L^\infty} + \underbrace{\partial_y \psi \circ \phi}_{L^\infty} \underbrace{\partial_z^3 \phi}_{L^2}.$$

Fourth, omitting the composition with ϕ in every occurrence of ψ in order to alleviate
the notation,

$$
\begin{aligned}
- (\partial_z \phi)^4 \partial_y^4 \psi \\
= 6 \underbrace{\partial_y^3 \psi}_{L^2} \underbrace{(\partial_z \phi)^2 \partial_z^2 \phi}_{L^\infty} + \underbrace{\partial_y^2 \psi}_{L_x^2 H_y^1} \left(3 \underbrace{(\partial_z^2 \phi)^2}_{L^\infty} + 4 \underbrace{(\partial_z \phi)}_{L^\infty} \underbrace{(\partial_z^3 \phi)}_{L_x^\infty L_z^2}\right) + \underbrace{\partial_y \psi}_{L^\infty} \underbrace{\partial_z^4 \phi}_{L^2}.
\end{aligned}
$$

Remembering that $1/(\partial_z \phi) \in L^\infty$, we conclude that

$$\|\psi(x, y) - y\|_{L_x^2 H_y^4} \lesssim \|\phi(x, z) - z\|_{Q^1} + \|\phi(x, z) - z\|_{L_x^2 H_z^4}.$$

Step 2. Integer horizontal regularity of ψ. This step uses that $\partial_x \phi \in L_x^\infty L_z^2$ which
follows from $\phi \in H_x^{5/3} L_z^2$. Note however that, even for $\phi \in Q^1 \cap L_x^2 H_y^4$, one does
not have $\partial_x \phi \in L^\infty$. We proceed similarly for the integer horizontal regularity. First,

$$-\partial_x \psi \circ \phi = \underbrace{\partial_y \psi \circ \phi}_{L^\infty} \underbrace{\partial_x \phi}_{L^2}.$$

Second,

$$-(\partial_z\phi)\partial_{xy}\psi\circ\phi = \underbrace{\partial_y^2\psi\circ\phi}_{L_x^2 H_y^1}\,\underbrace{\partial_z\phi}_{L^\infty}\,\underbrace{\partial_x\phi}_{L_x^\infty L_z^2} + \underbrace{\partial_y\psi\circ\phi}_{L^\infty}\,\underbrace{\partial_{xz}\phi}_{L^2}\,.$$

Third,

$$-(\partial_z\phi)^2\partial_{xyy}\psi\circ\phi = 2\partial_y^2\psi\circ\phi\,\partial_z\phi\,\partial_{xz}\phi + h, \tag{B.5}$$

where, omitting once again the composition with ϕ in the derivatives of ψ,

$$h = \underbrace{\partial_y^3\psi}_{L_x^2 H_y^1}\,\underbrace{\partial_x\phi}_{L_x^\infty L_z^2}\,\underbrace{(\partial_z\phi)^2}_{L^\infty} + \underbrace{\partial_{xy}\psi}_{L^2}\,\underbrace{\partial_z^2\phi}_{L^\infty} + \underbrace{\partial_y^2\psi}_{L_x^2 H_y^1}\,\underbrace{\partial_x\phi}_{L_x^\infty L_z^2}\,\underbrace{\partial_z^2\phi}_{L^\infty} + \underbrace{\partial_z\psi}_{L^\infty}\,\underbrace{\partial_{xzz}\phi}_{L^2}\,.$$

From (B.5) and $h \in L^2$, we obtain that

$$\|\partial_{xyy}\psi\|_{L^2} \lesssim \|h\|_{L^2} + \|\partial_{xz}\phi\|_{L_x^2 H_z^1}\|\partial_{yy}\psi\|_{L_x^\infty L_z^2}$$
$$\lesssim \|h\|_{L^2} + \|\phi(x,z)-z\|_{H_x^1 H_z^2}(\|\partial_{yy}\psi\|_{L^2} + \|\partial_{xyy}\psi\|_{L^2}).$$

Hence, using the smallness of $\|\phi(x,z)-z\|_{H_x^1 H_z^2}$, we conclude that

$$\|\psi(x,y)-y\|_{H_x^1 H_y^2} \lesssim \|\phi(x,z)-z\|_{Q^1} + \|\phi(x,z)-z\|_{L_x^2 H_z^4}.$$

Step 3. Fractional horizontal regularity of ψ. Eventually, to obtain the $H_x^{5/3} L_y^2$ regularity, we write

$$\partial_x\psi(x,y) = -\frac{\partial_x\phi}{\partial_z\phi}(x,\psi(x,y))$$

and we apply Lemma B.5 with $\sigma = 2/3$. Let us first assume that ϕ is smooth (so that ψ is smooth as well by usual results) and then argue by density. Estimate (B.2) yields

$$\|\psi_x\|_{H_x^{2/3} L_y^2} \lesssim \|\partial_x\phi/\partial_z\phi\|_{H_x^{2/3} L_z^2} + (1 + \|\psi\|_{Q^1}^{2/3})\|\partial_x\phi/\partial_z\phi\|_{L_x^2 H_z^2}.$$

Since we already know that ψ can be estimated in $H_x^1 H_y^2$, we can use (the Peter–Paul version) of Young's inequality to obtain

$$\|\psi_x\|_{H_x^{2/3} L_y^2} \lesssim \|\partial_x\phi/\partial_z\phi\|_{H_x^{2/3} L_z^2} + \|\partial_x\phi/\partial_z\phi\|_{L_x^2 H_z^2} + \|\partial_x\phi/\partial_z\phi\|_{L_x^2 H_z^2}^{3/2}.$$

Moreover, one easily proves, using standard product rules, that

$$\|\partial_x\phi/\partial_z\phi\|_{H_x^{2/3} L_z^2} + \|\partial_x\phi/\partial_z\phi\|_{L_x^2 H_z^2} \lesssim \|\phi(x,z)-z\|_{Q^1} + \|\phi(x,z)-z\|_{L_x^2 H_z^4} \ll 1.$$

Hence we obtain

$$\|\psi_x\|_{H_x^{2/3} L_y^2} \lesssim \|\phi(x,z)-z\|_{Q^1} + \|\phi(x,z)-z\|_{L_x^2 H_z^4}$$

when ϕ is smooth and $\|\phi(x,z)-z\|_{Q^1} + \|\phi(x,z)-z\|_{L_x^2 H_z^4} \ll 1$. We conclude by density. ∎

B.3 Extension operators

We start with Lemma 1.6, which allows extending functions from $Z^0(\Omega)$ to $Z^0(\mathbb{R}^2)$.

Proof of Lemma 1.6. Up to translation and rescaling, we assume that $(x_0, x_1) = (0, 1)$.

We start by constructing a continuous horizontal extension operator denoted P_x from $Z^0((0,1) \times (-1,1))$ to $Z^0(\mathbb{R} \times (-1,1))$. Let $\chi \in C^\infty(\mathbb{R}; [0,1])$ such that $\chi \equiv 1$ on $(0,1)$ and supp $\chi \subset (-1,2)$. Let $\phi \in Z^0((0,1) \times (-1,1))$. For $x \in (-1,2)$ and $z \in (-1,1)$, let

$$(Q_x\phi)(x,z) := \begin{cases} \phi(-x,z) & \text{if } x \in (-1,0), \\ \phi(x,z) & \text{if } x \in (0,1), \\ \phi(2-x,z) & \text{if } x \in (1,2), \end{cases}$$

$$(P_x\phi)(x,z) := \chi(x)(Q_x\phi)(x,z).$$

First, $\|P_x\|_{L^2_{x,z} \to L^2_{x,z}} \le 3$. Moreover, $\partial_z^k (P_x\phi) = P_x \partial_z^k \phi$ for $k = 1$ and $k = 2$. Hence $\|P_x\|_{L^2_x H^2_z \to L^2_x H^2_z} \le 3$. Eventually,

$$\|z\partial_x(Q_x\phi)\|_{L^2((-1,2)\times(-1,1))} \le 3\|z\partial_x\phi\|_{L^2((0,1)\times(-1,1))},$$

so that

$$\|z\partial_x(P_x\phi)\|_{L^2} \le 3\|z\partial_x\phi\|_{L^2} + 2\|\chi'\|_{L^\infty}\|\phi\|_{L^2}.$$

Thus P_x is a continuous extension map from $Z^0((0,1) \times (-1,1))$ to $Z^0(\mathbb{R} \times (-1,1))$.

We now construct a continuous upwards vertical extension operator denoted by P_+ from $Z^0(\mathbb{R} \times (-1,1))$ to $Z^0(\mathbb{R} \times (-1,+\infty))$. We proceed in a classical manner (see, e.g., [4]), by considering a weighted linear combination of rescaled reflections. For $\phi \in Z^0(\mathbb{R} \times (-1,1))$, $x \in \mathbb{R}$ and $z \in (-1,\infty)$, let

$$(Q_+\phi)(x,z) := \begin{cases} \phi(x,z) & \text{if } z \in (-1,1), \\ 3\phi(x, 2-z) - 2\phi(x, 3-2z) & \text{if } z \in (1,2), \end{cases}$$

$$(P_+\phi)(x,z) := \chi_+(z)(Q_+\phi)(x,z),$$

where $\chi_+ \in C^\infty(\mathbb{R}; [0,1])$ is such that $\chi_+ \equiv 1$ on $(-1,1)$ and supp $\chi_+ \subset (-2, 1 + \frac{1}{4})$. The chosen coefficients ensure that both $Q_+\phi$ and $\partial_z(Q_+\phi)$ are continuous at $z = 1$. Hence $P_+\phi \in L^2_x H^2_z$ and

$$\|P_+\phi\|_{L^2_x(\mathbb{R}; H^2_z(-1,+\infty))}$$
$$= \|P_+\phi\|_{L^2_x(\mathbb{R}; H^2_z(-1,1))} + \|P_+\phi\|_{L^2_x(\mathbb{R}; H^2_z(1,+\infty))} \le C_+\|\phi\|_{L^2_x H^2_z},$$

for some constant C_+ depending only on $\|\chi_+\|_{W^{2,\infty}}$. Moreover, using that $\chi(z) = 0$ for $z > 1 + \frac{1}{4}$,

$$
\begin{aligned}
\|z\partial_x(P_+\phi)\|_{L^2_x(\mathbb{R};L^2(1,+\infty))} &= \|z\partial_x(P_+\phi)\|_{L^2_x(\mathbb{R};L^2(1,1+\frac{1}{4}))} \\
&\lesssim \|\partial_x\phi\|_{L^2_x(\mathbb{R};L^2(\frac{1}{2},1))} \\
&\lesssim \|z\partial_x\phi\|_{L^2_x(\mathbb{R};L^2(\frac{1}{2},1))}.
\end{aligned}
$$

Hence P_+ is a continuous extension map from $Z^0(\mathbb{R}\times(-1,1))$ to $Z^0(\mathbb{R}\times(-1,+\infty))$.

The extension for $z < -1$ is performed in a similar fashion and left to the reader. ∎

B.4 Embeddings

We collect in this paragraph various embedding results used throughout the memoir.

B.4.1 Embedding of the Pagani space Z^0 in $H_x^{2/3}L_z^2$

We start with an easy one-dimensional inequality.

Lemma B.7. *For $\psi \in C_c^\infty(\mathbb{R})$,*

$$
\|\psi\|_{L^2} \lesssim \|z\psi\|_{L^2} + \|\partial_{zz}\psi\|_{L^2}.
$$

Proof. On the one hand, for $|z| \geq 1$,

$$
\int_{|z|\geq 1}\psi^2 \leq \|z\psi\|_{L^2}^2.
$$

On the other hand, for every $(z_0, z) \in (-2, 2)$,

$$
|\partial_z\psi(z)| \leq |\partial_z\psi(z_0)| + 2\|\partial_{zz}\psi\|_{L^2}. \tag{B.6}
$$

Moreover, by classical Sobolev embeddings,

$$
\|\partial_z\psi\|_{L^2(1,2)} \lesssim \|\psi\|_{L^2(1,2)} + \|\partial_{zz}\psi\|_{L^2(1,2)} \leq \|z\psi\|_{L^2(\mathbb{R})} + \|\partial_{zz}\psi\|_{L^2(\mathbb{R})}.
$$

Thus, integrating (B.6) for $z_0 \in (1, 2)$,

$$
\|\partial_z\psi\|_{L^\infty(-2,2)} \lesssim \|z\psi\|_{L^2(\mathbb{R})} + \|\partial_{zz}\psi\|_{L^2(\mathbb{R})}.
$$

Now, writing $\psi(z) = \psi(z_0) + \int_{z_0}^z \psi'$ and integrating for $z_0 \in (1, 2)$ yields

$$
\|\psi\|_{L^2(-1,1)} \lesssim \|\psi\|_{L^2(1,2)} + \|z\psi\|_{L^2(\mathbb{R})} + \|\partial_{zz}\psi\|_{L^2(\mathbb{R})} \lesssim \|z\psi\|_{L^2(\mathbb{R})} + \|\partial_{zz}\psi\|_{L^2(\mathbb{R})},
$$

which concludes the proof. ∎

We then turn towards the proof of the key result $Z^0(\mathbb{R}^2) \hookrightarrow H_x^{2/3}L_z^2(\mathbb{R}^2)$.

Proof of Proposition 1.7. Let $\psi \in C_c^\infty(\mathbb{R})$. By Lemma B.7, one has

$$\|\psi\|_{L^2} \lesssim \|z\psi\|_{L^2} + \|\partial_{zz}\psi\|_{L^2}. \tag{B.7}$$

Using standard dimensional analysis arguments (e.g., by introducing the rescaled function $\psi_\lambda : z \mapsto \psi(\lambda z)$ for $\lambda > 0$ and optimizing the choice of λ), one deduces from (B.7) that

$$\|\psi\|_{L^2} \lesssim \|z\psi\|_{L^2}^{\frac{2}{3}} \|\partial_{zz}\psi\|_{L^2}^{\frac{1}{3}}. \tag{B.8}$$

Let $\phi \in C_c^\infty(\mathbb{R}^2)$. Let $\hat{\phi}(\xi, z)$ denote the Fourier-transform of ϕ in the horizontal direction. Then using (B.8) and Hölder's inequality,

$$\begin{aligned}
\|\phi\|_{H_x^{2/3}L_z^2}^2 &= \int_{\mathbb{R}^2} (1 + |\xi|^2)^{\frac{2}{3}} |\hat{\phi}(\xi, z)|^2 \, \mathrm{d}\xi \, \mathrm{d}z \\
&\lesssim \|\phi\|_{L^2}^2 + \int_{\mathbb{R}} |\xi|^{\frac{4}{3}} \|z\hat{\phi}(\xi, z)\|_{L_z^2}^{\frac{4}{3}} \|\partial_{zz}\hat{\phi}(\xi, z)\|_{L_z^2}^{\frac{2}{3}} \, \mathrm{d}\xi \\
&\lesssim \|\phi\|_{L^2}^2 + \left(\int_{\mathbb{R}^2} |\xi|^2 z^2 |\hat{\phi}(\xi, z)|^2 \, \mathrm{d}z \, \mathrm{d}\xi\right)^{\frac{2}{3}} \left(\int_{\mathbb{R}^2} |\partial_{zz}\hat{\phi}(\xi, z)|^2 \, \mathrm{d}z \, \mathrm{d}\xi\right)^{\frac{1}{3}} \\
&\lesssim \|\phi\|_{L^2}^2 + \|z\partial_x\phi\|_{L^2}^{\frac{4}{3}} \|\partial_{zz}\phi\|_{L^2}^{\frac{2}{3}}.
\end{aligned}$$

Hence $\|\phi\|_{H_x^{2/3}L_z^2} \lesssim \|\phi\|_{Z^0}$. This concludes the proof, by density of $C_c^\infty(\mathbb{R}^2)$ in $Z^0(\mathbb{R}^2)$. ∎

B.4.2 Embedding of the Baouendi–Grisvard space $\mathcal{B}$ in $H_x^{1/3}L_z^2$

Once again, we start with a one-dimensional inequality of Hardy type.

Lemma B.8. *For* $\phi \in L^2(0, 1)$,

$$\int_0^1 \left(\frac{1}{z^2} \int_0^z s\phi(s) \, \mathrm{d}s\right)^2 \mathrm{d}z \le \frac{4}{5} \|\phi\|_{L^2}^2. \tag{B.9}$$

Proof. For $z \in (0, 1)$, by the Cauchy–Schwarz inequality

$$\left(\int_0^z s\phi(s) \, \mathrm{d}s\right)^2 \le \left(\int_0^z \phi^2(s) s^{\frac{1}{2}} \, \mathrm{d}s\right)\left(\int_0^z s^{2-\frac{1}{2}} \, \mathrm{d}s\right) = \frac{2}{5} z^{\frac{5}{2}} \left(\int_0^z \phi^2(s) s^{\frac{1}{2}} \, \mathrm{d}s\right).$$

Hence, by Fubini,

$$\begin{aligned}
\int_0^1 \left(\frac{1}{z^2} \int_0^z s\phi(s) \, \mathrm{d}s\right)^2 \mathrm{d}z &\le \frac{2}{5} \int_0^1 z^{-4+\frac{5}{2}} \left(\int_0^z \phi^2(s) s^{\frac{1}{2}} \, \mathrm{d}s\right) \mathrm{d}z \\
&= \frac{2}{5} \int_0^1 \phi^2(s) s^{\frac{1}{2}} \left(\int_s^1 z^{-\frac{3}{2}} \, \mathrm{d}z\right) \mathrm{d}s \\
&= \frac{2}{5} \int_0^1 \phi^2(s) s^{\frac{1}{2}} (2(s^{-\frac{1}{2}} - 1)) \mathrm{d}s = \frac{4}{5} \int_0^1 \phi^2(s)(1 - s^{\frac{1}{2}}) \mathrm{d}s,
\end{aligned}$$

which implies (B.9). ∎

We then turn towards the proof of the embedding.

Proof of Lemma 1.13. *Step 1. Extension to $(x_0, x_1) \times \mathbb{R}$ with compact vertical support.*

Let $u \in L^2((x_0, x_1), H_0^1(-1, 1))$ such that $z\partial_x u \in L^2((x_0, x_1)), H^{-1}(-1, 1)$. We first extend u to $(x_0, x_1) \times (-3, 3)$ by setting, for all $x \in (x_0, x_1)$ and $z' \in (0, 2)$,

$$u(x, 1 + z') = -u(x, 1 - z'),$$
$$u(x, -1 - z') = -u(x, -1 + z').$$

It is clear that the above extension belongs to $L^2((x_0, x_1), H_0^1(-3, 3))$, and we further extend u by zero on $(x_0, x_1) \times \{z \in \mathbb{R}, |z| \geq 3\}$. We then take $\chi \in C_c^\infty(\mathbb{R})$ such that supp $\chi \subset (-3/2, 3/2)$, and $\chi \equiv 1$ on $(-1, 1)$, and we prove that $u\chi \in \mathcal{B}((x_0, x_1) \times \mathbb{R})$. Using a partition of unity, we write $\chi = \chi_{-1} + \chi_0 + \chi_1$, where supp $\chi_{\pm 1} \subset (\pm 1/2, \pm 3/2)$, and supp $\chi_0 \subset (-3/4, 3/4)$. It is clear that $\chi_0 u \in \mathcal{B}((x_0, x_1) \times \mathbb{R})$, and therefore by symmetry is sufficient to prove the result for $\chi_1 u$.

Let us take $\phi \in H_0^1((x_0, x_1) \times \mathbb{R})$ be arbitrary, and compute

$$I := -\int_{x_0}^{x_1} \int_{\mathbb{R}} z\chi_1 u \partial_x \phi.$$

By definition of u on $(x_0, x_1) \times (1, 2)$,

$$
\begin{aligned}
I = &-\int_{x_0}^{x_1} \int_0^1 zu(x, z)\chi_1(z)\partial_x\phi(x, z) \, dx \, dz \\
&+ \int_{x_0}^{x_1} \int_0^1 (1 + z')u(x, 1 - z')\chi_1(1 + z')\partial_x\phi(x, 1 + z') \, dx \, dz' \\
= &-\int_{x_0}^{x_1} \int_0^1 zu(x, z)\chi_1(z)\partial_x\phi(x, z) \, dx \, dz \\
&+ \int_{x_0}^{x_1} \int_0^1 \frac{(1 + z')}{1 - z'}(1 - z')u(x, 1 - z')\chi_1(1 + z')\partial_x\phi(x, 1 + z') \, dx \, dz'.
\end{aligned}
$$

(B.10)

Since $z\partial_x u \in L^2((x_0, x_1), H^{-1}(-1, 1))$, we may write $z\partial_x u = f + \partial_z g$, with $f, g \in L^2((x_0, x_1) \times (-1, 1))$. Then

$$I = \int_{x_0}^{x_1} \int_0^1 (f + \partial_z g)(x, z)\left[\chi_1(z)\phi(x, z) - \frac{2 - z}{z}\chi_1(2 - z)\phi(x, 2 - z)\right] dx \, dz.$$

The assumptions on supp χ_1 ensure that the quantity within the brackets belongs to $L^2((x_0, x_1), H_0^1(0, 1))$. We conclude that for all $\phi \in H_0^1((x_0, x_1) \times \mathbb{R})$,

$$\left|-\int_{x_0}^{x_1} \int_{\mathbb{R}} z\chi_1 u \partial_x \phi\right| \lesssim \|u\|_{\mathcal{B}((x_0, x_1) \times (0, 1))} \|\phi\|_{L_x^2 H_z^1}.$$

It follows that $z\chi_1\partial_x u \in L^2((x_0, x_1), H^1(\mathbb{R}))$, and

$$\|\chi u\|_{\mathcal{B}((x_0,x_1)\times\mathbb{R})} \lesssim \|u\|_{\mathcal{B}((x_0,x_1)\times(0,1))}.$$

Step 2. The vertical anti-derivative of χu belongs to Z^0.

We now work with the extension of the previous step, and we set $U := -\int_z^\infty \chi u$. Let us prove that $U \in Z^0((x_0, x_1) \times \mathbb{R}_+)$. Since $\partial_z^2 U = \partial_z(\chi u) \in L^2((x_0, x_1) \times \mathbb{R}_+)$, it suffices to prove that $z\partial_x U \in L^2((x_0, x_1) \times \mathbb{R}_+)$. Hence we take $\phi \in L^2((x_0, x_1) \times \mathbb{R}_+)$ arbitrary, and we compute, after observing that U is supported in $\{z \leq 3/2\}$,

$$\int_{x_0}^{x_1} \int_0^\infty s\partial_x U(x, s)\phi(x, s)\, dx\, ds$$
$$= -\int_{x_0}^{x_1} \int_0^\infty s\left(\int_s^{3/2} \partial_x \chi u(x, z)\, dz\right)\phi(x, s)\, ds\, dx$$
$$= -\int_{x_0}^{x_1} \int_0^\infty \frac{1}{z}\left(\int_0^z s\mathbf{1}_{0<s<3/2}\phi(x, s)\, ds\right)z\partial_x \chi u(x, z)\, dx\, dz.$$

Therefore

$$\|s\partial_x U\|_{L^2((x_0,x_1)\times\mathbb{R}_+)}$$
$$\lesssim \|\chi u\|_{\mathcal{B}} \sup_{\substack{\phi \in L^2, \\ \|\phi\|_{L^2} \leq 1}} \left\|\frac{1}{z}\left(\int_0^z s\mathbf{1}_{0<s<3/2}\phi(x, s)\, ds\right)\right\|_{L_x^2(H_0^1(0,+\infty))}.$$

The claim therefore follows from the following result, which is postponed to the third and last step.

Lemma B.9. *For all $z_0 > 0$, there exists a constant C_{z_0} such that for all $\psi \in L^2(\mathbb{R})$,*

$$\left\|\frac{1}{z}\left(\int_0^z s\mathbf{1}_{0<s<z_0}\psi(s)\, ds\right)\right\|_{H_0^1(0,+\infty)} \leq C_{z_0}\|\psi\|_{L^2(\mathbb{R})}. \tag{B.11}$$

From there, we infer that $U \in Z^0((x_0, x_1) \times \mathbb{R}_+)$, and $\|U\|_{Z^0} \lesssim \|u\|_{\mathcal{B}}$. Using the embedding $Z^0 \hookrightarrow H_x^{1/3}H_z^1$, we deduce that $\partial_z U = \chi u \in H_x^{1/3}L_z^2((x_0, x_1) \times \mathbb{R}_+)$. Since $\chi \equiv 1$ on $(-1, 1)$, we obtain the desired result.

Step 3. Proof of (B.11).

First, note that for all $s \in (0, +\infty)$,

$$\left|\int_0^z s\mathbf{1}_{0<s<z_0}\psi(s)\, ds\right| \leq C_{z_0}\inf(s^{3/2}, 1)\|\psi\|_{L^2}.$$

Thus we only need to prove that

$$\int_0^{z_0}\left(\frac{1}{z^2}\int_0^z s\psi(s)\, ds\right)^2 dz \leq C_{z_0}\|\psi\|_{L^2}^2.$$

This is a rescaling of inequality (B.9) of Lemma B.8. ∎