

Appendix C

Unconditional regularity away from lateral boundaries

Proof of Lemma 5.4. We proceed by (horizontal) viscous regularization to obtain uniform estimates which pass to the limit.

Let us extend the functions w_0, w_1 into H^2 functions on the whole interval (z_b, z_t) , such that $w_0(z_b) = w_b(x_0)$ and $w_1(z_t) = w_t(x_1)$. For $\varepsilon > 0$, consider the solution to the elliptic equation

$$\begin{cases} -\varepsilon \partial_x^2 W^\varepsilon - \partial_z^2(\alpha W^\varepsilon) + z \partial_x W^\varepsilon + \beta \partial_z W^\varepsilon = h, \\ W^\varepsilon|_{x=x_i} = w_i, \\ W^\varepsilon|_{z=z_j} = w_j. \end{cases} \quad (\text{C.1})$$

Let us recall that $|z_b|, z_t \leq z_0$ for some small constant z_0 depending only on α . Classical results on elliptic equations ensure that if z_0 is small enough, (C.1) has a unique solution in $H^1(\Omega)$ for all $\varepsilon > 0$, which satisfies the energy estimate

$$\sqrt{\varepsilon} \|\partial_x W^\varepsilon\|_{L^2} + \|\partial_z W^\varepsilon\|_{L^2} \lesssim \|h\|_{L^2} + \|w_0\|_{H_z^2} + \|w_1\|_{H_z^2} + \|w_t\|_{H_x^2} + \|w_b\|_{H_x^2}. \quad (\text{C.2})$$

Hence W^ε is uniformly bounded in $L_x^2 H_z^1$. It follows that $W^\varepsilon \rightharpoonup W$ in $L_x^2 H_z^1$, where $W \in Z^0$ is the unique solution to (5.12).

Furthermore, since $h \in L^2(\Omega)$, the compatibility conditions in the corners of the domain and the fact that Ω is a rectangle ensure that $W^\varepsilon \in H^2(\Omega)$ (see [28, Chapter 4]). Hence $\partial_x W^\varepsilon \in H^1(\Omega)$ is a weak solution to

$$-\varepsilon \partial_x^2 \partial_x W^\varepsilon - \partial_z^2(\alpha \partial_x W^\varepsilon) + z \partial_x^2 W^\varepsilon + \beta \partial_z \partial_x W^\varepsilon = \partial_x h + \partial_z^2(\partial_x \alpha W^\varepsilon) - \partial_x \beta \partial_z W^\varepsilon. \quad (\text{C.3})$$

Without loss of generality, we assume from now on that $w_t = w_b = 0$ in order to simplify the computations. This condition can always be satisfied up to a lift of the boundary conditions. Let $\rho(x) := (x - x_0)(x_1 - x)$. We multiply (C.3) by $\rho^2 \partial_x W^\varepsilon$, integrate by parts and obtain

$$\begin{aligned} & \varepsilon \int_{\Omega} \rho^2 (\partial_x^2 W^\varepsilon)^2 + \int_{\Omega} \alpha \rho^2 (\partial_{xz} W^\varepsilon)^2 \\ & \leq \varepsilon \int_{\Omega} \partial_x(\rho \partial_x \rho) (\partial_x W^\varepsilon)^2 - \int_{\Omega} \rho^2 \partial_x^2 W^\varepsilon (z \partial_x W^\varepsilon) \\ & \quad + (\|\rho \partial_x h\|_{L^2} + \|\rho \partial_x \beta \partial_z W^\varepsilon\|_{L^2}) \|\rho \partial_x W^\varepsilon\|_{L^2} \\ & \quad + \|\rho \partial_z(\partial_x \alpha W^\varepsilon)\|_{L^2} \|\rho \partial_{xz} W^\varepsilon\|_{L^2} \\ & \quad + \|\partial_z \alpha\|_{\infty} \|\rho \partial_x W^\varepsilon\|_{L^2} \|\rho \partial_{xz} W^\varepsilon\|_{L^2}. \end{aligned}$$

The first integral on the right-hand side is uniformly bounded thanks to (C.2). Let us focus momentarily on the second integral on the right-hand side. Using the equation satisfied by W^ε , we infer

$$\int_{\Omega} \rho^2 \partial_x^2 W^\varepsilon (z \partial_x W^\varepsilon) = \int_{\Omega} \rho^2 \partial_x^2 W^\varepsilon [h - \beta \partial_z W^\varepsilon + \partial_z^2 (\alpha W^\varepsilon) + \varepsilon \partial_x^2 W^\varepsilon].$$

We then perform integration by parts on the right-hand side, which is equal to

$$\begin{aligned} & \varepsilon \int_{\Omega} \rho^2 (\partial_x^2 W^\varepsilon)^2 + \int_{\Omega} \rho^2 \alpha (\partial_{xz} W^\varepsilon)^2 \\ & - 2 \int_{\Omega} \rho \partial_x \rho \partial_x W^\varepsilon (h - \beta \partial_z W^\varepsilon) - \int_{\Omega} \rho^2 \partial_x W^\varepsilon \partial_x (h - \beta \partial_z W^\varepsilon) \\ & + 2 \int_{\Omega} \rho \partial_x \rho \partial_{xz} W^\varepsilon \partial_z (\alpha W^\varepsilon) + \int_{\Omega} \rho^2 \partial_{xz} W^\varepsilon (\partial_x (\partial_z \alpha W^\varepsilon) + \partial_x \alpha \partial_z W^\varepsilon). \end{aligned}$$

Since $\partial_z \beta = 0$ and $w_t = w_b = 0$, we have

$$\int \rho^2 \partial_x W^\varepsilon \beta \partial_{xz} W^\varepsilon = 0.$$

We also recall that $\|\rho \partial_x W^\varepsilon\|_{L^2} \leq z_0 \|\rho \partial_x \partial_z W^\varepsilon\|_{L^2}$. Therefore, provided that z_0 is small enough, there exists $C > 0$ such that

$$\begin{aligned} \int_{\Omega} \rho^2 \partial_x^2 W^\varepsilon (z \partial_x W^\varepsilon) & \geq \varepsilon \int_{\Omega} \rho^2 (\partial_x^2 W^\varepsilon)^2 + \frac{1}{2} \int_{\Omega} \rho^2 \alpha (\partial_{xz} W^\varepsilon)^2 \\ & - C (\|h\|_{L^2}^2 + \|W^\varepsilon\|_{L_x^2 H_z^1}^2 + \|\rho \partial_x h\|_{L^2}). \end{aligned}$$

Gathering all the terms and using the $L_x^2 H_z^1$ estimate on W^ε of (C.2), for z_0 small enough,

$$\begin{aligned} \varepsilon \int_{\Omega} \rho^2 (\partial_x^2 W^\varepsilon)^2 + \int_{\Omega} \alpha \rho^2 (\partial_x \partial_z W^\varepsilon)^2 & \lesssim \|h\|_{L^2}^2 + \|\rho \partial_x h\|_{L^2}^2 \\ & + \sum_{i \in \{0,1\}} \|w_i\|_{H_z^2}^2 + \sum_{j \in \{t,b\}} \|w_j\|_{H_x^2}^2. \end{aligned}$$

Hence $\rho \partial_{xz} W^\varepsilon$ is uniformly bounded in L^2 . Passing to the limit, we obtain

$$\|\rho \partial_x W\|_{L_x^2 H_z^1} \lesssim \|h\|_{L^2} + \|\rho \partial_x h\|_{L^2} + \sum_{i \in \{0,1\}} \|w_i\|_{H_z^2} + \sum_{j \in \{t,b\}} \|w_j\|_{H_x^2}.$$

It follows that $\rho \partial_x W$ is a weak $L_x^2 H_z^1$ solution to

$$\begin{cases} z \partial_x (\rho \partial_x W) + \beta \partial_z (\rho \partial_x W) - \partial_z^2 (\alpha \rho \partial_x W) = g, \\ \rho \partial_x W|_{\Sigma_i} = 0, \\ \rho \partial_x W|_{z=z_j} = \rho \partial_x w_j, \end{cases}$$

where

$$g := \rho \partial_x h - \rho \partial_x \beta \partial_z W + \rho \partial_{zz}(\partial_x \alpha W) + (x_0 + x_1 - 2x)z \partial_x W.$$

Since $\alpha \in C^3(\bar{\Omega})$ and $W \in Z^0$, $g \in L^2$. Hence, according to Lemma 5.3, we obtain $\rho \partial_x W \in Z^0$. ■