## Chapter 1

# Introduction and main results

## 1.1 System and notation

In this memoir, we are interested in the following coupling between fluid and particles:

$$
\begin{cases}
\partial_t f + v \cdot \nabla_x f + \operatorname{div}_v[f\Gamma] = 0, \\
\partial_t(\alpha\varrho) + \operatorname{div}_x(\alpha\varrho u) = 0, \\
\partial_t(\alpha\varrho u) + \operatorname{div}_x(\alpha\varrho u \otimes u) + \nabla_x p(\varrho) - \operatorname{div}_x(\tau[u]) = -m_p \displaystyle\int_{\mathbb{R}^d} f\Gamma \, \mathrm{d}v.
\end{cases}
\tag{1.1}
$$

This system describes the evolution of a cloud of particles (e.g. droplets or dust specks) in an underlying compressible fluid (e.g. a gas). Such a suspension is commonly referred to as a *spray* [137]. More generally, the system (1.1) belongs to the broad family of "multiphase flows" equations [59].

In this work, we study (1.1) in the phase space $\mathbb{T}^d \times \mathbb{R}^d$, with $d \in \mathbb{N}\backslash\{0\}$. The first equation of (1.1) is a kinetic equation of Vlasov type on the particle distribution function $f(t, x, v) \in \mathbb{R}^+$ in the phase space (position-velocity), set for $t > 0$ and $(x, v) \in \mathbb{T}^d \times \mathbb{R}^d$. The other equations of (1.1) are set for $t > 0$ and $x \in \mathbb{T}^d$ and are barotropic compressible Navier–Stokes equations for the fluid density $\varrho(t, x) \in \mathbb{R}^+$ and the fluid velocity $u(t, x) \in \mathbb{R}^d$. Here, the function $\alpha(t, x) \in [0, 1]$ is the volume fraction of the fluid.

This type of models can be used to capture various natural phenomena and has widespread applications. Examples include, among others, combustion phenomena in engines [4, 108], evaporation of droplets [58, 113], aerosols for medical purposes [27, 28], marine and volcanic aerosols and their impact on the atmosphere [114, 134], as well as aerosols in the atmosphere of gas giants or exoplanets [70, 136].

Here, the system (1.1) describes the so-called *thick sprays* and has been introduced and derived by O'Rourke in [124]. Such coupling is appropriate for modeling two-phase mixtures where particles are small but occupy a non-negligible volume fraction of the whole suspension. The thick (or dense) spray regime is typically found in regions where droplets are injected in a carrying gas (see [64, 110, 124]). The system (1.1) has also been recognized as a set of equations linked to multifluid systems, which are thoroughly described in [103]. Further details can be found in the overview provided in Section 1.4.

Let us detail and close the previous equations, explaining the meaning of the different terms involved. For this physical discussion, we choose $d = 3$.

- The volume fraction $\alpha = \alpha(t,x) \in \mathbb{R}^+$ of the fluid is given by

$$\alpha(t,x) := 1 - m_p \int_{\mathbb{R}^3} f(t,x,v)\,\mathrm{d}v, \quad m_p := \frac{4\pi}{3} r_p^3,$$

where $r_p > 0$ is the radius of one droplet. In the *thick spray* regime, the quantity $\alpha$ is not assumed to be close to 1 (this concerns suspensions for which, typically, $\alpha$ is around the value 0.9), so that the volume fraction for the cloud of particles is not negligible compared to that of the fluid. This is in sharp contrast with the *thin spray* regime, that we shall recall below (see (VNS) in Section 1.4), where $\alpha$ is somehow directly set to 1 and thus does not explicitly appear in the system. Here, this quantity induces an extra coupling between both phases. We refer to [59, 64, 124] for comparisons between the thin and thick regimes.

- The pressure $p : \mathbb{R}^+ \to \mathbb{R}^+$ is a given $\mathcal{C}(\mathbb{R}^+) \cap \mathcal{C}^\infty(\mathbb{R}^+\backslash\{0\})$ function such that, in the barotropic regime, the pressure term is a function $p = p(\varrho)$ depending only on the fluid density. One usually assumes that $p(0) = 0$ and $p'(\rho) > 0$. For reasons that will appear later, we shall also assume that $p$ is such that

$$\rho \mapsto \rho p'(\rho) \text{ is nondecreasing on } \mathbb{R}^+. \tag{1.2}$$

A common example is, for instance, $p(\varrho) = \varrho^\gamma$ for some $\gamma > 1$.

- The viscous stress tensor $\tau[u]$ is given by

$$\tau[u] = 2\mu \mathrm{D}(u) + \lambda\,\mathrm{div}_x\, u\, \mathrm{I}_d$$

for some constants $\mu > 0$ and $\lambda \in \mathbb{R}$, where $\mathrm{D}(u)$ stands for the deformation tensor defined as $\mathrm{D}(u) = (\nabla_x u + (\nabla_x u)^\intercal)/2$. In this memoir (see the possible generalization in Section 1.3), we choose the constants $\mu$ and $\lambda$ so that

$$\mathrm{div}_x(\tau[u]) = \Delta_x u + \nabla_x\,\mathrm{div}_x\, u,$$

but we emphasize that this choice has been made solely for simplicity, and that no special algebraic property arises in this case.

- The force $\Gamma = \Gamma(t,x,v) \in \mathbb{R}^3$ acting on the cloud of particles is defined by

$$m_p \Gamma(t,x,v) = D r_p(u(t,x) - v) - m_p \nabla_x[p(\varrho)](t,x). \tag{1.3}$$

The first term $u(t,x) - v$, referred to as the drag force exerted by the fluid on the particles, is common to all fluid-kinetic couplings. Here, it is linear in the relative velocity between fluid and particles, and creates friction. Here, the drag coefficient $Dr_p$ (with $D > 0$) is taken proportional to the radius $r_p$ and is reminiscent of the Stokes law. The retroaction of this term in the momentum equation for the fluid is known as the *Brinkman force* and can be expressed as

$$-Dr_p \int_{\mathbb{R}^3} (u - v) f\,\mathrm{d}v.$$

The second term in $\Gamma$, which is a pressure gradient from the fluid, is a specific feature of thick spray models (see also [64, 124]) where the particles occupy a significant volume fraction of the two-phase mixture. Here, the constant density of the droplets has been taken to be equal to one. The presence of this term is consistent with the fact that the system (1.1) (or some of its variants) is formally linked to bifluid equations [61], where there is a common pressure gradient to both phases (see the overview below in Section 1.4). Note that the feedback of this term in the source term of the fluid momentum equation is

$$-m_p \int_{\mathbb{R}^3} f(-\nabla_x[p(\varrho)])\,\mathrm{d}v = (1-\alpha)\nabla_x[p(\varrho)].$$

**Remark 1.1.** After a suitable rescaling in the equations, it is actually possible to rewrite the force $\Gamma$ from (1.3) as

$$\Gamma = \frac{1}{\epsilon^2}(u-v) - \nabla_x[p(\varrho)],$$

together with

$$\alpha = 1 - \epsilon^3 \int_{\mathbb{R}^3} f\,\mathrm{d}v,$$

where $\epsilon > 0$ is proportional to the radius $r_p$ of a typical particle. Formally proceeding to the limit $\epsilon \to 0$, which corresponds to a small droplet size regime, we observe that $\alpha \to 1$ and that the drag term $u - v$ is dominant compared to the pressure gradient. This formally connects the *thick spray* equations to the *thin spray* equations (see the detailed overview made in Section 1.4). However, this formal link has not been made rigorous so far.

In what follows, we consider a normalization of all physical constants. Introducing the kinetic moments (of order 0 and 1) of the distribution function $f$ as

$$\rho_f(t,x) := \int_{\mathbb{R}^d} f(t,x,v)\,\mathrm{d}v \quad \text{(local density)},$$
$$j_f(t,x) := \int_{\mathbb{R}^d} f(t,x,v)v\,\mathrm{d}v \quad \text{(local current)},$$

so that $\alpha(t,x) = 1 - \rho_f(t,x)$, we are thus led to the following system:

$$\text{(TS)}\qquad \begin{cases} \partial_t f + v\cdot\nabla_x f + \mathrm{div}_v\big[f(u-v) - f\nabla_x p(\varrho)\big] = 0, \\ \partial_t((1-\rho_f)\varrho) + \mathrm{div}_x\big[(1-\rho_f)\varrho u\big] = 0, \\ \partial_t((1-\rho_f)\varrho u) + \mathrm{div}_x\big[(1-\rho_f)\varrho u\otimes u\big] + (1-\rho_f)\nabla_x p(\varrho) \\ \qquad\qquad = \Delta_x u + \nabla_x \mathrm{div}_x u + j_f - \rho_f u. \end{cases}$$

The unknowns of the problem are

$$f = f(t,x,v) \in \mathbb{R}^+, \quad \varrho = \varrho(t,x) \in \mathbb{R}^+, \quad u = u(t,x) \in \mathbb{R}^d,$$

and they are coupled through the drag term $u - v$, the pressure gradient $\nabla_x p(\varrho)$ and the volume fraction of the fluid $\alpha = 1 - \rho_f$. We finally prescribe initial conditions

$$f^{\mathrm{in}} = f^{\mathrm{in}}(x, v) \in \mathbb{R}^+, \quad \varrho^{\mathrm{in}} = \varrho^{\mathrm{in}}(x) \in \mathbb{R}^+, \quad u^{\mathrm{in}} = u^{\mathrm{in}}(x) \in \mathbb{R}^d.$$

We normalize the torus $\mathbb{T}^d = \mathbb{R}^d/(2\pi\mathbb{Z})^d$ endowed with the normalized Lebesgue measure, so that $\mathrm{Leb}(\mathbb{T}^d) = 1$.

In this work, we investigate the *local well-posedness* of the thick spray equations (TS). The main difficulty comes from the fact that rough energy estimates on the transport and kinetic part of (TS) seem to result in a *loss of several derivatives*. As a matter of fact, suppose that we have a smooth solution $(f, \varrho, u)$ with compact support to the system (TS). Since it is a coupling between a parabolic type equation for $u$ and two transport equations for $f$ and $\varrho$, the following observations can be made:

- a standard energy estimate for transport equations shows that a control of $k$ derivatives of $\varrho$ seems to require the control of $k + 1$ derivatives of $f$. This is due to the coupling with the volume fraction $1 - \rho_f$ in the mass conservation equation;
- a standard energy estimate for transport(-kinetic) equations shows that a control of $k$ derivatives of $f$ seems to require the control of $k + 1$ derivatives of $\varrho$. This comes from the pressure gradient in the force field of the Vlasov equation.

As a consequence, we obtain a control of $k$ derivatives of $(f, \varrho)$ by $k + 1$ derivatives of $(f, \varrho)$ for any $k \in \mathbb{N}$, and so on. Rough estimates of transport type therefore seem to involve a formal loss of one derivative on the unknown $(f, \varrho)$, which *makes the coupling singular*. Hence, standard techniques cannot be applied to obtain a solution to the system (apart from working with functions with analytic regularity). In [14], Baranger and Desvillettes conjectured anyway that the system is well posed in Sobolev spaces.

In this work, we partly confirm this conjecture by showing that (TS) is locally well posed in Sobolev spaces, when the initial data satisfy a *stability condition* of Penrose type. However, when this stability condition is violated, the system is actually ill posed in the sense of Hadamard (see [12]), which means, loosely speaking, that it indeed displays losses of derivatives in this case.

The rest of the introduction is structured as follows. In Section 1.2, we introduce the Penrose stability condition for thick sprays and state our main result on local well-posedness for (TS). In Section 1.3, we describe several generalizations of the thick spray equations (TS), taking into account possible density-dependent drag or collisions in the kinetic equation, as well as non-barotropic Navier–Stokes equations. These variants will be treated in Sections 7.1, 7.2 and 7.3. Section 1.4 is a general overview on fluid-kinetic systems, as well as on singular Vlasov equations. Finally, Section 1.5 provides a detailed outline of our method of proof.

## 1.2 Assumptions and main result

For $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}^d$, we write $|\alpha| = \sum_{i=1}^d \alpha_i$ and $\partial_x^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_d}^{\alpha_d}$. In this work, we will denote by $\mathrm{H}^k$ (or $\mathrm{H}^k(\mathbb{T}^d)$) the standard $\mathrm{L}^2$ Sobolev spaces for functions depending on $x \in \mathbb{T}^d$. When it is necessary, we will denote by $\mathrm{H}^k_{x,v}$ the same spaces for functions depending on $(x,v) \in \mathbb{T}^d \times \mathbb{R}^d$. To ease readability, we shall sometimes abbreviate $\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{T}^d))$ and $\mathrm{L}^2(0,T;\mathrm{H}^k(\mathbb{T}^d))$ as $\mathrm{L}^2_T\mathrm{L}^2$ and $\mathrm{L}^2_T\mathrm{H}^k$, respectively. We will also use *weighted* $\mathrm{L}^2_{x,v}$ Sobolev spaces. They are defined as follows:

**Definition 1.2.** For $k \in \mathbb{N}$, $r \geq 0$ and $\mathrm{f} : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}^+$, we define the *weighted (in velocity) Sobolev norms* as

$$\|\mathrm{f}\|_{\mathcal{H}^k_r} := \left( \sum_{|\alpha|+|\beta| \leq k} \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} \langle v \rangle^{2r} |\partial_x^\alpha \partial_v^\beta \mathrm{f}(x,v)|^2 \,\mathrm{d}x\,\mathrm{d}v \right)^{\frac{1}{2}},$$

where $\langle v \rangle = (1+|v|^2)^{\frac{1}{2}}$.

We now introduce the following Penrose function, which will be used to state the relevant corresponding Penrose stability condition.

**Definition 1.3.** For any distribution function $\mathrm{f}(x,v)$ and density $\rho(x)$, we define the *Penrose function*

$$\mathcal{P}_{\mathrm{f},\rho}(x,\gamma,\tau,k) := \frac{p'(\rho(x))\rho(x)}{1-\rho_{\mathrm{f}}(x)} \int_0^{+\infty} e^{-(\gamma+i\tau)s} \frac{ik}{1+|k|^2} \cdot (\mathcal{F}_v \nabla_v \mathrm{f})(x,ks)\,\mathrm{d}s \quad (1.4)$$

for $(x,\gamma,\tau,k) \in \mathbb{T}^d \times (0,+\infty) \times \mathbb{R} \in \mathbb{R}^d \backslash \{0\}$.

**Definition 1.4.** We say that the couple $(\mathrm{f}(x,v), \rho(x))$ satisfies the *c-Penrose stability condition* (for the thick spray equations (TS)) if there exists $c > 0$ such that

$$\text{(P)} \qquad \forall x \in \mathbb{T}^d, \quad \inf_{(\gamma,\tau,k) \in (0,+\infty)\times\mathbb{R}\times\mathbb{R}^d\backslash\{0\}} |1 - \mathcal{P}_{\mathrm{f},\rho}(x,\gamma,\tau,k)| > c.$$

When needed, we shall denote this condition by $(\mathrm{P})_c$.

As we shall explain later on, such a condition stems from the study of Vlasov equations in plasma physics. In the context of the Vlasov–Benney equation, a similar condition was the key to obtain local well-posedness in Sobolev spaces (and for its derivation in the quasineutral limit), see [90]. We refer to Section 1.5 for more details.

**Remark 1.5** (Sufficient conditions ensuring (P)). Let us describe a few classes of profiles $(\mathrm{f}(x,v), \rho(x))$ which satisfy the Penrose stability condition (P). Several criteria on the shape of $\mathrm{f}(x,\cdot)$ for every $x \in \mathbb{T}^d$ indeed provide sufficient conditions for (P) to hold (see for instance [121]). We shall assume that $\mathrm{f}(x,\cdot)$ is at least integrable and that the nonnegative prefactor in front of the integral in $\mathcal{P}_{\mathrm{f},\rho}$ is bounded from above on $\mathbb{T}^d$.

- First, any sufficiently small smooth profile f satisfies (P).
- When $d = 1$, the one-bump profiles in velocity, that is to say profiles such that, for all $x \in \mathbb{T}$, the function $v \mapsto \mathrm{f}(x, v)$ is increasing then decreasing, satisfy the Penrose condition.
- In any dimension $d \geq 1$, any profile such that, for all $x \in \mathbb{T}^d$, $\mathrm{f}(x, \cdot)$ is a radial nonincreasing function in velocity is Penrose stable. In particular, it includes the case of (local smooth) Maxwellians in velocity. If $d \geq 3$, any profile such that, for all $x \in \mathbb{T}^d$, $\mathrm{f}(x, \cdot)$ is a radial and strictly positive function in velocity satisfies the condition.
- More generally, the following criterion on the marginals of f has been devised in [121] and ensures the Penrose stability: for all $x \in \mathbb{T}^d$,

$$\forall (\omega_0, k) \in \mathbb{R} \times \mathbb{R}^d \backslash \{0\},$$
$$\frac{p'(\rho(x))\rho(x)}{(1 + |k|^2)(1 - \rho_{\mathrm{f}}(x))} \,\mathrm{p.v.} \int_{\mathbb{R}} \partial_y \mathrm{f}_{\frac{k}{|k|}}(x, y) \frac{1}{y + \omega_0} \,\mathrm{d}y < 1,$$

  where p.v. stands for the principal value on $\mathbb{R}$ and

$$\forall r \in \mathbb{R}, \quad \mathrm{f}_{\frac{k}{|k|}}(r) := \int_{\frac{k}{|k|}^{\perp}} \mathrm{f}\left( r\frac{k}{|k|} + w \right) \mathrm{d}w.$$

- Finally, any sufficiently small smooth perturbation of a Penrose stable profile will still satisfy (P). In particular, a slightly perturbed one-bump profile $\mathrm{f}(x, \cdot)$ remains Penrose stable.

Our *main result* reads as follows.

**Theorem 1.6.** *There exist $m_0 > 0$ and $r_0 > 0$, depending only on the dimension, such that the following holds for all $m \geq m_0$ and $r \geq r_0$. Let*

$$f^{\mathrm{in}} \in \mathcal{H}^m_r, \quad \varrho^{\mathrm{in}} \in \mathrm{H}^{m+1}, \quad u^{\mathrm{in}} \in \mathrm{H}^m$$

*such that $(f^{\mathrm{in}}, \varrho^{\mathrm{in}})$ satisfies the $c$-Penrose stability condition $(\mathrm{P})_c$ for some $c > 0$ and*

$$0 \leq f^{\mathrm{in}}, \quad \rho_{f^{\mathrm{in}}} < \Theta < 1, \quad 0 < \mu \leq \varrho^{\mathrm{in}}, \quad 0 < \underline{\theta} \leq (1 - \rho_{f^{\mathrm{in}}})\varrho^{\mathrm{in}} \leq \overline{\theta}$$

*for some constants $\Theta, \mu, \underline{\theta}, \overline{\theta}$. Then there exist $T > 0$ and a solution $(f, \varrho, u)$ to (TS) with initial condition $(f^{\mathrm{in}}, \varrho^{\mathrm{in}}, u^{\mathrm{in}})$ such that*

$$f \in \mathcal{C}([0, T]; \mathcal{H}^{m-1}_r), \quad \varrho \in \mathrm{L}^2(0, T; \mathrm{H}^m), \quad u \in \mathcal{C}([0, T]; \mathrm{H}^m) \cap \mathrm{L}^2(0, T; \mathrm{H}^{m+1}),$$

*and with $(f(t), \varrho(t))$ satisfying the $\frac{c}{2}$-Penrose stability condition $(\mathrm{P})_{c/2}$ for all $t \in [0, T]$. In addition, this solution is unique in this regularity class.*

In short, our result yields the local well-posedness for the thick spray equations, in the class of Penrose stable initial data. On the other hand, as already mentioned earlier, outside of this class, the system is ill posed in the sense of Hadamard. We refer to [12], in the spirit of [11, 87] for singular Vlasov equations (see also the discussion in Section 1.4).

**Remark 1.7.** The uniqueness part in the previous statement must be understood as follows: if $(f_1, \varrho_1, u_1)$ and $(f_2, \varrho_2, u_2)$ are two solutions to (TS) on $[0, T]$ with the previous regularity and with the same initial condition $(f^{\mathrm{in}}, \varrho^{\mathrm{in}}, u^{\mathrm{in}})$ (satisfying (P)), then $(f_1, \varrho_1, u_1) = (f_2, \varrho_2, u_2)$ if $t \mapsto (f_1(t), \varrho_1(t))$ satisfies the Penrose stability condition (P) on $[0, T]$.

**Remark 1.8.** Let us point out the shift of one derivative in the regularity between $f$ and $\varrho$, which is reminiscent of the formal loss of derivative that was evoked earlier.

**Remark 1.9.** In [39], the authors consider the linearization of (TS) around a radially nonincreasing and homogeneous profile (for the kinetic part). This can be seen as a particular case of the Penrose stability condition (P). They obtain a stability estimate in $\mathrm{L}^2$ around the solution generated by such particular data. However, it is not sufficient to provide a well-posedness result for the full nonlinear equations. As a matter of fact, since the equations are quasilinear, one would need to prove the analog of such stability estimates for all functions in a neighborhood of the aforementioned solution.

**Remark 1.10.** (1) It appears to be more natural (see, in particular, the proof of uniqueness in Section 6.3) to consider the optimal Penrose function

$$\mathscr{P}_{\mathrm{f},\rho}(x,\gamma,\tau,k) := \frac{p'(\rho(x))\rho(x)}{1-\rho_f(x)} \int_0^{+\infty} e^{-(\gamma+i\tau)s} ik \cdot (\mathcal{F}_v \nabla_v \mathrm{f})(x, ks)\,\mathrm{d}s,$$

instead of $\mathcal{P}_{\mathrm{f},\rho}$, as well as the related stability condition

$$\text{(Opt-P)} \qquad \forall x \in \mathbb{T}^d, \quad \inf_{(\gamma,\tau,k)\in(0,+\infty)\times\mathbb{R}\times\mathbb{R}^d\setminus\{0\}} \left|1 - \mathscr{P}_{\mathrm{f},\rho}(x,\gamma,\tau,k)\right| > c.$$

We refer to (Opt-P) as the *optimal Penrose stability condition*. The condition (Opt-P) is weaker than (P). Indeed, using a homogeneity argument combined with a continuity argument, it is possible to prove that if (P) holds for some $c > 0$ for $(\mathrm{f}, \rho)$, then (Opt-P) holds as well.[1] However, our strategy in this work will be based on a regularization of the force field in the Vlasov equation, and requires the stability condition (P). It is likely that, using the techniques of [41], one could assume (Opt-P) instead of (P) on

[1] We refer to the uniqueness part of the proof in Section 6.3 for more details.

$(f^{\text{in}}, \varrho^{\text{in}})$ to prove the well-posedness of thick spray equations. However, this would require substantial work.

(2) The factor $\frac{1}{1+|k|^2}$ in the Penrose function $\mathcal{P}_{\mathrm{f},\rho}$ could appear as arbitrary. It is actually related to the explicit regularization procedure of the force field that will be made clearer later on. By a homogeneity argument, it is however possible to prove that the condition (P) is equivalent to

$$\forall x \in \mathbb{T}^d, \qquad \inf_{(\gamma,\tau,k,\lambda)\in S^+\times(0,1]} \left|1 - \lambda \mathcal{P}_{\mathrm{f},\rho}(x,\gamma,\tau,k)\right| > c, \tag{1.5}$$

where

$$S^+ := \left\{(\gamma,\tau,k) \in (0,+\infty)\times\mathbb{R}\times\mathbb{R}^d\backslash\{0\} \mid \gamma^2+\tau^2+k^2 = 1\right\}.$$

This implies, in particular, that the factor $\frac{1}{1+|k|^2}$ could for instance be replaced by $\frac{1}{(1+|k|^2)^\alpha}$ for any $\alpha > 0$ in (1.4). We refer to Remark 2.12 in Section 2.2 for the use of such reformulation of (P) in the regularization procedure.

**Remark 1.11.** In the Euler case for the fluid, that is, for the same system on $(f, \varrho, u)$ but *without* the term $-\Delta_x u - \nabla_x \operatorname{div}_x u$ in the equation for $u$, the question of well-posedness remains open.

## 1.3 Generalization to several variants

We are also interested in more complex versions of the system (TS) that take into account more physical effects. In this section, several such variants are presented. The main strategy of proof adopted in this work for (TS) will be robust enough to handle such models: we will show how to obtain their local well-posedness in Sections 7.1, 7.2 and 7.3.

### 1.3.1 Non-barotropic Navier–Stokes equations

If one wants to get rid of the barotropic-type assumption on the fluid, one has to consider its internal energy $e = e(t,x) \in \mathbb{R}^+$ as an additional unknown. This leads to the following system of equations:

$$\begin{cases} \partial_t f + v\cdot\nabla_x f + \operatorname{div}_v\left[f(u-v) - f\nabla_x p(\varrho,e)\right] = 0, \\ \partial_t(\alpha\varrho) + \operatorname{div}_x(\alpha\varrho u) = 0, \\ \partial_t(\alpha\varrho u) + \operatorname{div}_x(\alpha\varrho u\otimes u) + \alpha\nabla_x p(\varrho,e) - \Delta_x u - \nabla_x \operatorname{div}_x u = j_f - \rho_f u, \\ \partial_t(\alpha\varrho e) + \operatorname{div}_x(\alpha\varrho e u) + p(\varrho,e)(\partial_t\alpha + \operatorname{div}_x(\alpha u)) = \displaystyle\int_{\mathbb{R}^d} |u-v|^2 f \,\mathrm{d}v, \\ \alpha = 1 - \rho_f. \end{cases} \tag{1.6}$$

Here, $p : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ is a given $\mathcal{C}(\mathbb{R}^+ \times \mathbb{R}^+) \cap \mathcal{C}^\infty(\mathbb{R}^+ \backslash \{0\} \times \mathbb{R}^+ \backslash \{0\})$ function such that the pressure term is a function $p = p(\varrho, e)$ depending on the fluid density and the internal energy. For instance, a relation of the type $p(\varrho, e) = b\varrho e$ (for some $b > 0$) is a perfect gas pressure law.

The apparent loss of derivative is still present in such model and occurs between $f$ and $e$. In Section 7.1, we shall also justify how to build local-in-time solutions for (1.6) thanks to an adapted Penrose stability condition.

### 1.3.2 Kinetic equation with collisions

In the thick spray regime, it is physically relevant to take into account collisions between droplets. A collision operator is therefore sometimes added in the kinetic equation, which turns into a Vlasov–Boltzmann type:

$$\partial_t f + v \cdot \nabla_x f + \mathrm{div}_v \big[ f(u-v) - f \nabla_x p(\varrho) \big] = \mathcal{Q}(f, f).$$

The quadratic operator $\mathcal{Q}(f, f) = \mathcal{Q}_\lambda(f, f)$ stands for some Boltzmann collision operator for (in)elastic hard-spheres. Here $\lambda \in (0, 1]$ is given and is called the *restitution coefficient*, which is involved in the microscopic laws defining collisions between particles. We refer to Section 7.2 for some details about the precise definition of the collision operator $\mathcal{Q}_\lambda$ and some of the main basic features of inelastic collisions. Let us mention that the case $\lambda = 1$ corresponds to standard perfectly elastic collisions and that the inelastic case $\lambda \in (0, 1)$ leads to a loss of kinetic energy along collisions (while mass and momentum are always conserved).

We will also explain how to include a collision operator in the kinetic equation for $f$ and still obtain an analog of Theorem 1.6.

### 1.3.3 Density-dependent drag force

In many applications, the force $\Gamma = \Gamma(t, x, v) \in \mathbb{R}^d$ acting on the particle should actually present a density-dependent drag force, for instance of the form

$$\Gamma(t, x, v) = \varrho(t, x)(u(t, x) - v) - \nabla_x [p(\varrho)](t, x).$$

Compared to that of the system (TS), this force displays an additional nonlinearity.[2] The Brinkman force in the Navier–Stokes equations also becomes

$$-\int_{\mathbb{R}^d} \varrho(u - v) f \, \mathrm{d}v = \varrho(j_f - \rho_f u).$$

[2] A more physical model should deal with a nonlinear drag of the form $\mathrm{C}[\varrho, |u - v|](u - v)$, which is for the moment out of the scope of a rigorous mathematical analysis. However, our approach can for instance allow the treatment of a drag of the form $\varrho[(u - v) + \gamma(u - v)]$ where $\gamma \in \mathcal{C}^\infty(\mathbb{R}^d; \mathbb{R}^d)$ is such that $\gamma(0) = 0$ and $\gamma'(0) = 0$.

Because of the modified term $\varrho(t,x)v \cdot \nabla_v f$ in the kinetic equation, this induces a potential growth in velocity which could become out of control. In Section 7.3, we will deal with the case of such a density-dependent drag term, up to the additional assumption that the initial data $f^{\mathrm{in}}$ has a compact support in velocity.

### 1.3.4 Density-dependent viscosities

It is also possible to consider more general viscosity coefficients in the Navier–Stokes equations of (TS), that is, replacing the differential operator $-\Delta_x u - \nabla_x \operatorname{div}_x u$ in the equation for $u$ by

$$-\operatorname{div}_x\big(2\mu[\varrho]\mathrm{D}(u) + \lambda[\varrho]\operatorname{div}_x u\,\mathrm{I}_d\big),$$

for smooth nonnegative coefficients $\mu, \lambda : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\mu(0) = \lambda(0) = 0$. For the sake of simplicity, we restrict ourselves in this work to the case $\mu = 1$ and $\lambda = 0$. We claim however that our analysis applies *mutatis mutandis* to this more general situation. As a matter of fact, we will consider local-in-time strong solutions for which $\varrho$ is non-vanishing.

## 1.4 Overview on fluid-kinetic systems and related models

Let us provide an overview on couplings between fluid and kinetic equations, both from the modeling and the theoretical points of view. In particular, we want to highlight the differences between the main regimes for the description of sprays (namely, *thick* versus *thin*). We also review some existing works on singular Vlasov equations coming from plasma physics, our strategy for thick sprays being inspired by the study of such systems (see Section 1.5).

### 1.4.1 Fluid-kinetic description

We can trace back the introduction of fluid-kinetic couplings for the description of sprays involving a large number of particles to [124, 137]. We also refer to [59] for a general overview on the description of multiphase flows, as well as to [129]. Note that, compared to an Eulerian-Eulerian description for both gas and droplets (where both phases are described at the macroscopic level, with $(t,x)$ variables), a fluid-kinetic point of view seems to be well-suited for polydispersed flows (i.e., when the size of the droplets can vary). Indeed, no average is taken to compute the drag force, for instance. Of course, many other physical effects, such as coalescence, breakup, vaporization or chemical reactions, can be included in the models (see e.g. [13, 108]).

### 1.4.2 Thick spray case

Models for *thick sprays* have been explicitly introduced and derived at the formal level in [64] and in [124, Chapter 2]. In particular, the pressure gradient acting on the dispersed phase is already present in [64], while [124, Chapter 2] also considers the additional contribution of collisions. The use of such complex models was subsequently pursued by O'Rourke's team at the Los Alamos National Laboratory to develop the Kiva code [4, 125, 126]. From the numerical and modeling point of view, let us also refer to the works [20, 26].

As explained above, the coupling between both phases makes the rigorous study of thick spray equations reputedly challenging. The mathematical study of such type of models is still in its infancy and only a few formal results are available.

In [61], the authors consider a formal hydrodynamic limit starting from a thick spray system of the type (1.6) (without fluid dissipation and with an additional energy variable for $f$) with an *inelastic collision operator*. The limit of Knudsen number tending to 0 allows one to derive (at least formally) a two-fluid coupled system. The latter turns out to be a standard model of multiphase flow theory where the volume fraction is now an unknown (see the book [103]). It formally connects thick spray models to multifluid systems, where the presence of a common pressure term is standard. This somehow *a posteriori* explains the additional pressure gradient in the force field acting in the kinetic equation. A standard feature of this bifluid limiting system seems to be a lack of hyperbolicity [92, 107, 123], so that its behavior is *a priori* highly unstable. Note that preliminary computations performed in [128] tend to indicate that adding some viscous term along some directions makes this type of system better behaved.

A new understanding of thick sprays has been obtained in [39], where the linear stability for (TS) and (1.6) is investigated. More precisely, the $\mathrm{L}^2$ linear stability around a family of particular space-homogeneous profiles (for the kinetic phase) is obtained thanks to a suitable Lyapunov functional. The profiles in velocity are required to satisfy a property of monotonicity, this condition being a special example of the Penrose stability condition (P) that we shall impose on the initial condition (recall Remark 1.5 above).

Very recently, [69] has proposed a new averaged version of thick spray models, where the pressure gradient $-\nabla_x p(\varrho)$ in the kinetic equation and the volume fraction $\alpha$ are regularized by including an extra convolution operator. Local existence in Sobolev spaces for this new version of the original thick spray model is obtained in the Euler case for the fluid, using tools from symmetrizable hyperbolic systems (see [14, 115]).

### 1.4.3 Barotropic compressible Navier–Stokes equations

When $f = 0$ (and thus $\alpha = 1$), the system (TS) reduces to the standard compressible Navier–Stokes equations for $(\varrho, u)$, in the barotropic-type regime. These equations have given rise to an abundant literature for more than half a century. Global weak solutions of finite energy have been built for the first time in [109] for constant viscosity coefficients, a result extended in [68] for more general pressure power laws. Another notion of weak solutions was also considered in Hoff [100]. For more recent results allowing one to include degenerate viscosities and more general pressure laws, see e.g. [35–38, 118, 133].

In the framework of strong solutions (local in time or global with assumption on the data), we can for instance refer to [122, 131] for classical ones, to [132] for mild solutions, to [116, 117] for solutions with high Sobolev regularity near equilibria, and to [101] for the fine description of the time asymptotics of the system. Let us also mention the more recent works [43, 50–52, 54, 56, 57] for the study of the system in critical spaces.

### 1.4.4 Thin spray case: The Vlasov–Navier–Stokes system

Unlike the *thick spray* regime corresponding to (TS), there exists a rich literature on the so-called *thin spray* models. This corresponds to a regime where the particle volume fraction is small compared to that of the surrounding fluid; the quantity $\alpha$ is directly set to 1 and does not appear in the system (see Remark 1.1). The main term of the coupling which is retained is the drag force (that is, $\Gamma(t, x, v) = u(t, x) - v$), and its feedback in the fluid equation.

An important fluid-kinetic model in this class is the so-called Vlasov–Navier–Stokes system, which describes a monodisperse phase of small particles flowing in an ambient, incompressible, homogeneous, viscous fluid. It takes the form

$$\text{(VNS)} \qquad \begin{cases} \partial_t f + v \cdot \nabla_x f + \operatorname{div}_v[f(u - v)] = 0, \\ \partial_t u + (u \cdot \nabla_x) u - \Delta_x u + \nabla_x p = j_f - \rho_f u, \\ \operatorname{div}_x u = 0. \end{cases}$$

This system has been for instance shown to provide a good description of medical aerosols in the upper part of the lung (see e.g. [28, 30]).

From the mathematical point of view, many directions of research have been explored about (VNS) (and its variants) over the past twenty years. The Cauchy theory, addressing the existence of global weak solutions for (VNS) on a large class of domains in dimension 2 or 3, is by now well developed (see e.g. [5, 25, 77]), and also allows for more complex physics in the model (see [29, 31]). It mainly consists in obtaining a Leray weak solution for $u$ and a renormalized weak solution (in the sense of DiPerna and Lions [63]) for $f$, using a remarkable energy-dissipation identity that is satisfied by solutions to the system. In dimension 2, the uniqueness of such solutions has been shown in [85].

More recently, several asymptotic behaviors of (VNS) have attracted considerable attention. The question of the large-time dynamics for (VNS) has seen significant advances over the past few years. Roughly speaking, it is expected that the cloud of particles aligns its velocity with that of the fluid, that is,

$$u(t) \xrightarrow[t\to+\infty]{} v^\infty, \quad f(t) \xrightarrow[t\to+\infty]{} \rho^\infty \otimes \delta_{v=v^\infty}$$

for some asymptotic velocity $v^\infty \in \mathbb{R}^3$ and profile $\rho^\infty(t,x)$.

The first complete answer justifying such singular asymptotics has been obtained in [86] for Fujita–Kato-type solutions, in the 3D torus case. In the whole space $\mathbb{R}^3$, this question is studied in [55, 80], while the case of a 3D bounded domain (with absorption boundary conditions for $f$) is investigated in [67]. We also refer to [65] for the half-space case, where the additional effect of a gravity force on the particles (combined with absorption at the boundary) leads to decay of the solution to 0.

Another asymptotic regime is the so-called *hydrodynamic limit* starting from (VNS), related to high-friction regimes. Considering some suitable scalings making a small parameter $\varepsilon$ appear in (VNS), one wants to obtain a hydrodynamic and effective system when $\varepsilon$ tends to 0. Here again, this issue is linked to a monokinetic behavior of the form $f_\varepsilon(t) \to \rho(t) \otimes \delta_{v=u(t)}$ and $u_\varepsilon(t) \to u(t)$, with $(\rho(t), u(t))$ satisfying transport-Navier–Stokes or inhomogeneous Navier–Stokes equations. We refer to [84] for the more complete and recent results on this question, and to [66, 96], where the gravity effect is taken into account, leading to macroscopic sedimentation couplings in the limit.

Let us finally mention the challenging open problem of the derivation of (VNS), starting from microscopic first principles. We refer to [42, 60, 93–95, 97] for a partial answer based on homogenization and the justification of the Brinkman force in the fluid equation, but without the complete dynamics of the particles. In the case of the Vlasov–Stokes system, let us mention the recent results obtained in [98, 99]. An alternative (but still formal) program has been proposed in [22, 23], starting from a system of coupled Boltzmann equations.

### 1.4.5 Several variants of (VNS)

The Vlasov–Navier–Stokes system can also be considered with inhomogeneous or compressible Navier–Stokes equations [45, 46, 49] and additional terms in kinetic equations [47, 48].

Note that the case of compressible Euler equations for the fluid, coupled to a kinetic equation, has also been investigated. We refer to [14] (for the thin spray case) and [115] (for the so-called moderately thick spray case when collisions between particles are not neglected), where local-in-time strong solutions are built, thanks to ideas coming from hyperbolic systems.

### 1.4.6 Singular Vlasov equations

As we shall explain later on (see Section 1.5 on our strategy of proof), we shall take our inspiration from a different problem coming from plasma physics, which is the so-called *quasineutral limit* problem. More specifically, let us look at the dynamics of ions described by the following Vlasov–Poisson system:

$$(\mathrm{VP}_\varepsilon) \qquad \begin{cases} \partial_t f_\varepsilon + v \cdot \nabla_x f_\varepsilon + E_\varepsilon \cdot \nabla_v f_\varepsilon = 0, \\ E_\varepsilon = -\nabla_x U_\varepsilon, \\ (\mathrm{Id} - \varepsilon^2 \Delta_x) U_\varepsilon = \displaystyle\int_{\mathbb{R}^d} f_\varepsilon(t,x,v)\,\mathrm{d}v - 1, \end{cases}$$

when $\varepsilon \ll 1$, corresponding to a small Debye length regime for the plasma. The issue at stake is the validity (or invalidity) of the formal limit $\varepsilon \to 0$, leading to

$$(\mathrm{VB}) \qquad \begin{cases} \partial_t f + v \cdot \nabla_x f - \nabla_x \rho_f \cdot \nabla_v f = 0, \\ \rho_f(t,x) = \displaystyle\int_{\mathbb{R}^d} f(t,x,v)\,\mathrm{d}v. \end{cases}$$

This system was named as the *Vlasov–(Dirac)–Benney* system by Bardos in [15]. As directly seen on (VB), the force field in this Vlasov equation is one derivative less regular than the distribution function $f$ itself, thus displaying an apparent loss of derivative.

The question of the justification of the quasineutral limit (from ($\mathrm{VP}_\varepsilon$) to (VB)) and of the well-posedness of the limiting system (VB) has given rise to a wealth of literature for more than twenty years. Preliminary results have investigated the limit, up to some defect measures [34, 74], and have been followed by a full justification in the analytic regime [75] (see also [82, 83]). We also refer to [33, 78, 112] for the case of singular monokinetic data leading to fluid equations. However, the quasineutral limit does not hold in general because instabilities for Vlasov–Poisson can take over (see [81]).

In general, there also exist unstable homogeneous equilibria of (VB) around which the linearized equations have unbounded unstable spectrum (typically two-bump profile in velocity, leading to the so-called *two-stream* instability). Therefore (VB) may be ill posed in the sense of Hadamard [11, 18, 87] in Sobolev spaces, even with arbitrary losses of derivatives and arbitrary small time.

A local theory for (VB) thus requires additional assumptions on the initial data. A Cauchy–Kovalevskaya-type theorem can be applied [24, 105, 121] to show that there is local existence of analytic solutions for analytic initial data. In dimension

$d = 1$, Sobolev initial data with a one-bump profile in velocity (for all $x$) leads to local-in-time solutions, as shown in [16] (see also [17] for more properties).

In any dimension, the quasineutral limit and the well-posedness in Sobolev spaces of (VB) have been obtained in [90] under a Penrose stability condition on the initial data $f^{\text{in}}$. In this work, the same type of condition as (P) is assumed, replacing the function $\mathcal{P}_{\mathrm{f},\rho}$ by $\mathcal{P}_{\mathrm{f}}^{\mathrm{VP}}$ defined as

$$\mathcal{P}_{\mathrm{f}}^{\mathrm{VP}}(x, \gamma, \tau, k) = \int_0^{+\infty} e^{-(\gamma + i\tau)s} \frac{ik}{1+|k|^2} \cdot (\mathcal{F}_v \nabla_v \mathrm{f})(x, ks)\, \mathrm{d}s.$$

Up to some prefactor coming from coupling for sprays, our assumption (P) in Theorem 1.6 is closely related to that of [90], and this work is the main inspiration of our analysis. We also refer to the recent work [44], where the existence of local-in-time solutions for mildly singular Vlasov equations is shown (without assuming any stability condition).

Note that the rescaling $(t, x, v) \mapsto (t/\varepsilon, x/\varepsilon, v)$ in ($\mathrm{VP}_\varepsilon$) leads to the same equation with $\varepsilon = 1$, hence connecting the quasineutral limit to an issue of large-time dynamics. The Penrose stability condition (P) appearing on a homogeneous profile $\mathrm{f}(v)$ is actually a necessary condition for its long-time stability in the Vlasov–Poisson equation ($\mathrm{VP}_\varepsilon$) with $\varepsilon = 1$. This last issue is also linked to the Landau damping effect, which has been proved to hold in a small (Gevrey) neighborhood of such stable profile. In the torus, we refer to the breakthrough work [121], as well as to [19, 76].

## 1.5 Strategy of the proof

We conclude this introduction by presenting a detailed outline of the proof. This will allow at the same time to explain the structure of the memoir. In order to ease readability and highlight the main features of the analysis, we deliberately state our results without specifying the precise assumptions.

As explained above, and for the sake of clarity, we shall focus on (TS). This system indeed retains the main features and difficulties of this work. Our result and proof will be generalized to the more complete systems presented in Section 1.3 (see Sections 7.1, 7.2 and 7.3).

In the preliminary Chapter 2, we start by deriving several *a priori* energy estimates on the system (TS). We show in Proposition 2.3 that for all $t \in [0, T]$,

$$\|\varrho(t)\|_{\mathrm{H}^m} \leq \|\varrho^{\text{in}}\|_{\mathrm{H}^m} \Phi\big(T, \dots, \|u\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^{m+1})}, \|f\|_{\mathrm{L}^\infty(0,T;\mathcal{H}_r^{m+1})}\big), \tag{1.7}$$

where $\Phi$ is a continuous function which is increasing with respect to each of its arguments and '...' involves lower order terms. On the other hand, we have, for all

$t \in [0, T]$,

$$\|\rho_f(t)\|_{\mathrm{H}^m} + \|j_f(t)\|_{\mathrm{H}^m} \lesssim \|f(t)\|_{\mathcal{H}^m_r} \\ \leq \|f^{\mathrm{in}}\|^2_{\mathcal{H}^m_r} \Phi\big(T, \dots, \|u\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^m)}, \|\varrho\|_{\mathrm{L}^2(0,T;\mathrm{H}^{m+1})}\big). \tag{1.8}$$

The estimates (1.7) and (1.8) thus yield a loss of two derivatives for the fluid density $\varrho$. This formally prevents the use of standard techniques to obtain a (local-in-time) solution. The main goal of the analysis is to show that these losses are only apparent when the initial condition $(f^{\mathrm{in}}, \varrho^{\mathrm{in}})$ satisfies the Penrose stability condition (P).

To this end, we first consider the following regularization of the system (see also Remark 2.12), which includes a parameter $\varepsilon \in (0, 1)$ that is bound to go to 0:

$$\begin{cases} \partial_t f_\varepsilon + v \cdot \nabla_x f_\varepsilon + \mathrm{div}_v[f_\varepsilon E_\varepsilon - f_\varepsilon v] = 0, \\ \partial_t((1-\rho_{f_\varepsilon})\varrho_\varepsilon) + \mathrm{div}_x((1-\rho_{f_\varepsilon})\varrho_\varepsilon u_\varepsilon) = 0, \\ (1-\rho_{f_\varepsilon})\big(\varrho_\varepsilon[\partial_t u_\varepsilon + (u_\varepsilon \cdot \nabla_x) u_\varepsilon] + \nabla_x p(\varrho_\varepsilon)\big) \\ \qquad\qquad = \Delta_x u_\varepsilon + \nabla_x \mathrm{div}_x\, u_\varepsilon + j_{f_\varepsilon} - \rho_{f_\varepsilon} u_\varepsilon, \\ f_{\varepsilon|t=0} = f^{\mathrm{in}}, \quad \varrho_{\varepsilon|t=0} = \varrho^{\mathrm{in}}, \quad u_{\varepsilon|t=0} = u^{\mathrm{in}}, \end{cases} \tag{1.9}$$

where

$$\rho_{f_\varepsilon}(t,x) = \int_{\mathbb{R}^d} f_\varepsilon(t,x,v)\,\mathrm{d}v, \qquad j_{f_\varepsilon}(t,x) = \int_{\mathbb{R}^d} f_\varepsilon(t,x,v) v\,\mathrm{d}v,$$
$$E_\varepsilon = -p'(\varrho_\varepsilon)\nabla_x[\mathrm{J}_\varepsilon \varrho_\varepsilon] + u_\varepsilon, \qquad \mathrm{J}_\varepsilon = (\mathrm{Id} - \varepsilon^2 \Delta_x)^{-1}.$$

In the following, for the sake of readability, we systematically dismiss the subscripts $\varepsilon$ but keep in mind that all unknowns depend on $\varepsilon$. When $\varepsilon > 0$, the regularized system can be seen as a non-singular coupling between compressible Navier–Stokes and the Vlasov equation; as a result, classical energy methods allow us to build local-in-time solutions, away from vacuum (this is shown in Appendix B). However, the point is to obtain uniform in $\varepsilon$ estimates on some interval of time which has to be independent of $\varepsilon$. With this goal in mind, we set up a bootstrap argument that starts at the end of Section 2.2.

We introduce

$$\mathcal{N}_{m,r}(f, \varrho, u, T) \\ := \|f\|_{\mathrm{L}^\infty(0,T;\mathcal{H}^{m-1}_r)} + \|\varrho\|_{\mathrm{L}^2(0,T;\mathrm{H}^m)} + \|u\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^m) \cap \mathrm{L}^2(0,T;\mathrm{H}^{m+1})}$$

for $T > 0$, and we want to obtain a uniform (in $\varepsilon$) estimate for this quantity. This will pave the way for a compactness argument allowing us to pass to the limit in the previous regularized system when $\varepsilon \to 0$. Observe the shift of one derivative between the norm on $f$ and that on $\varrho$. By (1.8), a control on $\|\varrho\|_{\mathrm{L}^2(0,T;\mathrm{H}^m)}$ and $\|u\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^m)}$ implies a control on $\|f\|_{\mathrm{L}^\infty(0,T;\mathcal{H}^{m-1}_r)}$. Hence, the main challenge is to derive an estimate for $\|\varrho\|_{\mathrm{L}^2(0,T;\mathrm{H}^m)}$.

Our main observation is that, using the definition of $\alpha$ in the equation of conservation of mass, the fluid density satisfies a transport equation of the type

$$\partial_t \varrho + u \cdot \nabla_x \varrho + \frac{\varrho}{1-\rho_f} \operatorname{div}_x\big[j_f - \rho_f u\big] = \text{lower order terms}; \tag{1.10}$$

therefore $\varrho$ depends on $f$ only through its moments in velocity $\rho_f$ and $j_f$. The goal of Chapters 3 and 4 is thus to relate these two moments to the fluid density $\varrho$ itself.

To do so, we initiate in Section 2.3 the study of the Vlasov equation satisfied by $f$ with a Lagrangian point of view. We study the characteristic curves for the kinetic dynamics with friction

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}s}\mathrm{X}^{s;t}(x,v) = \mathrm{V}^{s;t}(x,v), \quad \mathrm{X}^{t;t}(x,v) = x, \\ \dfrac{\mathrm{d}}{\mathrm{d}s}\mathrm{V}^{s;t}(x,v) = -\mathrm{V}^{s;t}(x,v) + E(s, \mathrm{X}^{s;t}(x,v), \mathrm{V}^{s;t}(x,v)), \quad \mathrm{V}^{t;t}(x,v) = v, \end{cases}$$

stemming from the Vlasov equation in (1.9). The term $-v$ in the force field is responsible for the friction dynamics. To simplify its study, we want to straighten the total kinetic operator

$$\partial_t + v \cdot \nabla_x + \operatorname{div}_v((E - v)\cdot)$$

into

$$\partial_t + v \cdot \nabla_x - v \cdot \nabla_v \tag{1.11}$$

for short times. The operator in (1.11) corresponds to the free dynamics with friction.

More precisely, we prove in Lemma 2.26 that for $T$ small enough (independent of $\varepsilon$), $x \in \mathbb{T}^d$ and $s, t \in [0, T]$, there exists a diffeomorphism $\psi_{s,t}(x, \cdot) : \mathbb{R}^d \to \mathbb{R}^d$ satisfying, for all $v \in \mathbb{R}^d$,

$$\mathrm{X}^{s;t}(x, \psi_{s,t}(x,v)) = x + (1 - e^{t-s})v. \tag{1.12}$$

In addition, we provide several useful Sobolev estimates on $\psi$. We call this diffeomorphism the *straightening change of variable* in velocity.

The heart of the proof appears in the remaining chapters. In Chapter 3, we study some smoothing averaging operators that will be crucial for the subsequent analysis of Chapter 4. In short, the study of these operators will enable us to split all the quantities exhibiting a loss of derivative into a leading term and a good remainder which will be controlled.

Let us introduce the following kernel operator, already considered in [90]:

$$\mathrm{K}_G^{\text{free}}[H](t,x) = \int_0^t \int_{\mathbb{R}^d} [\nabla_x H](s, x - (t-s)v) \cdot G(t,s,x,v) \, \mathrm{d}v \, \mathrm{d}s.$$

Despite the apparent loss of derivative, it is proved in [90] that this operator is bounded in $\mathrm{L}^2_T \mathrm{L}^2_x$ as soon as the kernel $G$ is sufficiently smooth and decaying in velocity, a result related to the classical averaging lemmas [71]. We refer to the introduction of Chapter 3 for more references and explanations about this aspect. We shall provide here extensions of this result to the natural averaging operator for the dynamics with friction (associated with (1.11)), namely

$$\mathrm{K}_G^{\mathrm{fric}}[H](t,x) = \int_0^t \int_{\mathbb{R}^d} [\nabla_x H](s, x + (1-e^{t-s})v) \cdot G(t,s,x,v)\,\mathrm{d}v\,\mathrm{d}s.$$

We shall see in Proposition 3.4 that $\mathrm{K}_G^{\mathrm{fric}}$ is also bounded in $\mathrm{L}^2_T\mathrm{L}^2_x$ under similar smoothness and decay assumptions for the kernel $G$. It was also observed in [91] that when the kernel cancels out on the diagonal $s = t$, the operator $\mathrm{K}_G^{\mathrm{free}}$ becomes bounded from $\mathrm{L}^2_T\mathrm{L}^2_x$ to $\mathrm{L}^2_T\mathrm{H}^1_x$, i.e., we gain one extra derivative in $x$; the same holds as well for $\mathrm{K}_G^{\mathrm{fric}}$, see Proposition 3.5.

A key result we prove (see Proposition 3.7) is the fact that the *difference* between the two latter operators also allows us to gain a derivative in $x$, namely $\mathrm{K}_G^{\mathrm{free}} - \mathrm{K}_G^{\mathrm{fric}}$ is bounded from $\mathrm{L}^2_T\mathrm{L}^2_x$ to $\mathrm{L}^2_T\mathrm{H}^1_x$. Propositions 3.4, 3.5 and 3.7 will be used at multiple times in this work.

Chapter 4 is dedicated to the proper analysis of the kinetic moments $\rho_f$ and $j_f$. The main result provided in Proposition 4.1 is the fact that for all $|I| \leq m$ we can write

$$\begin{aligned}
&\partial_x^I \rho_f(t,x) \\
&\quad = p'(\varrho(t,x)) \int_0^t \int_{\mathbb{R}^d} \nabla_x[\mathrm{J}_\varepsilon \partial_x^I \varrho](s, x-(t-s)v) \cdot \nabla_v f(t,x,v)\,\mathrm{d}v\,\mathrm{d}s + R, \\
&\partial_x^I j_f(t,x) \\
&\quad = p'(\varrho(t,x)) \int_0^t \int_{\mathbb{R}^d} v \nabla_x[\mathrm{J}_\varepsilon \partial_x^I \varrho](s, x-(t-s)v) \cdot \nabla_v f(t,x,v)\,\mathrm{d}v\,\mathrm{d}s + R,
\end{aligned} \tag{1.13}$$

where $R$ stands for a well-controlled remainder in $\mathrm{L}^2_T\mathrm{H}^1_x$. Combining with the continuity results for the averaging operator $\mathrm{K}^{\mathrm{free}}_{p'(\varrho)\nabla_v f}$, this proves that the loss of derivative for $\rho_f$ and $j_f$ in (1.8) was only apparent.

To obtain these identities, the first step is to derive a good equation satisfied by $\partial_x^I f$; to this end, it is natural to apply the operator $\partial_x^I$ to the Vlasov equation. We readily obtain

$$\partial_t \partial_x^I f + v \cdot \nabla_x \partial_x^I f + \mathrm{div}_v(\partial_x^I f (E - v)) + \mathrm{div}_v([\partial_x^I, E] f) = 0,$$

and we observe that the commutator involves

- the main order term
$$\mathrm{div}_v(\partial_x^I(E) f) \tag{1.14}$$
that will account for the leading term in the identities (1.13),
- low order terms that can be controlled,

- but also terms of the form

$$\begin{aligned} &\text{(I)} \quad \partial_x E \cdot \partial_x^J \nabla_v f, \quad |J| = m-1,\\ &\text{(II)} \quad \partial_x^2 E \cdot \partial_x^J \nabla_v f, \quad |J| = m-2. \end{aligned}$$

The terms of type (I) clearly cannot be considered as remainders, since they involve $m$ derivatives of $f$, which we do not uniformly control. The terms of type (II) are not remainders either, since we expect to plug in the identities (1.13) in the equation for $\varrho$, and this involves an extra derivative in $x$, thus also resulting in terms with $m$ derivatives of $f$. To overcome this difficulty, we argue as in [90] and consider an augmented unknown $\mathcal{F} = (\partial_{x,v}^I f)_{|I|=m-1,m}$ which satisfies a system of the form

$$\partial_t \mathcal{F} + v \cdot \nabla_x \mathcal{F} - v \cdot \nabla_v \mathcal{F} + \mathrm{div}_v(E\mathcal{F}) + \mathcal{M}\mathcal{F} + \mathcal{L} = \mathcal{R},$$

where $\mathcal{M}$ is a bounded linear map, $\mathcal{L}$ stands for the terms like (1.14), and $\mathcal{R}$ is a well-controlled remainder. Note though that in [90], only the terms of type (I) are relevant and the augmented unknown only involves derivatives of order $m$.

Controlling the averages in velocity of the whole family $(\partial_{x,v}^I f)_{|I|=m-1,m}$ allows us to recover derivatives of $\rho_f$ and $j_f$ in $\mathrm{H}^m$. We finally rely on the Duhamel formula combined with an integration in velocity along the characteristics, on the straightening change of variable in velocity $\psi_{s,t}$ satisfying (1.12), and on the crucial gain of derivatives provided by the kernel operators $\mathrm{K}^{\mathrm{free}}_{\nabla_v f}$ and $\mathrm{K}^{\mathrm{fric}}_{\nabla_v f}$ to deduce (1.13).

We refer to this approach as a *semi-Lagrangian* one, in the sense that we first apply derivatives on the kinetic equation and then integrate along the characteristics to obtain equations bearing on moments.

Chapter 5 is then devoted to obtaining an estimate for $\|\varrho\|_{\mathrm{L}^2(0,T;\mathrm{H}^m)}$. Taking derivatives in the transport equation (1.10) for $\varrho$ and using (1.13), one can write an equation for $\partial_x^I \varrho$ for all $|I| = m$ under the form

$$\begin{aligned} &\partial_t \partial_x^I \varrho + u \cdot \nabla_x \partial_x^I \varrho + \frac{\varrho}{1-\rho_f} \mathrm{div}_x \big[\mathrm{K}^{\mathrm{free}}_{vp'(\varrho)\nabla_v f}(\mathrm{J}_\varepsilon \partial_x^I \varrho) - \mathrm{K}^{\mathrm{free}}_{p'(\varrho)\nabla_v f}(\mathrm{J}_\varepsilon \partial_x^I \varrho) u\big]\\ &\qquad = \text{lower order terms.} \end{aligned}$$

Based on this equation, and using some commutation properties relating the operators $\mathrm{div}_x$ and $\mathrm{K}^{\mathrm{free}}_{vp'(\varrho)\nabla_v f}$, it is then possible to prove (see Proposition 5.1) that for all $|I| = m$ the function $\partial_x^I \varrho$ satisfies

$$\begin{aligned} \left(\mathrm{Id} - \frac{\varrho}{1-\rho_f}\mathrm{K}^{\mathrm{free}}_G \circ \mathrm{J}_\varepsilon\right)\big[\partial_t \partial_x^I \varrho + u \cdot \nabla_x \partial_x^I \varrho\big] = \mathcal{R}, \qquad (1.15)\\ G(t,x,v) = p'(\varrho(t,x))\nabla_v f(t,x,v), \end{aligned}$$

where $\mathcal{R}$ is a well-controlled remainder. The equality (1.15) has to be seen as a structural *factorization* of the equation for $\partial_x^I \varrho$, between the operators

$$\mathrm{Id} - \frac{\varrho}{1-\rho_f}\mathrm{K}_G^{\mathrm{free}} \circ \mathrm{J}_\varepsilon$$

and

$$\partial_t + u \cdot \nabla_x.$$

This relation is fully based on the coupling with the kinetic part.

The main goal is then to derive some good $\mathrm{L}^2_T\mathrm{L}^2_x$ estimates on $\partial_x^I \varrho$. Again following [90], the idea is to relate $\frac{\varrho}{1-\rho_f}\mathrm{K}_G^{\mathrm{free}} \circ \mathrm{J}_\varepsilon$ to a pseudodifferential operator and use pseudodifferential calculus to derive a suitable estimate. This is where the Penrose stability condition steps in and plays a crucial role: it will allow us to obtain estimates without loss.

Compared to the analysis of [90], the extra derivative due to the transport operator in (1.15) forces us to consider time-dependent symbols; this requires an extension on the whole line $\mathbb{R}$ of all functions, ensuring in the process that the Penrose stability condition still holds globally, see Section 5.3. For any $\gamma > 0$, one has (see Lemma 5.14)

$$e^{-\gamma t}\mathrm{K}_G^{\mathrm{free}}\big[e^{\gamma\cdot}H\big](t,x) := \mathrm{Op}^\gamma(a_{f,\varrho})(H)(t,x) \quad \text{on } (0,T)\times\mathbb{T}^d,$$

with

$$a_{f,\varrho}(t,x,\gamma,\tau,k) := p'(\varrho(t,x))\int_0^{+\infty} e^{-(\gamma+i\tau)s} ik\cdot(\mathcal{F}_v\nabla_v f)(t,x,ks)\,\mathrm{d}s,$$

and thus (1.15) turns into the pseudodifferential equation

$$\left(\mathrm{Id} - \frac{\varrho}{1-\rho_f}\mathrm{Op}^\gamma(a_{f,\varrho})\circ \mathrm{J}_\varepsilon\right)\big[\partial_t\partial_x^I\varrho + u\cdot\nabla_x\partial_x^I\varrho\big] = \mathcal{R}. \tag{1.16}$$

Here, $\mathrm{Op}^\gamma$ refers to a pseudodifferential quantization on $\mathbb{R}\times\mathbb{T}^d$ and with parameter $\gamma > 0$ (see Appendix C for more details). By observing that

$$\frac{\varrho}{1-\rho_f}a_{f,\varrho} = \mathcal{P}_{f,\varrho},$$

where

$$\mathcal{P}_{f(t),\varrho(t)}(x,\gamma,\tau,k) := \frac{p'(\varrho(t,x))\varrho(t,x)}{1-\rho_f(t,x)}\int_0^{+\infty} e^{-(\gamma+i\tau)s}\frac{ik}{1+|k|^2}\cdot(\mathcal{F}_v\nabla_v f)(t,x,ks)\,\mathrm{d}s,$$

we discover that the Penrose stability condition

$$\forall t\in\mathbb{R},\qquad \inf_{(x,\gamma,\tau,k)}|1-\mathcal{P}_{f(t),\varrho(t)}(x,\gamma,\tau,k)| > 0$$

thus asserts the *ellipticity* of the symbol involved in the equation (1.16). Roughly speaking, the Penrose stability condition shows that the equation (1.15) can be seen as a factorization between an *elliptic* part and a *hyperbolic* part.

If we have an $\mathrm{L}^2_T\mathrm{L}^2_x$ bound on

$$H = \partial_t \partial_x^I \varrho + u \cdot \nabla_x \partial_x^I \varrho,$$

a standard hyperbolic energy estimate associated to the transport part $\partial_t + u \cdot \nabla_x$ directly leads to a suitable estimate on $\partial_x^I \varrho$ (see Corollary 5.22).

Obtaining an $\mathrm{L}^2_T\mathrm{L}^2_x$ control on $H$ as the solution to the previous pseudodifferential equation (1.16) is thus enough to conclude. To do so, as in [90], we rely on a semiclassical (with small parameter $\varepsilon$) pseudodifferential calculus (with large parameter $\gamma$) whose aim is to invert the equation for $H$ up to some small remainder.[3] The key is that one can consider the symbol $(1 - \mathcal{P}_{f,\varrho})^{-1}$. This yields an $\mathrm{L}^2_T\mathrm{L}^2_x$ estimate for $H$ in terms of the remainder $\mathcal{R}$ (see Corollary 5.20).

Chapter 6 is dedicated to the conclusion of the proof, gathering all the previous steps and estimates of the bootstrap analysis. We obtain the desired uniform estimate for the quantity $\mathcal{N}_{m,r}(f_\varepsilon, \varrho_\varepsilon, u_\varepsilon, T)$, which is valid for some time $T > 0$ independent of $\varepsilon$. The existence part of Theorem 1.6 is then easily deduced by a compactness argument on $[0, T]$. The uniqueness part requires an additional argument, in the same spirit as the strategy previously devised.

We provide in Chapter 7 several generalizations of our analysis to the more complex models introduced in Section 1.3. In Section 7.1, we show how one can easily adapt the strategy developed in this work to treat the case of a non-barotropic fluid with an additional equation for the fluid's internal energy.

In Section 7.2, we describe how one can include an inelastic collision operator of Boltzmann type in the kinetic equation (see (7.3)). Note that our method follows an idea used in [115], which allows us to overcome the loss of weight in velocity from the collision operator *thanks to* the friction term in the original Vlasov equation.

In Section 7.3, we consider the case of a density-dependent drag force, for which one can also prove a local well-posedness result, with the limitation that the initial data $f^{\mathrm{in}}$ has a compact support in velocity.

We refer to the precise statements of Sections 7.1, 7.2 and 7.3 for more details about the corresponding existence results.

**Remark 1.12.** Let us mention a possible simplified variant of our strategy (that will not be developed in this work), which would only allow one to treat the barotropic

[3] This part of the analysis (involving a large parameter) is reminiscent of the use of the Lopatinskii determinant or Evans functions to obtain good estimates in hyperbolic boundary value problems or singular stable boundary layer problems (see e.g. [119, 120, 130]).

case. The idea is to introduce the unknown

$$\mathfrak{m} := (1 - \varrho_f)\varrho$$

so that, using the relation

$$\nabla_x \varrho = \frac{\varrho}{1-\rho_f}\nabla_x \rho_f + \frac{1}{1-\rho_f}\nabla_x \mathfrak{m},$$

the Vlasov equation and the equation for the mass conservation of the fluid can be rewritten as

$$\partial_t \mathfrak{m} + \mathrm{div}_x(\mathfrak{m} u) = 0,$$

$$\partial_t f + v \cdot \nabla_x f + \mathrm{div}_v[(u-v)f] - \frac{p'(\varrho)\varrho}{1-\rho_f}\nabla_x \rho_f \cdot \nabla_v f = \frac{p'(\varrho)}{1-\rho_f}\nabla_x \mathfrak{m} \cdot \nabla_v f,$$

where $u$ is the solution of the associated equation for the momentum balance in the Navier–Stokes system. Since $u$ is expected to be more regular than $f$ and $\mathfrak{m}$, the only loss of derivative displayed by the estimates satisfied by $(f, \mathfrak{m}, u)$ is now that stemming from the force field $\nabla_x \rho_f$ in the former equation for $f$. We can then focus on the distinguished quantity $\rho_f$ and rely on an analysis very akin to that of [90] to prove local well-posedness under the Penrose condition (P).

The strategy which we develop in this work turns out to be more robust and appears in particular effective in handling more complex cases, in particular that of *non-barotropic fluids* (see Section 1.3.1 and Section 7.1), which is relevant in physical applications.

Finally, we describe the content of the appendices at the end of this work.

In Appendix A, we state several useful functional inequalities for commutators, products and composition on $\mathbb{T}^d$ and $\mathbb{T}^d \times \mathbb{R}^d$. In Appendix B, we justify the main steps providing the existence of a local-in-time solution $(f_\varepsilon, \varrho_\varepsilon, u_\varepsilon)$ to the regularized system (1.9) when $\varepsilon > 0$ is fixed. In Appendix C, we recall and give the main notions on pseudodifferential calculus (with parameter) that we shall need in this work.

In the rest of the work, we use the standard notation $A \lesssim B$ to mean $A \leq cB$ for some $c > 0$ that is independent of $A$, $B$ and $\varepsilon$, but that may change from line to line. Furthermore, $\Lambda$ will stand for a nonnegative continuous function which is independent of $\varepsilon$, nondecreasing with respect to each of its arguments, that may depend implicitly on the initial data and that may change from line to line. Finally, we denote by $[P, Q] = PQ - QP$ the commutator between two operators $P$ and $Q$.