Chapter 2

Preliminaries

In this chapter, we initiate the bootstrap strategy that will be used to prove Theorem 1.6. Throughout this work, we will constantly rely on the following lemma (which is a straightforward consequence of the Cauchy–Schwarz inequality).

Lemma 2.1. *For any nonnegative measurable function* $\mathrm{f}(x, v)$ *and* $k \in \mathbb{N}$*, we have*

$$\forall \ell \in \mathbb{N},\ \forall r > \frac{d}{2} + k, \quad \left\| \int_{\mathbb{R}^d} |v|^k \mathrm{f}(\cdot, v)\, \mathrm{d}v \right\|_{\mathrm{H}^\ell} \lesssim \|\mathrm{f}\|_{\mathcal{H}^\ell_r}.$$

In particular, we have

$$\forall \ell \in \mathbb{N},\ \forall r > \frac{d}{2}, \quad \|\rho_{\mathrm{f}}\|_{\mathrm{H}^\ell} \lesssim \|\mathrm{f}\|_{\mathcal{H}^\ell_r},$$
$$\forall \ell \in \mathbb{N},\ \forall r > \frac{d}{2} + 1, \quad \|j_{\mathrm{f}}\|_{\mathrm{H}^\ell} \lesssim \|\mathrm{f}\|_{\mathcal{H}^\ell_r}.$$

2.1 Energy estimates

Our aim is to obtain some *a priori* estimates for smooth solutions to the system (TS). We first study the fluid density ϱ, which is shown to satisfy a hyperbolic-type equation.

Lemma 2.2. *Let* $T > 0$*,* $c > 0$ *and* (f, ϱ, u) *satisfying* (TS) *on* $[0, T]$ *with* $1 - \rho_f \geq c$ *on* $[0, T]$*. Defining the operator* $\mathcal{L}^{u,f}$ *as*

$$\mathcal{L}^{u,f} := \partial_t + u \cdot \nabla_x + B^{u,f},$$

where

$$B^{u,f} := \frac{1}{1 - \rho_f} \mathrm{div}_x[F + u]\mathrm{Id}, \quad F(t, x) := (j_f - \rho_f u)(t, x),$$

the fluid density ϱ *satisfies*

$$\mathcal{L}^{u,f} \varrho = 0 \quad \textit{on } [0, T].$$

Proof. The transport equation for ϱ in (TS) can be rewritten as

$$(1 - \rho_f)\,(\partial_t \varrho + u \cdot \nabla_x \varrho) + \varrho\big(\partial_t(1 - \rho_f) + u \cdot \nabla_x(1 - \rho_f)\big) + (1 - \rho_f)\varrho\, \mathrm{div}_x\, u = 0.$$

Integrating the Vlasov equation in the v variable, we obtain the equation of conservation

$$\partial_t \rho_f + \operatorname{div}_x j_f = 0;$$

therefore $\partial_t(1-\rho_f) = \operatorname{div}_x j_f$ and we get

$$\begin{aligned}
0 &= \partial_t \varrho + u \cdot \nabla_x \varrho + \frac{\varrho}{1-\rho_f}\big[\operatorname{div}_x j_f - u \cdot \nabla_x \rho_f\big] + \varrho \operatorname{div}_x u \\
&= \partial_t \varrho + u \cdot \nabla_x \varrho + \frac{\varrho}{1-\rho_f} \operatorname{div}_x [j_f - \rho_f u] + \frac{\varrho}{1-\rho_f} \rho_f \operatorname{div}_x u + \varrho \operatorname{div}_x u \\
&= \partial_t \varrho + u \cdot \nabla_x \varrho + \frac{\varrho}{1-\rho_f} \operatorname{div}_x [j_f - \rho_f u] + \frac{\varrho}{1-\rho_f} \operatorname{div}_x u.
\end{aligned}$$

We recognize the expression of the Brinkman force $F := j_f - \rho_f u$ and therefore ϱ satisfies the equation

$$\partial_t \varrho + u \cdot \nabla_x \varrho + \frac{1}{1-\rho_f} \operatorname{div}_x [F + u] \varrho = 0,$$

which is the claimed result. ∎

We are now able to derive a Sobolev estimate bearing on the fluid density ϱ, in which we control ℓ derivatives of ϱ by $\ell + 1$ derivatives of f and u.

Proposition 2.3. *For all $\ell, r > 1 + d/2$, $c > 0$, $T > 0$, and all smooth functions (f, ϱ, u) such that $1 - \rho_f \geq c$ and*

$$\mathcal{L}^{u,f} \varrho = 0 \quad \textit{on } [0, T],$$

we have the estimate

$$\forall t \in [0,T], \quad \|\varrho(t)\|_{\mathrm{H}^\ell} \leq \|\varrho^{\mathrm{in}}\|_{\mathrm{H}^\ell} e^{C_T(u,f)T} \exp\big[T e^{C_T(u,f)T} \mathcal{Q}_\ell(T,u,f)\big],$$

where

$$\begin{aligned}
C_T(u,f) &= \|\operatorname{div}_x u\|_{\mathrm{L}^\infty((0,T)\times\mathbb{T}^d)} + 2\|B^{u,f}\|_{\mathrm{L}^\infty((0,T)\times\mathbb{T}^d)}, \\
\mathcal{Q}_\ell(T,u,f) &= \|u\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^\ell)} \\
&\quad + \left\| \frac{1}{1-\rho_f} \right\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^\ell)} \Big(\|f\|^2_{\mathrm{L}^\infty(0,T;\mathcal{H}^{\ell+1}_r)} + \|u\|^2_{\mathrm{L}^\infty(0,T;\mathrm{H}^{\ell+1})} \Big).
\end{aligned}$$

Remark 2.4. Note that for the same exponents ℓ and r, one has

$$\begin{aligned}
&C_T(u,f) \\
&\lesssim \|u\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^\ell)} + \left\| \frac{1}{1-\rho_f} \right\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty)} \Big(\|f\|^2_{\mathrm{L}^\infty(0,T;\mathcal{H}^{\ell+1}_r)} + \|u\|^2_{\mathrm{L}^\infty(0,T;\mathrm{H}^{\ell+1})} \Big).
\end{aligned}$$

Remark 2.5. Let us mention that, for any smooth solution (f, ϱ, u) to (TS), the function $(1-\rho_f)\varrho$ satisfies

$$\mathcal{L}^{u,0}[(1-\rho_f)\varrho] = 0.$$

As a consequence, the result of Proposition 2.3 holds for $(1-\rho_f)\varrho$ instead of ϱ, by considering $\mathcal{Q}_\ell(T, u, 0)$ and $C_T(u, 0)$.

Proof of Proposition 2.3. The proof is standard but for the sake of completeness and for highlighting the dependence on f and u, let us write it. First, suppose we have a smooth function h satisfying

$$\mathcal{L}^{u,f}(h) = S,$$

where S is a given smooth source term. Performing an L^2_x-energy estimate, we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|h\|^2_{\mathrm{L}^2} &= -2\langle u\cdot\nabla_x h, h\rangle_{\mathrm{L}^2} - 2\langle B^{u,f}h, h\rangle_{\mathrm{L}^2} + 2\langle S, h\rangle_{\mathrm{L}^2}\\ &= \int_{\mathbb{T}^d}\big(\mathrm{div}_x\, u - 2B^{u,f}\big)|h|^2\,\mathrm{d}x + 2\langle S, h\rangle_{\mathrm{L}^2}.\end{aligned}$$

By the Cauchy–Schwarz inequality, this yields, for all $t\in(0,T)$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|h(t)\|^2_{\mathrm{L}^2} \le C_T(u,f)\|h(t)\|^2_{\mathrm{L}^2} + 2\|S(t)\|_{\mathrm{L}^2}\|h(t)\|_{\mathrm{L}^2},$$

where

$$C_T(u,f) := \|\mathrm{div}_x\, u\|_{\mathrm{L}^\infty((0,T)\times\mathbb{T}^d)} + 2\|B^{u,f}\|_{\mathrm{L}^\infty((0,T)\times\mathbb{T}^d)}.$$

By Grönwall's inequality, we deduce that

$$\|h(t)\|_{\mathrm{L}^2} \le \|h(0)\|_{\mathrm{L}^2}e^{C_T(u,f)t/2} + \int_0^t e^{C_T(u,f)(t-\tau)/2}\|S(\tau)\|_{\mathrm{L}^2}\,\mathrm{d}\tau.$$

Now, let us assume that ϱ is such that $\mathcal{L}^{u,f}(\varrho) = 0$. Let $\beta\in\mathbb{N}^d$ such that $|\beta|\le\ell$. Since

$$\mathcal{L}^{u,f}\big(\partial_x^\beta\varrho\big) = -\big[\partial_x^\beta, u\cdot\nabla_x + B^{u,f}\big]\varrho,$$

the first part of the proof with $h = \partial_x^\beta\varrho$ and $S = -\big[\partial_x^\beta, u\cdot\nabla_x + B^{u,f}\big]\varrho$ leads to

$$\begin{aligned}\|\partial_x^\beta\varrho(t)\|_{\mathrm{L}^2} \le{}& \|\partial_x^\beta\varrho(0)\|_{\mathrm{L}^2}e^{C_T(u,f)t/2}\\ &+ \int_0^t e^{C_T(u,f)(t-\tau)/2}\big\|\big[\partial_x^\beta, u\cdot\nabla_x + B^{u,f}\big]\varrho(\tau)\big\|_{\mathrm{L}^2}\,\mathrm{d}\tau.\end{aligned}$$

We can estimate the commutator

$$\big[\partial_x^\beta, u\cdot\nabla_x + B^{u,f}\big]\varrho = \big[\partial_x^\beta, u\cdot\big](\nabla_x\varrho) + \big[\partial_x^\beta, B^{u,f}\big](\varrho)$$

thanks to Proposition A.1. This gives

$$\begin{aligned}
&\sum_{0\le|\beta|\le\ell} \big\| \big[\partial_x^\beta, u\cdot\nabla_x + B^{u,f}\big]\varrho \big\|_{\mathrm{L}^2} \\
&\quad \le \sum_{0\le|\beta|\le\ell} \big\| \big[\partial_x^\beta, u\cdot\big](\nabla_x \varrho) \big\|_{\mathrm{L}^2} + \sum_{0\le|\beta|\le\ell} \big\| \big[\partial_x^\beta, B^{u,f}\big](\varrho) \big\|_{\mathrm{L}^2} \\
&\quad \lesssim \|\nabla_x u\|_{\mathrm{L}^\infty} \|\nabla_x \varrho\|_{\mathrm{H}^{\ell-1}} + \|u\|_{\mathrm{H}^\ell} \|\nabla_x \varrho\|_{\mathrm{L}^\infty} \\
&\qquad + \|\nabla_x B^{u,f}\|_{\mathrm{L}^\infty} \|\varrho\|_{\mathrm{H}^{\ell-1}} + \|B^{u,f}\|_{\mathrm{H}^\ell} \|\varrho\|_{\mathrm{L}^\infty} \\
&\quad \lesssim \|u\|_{\mathrm{H}^\ell} \|\varrho\|_{\mathrm{H}^\ell} + \|B^{u,f}\|_{\mathrm{H}^\ell} \|\varrho\|_{\mathrm{H}^\ell},
\end{aligned}$$

by Sobolev embedding, since $\ell > 1 + d/2$. By summing on β, we eventually get, for all $t \in [0, T]$,

$$\|\varrho(t)\|_{\mathrm{H}^\ell} \le e^{C_T(u,f)t/2}\left(\|\varrho^{\mathrm{in}}\|_{\mathrm{H}^\ell} + \int_0^t \big[\|u(\tau)\|_{\mathrm{H}^\ell} + \|B(\tau)\|_{\mathrm{H}^\ell}\big] \|\varrho(\tau)\|_{\mathrm{H}^\ell} \,\mathrm{d}\tau \right).$$

Again by Grönwall's inequality, we get, for all $t \in [0, T]$,

$$\begin{aligned}
&\|\varrho(t)\|_{\mathrm{H}^\ell} \\
&\quad \le e^{C_T(u,f)T/2} \|\varrho^{\mathrm{in}}\|_{\mathrm{H}^\ell} \exp\Big[T e^{C_T(u,f)T/2} \big(\|u\|_{\mathrm{L}^\infty(0,T,\mathrm{H}^\ell)} + \|B^{u,f}\|_{\mathrm{L}^\infty(0,T,\mathrm{H}^\ell)} \big) \Big].
\end{aligned}$$

Finally, we write, for all $\tau \in (0, T)$,

$$\begin{aligned}
&\|u\|_{\mathrm{L}^\infty(0,T,\mathrm{H}^\ell)} + \|B^{u,f}\|_{\mathrm{L}^\infty(0,T,\mathrm{H}^\ell)} \\
&\quad \le \|u\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^\ell)} + \left\| \frac{1}{1-\rho_f} \right\|_{\mathrm{L}^\infty((0,T);\mathrm{H}^\ell)} \big\| \mathrm{div}_x \big[j_f - \rho_f u + u \big] \big\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^\ell)},
\end{aligned}$$

and use

$$\big\| \mathrm{div}_x \big[j_f - \rho_f u + u \big] \big\|_{\mathrm{H}^\ell} \lesssim \|f\|_{\mathcal{H}_r^{\ell+1}} + \|f\|_{\mathcal{H}_r^{\ell+1}} \|u\|_{\mathrm{H}^{\ell+1}} + \|u\|_{\mathrm{H}^{\ell+1}},$$

via $\ell + 1 > d/2$ and Lemma 2.1 with $r > 1 + d/2$. This concludes the proof. ∎

Let us now start the study of the kinetic equation satisfied by f.

Definition 2.6. For any vector field $u(t, x)$ and function $\varrho(t, x)$, we define the *kinetic transport operator* $\mathcal{T}^{u,\varrho}$ as

$$\mathcal{T}^{u,\varrho} := \partial_t + v\cdot\nabla_x - v\cdot\nabla_v + E^{u,\varrho}(t,x)\cdot\nabla_v - d\,\mathrm{Id},$$

where

$$E^{u,\varrho}(t,x) := u(t,x) - p'(\varrho)\nabla_x \varrho(t,x).$$

By developing the divergence (in v) term in the kinetic equation, the Vlasov equation for f in (TS) can be rewritten as

$$\mathcal{T}^{u,\varrho} f = 0.$$

We now state several standard useful estimates to handle the force field $E^{u,\varrho}$.

Lemma 2.7. *Let $T > 0$. If $\varrho \geq c > 0$ on $[0, T]$ for some given constant c, then the following hold:*

- *For all $k \geq 0$ and $t \in [0, T]$, we have*

$$\|p'(\varrho(t))\|_{\mathrm{H}^k} \leq \Lambda(\|\varrho(t)\|_{\mathrm{L}^\infty})\|\varrho(t)\|_{\mathrm{H}^k}. \tag{2.1}$$

- *For all $k > 3 + d/2$ and $t \in [0, T]$, we have*

$$\|E^{u,\varrho}(t)\|_{\mathrm{H}^k} \lesssim \|u(t)\|_{\mathrm{H}^k} + \Lambda\big(\|\varrho(t)\|_{\mathrm{H}^{k-2}}\big)\|\varrho(t)\|_{\mathrm{H}^{k+1}}. \tag{2.2}$$

Proof. To prove (2.1), we rely on the paralinearization theorem of Bony applied to p' (see Proposition A.3 and Remark A.4), thanks to the assumption on the pressure p and the lower bound on ϱ.

To prove (2.2), we only have to estimate the term $p'(\varrho)\nabla_x\varrho$. We rely on the following tame estimate for products (see Proposition A.2):

$$\begin{aligned}\|p'(\varrho(t))\nabla_x\varrho(t)\|_{\mathrm{H}^k} &\lesssim \|p'(\varrho(t))\|_{\mathrm{L}^\infty}\|\nabla_x\varrho(t)\|_{\mathrm{H}^k} + \|p'(\varrho(t))\|_{\mathrm{H}^k}\|\nabla_x\varrho(t)\|_{\mathrm{L}^\infty} \\ &\lesssim \|p'(\varrho(t))\|_{\mathrm{L}^\infty}\|\varrho(t)\|_{\mathrm{H}^{k+1}} + \|p'(\varrho(t))\|_{\mathrm{H}^k}\|\nabla_x\varrho(t)\|_{\mathrm{L}^\infty}.\end{aligned}$$

Therefore by Sobolev embedding we have, for $s_1 > d/2$ and $s_2 > 1 + d/2$,

$$\|E^{u,\varrho}(t)\|_{\mathrm{H}^k} \lesssim \|u(t)\|_{\mathrm{H}^k} + \|p'(\varrho(t))\|_{\mathrm{H}^{s_1}}\|\varrho(t)\|_{\mathrm{H}^{k+1}} + \|p'(\varrho(t))\|_{\mathrm{H}^k}\|\varrho(t)\|_{\mathrm{H}^{s_2}}.$$

With $s_1 = s_2 = k - 2$, this yields

$$\|E^{u,\varrho}(t)\|_{\mathrm{H}^k} \lesssim \|u(t)\|_{\mathrm{H}^k} + \|p'(\varrho(t))\|_{\mathrm{H}^{k-2}}\|\varrho(t)\|_{\mathrm{H}^{k+1}} + \|p'(\varrho(t))\|_{\mathrm{H}^k}\|\varrho(t)\|_{\mathrm{H}^{k-2}}.$$

In view of (2.1), there exists a continuous nondecreasing function $\mathrm{C}_{k,p'} : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\begin{aligned}\|E^{u,\varrho}(t)\|_{\mathrm{H}^k} \lesssim \|u(t)\|_{\mathrm{H}^k} &+ \mathrm{C}_{k,p'}(\|\varrho(t)\|_{\mathrm{L}^\infty})\|\varrho(t)\|_{\mathrm{H}^{k-2}}\|\varrho(t)\|_{\mathrm{H}^{k+1}} \\ &+ \mathrm{C}_{k,p'}(\|\varrho(t)\|_{\mathrm{L}^\infty})\|\varrho(t)\|_{\mathrm{H}^k}\|\varrho(t)\|_{\mathrm{H}^{k-2}},\end{aligned}$$

and finally, by using Sobolev embedding (with $k - 2 > d/2$), we get

$$\|E^{u,\varrho}(t)\|_{\mathrm{H}^k} \lesssim \|u(t)\|_{\mathrm{H}^k} + \widetilde{\mathrm{C}}_{k,p'}(\|\varrho(t)\|_{\mathrm{H}^{k-2}})\|\varrho(t)\|_{\mathrm{H}^{k+1}}$$

for another function $\widetilde{\mathrm{C}}_{k,p'}$ of the same type. This concludes the proof. ∎

Notation 2.8. Let $k \in [[1, d]]$. For $\beta \in \mathbb{N}^d$, let $\widehat{\beta}^k \in \mathbb{N}^d$ and $\overline{\beta}^k \in (\mathbb{N} \cup \{-1\})^d$ be defined as

$$\begin{aligned}\widehat{\beta}^k &:= (\beta_1, \dots, \beta_{k-1}, \beta_k + 1, \beta_{k+1}, \dots, \beta_d),\\ \overline{\beta}^k &:= (\beta_1, \dots, \beta_{k-1}, \beta_k - 1, \beta_{k+1}, \dots, \beta_d).\end{aligned}$$

We have the following straightforward lemma of commutation for the kinetic equation.

Lemma 2.9. *For any $\alpha, \beta \in \mathbb{N}^d$ and for any smooth function $f(t, x, v)$, we have*

$$\left[\partial_x^\alpha \partial_v^\beta, \mathcal{T}^{u,\varrho}\right] f = \sum_{\substack{i=1\\ \beta_i \neq 0}}^{d} \left(\partial_x^{\widehat{\alpha}^i} \partial_v^{\overline{\beta}^i} f - \partial_x^\alpha \partial_v^\beta f\right) + \left[\partial_x^\alpha \partial_v^\beta, E^{u,\varrho}(t, x) \cdot \nabla_v\right] f.$$

The Sobolev estimate for the kinetic equation goes as follows, showing that we can control m derivatives of f by $m+1$ derivatives of ϱ and m derivatives of u.

Proposition 2.10. *For all $r \geq 0$, $m > 3 + d/2$, $c > 0$, there exists $C > 0$ such that, for all $T > 0$ and all smooth functions (f, ϱ, u) satisfying*

$$\mathcal{T}^{u,\varrho}(f) = 0 \quad \textit{on } [0, T]$$

and $\varrho \geq c$ on $[0, T]$, we have, for all $t \in [0, T]$,

$$\begin{aligned}\|f(t)\|^2_{\mathcal{H}^m_r} \leq \|f(0)\|^2_{\mathcal{H}^m_r} \exp\Big[C\Big(&(1 + \|u\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^m)})T\\ &+ \sqrt{T}\Lambda\left(\|\varrho\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^{m-2})}\right)\|\varrho\|_{\mathrm{L}^2(0,T;\mathrm{H}^{m+1})}\Big)\Big].\end{aligned}$$

Proof. By Lemma 2.9, we have

$$\mathcal{T}^{u,\varrho}\left(\partial_x^\alpha \partial_v^\beta f\right) = -\sum_{\substack{i=1\\ \beta_i \neq 0}}^{d} \left(\partial_x^{\widehat{\alpha}^i} \partial_v^{\overline{\beta}^i} f - \partial_x^\alpha \partial_v^\beta f\right) - \left[\partial_x^\alpha \partial_v^\beta, E^{u,\varrho}(t, x) \cdot \nabla_v\right] f$$

for all $\alpha, \beta \in \mathbb{N}^d$. We take the scalar product of this equality with $(1 + |v|^2)^r \partial_x^\alpha \partial_v^\beta f$, sum for all $|\alpha| + |\beta| \leq m$ and then integrate on $\mathbb{T}^d \times \mathbb{R}^d$. For the left-hand side, we observe that

$$\begin{aligned}&\sum_{|\alpha|+|\beta|\leq m} \int_{\mathbb{T}^d\times\mathbb{R}^d} (1 + |v|^2)^r \mathcal{T}^{u,\varrho}\left(\partial_x^\alpha \partial_v^\beta f\right) \partial_x^\alpha \partial_v^\beta f\\ &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|f(t)\|^2_{\mathcal{H}^m_r} - d\|f(t)\|^2_{\mathcal{H}^m_r}\\ &\quad + \sum_{|\alpha|+|\beta|\leq m} \int_{\mathbb{T}^d\times\mathbb{R}^d} (1 + |v|^2)^r \partial_x^\alpha \partial_v^\beta f\\ &\qquad\qquad \times \left[v \cdot \nabla_x - v \cdot \nabla_v + E^{u,\varrho}(t, x) \cdot \nabla_v\right]\left(\partial_x^\alpha \partial_v^\beta f\right)\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|f(t)\|^2_{\mathcal{H}^m_r} - d\|f(t)\|^2_{\mathcal{H}^m_r} \\
&\quad + \sum_{|\alpha|+|\beta|\le m} \int_{\mathbb{T}^d\times\mathbb{R}^d} (1+|v|^2)^r \operatorname{div}_x\left(v\frac{|\partial_x^\alpha\partial_v^\beta f|^2}{2}\right) \\
&\quad + \sum_{|\alpha|+|\beta|\le m} \int_{\mathbb{T}^d\times\mathbb{R}^d} (1+|v|^2)^r \operatorname{div}_v\left((E^{u,\varrho}-v)\frac{|\partial_x^\alpha\partial_v^\beta f|^2}{2}\right) + d\|f(t)\|^2_{\mathcal{H}^m_r} \\
&= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|f(t)\|^2_{\mathcal{H}^m_r} - \sum_{|\alpha|+|\beta|\le m} \int_{\mathbb{T}^d\times\mathbb{R}^d} \nabla_v(1+|v|^2)^r\cdot(E^{u,\varrho}-v)\frac{|\partial_x^\alpha\partial_v^\beta f|^2}{2},
\end{aligned}
$$

and that the last term satisfies

$$
\begin{aligned}
&\sum_{|\alpha|+|\beta|\le m} \int_{\mathbb{T}^d\times\mathbb{R}^d} \nabla_v(1+|v|^2)^r\cdot(E^{u,\varrho}-v)\frac{|\partial_x^\alpha\partial_v^\beta f|^2}{2} \\
&\qquad \le (1+\|E^{u,\varrho}(t)\|_{\mathrm{L}^\infty})\|f(t)\|^2_{\mathcal{H}^m_r}.
\end{aligned}
$$

We now look at the two terms on the right-hand side. For the first one, we have

$$
-\sum_{|\alpha|+|\beta|\le m} \int_{\mathbb{T}^d\times\mathbb{R}^d} (1+|v|^2)^r \sum_{\substack{i=1\\ \beta_i\neq 0}}^{d} \left(\partial_x^{\widehat{\alpha}^i}\partial_v^{\overline{\beta}^i} f - \partial_x^\alpha\partial_v^\beta f\right)\partial_x^\alpha\partial_v^\beta f \lesssim \|f(t)\|^2_{\mathcal{H}^m_r},
$$

while for the second one, we write

$$
\begin{aligned}
&\sum_{|\alpha|+|\beta|\le m} \int_{\mathbb{T}^d\times\mathbb{R}^d} (1+|v|^2)^r\left[\partial_x^\alpha\partial_v^\beta, E^{u,\varrho}(t,x)\cdot\nabla_v\right]f\,\partial_x^\alpha\partial_v^\beta f \\
&\qquad \lesssim \left\|(1+|v|^2)^{r/2}\left[\partial_x^\alpha\partial_v^\beta, E^{u,\varrho}(t,x)\cdot\nabla_v\right]f\right\|_{\mathrm{L}^2_{x,v}}\|f(t)\|_{\mathcal{H}^m_r} \\
&\qquad \lesssim \|E^{u,\varrho}(t)\|_{\mathrm{H}^m}\|f(t)\|^2_{\mathcal{H}^m_r},
\end{aligned}
$$

by the Cauchy–Schwarz inequality and the product law (A.3) in Sobolev spaces (since $m > 1 + d$) of Lemma A.6. All in all, we obtain

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\|f(t)\|^2_{\mathcal{H}^m_r} &\lesssim (1+\|E^{u,\varrho}(t)\|_{\mathrm{L}^\infty}+\|E^{u,\varrho}(t)\|_{\mathrm{H}^m})\|f(t)\|^2_{\mathcal{H}^m_r} \\
&\lesssim (1+\|E^{u,\varrho}(t)\|_{\mathrm{H}^m})\|f(t)\|^2_{\mathcal{H}^m_r}
\end{aligned}
\tag{2.3}
$$

if $m > d/2$. Invoking the estimate (2.2) of Lemma 2.7, and by integrating in time the inequality (2.3), we obtain

$$
\begin{aligned}
&\|f(t)\|^2_{\mathcal{H}^m_r} \\
&\quad \le \|f(0)\|^2_{\mathcal{H}^m_r} \\
&\qquad + C\int_0^t \left(1+\|u(s)\|_{\mathrm{H}^m} + \Lambda\left(\|\varrho\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^{m-2})}\right)\|\varrho(s)\|_{\mathrm{H}^{m+1}}\right)\|f(s)\|^2_{\mathcal{H}^m_r}\,\mathrm{d}s
\end{aligned}
$$

for all $t \in [0,T)$ and for some constant $C>0$. Using the Cauchy–Schwarz inequality and Grönwall's inequality, this implies, for all $t \in [0,T)$,

$$\|f(t)\|^2_{\mathcal{H}^m_r} \leq \|f(0)\|^2_{\mathcal{H}^m_r} \exp\Big[C\big((1+\|u\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^m)})T + \sqrt{T}\Lambda\big(\|\varrho\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^{m-2})}\big)\|\varrho\|_{\mathrm{L}^2(0,T;\mathrm{H}^{m+1})}\big)\Big],$$

and this concludes the proof. ∎

The estimates given by Proposition 2.3 and Proposition 2.10 show a possible loss of derivative between f and ϱ. This constitutes the main obstacle of the analysis.

2.2 Regularization of the system and setup of the bootstrap

To (temporarily) bypass this problem, we introduce the following regularized version of the equations. Since the pressure gradient in the force field of the Vlasov equation seems to cause estimates with a loss of derivative, we smooth out this precise term. For all $\varepsilon>0$, we consider

$$(\mathrm{S}_\varepsilon)\quad \begin{cases} \partial_t f_\varepsilon + v\cdot\nabla_x f_\varepsilon + \mathrm{div}_v[f_\varepsilon(u_\varepsilon - v)] - p'(\varrho_\varepsilon)\nabla_x\big[(I-\varepsilon^2\Delta_x)^{-1}\varrho_\varepsilon\big]\cdot\nabla_v f_\varepsilon = 0,\\ \partial_t\varrho_\varepsilon + u_\varepsilon\cdot\nabla_x\varrho_\varepsilon + \dfrac{\varrho_\varepsilon}{1-\rho_{f_\varepsilon}}\mathrm{div}_x[j_{f_\varepsilon} - \rho_{f_\varepsilon}u_\varepsilon] = -\dfrac{\varrho_\varepsilon}{1-\rho_{f_\varepsilon}}\mathrm{div}_x u_\varepsilon,\\ \partial_t u_\varepsilon + (u_\varepsilon\cdot\nabla_x)u_\varepsilon + \dfrac{1}{\varrho_\varepsilon}\nabla_x p(\varrho_\varepsilon) - \dfrac{1}{\varrho_\varepsilon(1-\rho_{f_\varepsilon})}[\Delta_x + \nabla_x\,\mathrm{div}_x]u_\varepsilon \\ \qquad\qquad = \dfrac{1}{\varrho_\varepsilon(1-\rho_{f_\varepsilon})}(j_{f_\varepsilon} - \rho_{f_\varepsilon}u_\varepsilon),\\ f_{\varepsilon|t=0} = f^{\mathrm{in}},\quad \varrho_{\varepsilon|t=0} = \varrho^{\mathrm{in}},\quad u_{\varepsilon|t=0} = u^{\mathrm{in}}, \end{cases}$$

where

$$\rho_{f_\varepsilon}(t,x) := \int_{\mathbb{R}^d} f_\varepsilon(t,x,v)\,\mathrm{d}v,\quad j_{f_\varepsilon}(t,x) := \int_{\mathbb{R}^d} f_\varepsilon(t,x,v)v\,\mathrm{d}v.$$

Let us highlight that we have used the rewriting of the transport equation for ϱ_ε based on Lemma 2.2.

Definition 2.11. For all $\varepsilon>0$, we define the *regularized kinetic transport operator* $\mathcal{T}^{u_\varepsilon,\varrho_\varepsilon}_{\mathrm{reg},\varepsilon}$ as

$$\mathcal{T}^{u_\varepsilon,\varrho_\varepsilon}_{\mathrm{reg},\varepsilon} := \partial_t + v\cdot\nabla_x - v\cdot\nabla_v + E^{u_\varepsilon,\varrho_\varepsilon}_{\mathrm{reg},\varepsilon}(t,x)\cdot\nabla_v - d\,\mathrm{Id},$$

where

$$E^{u_\varepsilon,\varrho_\varepsilon}_{\mathrm{reg},\varepsilon}(t,x) := u_\varepsilon(t,x) - p'(\varrho_\varepsilon)\nabla_x[\mathrm{J}_\varepsilon\varrho_\varepsilon](t,x),\quad \mathrm{J}_\varepsilon := (I-\varepsilon^2\Delta_x)^{-1}.$$

Remark 2.12. The regularization through the operator J_ε could appear as quite arbitrary. In view of the equivalent Penrose condition (1.5) appearing in Remark 1.10, it is actually possible to consider a more general Fourier multiplier

$$\mathcal{J}_\varepsilon = m(\varepsilon D),$$

associated to a smooth function $m : \mathbb{R}^d \to (0, 1]$ such that

$$\forall k \in \mathbb{R}^d, \quad m(k) \le \frac{C}{1 + |k|^2}.$$

For all $\varepsilon > 0$, the Vlasov equation satisfied by f in (S_ε) can be recast as

$$\mathcal{T}^{u_\varepsilon,\varrho_\varepsilon}_{\mathrm{reg},\varepsilon} f = 0.$$

Relying on the elliptic regularity provided by J_ε, we now have the following estimates for the regularized force field $E^{u_\varepsilon,\varrho_\varepsilon}_{\mathrm{reg},\varepsilon}$.

Lemma 2.13. *Let $\varepsilon > 0$ and $T > 0$. If $\varrho \ge c > 0$ on $[0, T]$, then, for all $k > 3 + d/2$ and $t \in [0, T]$,*

$$\|E^{u_\varepsilon,\varrho_\varepsilon}_{\mathrm{reg},\varepsilon}(t)\|_{\mathrm{H}^k} \lesssim \|u(t)\|_{\mathrm{H}^k} + \frac{1}{\varepsilon}\Lambda\left(\|\varrho(t)\|_{\mathrm{H}^{k-2}}\right)\|\varrho(t)\|_{\mathrm{H}^k}. \tag{2.4}$$

Thanks to the regularization, we can overcome the loss of derivative exhibited by the estimate of Proposition 2.10, up to some factor which is diverging when $\varepsilon \to 0$.

Proposition 2.14. *For all $r \ge 0$, $m > 3 + d/2$, $c > 0$, there exists $C > 0$ such that, for all $\varepsilon > 0$, $T > 0$ and all smooth functions (f, ϱ, u) satisfying*

$$\mathcal{T}^{u,\varrho}_{\mathrm{reg},\varepsilon}(f) = 0 \quad \text{on } [0, T]$$

and $\varrho \ge c$ on $[0, T]$, the following holds for all $t \in [0, T]$:

$$\|f(t)\|^2_{\mathcal{H}^m_r} \le \|f(0)\|^2_{\mathcal{H}^m_r} \exp\Big[C\Big((1 + \|u\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^m)})T + \frac{\sqrt{T}}{\varepsilon}\Lambda\big(\|\varrho\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^{m-2})}\big)\|\varrho\|_{\mathrm{L}^2(0,T;\mathrm{H}^m)}\Big)\Big].$$

Proof. The proof is the same as for Proposition 2.10, using (2.4) instead of (2.2) to conclude. ■

We shall also need the following condition about pointwise bounds for the densities.

Definition 2.15. Let $T > 0$. For any nonnegative functions $f(t, x, v)$ and $\varrho(t, x)$ on $[0, T]$, we define the property

$$(\mathrm{B}^{\mu,\theta}_{\Theta}(T)) \quad \forall t \in [0, T],\ \rho_f(t) \le \frac{\Theta + 1}{2},\ \frac{\mu}{2} \le \varrho(t),\ \frac{\underline{\theta}}{2} \le (1 - \rho_f(t))\varrho(t) \le 2\overline{\theta},$$

where $\Theta, \mu, \underline{\theta}, \overline{\theta}$ are given in the statement of Theorem 1.6.

We will be able to propagate the condition ($\mathrm{B}^{\mu,\theta}_{\Theta}(T)$) thanks to the following lemmas, giving some rough pointwise control for the local particle density and the fluid density.

Lemma 2.16. *Assume that $\rho_{f^{\mathrm{in}}} < \Theta$. For any $\ell, r > 1 + d/2$, any smooth solution $f(t,x,v)$ to the Vlasov equation in (S_ε) satisfies, for all $T > 0$,*

$$\begin{aligned}\|\rho_f\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\mathbb{T}^d))} &\leq \Theta + CT\|f\|_{\mathrm{L}^\infty(0,T;\mathcal{H}^\ell_r)},\\ \|1-\rho_f\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\mathbb{T}^d))} &\leq \Theta + CT\|f\|_{\mathrm{L}^\infty(0,T;\mathcal{H}^\ell_r)},\end{aligned}$$

where Θ has been introduced in the statement of Theorem 1.6 and $C = C_\ell > 0$ only depends on ℓ. Furthermore, if $\Theta + CT\|f\|_{\mathrm{L}^\infty(0,T;\mathcal{H}^\ell_r)} < 1$, then, for all $t \in [0,T]$ and $x \in \mathbb{T}^d$,

$$\begin{aligned}\frac{1}{1-\rho_f(t,x)} &\lesssim \frac{1}{1-\Theta-CT\|f\|_{\mathrm{L}^\infty(0,T;\mathcal{H}^\ell_r)}},\\ \frac{1}{1-\rho_f(t,x)} &\gtrsim \frac{1}{\Theta+CT\|f\|_{\mathrm{L}^\infty(0,T;\mathcal{H}^\ell_r)}}.\end{aligned}$$

Proof. Integrating the Vlasov equation with respect to the velocity, one gets the conservation law $\partial_t \rho_f + \mathrm{div}_x\, j_f = 0$. We thus have

$$\rho_f(t) = \rho_{f^{\mathrm{in}}} + \int_0^t \mathrm{div}_x(j_f)(s)\,\mathrm{d}s.$$

Therefore, by using Sobolev embedding, we get, for all $t \in [0,T]$,

$$\begin{aligned}\|\rho_f(t)\|_{\mathrm{L}^\infty(\mathbb{T}^d)} &\leq \|\rho_{f^{\mathrm{in}}}\|_{\mathrm{L}^\infty(\mathbb{T}^d)} + \int_0^t \|\mathrm{div}_x(j_f)(s)\|_{\mathrm{L}^\infty(\mathbb{T}^d)}\,\mathrm{d}s\\ &\leq \Theta + CT\|j_f\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^\ell(\mathbb{T}^d))},\end{aligned}$$

for some $C = C_\ell > 0$, provided that $\ell > 1 + d/2$. We obtain the conclusion by using Lemma 2.1. The last estimates stated in the lemma then follow directly. ∎

Lemma 2.17. *Assume that $0 < \underline{\theta} \leq (1-\rho_{f^{\mathrm{in}}})\varrho^{\mathrm{in}} \leq \overline{\theta}$. Let $T > 0$. For $\ell > 1 + d/2$, any smooth solution $(f(t,x,v), \varrho(t,x), u(t,x))$ to (S_ε) satisfies, for all $t \in [0,T]$,*

$$\underline{\theta}\exp\big(-T\|u\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^\ell)}\big) \leq (1-\rho_f(t))\varrho(t) \leq \overline{\theta}\exp\big(T\|u\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^\ell)}\big),$$

where $\underline{\theta}, \overline{\theta}$ have been introduced in the statement of Theorem 1.6.

Proof. The proof is a straightforward application of the method of characteristics applied to $(1-\rho_f)\varrho$, as the solution to the continuity equation

$$\partial_t((1-\rho_f)\varrho) + \mathrm{div}_x((1-\rho_f)\varrho u) = 0.$$

We obtain the conclusion in view of the assumption on $(1-\rho_{f^{\mathrm{in}}})\varrho^{\mathrm{in}}$. ∎

The following proposition then shows that, thanks to the regularization, we can actually build for all $\varepsilon > 0$ a local solution to the regularized system (S_ε).

Proposition 2.18. *There exist $r_0 > 0$ and $m_0 > 0$ such that the following holds. For all $\varepsilon > 0$, the system (S_ε) is locally well posed in Sobolev spaces, that is, if $r > r_0$, $m > m_0$ and*

$$(f^{\mathrm{in}}, \varrho^{\mathrm{in}}, u^{\mathrm{in}}) \in \mathcal{H}^m_r \times \mathrm{H}^m \times \mathrm{H}^m$$

satisfies

$$0 \le f^{\mathrm{in}}, \quad \rho_{f^{\mathrm{in}}} < \Theta < 1, \quad 0 < \mu \le \varrho^{\mathrm{in}}, \quad 0 < \underline{\theta} \le (1-\rho_{f^{\mathrm{in}}})\varrho^{\mathrm{in}} \le \overline{\theta}$$

for some fixed constants $\Theta, \mu, \underline{\theta}, \overline{\theta}$, then there exist $T = T(\varepsilon) > 0$ and a unique solution $(f_\varepsilon, \varrho_\varepsilon, u_\varepsilon)$ to the regularized system (S_ε) on $(0, T(\varepsilon)]$ such that

$$(f_\varepsilon, \varrho_\varepsilon, u_\varepsilon) \in \mathcal{C}(0,T;\mathcal{H}^m_r) \times \mathcal{C}(0,T;\mathrm{H}^m) \times \mathcal{C}(0,T;\mathrm{H}^m) \cap \mathrm{L}^2(0,T;\mathrm{H}^{m+1}),$$

and starting at $(f^{\mathrm{in}}, \varrho^{\mathrm{in}}, u^{\mathrm{in}})$. Furthermore, the condition ($\mathrm{B}^{\mu,\theta}_{\Theta}(T)$) is satisfied by $(f_\varepsilon, \varrho_\varepsilon)$ with $T = T(\varepsilon)$.

Proof. The proof is mainly based on the *a priori* estimates we have just derived, through a classical approximation procedure. Because of the regularization on the gradient of ϱ_ε in the Vlasov equation, the procedure is fairly standard. For the reader's convenience, we write the proof in Appendix B. ■

Hereafter and until the end of this work, we consider exponents $r > 0$ and $m > 0$ which can be taken large enough. Their values will be specified later, at the end of the proof.

We now introduce the following quantity, in view of the expected result of Theorem 1.6.

Notation 2.19. For any functions $f(t,x,v)$, $\varrho(t,x)$ and $u(t,x)$, we set

$$\begin{aligned}\mathcal{N}_{m,r}(f,\varrho,u,T) := {} & \|f\|_{\mathrm{L}^\infty(0,T;\mathcal{H}^{m-1}_r)} + \|\varrho\|_{\mathrm{L}^2(0,T;\mathrm{H}^m)} \\ & + \|u\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^m)\cap \mathrm{L}^2(0,T;\mathrm{H}^{m+1})},\end{aligned}$$

where $T > 0$.

The proof of Theorem 1.6 will rely on a bootstrap argument. Let $\varepsilon > 0$. From Proposition 2.18, consider the maximal time of existence T^*_ε for the system (S_ε). By definition, Proposition 2.18 ensures that

$$\forall T < T^*_\varepsilon, \qquad \mathcal{N}_{m,r}(f_\varepsilon, \varrho_\varepsilon, u_\varepsilon, T) < +\infty \quad \text{and} \quad (\mathrm{B}^{\mu,\theta}_{\Theta}(T)) \text{ holds.}$$

So we can consider the time

$$T_\varepsilon = T_\varepsilon(R) := \sup\left\{T \in [0, T^*_\varepsilon),\ \mathcal{N}_{m,r}(f_\varepsilon, \varrho_\varepsilon, u_\varepsilon, T) \le R \text{ and } (\mathrm{B}^{\mu,\theta}_{\Theta}(T)) \text{ holds}\right\},$$

where $R > 0$ will be chosen large enough and independent of ε. By continuity, we observe that $T_\varepsilon > 0$ if R is taken large enough and independent of ε. In particular, for all $t \in [0, T_\varepsilon]$ we have

$$0 < \frac{1-\Theta}{2} \le 1 - \rho_f(t), \qquad \frac{1}{1-\rho_f(t)} \le \frac{2}{1-\Theta}. \tag{2.5}$$

Our main goal is to prove that R can be chosen large enough so that there exists $T(R) > 0$ independent of ε such that

$$\forall \varepsilon \in (0,1), \quad T(R) \le T_\varepsilon(R).$$

Such a lower bound independent of ε will pave the way for a compactness argument when $\varepsilon \to 0$, leading to the existence of a solution for (TS) on $[0, T(R)]$. In what follows, we will work on the interval of time $[0, T_\varepsilon]$.

We have the following trichotomy:

- Either $T_\varepsilon^* = +\infty$ and $T_\varepsilon = T_\varepsilon^*$ – in this case there is nothing to do, because we have the control $\mathcal{N}_{m,r}(f_\varepsilon, \varrho_\varepsilon, u_\varepsilon, T) \le R$ for all times $T > 0$;
- or $T_\varepsilon^* < +\infty$ and $T_\varepsilon = T_\varepsilon^*$ – we shall soon see that this case is impossible, as it leads to a contradiction;
- otherwise, $T_\varepsilon < T_\varepsilon^*$ and $\mathcal{N}_{m,r}(f_\varepsilon, \varrho_\varepsilon, u_\varepsilon, T_\varepsilon) = R$.

Let us show how to exclude the second case. We need the following lemma.

Lemma 2.20. *Let $\varepsilon > 0$. If $m > 3 + d/2$ and $r \ge 1 + d/2$, we have, for all $T \in [0, T_\varepsilon)$,*

$$\|\varrho_\varepsilon\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty)} \lesssim \|\varrho_\varepsilon\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^{m-2})} \le \Lambda\big(T, R, \|\varrho^{\mathrm{in}}\|_{\mathrm{H}^{m-2}}\big).$$

Proof. From Proposition 2.3, we know that for all $T \in [0, T_\varepsilon)$ and $t \in [0, T]$,

$$\|\varrho_\varepsilon(t)\|_{\mathrm{H}^{m-2}} \le \|\varrho^{\mathrm{in}}\|_{\mathrm{H}^{m-2}} e^{C_{m-2}(T,u_\varepsilon,f_\varepsilon)T} \exp\big[T e^{C_{m-2}(T,u_\varepsilon,f_\varepsilon)T} \mathcal{Q}_{m-2}(T, u_\varepsilon, f_\varepsilon)\big],$$

provided that $m-2, r > 1 + d/2$, where

$$C_{m-2}(T, u_\varepsilon, f_\varepsilon) \le R + 2R^2 \left\| \frac{1}{1-\rho_{f_\varepsilon}} \right\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty)},$$

$$\mathcal{Q}_{m-2}(T, u_\varepsilon, f_\varepsilon) \le R + 2R^2 \left\| \frac{1}{1-\rho_{f_\varepsilon}} \right\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^{m-2})},$$

because $\mathcal{N}_{m,r}(f_\varepsilon, \varrho_\varepsilon, u_\varepsilon, T) \le R$ for all $T \in [0, T_\varepsilon)$. By Sobolev embedding and the bound (2.5), this means that for $r > 1 + d/2$ and $m > 1 + d/2 + 2$ we have, for all $T \in [0, T_\varepsilon)$,

$$\begin{aligned}\|\varrho_\varepsilon\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty)} &\lesssim \|\varrho_\varepsilon\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^{m-2})} \\ &\le \Lambda\left(T, R, \|\varrho^{\mathrm{in}}\|_{\mathrm{H}^{m-2}}, \left\| \frac{1}{1-\rho_{f_\varepsilon}} \right\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^{m-2})}\right).\end{aligned}$$

To conclude, we only have to understand the last term in the previous function Λ: by Lemma A.5, there exists a continuous nonnegative nondecreasing function C_m such that

$$\left\| \frac{1}{1-\rho_{f_\varepsilon}} \right\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^{m-2})} \leq 1 + \mathrm{C}_m\big(\|\rho_{f_\varepsilon}\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty)}\big)\|\rho_{f_\varepsilon}\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^{m-2})} \lesssim \Lambda(R),$$

thanks to Lemma 2.1 and the fact that $\mathcal{N}_{m,r}(f_\varepsilon, \varrho_\varepsilon, u_\varepsilon, T) \leq R$. This concludes the proof. ∎

Remark 2.21. A careful inspection of the proof of Proposition 2.3 reveals that for all $k \leq m-2$ with $k > 3 + d/2$, $r > 1 + d/2$ and $T \in [0, T_\varepsilon)$,

$$\|\varrho_\varepsilon\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^k)} \lesssim \Lambda(\|\varrho^{\mathrm{in}}\|_{\mathrm{H}^k}) + T\Lambda(T, R).$$

As a corollary, we can now exclude the second case written above: if $T_\varepsilon^* < +\infty$ and $T_\varepsilon = T_\varepsilon^*$, then according to Proposition 2.14 and Lemma 2.20,

$$\sup_{t\in[0,T_\varepsilon^*)} \|f_\varepsilon(t)\|^2_{\mathcal{H}^m_r} \leq \|f^{\mathrm{in}}\|^2_{\mathcal{H}^m_r} \exp\left[C\left((1+R)T_\varepsilon + \frac{\sqrt{T_\varepsilon}}{\varepsilon}\Lambda\big(T, R, \|\varrho^{\mathrm{in}}\|_{\mathrm{H}^{m-2}}\big)\right)\right] < +\infty.$$

The previous inequality means that the solution could be continued beyond T_ε^*, which is impossible by the maximality of T_ε^*. This case is thus impossible.

From now on, we assume that $T_\varepsilon < T_\varepsilon^*$ and $\mathcal{N}_{m,r}(f_\varepsilon, \varrho_\varepsilon, u_\varepsilon, T_\varepsilon) = R$. In view of our bootstrap strategy, we need to estimate $\mathcal{N}_{m,r}(f_\varepsilon, \varrho_\varepsilon, u_\varepsilon, T)$ for all $T < T_\varepsilon$.

At the end of this section, we show that the term $\|f_\varepsilon\|_{\mathrm{L}^\infty(0,T;\mathcal{H}^{m-1}_r)}$ and the term $\|u_\varepsilon\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^m)\cap\mathrm{L}^2(0,T;\mathrm{H}^{m+1})}$ can be handled by energy estimates. The main part of the upcoming analysis will be to provide a uniform control in ε for $\|\varrho\|_{\mathrm{L}^2(0,T;\mathcal{H}^m_r)}$.

In the following lemma, we give an estimate, independent of ε, for the term $\|f_\varepsilon\|_{\mathrm{L}^\infty(0,T;\mathcal{H}^{m-1}_r)}$ in $\mathcal{N}_{m,r}(f_\varepsilon, \varrho_\varepsilon, u_\varepsilon, T)$, for all $T < T_\varepsilon$.

Lemma 2.22. *For $m > 1 + d/2 + 2$ and $r > 1 + d/2$, the solution $(f_\varepsilon, \varrho_\varepsilon, u_\varepsilon)$ to (S_ε) satisfies, for all $T \in [0, T_\varepsilon)$,*

$$\|f_\varepsilon\|_{\mathrm{L}^\infty(0,T;\mathcal{H}^{m-1}_r)} \leq \|f^{\mathrm{in}}\|_{\mathcal{H}^{m-1}_r} + T^{\frac{1}{4}}\Lambda(T, R).$$

Proof. Following the same steps leading to (2.3) in the proof of Proposition 2.10, we have, for all $t \in [0, T_\varepsilon)$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|f_\varepsilon(t)\|^2_{\mathcal{H}^{m-1}_r} \lesssim \big(1 + \|E^{u_\varepsilon,\varrho_\varepsilon}_{\mathrm{reg},\varepsilon}(t)\|_{\mathrm{H}^{m-1}}\big)\|f_\varepsilon(t)\|^2_{\mathcal{H}^{m-1}_r},$$

since $m-1>d/2$; therefore,

$$\begin{aligned}&\|f_\varepsilon\|^2_{\mathrm{L}^\infty(0,T;\mathcal{H}^{m-1}_r)}\\&\qquad\le \|f^{\mathrm{in}}\|^2_{\mathcal{H}^{m-1}_r}+\|f_\varepsilon\|^2_{\mathrm{L}^\infty(0,T;\mathcal{H}^{m-1}_r)}\Big(T+\int_0^T\|E^{u_\varepsilon,\varrho_\varepsilon}_{\mathrm{reg},\varepsilon}(s)\|_{\mathrm{H}^{m-1}}\,\mathrm{d}s\Big).\end{aligned}$$

Using now the estimate (2.2) from Lemma 2.7, we get

$$\begin{aligned}&\int_0^T\|\mathrm{E}^{u_\varepsilon,\varrho_\varepsilon}_{\mathrm{reg},\varepsilon}(s)\|_{\mathrm{H}^{m-1}}\,\mathrm{d}s\\&\qquad\lesssim\sqrt{T}\|u_\varepsilon\|_{\mathrm{L}^2(0,T;\mathrm{H}^{m-1})}+\sqrt{T}\Lambda\left(\|\varrho_\varepsilon\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^{m-2})}\right)\|\varrho_\varepsilon\|_{\mathrm{L}^2(0,T;\mathrm{H}^m)}.\end{aligned}$$

We thus infer that for all $T\in[0,T_\varepsilon)$,

$$\begin{aligned}&\|f_\varepsilon\|^2_{\mathrm{L}^\infty(0,T;\mathcal{H}^{m-1}_r)}\\&\quad\le\|f^{\mathrm{in}}\|^2_{\mathcal{H}^{m-1}_r}\\&\qquad+C\|f_\varepsilon\|^2_{\mathrm{L}^\infty(0,T;\mathcal{H}^{m-1}_r)}\\&\qquad\quad\times\Big(T+\sqrt{T}\|u_\varepsilon\|_{\mathrm{L}^2(0,T;\mathrm{H}^{m-1})}+\sqrt{T}\Lambda\left(\|\varrho_\varepsilon\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^{m-2})}\right)\|\varrho_\varepsilon\|_{\mathrm{L}^2(0,T;\mathrm{H}^m)}\Big),\end{aligned}$$

where $C>0$ is independent of ε. Since $\mathcal{N}_{m,r}(f_\varepsilon,\varrho_\varepsilon,u_\varepsilon,T)\le R$ for all $T\in[0,T_\varepsilon)$, this gives

$$\begin{aligned}&\|f_\varepsilon\|^2_{\mathrm{L}^\infty(0,T;\mathcal{H}^{m-1}_r)}\\&\qquad\le\|f^{\mathrm{in}}\|^2_{\mathcal{H}^{m-1}_r}+CR^2\Big(T+\sqrt{T}R+\sqrt{T}\Lambda\big(\|\varrho_\varepsilon\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^{m-2})}\big)R\Big).\end{aligned}$$

In order to obtain a uniform bound in ε for the term $\Lambda\big(\|\varrho_\varepsilon\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^{m-2})}\big)$, we apply Lemma 2.20, leading to the conclusion of the lemma. ∎

We conclude this section by estimating the term $\|u_\varepsilon\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^m)\cap\mathrm{L}^2(0,T;\mathrm{H}^{m+1})}$ in $\mathcal{N}(f_\varepsilon,\varrho_\varepsilon,u_\varepsilon,T)$. To this end, we will crucially rely on the smoothing provided by the differential operator $\Delta_x+\nabla_x\operatorname{div}_x$ from the Navier–Stokes equation for u_ε. Indeed, we have the following classical lemma.

Lemma 2.23. *The differential operator $-\Delta_x-\nabla_x\operatorname{div}_x$ is elliptic.*

Proof. The operator $-\Delta_x-\nabla_x\operatorname{div}_x$ is associated to the matrix Fourier multiplier

$$L(k)=|k|^2\mathrm{I}_d+k\otimes k\quad(k\in\mathbb{Z}^d).$$

One can then prove (see e.g. [57]) that

$$2^{-3\frac{d}{2}}|k|^{2d}\le|\det L(k)|,$$

which yields the desired ellipticity. ∎

We are now able to prove the following proposition.

Proposition 2.24. *For $m > 2 + d/2$ and $r \geq 0$, for all $\varepsilon > 0$, the solution $(f_\varepsilon, \varrho_\varepsilon, u_\varepsilon)$ to (S_ε) satisfies, for all $T \in [0, T_\varepsilon)$,*

$$\begin{aligned}&\|u_\varepsilon\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^m)} + \|u_\varepsilon\|_{\mathrm{L}^2(0,T;\mathrm{H}^{m+1})} \\ &\qquad \lesssim \big(1 + T^{1/2}\Lambda(T,R)\big)\big(\|u^{\mathrm{in}}\|_{\mathrm{H}^m} + T^{1/2}\Lambda(T,R) + \Lambda(T,R)\|\varrho_\varepsilon\|_{\mathrm{L}^2(0,T;\mathrm{H}^m)}\big).\end{aligned}$$

Proof. First, we rewrite the equation for u_ε as

$$\partial_t u_\varepsilon - \sigma\big((1-\rho_{f_\varepsilon})\varrho_\varepsilon\big)(\Delta_x + \nabla_x \operatorname{div}_x)u_\varepsilon = F,$$

with

$$F := -(u_\varepsilon \cdot \nabla_x)u_\varepsilon - \nabla_x \pi(\varrho_\varepsilon) + \sigma\big((1-\rho_{f_\varepsilon})\varrho_\varepsilon\big)(j_{f_\varepsilon} - \rho_{f_\varepsilon}u_\varepsilon),$$

where $\sigma(s) := \frac{1}{s}$, $\pi(s) = \int_0^s \frac{p'(\tau)}{\tau}\,\mathrm{d}\tau$. We then apply ∂_x^β in the equation for $|\beta| \leq m-1$, which gives

$$\begin{aligned}&\partial_t(\partial_x^\beta u_\varepsilon) - \sigma\big((1-\rho_{f_\varepsilon})\varrho_\varepsilon\big)(\Delta_x + \nabla_x \operatorname{div}_x)(\partial_x^\beta u_\varepsilon) \\ &\qquad = \partial_x^\beta F + \big[\partial_x^\beta, \sigma\big((1-\rho_{f_\varepsilon})\varrho_\varepsilon\big)\big]((\Delta_x + \nabla_x \operatorname{div}_x)u_\varepsilon),\end{aligned}$$

and then multiply the equation with $-(\Delta_x + \nabla_x \operatorname{div}_x)(\partial_x^\beta u)$ so that, by integrating on $\mathbb{T}^d$ and with an integration by parts,

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^d}\big(|\nabla_x\partial_x^\beta u_\varepsilon|^2 + |\operatorname{div}_x \partial_x^\beta u_\varepsilon|^2\big)\,\mathrm{d}x \\ &+ \int_{\mathbb{T}^d}\sigma\big((1-\rho_{f_\varepsilon})\varrho_\varepsilon\big)\big|(\Delta_x + \nabla_x \operatorname{div}_x)(\partial_x^\beta u_\varepsilon)\big|^2\,\mathrm{d}x \\ &\quad = -\int_{\mathbb{T}^d}(\Delta_x + \nabla_x \operatorname{div}_x)(\partial_x^\beta u_\varepsilon)\cdot\partial_x^\beta F\,\mathrm{d}x \\ &\qquad - \int_{\mathbb{T}^d}\big[\partial_x^\beta, \sigma\big((1-\rho_{f_\varepsilon})\varrho_\varepsilon\big)\big]\big((\Delta_x + \nabla_x \operatorname{div}_x)u_\varepsilon\big)\cdot(\Delta_x + \nabla_x \operatorname{div}_x)(\partial_x^\beta u_\varepsilon)\,\mathrm{d}x.\end{aligned}$$

Thanks to the Cauchy–Schwarz and Young inequalities, and after integration in time, we get, for all $\eta > 0$ and $t \in (0,T)$,

$$\begin{aligned}&\frac{1}{2}\int_{\mathbb{T}^d}\big|\nabla_x\partial_x^\beta u_\varepsilon(t)\big|^2\,\mathrm{d}x \\ &+ \int_0^t\int_{\mathbb{T}^d}\big(\sigma\big((1-\rho_{f_\varepsilon})\varrho_\varepsilon\big) - \eta\big)\,\big|(\Delta_x + \nabla_x \operatorname{div}_x)\partial_x^\beta u_\varepsilon\big|^2\,\mathrm{d}x\,\mathrm{d}s \\ &\qquad \leq \frac{1}{2}\int_{\mathbb{T}^d}\Big(\big|\nabla_x\partial_x^\beta u^{\mathrm{in}}\big|^2 + \big|\operatorname{div}_x \partial_x^\beta u^{\mathrm{in}}\big|^2\Big)\,\mathrm{d}x\end{aligned}$$

$$+ \frac{1}{2\eta} \int_0^t \int_{\mathbb{T}^d} |\partial_x^\beta F|^2 \,\mathrm{d}x\,\mathrm{d}s$$
$$+ \frac{1}{2\eta} \int_0^t \int_{\mathbb{T}^d} \left|\left[\partial_x^\beta, \sigma\big((1-\rho_{f_\varepsilon})\varrho_\varepsilon\big)\right]((\Delta_x + \nabla_x \operatorname{div}_x)u_\varepsilon)\right|^2 \mathrm{d}x\,\mathrm{d}s.$$

Let us deal with the last term: by the commutator inequality from Proposition A.1, we have

$$\begin{aligned}
&\int_{\mathbb{T}^d} \left|\left[\partial_x^\beta, \sigma\big((1-\rho_{f_\varepsilon})\varrho_\varepsilon\big)\right]((\Delta_x + \nabla_x \operatorname{div}_x)u_\varepsilon)\right|^2 \mathrm{d}x \\
&\quad\le M\Big(\left\|\nabla_x \sigma\big((1-\rho_{f_\varepsilon})\varrho_\varepsilon\big)\right\|_{\mathrm{L}^\infty}^2 \|(\Delta_x + \nabla_x \operatorname{div}_x)u_\varepsilon\|_{\mathrm{H}^{m-2}}^2 \\
&\qquad\qquad + \left\|\sigma\big((1-\rho_{f_\varepsilon})\varrho_\varepsilon\big)\right\|_{\mathrm{H}^{m-1}}^2 \|(\Delta_x + \nabla_x \operatorname{div}_x)u_\varepsilon\|_{\mathrm{L}^\infty}^2 \Big) \\
&\quad\le M\Big(\left\|\sigma'\big((1-\rho_{f_\varepsilon})\varrho_\varepsilon\big)\nabla_x((1-\rho_{f_\varepsilon})\varrho_\varepsilon)\right\|_{\mathrm{L}^\infty}^2 \|u\|_{\mathrm{H}^m}^2 \\
&\qquad\qquad + \left\|\sigma\big((1-\rho_{f_\varepsilon})\varrho_\varepsilon\big)\right\|_{\mathrm{H}^{m-1}}^2 \|(\Delta_x + \nabla_x \operatorname{div}_x)u_\varepsilon\|_{\mathrm{L}^\infty}^2 \Big)
\end{aligned}$$

for some constant $M > 0$ independent of time. Combining the Sobolev embedding (with $m > 2 + d/2$) and Remark A.4, we get

$$\begin{aligned}
&\frac{1}{2\eta} \int_0^t \int_{\mathbb{T}^d} \left|\left[\partial_x^\beta, \sigma\big((1-\rho_{f_\varepsilon})\varrho_\varepsilon\big)\right]((\Delta_x + \nabla_x \operatorname{div}_x)u_\varepsilon)\right|^2 \mathrm{d}x\,\mathrm{d}s \\
&\quad\le M \frac{1}{2\eta} \Lambda\Big(\left\|(1-\rho_{f_\varepsilon})\varrho_\varepsilon\right\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^{m-1})} \Big) \int_0^t \|u_\varepsilon(\tau)\|_{\mathrm{H}^m}^2 \,\mathrm{d}\tau
\end{aligned}$$

for another constant $M > 0$. Note that by Remark 2.5, we have

$$\left\|(1-\rho_{f_\varepsilon})\varrho_\varepsilon\right\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^{m-1})} \le \Lambda(T, R, \|u_\varepsilon\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^m)}) \le \Lambda(T, R).$$

All in all, we get, for all $\eta > 0$ and $t \in (0, T)$,

$$\begin{aligned}
&\frac{1}{2}\left\|\nabla_x \partial_x^\beta u_\varepsilon(t)\right\|_{\mathrm{L}^2}^2 + \int_0^t \int_{\mathbb{T}^d} \Big(\sigma\big((1-\rho_{f_\varepsilon})\varrho_\varepsilon\big) - \eta\Big)\left|(\Delta_x + \nabla_x \operatorname{div}_x)\partial_x^\beta u_\varepsilon\right|^2 \mathrm{d}x\,\mathrm{d}s \\
&\quad\le \|u^{\mathrm{in}}\|_{\mathrm{H}^m}^2 + \frac{1}{\eta}\|F\|_{\mathrm{L}^2(0,T;\mathrm{H}^{m-1})}^2 + M\frac{\Lambda(T,R)}{2\eta} \int_0^t \|u_\varepsilon(\tau)\|_{\mathrm{H}^m}^2 \,\mathrm{d}\tau.
\end{aligned}$$

Thanks to the condition ($\mathrm{B}_\Theta^{\mu,\theta}(T)$), we can choose $\eta = 1/4\overline{\theta}$ so that

$$\forall (t,x) \in [0,T] \times \mathbb{T}^d, \qquad \frac{1}{4\overline{\theta}} < \sigma\big((1-\rho_{f_\varepsilon})\varrho_\varepsilon\big)(t,x) - \eta.$$

Summing for all $|\beta| = m$ and invoking the elliptic regularity for the operator $-\Delta_x - \nabla_x \operatorname{div}_x$ given by Lemma 2.23, we get, for all $t \in (0, T)$,

$$\begin{aligned}
\|u_\varepsilon(t)\|_{\mathrm{H}^m}^2 &\le \|u_\varepsilon(t)\|_{\mathrm{H}^m}^2 + \|u_\varepsilon\|_{\mathrm{L}^2(0,T;\mathrm{H}^{m+1})}^2 \\
&\lesssim \|u^{\mathrm{in}}\|_{\mathrm{H}^m}^2 + \|F\|_{\mathrm{L}^2(0,T;\mathrm{H}^{m-1})}^2 + \Lambda(T,R)\int_0^t \|u_\varepsilon(\tau)\|_{\mathrm{H}^m}^2 \,\mathrm{d}\tau.
\end{aligned}$$

By Grönwall's lemma, we deduce that for all $t \in (0,T)$,

$$\|u_\varepsilon(t)\|_{\mathrm{H}^m}^2 \leq \Lambda(T,R)\big(\|u^{\mathrm{in}}\|_{\mathrm{H}^m}^2 + \|F\|^2_{\mathrm{L}^2(0,T;\mathrm{H}^{m-1})}\big),$$

which then implies, by using again the previous inequality, that for all $t \in (0,T)$,

$$\begin{aligned}&\|u_\varepsilon\|^2_{\mathrm{L}^\infty(0,T;\mathrm{H}^m)} + \|u_\varepsilon\|^2_{\mathrm{L}^2(0,T;\mathrm{H}^{m+1})}\\ &\qquad\lesssim (1+T\Lambda(T,R))\big(\|u^{\mathrm{in}}\|_{\mathrm{H}^m}^2 + \|F\|^2_{\mathrm{L}^2(0,T;\mathrm{H}^{m-1})}\big).\end{aligned}$$

To conclude, let us now estimate the norm of the source term F. We have

$$\begin{aligned}\|F\|^2_{\mathrm{L}^2(0,T;\mathrm{H}^{m-1})} &\leq \int_0^T \Bigg(\|(u_\varepsilon\cdot\nabla_x)u_\varepsilon(\tau)\|^2_{\mathrm{H}^{m-1}} + \|\nabla_x \pi(\varrho_\varepsilon(\tau))\|^2_{\mathrm{H}^{m-1}}\\ &\qquad + \left\|\frac{1}{(1-\rho_{f_\varepsilon})\varrho_\varepsilon}(j_{f_\varepsilon}-\rho_{f_\varepsilon}u_\varepsilon)(\tau)\right\|^2_{\mathrm{H}^{m-1}}\Bigg)\,\mathrm{d}\tau\\ &\leq \int_0^T \left(\|u_\varepsilon(\tau)\|^2_{\mathrm{H}^{m-1}}\|u_\varepsilon(\tau)\|^2_{\mathrm{H}^m} + \|\pi(\varrho_\varepsilon(\tau))\|^2_{\mathrm{H}^m}\right)\,\mathrm{d}\tau\\ &\quad + T\left\|\frac{1}{(1-\rho_{f_\varepsilon})\varrho_\varepsilon}(j_{f_\varepsilon}-\rho_{f_\varepsilon}u_\varepsilon)\right\|^2_{\mathrm{L}^\infty(0,T;\mathrm{H}^{m-1})}.\end{aligned}$$

In the rest of the proof, we shall make a constant use of the condition ($\mathrm{B}^{\mu,\theta}_{\Theta}(T)$). By Proposition A.3, we have

$$\|\pi(\varrho_\varepsilon)\|_{\mathrm{H}^m} \lesssim \Lambda(\|\varrho_\varepsilon\|_{\mathrm{L}^\infty})\|\varrho_\varepsilon\|_{\mathrm{H}^m},$$

from which we infer, thanks to Sobolev embedding (taking $m > 2+\frac{d}{2}$), that

$$\begin{aligned}\|F\|^2_{\mathrm{L}^2(0,T;\mathrm{H}^{m-1})} &\lesssim T\,\|u_\varepsilon\|^4_{\mathrm{L}^\infty(0,T;\mathrm{H}^m)} + \Lambda(\|\varrho\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty)})\|\varrho_\varepsilon\|^2_{\mathrm{L}^2(0,T;\mathrm{H}^m)}\\ &\quad + T\left\|\frac{1}{(1-\rho_{f_\varepsilon})\varrho_\varepsilon}(j_{f_\varepsilon}-\rho_{f_\varepsilon}u_\varepsilon)\right\|^2_{\mathrm{L}^\infty(0,T;\mathrm{H}^{m-1})}\\ &\lesssim TR^4 + \Lambda(\|\varrho_\varepsilon\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^{m-2})})\|\varrho_\varepsilon\|^2_{\mathrm{L}^2(0,T;\mathrm{H}^m)}\\ &\quad + T\left\|\frac{1}{(1-\rho_{f_\varepsilon})\varrho_\varepsilon}(j_{f_\varepsilon}-\rho_{f_\varepsilon}u_\varepsilon)\right\|^2_{\mathrm{L}^\infty(0,T;\mathrm{H}^{m-1})},\end{aligned}$$

since $\mathcal{N}_{m,r}(f_\varepsilon,\varrho_\varepsilon,u_\varepsilon,T) \leq R$ for all $T \in [0,T_\varepsilon)$. The first term is then addressed thanks to Lemma 2.20 and it remains to estimate the last term. For this one, we have

$$\begin{aligned}&\left\|\frac{1}{(1-\rho_{f_\varepsilon})\varrho_\varepsilon}(j_{f_\varepsilon}-\rho_{f_\varepsilon}u_\varepsilon)\right\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^{m-1})}\\ &\qquad\leq \left\|\frac{1}{(1-\rho_{f_\varepsilon})\varrho_\varepsilon}\right\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^{m-1})}\|(j_{f_\varepsilon}-\rho_{f_\varepsilon}u_\varepsilon)\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^{m-1})}\\ &\qquad\leq \Lambda(T,R)\left\|\frac{1}{(1-\rho_{f_\varepsilon})\varrho_\varepsilon}\right\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^{m-1})},\end{aligned}$$

thanks to Lemma 2.1 and $\mathcal{N}_{m,r}(f_\varepsilon, \varrho_\varepsilon, u_\varepsilon, T) \leq R$ for all $T \in [0, T_\varepsilon)$. By Remark A.4, we then have by Sobolev embedding

$$\begin{aligned}
&\left\| \frac{1}{(1-\rho_{f_\varepsilon})\varrho_\varepsilon}(j_f - \rho_f u) \right\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^{m-1})} \\
&\quad \leq \Lambda(T, R, \|(1-\rho_{f_\varepsilon})\varrho_\varepsilon\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty)}) \|(1-\rho_{f_\varepsilon})\varrho_\varepsilon\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^{m-1})} \\
&\quad \leq \Lambda(T, R, \|(1-\rho_{f_\varepsilon})\varrho_\varepsilon\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^{m-1})}) \\
&\quad \leq \Lambda(T, R, \|u_\varepsilon\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^{m})}) \\
&\quad \leq \Lambda(T, R),
\end{aligned}$$

where we have also used Remark 2.5. This concludes the proof. ■

Remark 2.25. By looking at the previous proof, we have, for $T \in [0, T_\varepsilon)$,

$$\begin{aligned}
&\|u_\varepsilon\|^2_{\mathrm{L}^\infty(0,T;\mathrm{H}^k)} + \|u\|^2_{\mathrm{L}^2(0,T;\mathrm{H}^{k+1})} \\
&\qquad \leq \|u^{\mathrm{in}}\|^2_{\mathrm{H}^k} + T\Lambda(T, R) + \Lambda(T, R)\|\varrho_\varepsilon\|^2_{\mathrm{L}^2(0,T;\mathrm{H}^k)} \\
&\qquad \leq \|u^{\mathrm{in}}\|^2_{\mathrm{H}^k} + T\Lambda(T, R) + T\Lambda(T, R)\|\varrho_\varepsilon\|^2_{\mathrm{L}^\infty(0,T;\mathrm{H}^k)}
\end{aligned}$$

for all $k > 2 + d/2$ such that $k \leq m - 2$; therefore, by Remark 2.21, we obtain

$$\begin{aligned}
&\|u_\varepsilon\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^k)} + \|u_\varepsilon\|_{\mathrm{L}^2(0,T;\mathrm{H}^{k+1})} \\
&\qquad \lesssim \big(1 + T^{1/2}\Lambda(T, R)\big)\big(\|u^{\mathrm{in}}\|_{\mathrm{H}^k} + \|\varrho^{\mathrm{in}}\|_{\mathrm{H}^k}\big) + T^{1/2}\Lambda(T, R).
\end{aligned}$$

So far, Lemma 2.22 and Proposition 2.24 show that it remains to control the second term in $\mathcal{N}(f_\varepsilon, \varrho_\varepsilon, u_\varepsilon, T)$, that is, $\|\varrho_\varepsilon\|_{\mathrm{L}^2(0,T;\mathrm{H}^m)}$ to carry out a bootstrap argument. This will constitute the heart of our analysis and will be the purpose of the remaining chapters.

2.3 Trajectories and straightening change of variable

In this last preliminary section, we study the trajectories associated to a Vlasov equation with friction and force field $\mathrm{F}(t, x)$. We show that for small times their geometry can be simplified thanks to a straightening change of variable in velocity. Loosely speaking, this allows us to boil down the dynamics to that associated with free transport with friction. This procedure will be useful in Chapter 4.

Let $T > 0$. Given $\mathrm{F}(s, x) \in \mathbb{R}^d$ a given vector field defined on $[0, T] \times \mathbb{T}^d$ and satisfying

$$\mathrm{F} \in \mathrm{L}^2(0, T; \mathrm{W}^{1,\infty}(\mathbb{T}^d)),$$

we can consider, thanks to the Cauchy–Lipschitz theorem, the solution

$$s \mapsto \big(\mathrm{X}^{s;t}(x, v), \mathrm{V}^{s;t}(x, v)\big) \in \mathbb{T}^d \times \mathbb{R}^d$$

of the following system of ODEs:

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}s}\mathrm{X}^{s;t}(x,v) = \mathrm{V}^{s;t}(x,v), \\ \dfrac{\mathrm{d}}{\mathrm{d}s}\mathrm{V}^{s;t}(x,v) = -\mathrm{V}^{s;t}(x,v) + \mathrm{F}(s, \mathrm{X}^{s;t}(x,v)), \\ \mathrm{X}^{t;t}(x,v) = x, \\ \mathrm{V}^{t;t}(x,v) = v. \end{cases}$$

Later on, we will apply this to $\mathrm{F} = E^{u_\varepsilon, \varrho_\varepsilon}_{\mathrm{reg},\varepsilon}$, which has been defined at the beginning of Section 2.2 (see the end of the current section). Integrating the previous system of ODEs, we have

$$\begin{aligned} \mathrm{X}^{s;t}(x,v) &= x + (1-e^{t-s})v + \int_t^s (1-e^{\tau-s})\mathrm{F}\big(\tau, \mathrm{X}^{\tau;t}(x,v)\big)\,\mathrm{d}\tau, \qquad (2.6) \\ \mathrm{V}^{s;t}(x,v) &= e^{t-s}v + \int_t^s e^{\tau-s}\mathrm{F}\big(\tau, \mathrm{X}^{\tau;t}(x,v)\big)\,\mathrm{d}\tau. \end{aligned}$$

Considering the full kinetic transport operator

$$\mathcal{T}_{\mathrm{F}} = \partial_t + v\cdot\nabla_x - v\cdot\nabla_v + \mathrm{F}(t,x)\cdot\nabla_v - d\,\mathrm{Id},$$

the method of characteristics shows that a smooth function $f(t,x,v)$ satisfying

$$\begin{cases} \mathcal{T}_{\mathrm{F}} f = 0, \\ f_{|t=0} = f^{\mathrm{in}} \end{cases}$$

can be represented as

$$f(t,x,v) = e^{dt} f^{\mathrm{in}}\big(\mathrm{X}^{0;t}(x,v), \mathrm{V}^{0;t}(x,v)\big).$$

Note also that for all $t, s \in [0,T]$, the map

$$(x,v) \mapsto \big(\mathrm{X}^{s;t}(x,v), \mathrm{V}^{s;t}(x,v)\big)$$

is a diffeomorphism from $\mathbb{T}^d \times \mathbb{R}^d$ to itself, whose Jacobian value is $e^{d(s-t)}$.

The main goal of this section is to prove that for short times, and modulo a *straightening change of variable* in velocity, it is possible to come down to the free dynamics with friction associated to the transport operator

$$\mathcal{T}^{\mathrm{fric}} = \partial_t + v\cdot\nabla_x - v\cdot\nabla_v - d\,\mathrm{Id}.$$

This corresponds to the previous system of ODEs with $\mathrm{F}=0$, and for which the solution $(\mathrm{X}^{s;t}_{\mathrm{fric}}, \mathrm{V}^{s;t}_{\mathrm{fric}})$ is

$$\mathrm{X}^{s;t}_{\mathrm{fric}} = x + (1-e^{t-s})v, \quad \mathrm{V}^{s;t}_{\mathrm{fric}} = e^{t-s}v.$$

Namely, we have the following lemma.

Lemma 2.26. *Let $T > 0$ and $k \geq 1$. Let $\mathrm{F} \in \mathrm{L}^2(0,T;\mathrm{W}^{k,\infty}(\mathbb{T}^d))$ be a vector field such that*

$$\|\mathrm{F}\|_{\mathrm{L}^2(0,T;\mathrm{W}^{k,\infty}(\mathbb{T}^d))} \leq \Lambda(T,\mathrm{R})$$

for some $\mathrm{R} > 0$. There exists $\overline{T}(\mathrm{R}) > 0$ such that, for all $x \in \mathbb{T}^d$ and all times $s,t \in [0, \min(\overline{T}(\mathrm{R}), T)]$, there exists a diffeomorphism $\psi_{s,t}(x,\cdot) : \mathbb{R}^d \to \mathbb{R}^d$ satisfying, for all $v \in \mathbb{R}^d$,

$$\mathrm{X}^{s;t}\,(x, \psi_{s,t}(x,v)) = x + (1 - e^{t-s})v,$$

which furthermore satisfies the estimates

$$\frac{1}{C} \leq \det(\mathrm{D}_v \psi_{s,t}(x,v)) \leq C, \tag{2.7}$$

$$\sup_{s,t \in [0,T]} \left\| \partial^\beta_{x,v}\, (\psi_{s,t}(x,v) - v) \right\|_{\mathrm{L}^\infty(\mathbb{T}^d \times \mathbb{R}^d)} \leq \varphi(T)\Lambda(T,\mathrm{R}), \quad |\beta| \leq k, \tag{2.8}$$

$$\sup_{s,t \in [0,T]} \left\| \partial^\beta_{x,v} \partial_s \psi_{s,t}(x,v) \right\|_{\mathrm{L}^\infty(\mathbb{T}^d \times \mathbb{R}^d)} \leq \varphi(T)\Lambda(T,\mathrm{R}), \quad |\beta| \leq k-1, \tag{2.9}$$

for some $C > 0$ and some nondecreasing continuous function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ vanishing at 0.

Proof. We follow the approach of [89]. We observe that if we set

$$\widetilde{\mathrm{X}}^{s;t}(x,v) := \frac{1}{1 - e^{t-s}}\big[\mathrm{X}^{s;t}(x,v) - x - (1 - e^{t-s})v\big],$$

it suffices to prove that for s,t small enough, the mapping $\phi^{s,t,x} : v \mapsto v + \widetilde{\mathrm{X}}^{s;t}(x,v)$ is a small Lipschitz perturbation of the identity: denoting its inverse by $\psi_{s,t}(x,\cdot)$, it will satisfy

$$v = \psi_{s,t}(x,v) + \widetilde{\mathrm{X}}^{s,t}(x, \psi_{s,t}(x,v)), \tag{2.10}$$

and the first conclusion of the lemma will follow. We introduce the remainder

$$\mathrm{Y}^{s;t}(x,v) = \mathrm{X}^{s;t}(x,v) - x - (1 - e^{t-s})v,$$

which, in view of (2.6), satisfies, for all $s,t \in [0,T]$,

$$\begin{aligned} \mathrm{Y}^{s;t}(x,v) &= \int_s^t (e^{\tau-s} - 1)\mathrm{F}\big(\tau, \mathrm{X}^{\tau;t}(x,v)\big)\,\mathrm{d}\tau \\ &= \int_s^t (e^{\tau-s} - 1)\mathrm{F}\big(\tau, x + (1 - e^{t-\tau})v + \mathrm{Y}^{\tau;t}(x,v)\big)\,\mathrm{d}\tau. \end{aligned} \tag{2.11}$$

We now have

$$\widetilde{\mathrm{X}}^{s;t}(x,v) = \frac{1}{e^{t-s} - 1} \int_s^t (e^{\tau-s} - 1)\mathrm{F}\left(\tau, x + (1 - e^{t-\tau})v + \mathrm{Y}^{\tau;t}(x,v)\right)\,\mathrm{d}\tau. \tag{2.12}$$

Thus, estimates on Y and its derivatives obtained thanks to (2.11) shall provide estimates on $\widetilde{X}$ and its derivatives via (2.12).

Let us assume that $s \leq t$ (the case $s \geq t$ can be treated similarly). First, we have

$$\begin{aligned}
&\|\nabla_x Y^{s;t}\|_{L^\infty_{x,v}} \\
&\leq \int_s^t (e^{\tau-s}-1)\|\nabla_x F\big(\tau, x+(1-e^{t-\tau})v + Y^{\tau;t}(x,v)\big)\|_{L^\infty_{x,v}}(1+\|\nabla_x Y^{\tau;t}\|_{L^\infty_{x,v}})\,d\tau \\
&\leq \int_s^t (e^{\tau-s}-1)\|\nabla_x F(\tau)\|_{L^\infty_x}(1+\|\nabla_x Y^{\tau;t}\|_{L^\infty_{x,v}})\,d\tau \\
&\leq (e^T-1)T^{1/2}\|\nabla_x F\|_{L^2(0,T;L^\infty)}\Big(1+\sup_{\tau\leq t}\|\nabla_x Y^{\tau;t}\|_{L^\infty_{x,v}}\Big).
\end{aligned}$$

By the assumption on the vector field F, we get

$$\|\nabla_x Y^{s;t}\|_{L^\infty_{x,v}} \leq (e^T-1)T^{1/2}\Lambda(T,R)\Big(1+\sup_{\tau\leq t}\|\nabla_x Y^{\tau;t}\|_{L^\infty_{x,v}}\Big).$$

In the remainder of the proof, $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ will stand for a generic continuous function, vanishing at 0, that may change from line to line. This yields

$$\|\nabla_x Y^{s;t}\|_{L^\infty_{x,v}} \leq \sup_{\tau\leq t}\|\nabla_x Y^{\tau;t}\|_{L^\infty_{x,v}} \leq \frac{(e^T-1)T^{1/2}\Lambda(T,R)}{1-(e^T-1)T^{1/2}\Lambda(T,R)} \lesssim \varphi(T)\Lambda(T,R) \tag{2.13}$$

for T small enough. In a similar way, we have

$$\begin{aligned}
\|\nabla_v Y^{s;t}\|_{L^\infty_{x,v}} &\leq \int_s^t (e^{\tau-s}-1)\|\nabla_x F(\tau)\|_{L^\infty_{x,v}}(1-e^{t-\tau}+\|\nabla_v Y^{\tau;t}\|_{L^\infty_{x,v}})\,d\tau \\
&\leq (e^T-1)T^{1/2}\Lambda(T,R)\Big(1+\sup_{\tau\leq t}\|\nabla_v Y^{\tau;t}\|_{L^\infty_{x,v}}\Big);
\end{aligned}$$

therefore,

$$\|\nabla_v Y^{s;t}\|_{L^\infty_{x,v}} \lesssim \varphi(T)\Lambda(T,R) \tag{2.14}$$

for T small enough. We then deduce the following estimates:

$$\begin{aligned}
\|\nabla_x \widetilde{X}^{s;t}\|_{L^\infty_{x,v}} &\leq \frac{1}{e^{t-s}-1}\int_s^t (e^{\tau-s}-1)\|\nabla_x F(\tau)\|_{L^\infty_{x,v}}\big(1+\|\nabla_x Y^{\tau;t}\|_{L^\infty_{x,v}}\big)\,d\tau \\
&\lesssim T^{1/2}\Lambda(T,R)(1+\varphi(T)\Lambda(T,R))
\end{aligned}$$

thanks to (2.13), as well as

$$\begin{aligned}
\|\nabla_v \widetilde{X}^{s;t}\|_{L^\infty_{x,v}} &\leq \frac{1}{e^{t-s}-1}\int_s^t (e^{\tau-s}-1)\|\nabla_x F(\tau)\|_{L^\infty_{x,v}}\big(1-e^{\tau-t}+\|\nabla_v Y^{\tau;t}\|_{L^\infty_{x,v}}\big)\,d\tau \\
&\lesssim T^{1/2}\Lambda(T,R)(1+\varphi(T)\Lambda(T,R))
\end{aligned}$$

thanks to (2.14). This proves that for T small enough, we have

$$\|\nabla_x \widetilde{\mathrm{X}}^{s;t}\|_{\mathrm{L}^\infty_{x,v}} + \|\nabla_v \widetilde{\mathrm{X}}^{s;t}\|_{\mathrm{L}^\infty_{x,v}} \leq \varphi(T)\Lambda(T,\mathrm{R}). \tag{2.15}$$

For T small enough, we therefore obtain the existence of the desired diffeomorphism $\psi_{s,t}(x,\cdot)$. We also have

$$0 < \left|\det\left(\nabla_v \psi^x_{s,t}(v)\right)\right| = \left|\det\left(\mathrm{Id} + \nabla_v \widetilde{\mathrm{X}}^{s,t}(x, \psi^x_{s,t}(v))\right)\right|^{-1}.$$

We thus infer the uniform bound (2.7) from the estimate (2.15) and the continuity of $M \mapsto |\det(M)|$, reducing $\overline{T}(\mathrm{R})$ if necessary.

Let us finally prove the estimates (2.8) and (2.9). For (2.8), we proceed by induction on the length of α. In view of (2.10), we obtain the result for $|\alpha| = 0$. For the case $|\alpha| = 1$, we differentiate the identity (2.10) and get, with $\nabla = \nabla_x$ or $\nabla = \nabla_v$,

$$\nabla(v - \psi^x_{s,t}(v)) = \nabla\widetilde{\mathrm{X}}^{s;t}(x, \psi^x_{s,t}(v)) - \nabla\widetilde{\mathrm{X}}^{s;t}(x, \psi^x_{s,t}(v))\nabla(v - \psi^x_{s,t}(v));$$

therefore, thanks to (2.15) (reducing again $\overline{T}(\mathrm{R})$ if necessary), we have

$$\left\|\nabla\left(\psi^x_{s,t}(v) - v\right)\right\|_{\mathrm{L}^\infty_{x,v}} \leq \frac{\|\nabla\widetilde{\mathrm{X}}^{s;t}\|_{\mathrm{L}^\infty_{x,v}}}{1 - \|\nabla\widetilde{\mathrm{X}}^{s;t}\|_{\mathrm{L}^\infty_{x,v}}} \leq \varphi(T)\Lambda(T,\mathrm{R}).$$

This yields the result for $|\alpha| = 1$. If $1 < |\alpha| \leq k$ and if the result holds for all $|\widetilde{\alpha}| < |\alpha|$, we apply $\partial^\alpha_{x,v}$ in (2.10) and apply Faà di Bruno's formula:

$$\partial^\alpha_{x,v}\left(\psi^x_{s,t}(v) - v\right) = \sum_{\mu,\nu} C_{\mu,\nu}\partial^\mu_{x,v}\widetilde{\mathrm{X}}^{s,t}(\mathrm{z}(x,v)) \prod_{\substack{1\leq|\beta|\leq|\alpha| \\ 1\leq j\leq 2d}} (\partial^\beta_{x,v}\mathrm{z}(x,v)_j)^{\nu_{\beta_j}},$$
$$\mathrm{z}(x,v) := (x, \psi^x_{s,t}(v)),$$

where the sum is taken on (μ,ν) such that $1 \leq |\mu| \leq |\alpha|$ and $\nu_k \in \mathbb{N}\backslash\{0\}$ with

$$\forall 1 \leq j \leq 2d, \quad \sum_{1\leq|\beta|\leq|\alpha|} \nu_{\beta_j} = \mu_j \qquad \text{and} \qquad \sum_{\substack{1\leq|\beta|\leq|\alpha| \\ 1\leq j\leq 2d}} \nu_{\beta_j}\beta = \alpha.$$

We proceed as in the case $|\alpha| = 1$. We isolate the terms with multi-indices μ such that $|\mu| = 1$ (giving associated $\nu_{\beta_j} = 1$ for all $1 \leq j \leq 2d$): the terms $\partial^\alpha_{x,v}\psi_{s,t}$ (given by $\nu_{\alpha_j} = 1$) in the product are treated as above, while derivatives of order strictly less than $|\alpha|$ are bounded thanks to the induction hypothesis. This procedure is allowed, provided that uniform bounds (in time) of the same type for $\|\widetilde{\mathrm{X}}^{s,t}\|_{\mathrm{W}^{k,\infty}_{x,v}}$ $(k \leq |\alpha|)$ hold true.

Such bounds are obtained by performing the same induction at the level of $\mathrm{Y}^{s;t}$ first (using the same principle as before with (2.11)) and then for $\widetilde{\mathrm{X}}^{s,t}$ (arguing as before with (2.12)).

Concerning the estimate (2.9), we have by (2.10)

$$\partial_s \psi^x_{s,t}(v) = -\partial_s \widetilde{\mathrm{X}}^{s,t}(x, \psi^x_{s,t}(v)) - \nabla_v \widetilde{\mathrm{X}}^{s,t}(x, \psi^x_{s,t}(v))\partial_s \psi^x_{s,t}(v),$$

and by (2.12)

$$\begin{aligned}\partial_s \widetilde{\mathrm{X}}^{s;t}(x,v) = &-\frac{e^{t-s}}{1-e^{t-s}}\widetilde{\mathrm{X}}^{s;t}(x,v)\\ &-\frac{1}{1-e^{t-s}}\int_s^t e^{\tau-s}\mathrm{F}\left(\tau, x+(1-e^{t-\tau})v+\mathrm{Y}^{\tau;t}(x,v)\right)\,\mathrm{d}\tau.\end{aligned}$$

Since

$$\|\widetilde{\mathrm{X}}^{s;t}\|_{\mathrm{L}^\infty_{x,v}} \le T\Lambda(T,\mathrm{R}),$$

and $Te^T/(e^T-1)$ is bounded in a neighborhood of 0, we obtain an estimate on the term $\|\partial_s \widetilde{\mathrm{X}}^{s;t}(x,v)\|_{\mathrm{L}^\infty_{x,v}}$ and then on $\|\partial_s \psi^x_{s,t}(v)\|_{\mathrm{L}^\infty_{x,v}}$ as before. Using the same induction procedure as for (2.8), we finally obtain (2.9). ■

Remark 2.27. From the $\mathrm{W}^{k,\infty}_{x,v}$-bounds we have obtained on $\mathrm{Y}^{s,t}$ along the proof, and because

$$\mathrm{Y}^{s;t}(x,v) = \mathrm{X}^{s;t}(x,v) - x - (1-e^{t-s})v,$$

we can infer that

$$\sup_{s,t\in[0,T]} \left\|\partial^\beta_{x,v}\left(\mathrm{X}_{s,t}(x,v) - x - (1-e^{t-s})v\right)\right\|_{\mathrm{L}^\infty(\mathbb{T}^d_x\times\mathbb{R}^d_v)} \le \varphi(T)\Lambda(T,R), \quad |\beta|\le k. \tag{2.16}$$

Likewise, by considering

$$\mathrm{W}^{s,t}(x,v) = \mathrm{V}^{s,t}(x,v) - e^{t-s}v,$$

one can obtain the estimate

$$\sup_{s,t\in[0,T]} \left\|\partial^\beta_{x,v}\left(\mathrm{V}_{s,t}(x,v) - e^{t-s}v\right)\right\|_{\mathrm{L}^\infty(\mathbb{T}^d_x\times\mathbb{R}^d_v)} \le \varphi(T)\Lambda(T,R), \quad |\beta|\le k. \tag{2.17}$$

Let us conclude this section by showing that in Lemma 2.26 one can consider

$$(T,\mathrm{F}) = \left(T_\varepsilon, E^{u_\varepsilon,\varrho_\varepsilon}_{\mathrm{reg},\varepsilon}\right)$$

for a given $\varepsilon > 0$, where $(T_\varepsilon, E^{u_\varepsilon,\varrho_\varepsilon}_{\mathrm{reg},\varepsilon})$ have been defined in the set-up of the bootstrap argument of Section 2.2. Let us prove that the assumptions of Lemma 2.26 indeed hold with $k = \lfloor m-2-d/2\rfloor$. By the estimate (2.2) from Lemma 2.7, we have, for all $t\in(0,T_\varepsilon)$ and $\ell < m-1-d/2$,

$$\begin{aligned}\left\|E^{u_\varepsilon,\varrho_\varepsilon}_{\mathrm{reg},\varepsilon}(t)\right\|_{\mathrm{W}^{\ell,\infty}_x} &\lesssim \left\|E^{u_\varepsilon,\varrho_\varepsilon}_{\mathrm{reg},\varepsilon}(t)\right\|_{\mathrm{H}^{m-1}}\\ &\lesssim \|u_\varepsilon(t)\|_{\mathrm{H}^{m-1}} + \Lambda\left(\|\varrho_\varepsilon(t)\|_{\mathrm{H}^{m-2}}\right)\|\varrho_\varepsilon(t)\|_{\mathrm{H}^m}.\end{aligned}$$

Appealing to Lemma 2.20, and using $\mathcal{N}_{m,r}(f_\varepsilon, \varrho_\varepsilon, u_\varepsilon, T_\varepsilon) \leq R$ for all $T \leq T_\varepsilon$, we deduce that

$$\left\| E_{\mathrm{reg},\varepsilon}^{u_\varepsilon,\varrho_\varepsilon} \right\|_{\mathrm{L}^2(0,T;\mathrm{W}^{k,\infty}(\mathbb{T}^d))} \leq \Lambda(T, R)$$

for $k = \lfloor m - 2 - d/2 \rfloor$.