Chapter 3

Averaging operators related to the dynamics with friction

For any smooth vector field $G(t,s,x,v) \in \mathbb{R}^d$, we consider the following integral operator $\mathrm{K}_G^{\mathrm{free}}$ acting on functions $H(t,x)$:

$$\mathrm{K}_G^{\mathrm{free}}[H](t,x) := \int_0^t \int_{\mathbb{R}^d} [\nabla_x H](s, x-(t-s)v) \cdot G(t,s,x,v)\,\mathrm{d}v\,\mathrm{d}s.$$

This operator, featuring an apparent loss of derivative in space, was systematically studied in [90]. It was proved in [90, Proposition 5.1 and Remark 5.1] that this loss is only apparent, provided that the kernel is sufficiently smooth and decaying in velocity. The statement goes as follows.

Proposition 3.1. *Let $T > 0$. If $p > 1 + d$ and $\sigma > d/2$, then, for all functions $H \in \mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{T}^d))$,*

$$\left\| \mathrm{K}_G^{\mathrm{free}}[H] \right\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{T}^d))} \lesssim \sup_{0\le s,t\le T} \|G(t,s)\|_{\mathcal{H}^p_\sigma} \|H\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{T}^d))}\,.$$

As already noted in [90], this smoothing estimate is reminiscent of (but different from) the so-called kinetic averaging lemmas. Namely, Proposition 3.1 provides the gain of one full derivative.

Averaging lemmas are well known to provide powerful regularity and compactness results in the study of kinetic equations. Loosely speaking, moments in velocity of the solutions appear to gain some regularity compared to the solutions themselves, which are just transported along the flow of the equation. We refer to [1,71,72] for the introduction of the averaging lemmas, and to [6–8, 62, 73, 104, 106, 127] for several extensions of such results.

A thorough comparison between standard kinetic averaging lemmas and the estimate from Proposition 3.1 can be found in [79]. We finally refer to [88] for the use of Proposition 3.1 for a slightly different purpose, as well as to [44] for an extension of this proposition.

In this section, we prove crucial smoothing estimates adapted to kinetic equations with friction, in the spirit of Proposition 3.1. First, we define the corresponding integral operator.

Notation 3.2. For any smooth vector field $G(t,s,x,v)$, we define the following integral operator acting on functions $H(t,x)$ by

$$\mathrm{K}_G^{\mathrm{fric}}[H](t,x) := \int_0^t \int_{\mathbb{R}^d} [\nabla_x H](s, x + (1-e^{t-s})v) \cdot G(t,s,x,v)\, \mathrm{d}v\, \mathrm{d}s,$$

where $(t,x) \in \mathbb{R}^+ \times \mathbb{T}^d$.

Assuming that the kernel G is sufficiently smooth and decaying in velocity, we will prove several continuity and regularization estimates for $\mathrm{K}_G^{\mathrm{free}}$ and $\mathrm{K}_G^{\mathrm{fric}}$ (see Propositions 3.4, 3.5 and 3.7 below).

In what follows, $\mathcal{F}_{x,v}$ will refer to the Fourier transform on $\mathbb{T}^d \times \mathbb{R}^d$ defined as

$$\mathcal{F}_{x,v} h(k,\xi) = \int_{\mathbb{T}^d \times \mathbb{R}^d} e^{-i(k\cdot x + v\cdot \xi)} h(x,v)\, \mathrm{d}x\, \mathrm{d}v, \quad (k,\xi) \in \mathbb{Z}^d \times \mathbb{R}^d.$$

Our first result is the following.

Proposition 3.3. *There exists $C > 0$ such that the following holds. Suppose that $G_{[q]}(t,s,x,v)$ is a kernel of the form*

$$G_{[q]}(t,s,x,v) = (t-s)^q \mathcal{G}(t,s,x,v),$$

with $q \in \mathbb{N}$. For every $T > 0$ satisfying

$$\begin{aligned}
&\|\mathcal{G}\|_{T,s_1,s_2} \\
&\quad := \sup_{0\le t\le T} \left(\sum_{m\in\mathbb{Z}^d} \sup_{0\le s\le t} \sup_{\xi\in\mathbb{R}^d} \left\{ (1+|m|)^{s_2} (1+|\xi|)^{s_1} |(\mathcal{F}_{x,v}\mathcal{G})(t,s,m,\xi)| \right\}^2 \right)^{\frac{1}{2}} \\
&\quad < +\infty
\end{aligned}$$

for $s_1 > 1 + 2q$ and $s_2 > d/2 + 2q$, we have

$$\begin{aligned}
&\left\| \nabla_x^q \mathrm{K}_{G_{[q]}}^{\mathrm{free}}[H] \right\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{T}^d))} + \left\| \nabla_x^q \mathrm{K}_{G_{[q]}}^{\mathrm{fric}}[H] \right\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{T}^d))} \\
&\qquad \le C \|\mathcal{G}\|_{T,s_1,s_2} \|H\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{T}^d))}.
\end{aligned}$$

Proof. We only give the proof in the case of $\mathrm{K}_{G_{[q]}}^{\mathrm{fric}}$ (the proof is similar for $\mathrm{K}_{G_{[q]}}^{\mathrm{free}}$). Writing, for all $t \ge 0$,

$$H(t,x) := \sum_{k\in\mathbb{Z}^d} \widehat{H}_k(t) e^{ik\cdot x} \quad \text{in } \mathrm{L}^2(\mathbb{T}^d),$$

we have

$$\begin{aligned}
\mathrm{K}_{G_{[q]}}^{\mathrm{fric}}[H](t,x) &= \int_0^t \sum_{k\in\mathbb{Z}^d} \widehat{H}_k(s) e^{ik\cdot x} (ik) \cdot \int_{\mathbb{R}^d} e^{-ik\cdot(e^{t-s}-1)v} G_{[q]}(t,s,x,v)\, \mathrm{d}v\, \mathrm{d}s \\
&= \int_0^t \sum_{k\in\mathbb{Z}^d} \widehat{H}_k(s) e^{ik\cdot x} (ik) \cdot (\mathcal{F}_v G_{[q]})(t,s,x,k(e^{t-s}-1))\, \mathrm{d}s.
\end{aligned}$$

We now expand $G_{[q]}$ in Fourier series along the x variable so that

$$
\begin{aligned}
&\mathrm{K}^{\mathrm{fric}}_{G_{[q]}}[H](t,x) \\
&\quad = \sum_{k\in\mathbb{Z}^d}\sum_{l\in\mathbb{Z}^d} e^{i(k+\ell)\cdot x}\int_0^t \widehat{H}_k(s)(ik)\cdot(\mathcal{F}_{x,v}G_{[q]})(t,s,\ell,k(e^{t-s}-1))\,\mathrm{d}s \\
&\quad = \sum_{\ell'\in\mathbb{Z}^d} e^{i\ell'\cdot x}\sum_{k\in\mathbb{Z}^d}\int_0^t \widehat{H}_k(s)(ik)\cdot(\mathcal{F}_{x,v}G_{[q]})(t,s,\ell'-k,k(e^{t-s}-1))\,\mathrm{d}s,
\end{aligned}
$$

and then

$$
\begin{aligned}
&\nabla_x^q \mathrm{K}^{\mathrm{fric}}_{(t-s)^q\mathcal{G}}[H](t,x) \\
&= \sum_{\ell\in\mathbb{Z}^d} e^{i\ell\cdot x}(i\ell)^q\sum_{k\in\mathbb{Z}^d}\int_0^t (t-s)^q\,\widehat{H}_k(s)(ik)\cdot(\mathcal{F}_{x,v}\mathcal{G})(t,s,\ell-k,k(e^{t-s}-1))\,\mathrm{d}s.
\end{aligned}
$$

By the Parseval equality and the Cauchy–Schwarz inequality (in frequency and time), we get

$$
\begin{aligned}
&\left\|\nabla_x^q \mathrm{K}^{\mathrm{fric}}_{(t-s)\mathcal{G}}[F](t)\right\|^2_{\mathrm{L}^2(\mathbb{T}^d)} \\
&= \sum_{\ell\in\mathbb{Z}^d}|\ell|^{2q}\left|\sum_{k\in\mathbb{Z}^d}\int_0^t (t-s)^q\,\widehat{H}_k(s)(ik)\cdot(\mathcal{F}_{x,v}\mathcal{G})(t,s,\ell-k,k(e^{t-s}-1))\,\mathrm{d}s\right|^2 \\
&\le \sum_{\ell\in\mathbb{Z}^d}|\ell|^{2q}\left(\sum_{k\in\mathbb{Z}^d}\int_0^t (t-s)^{2q}\left|\widehat{H}_k(s)\right|^2\left|k\cdot(\mathcal{F}_{x,v}\mathcal{G})(t,s,\ell-k,k(e^{t-s}-1))\right|\mathrm{d}s\right) \\
&\qquad\qquad \times\left(\sum_{k\in\mathbb{Z}^d}\int_0^t \left|k\cdot(\mathcal{F}_{x,v}\mathcal{G})(t,s,\ell-k,k(e^{t-s}-1))\right|\mathrm{d}s\right).
\end{aligned}
$$

Integrating in time yields

$$
\begin{aligned}
&\left\|\nabla_x^q \mathrm{K}^{\mathrm{fric}}_{(t-s)\mathcal{G}}[F]\right\|^2_{\mathrm{L}^2(0,T;\mathrm{L}^2(\times\mathbb{T}^d))} \\
&\quad \le \sum_{\ell\in\mathbb{Z}^d}|\ell|^{2q} \\
&\qquad \times\int_0^T\int_0^t\sum_{k\in\mathbb{Z}^d}(t-s)^{2q}\left|\widehat{H}_k(s)\right|^2\left|k\cdot(\mathcal{F}_{x,v}\mathcal{G})(t,s,\ell-k,k(e^{t-s}-1))\right|\mathrm{d}s\,\mathrm{d}t \\
&\qquad \times\sup_{\ell\in\mathbb{Z}^d}\sup_{t\in(0,T)}\sum_{k\in\mathbb{Z}^d}\int_0^t\left|k\cdot(\mathcal{F}_{x,v}\mathcal{G})(t,s,\ell-k,k(e^{t-s}-1))\right|\mathrm{d}s \\
&\quad = (\mathrm{I})\times(\mathrm{II}).
\end{aligned}
$$

A first step is to note that

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}^d}\int_0^t |k\cdot(\mathcal{F}_{x,v}\mathcal{G})(t,s,\ell-k,k(e^{t-s}-1))|\,\mathrm{d}s\\
&\qquad\le \sum_{k\in\mathbb{Z}^d}\sup_{\substack{s\in(0,t)\\ \xi\in\mathbb{R}^d}}(1+|\xi|)^\alpha|\mathcal{F}_{x,v}\mathcal{G}(t,s,\ell-k,\xi)|\int_0^t\frac{|k|}{(1+|k|(e^{t-s}-1))^\alpha}\,\mathrm{d}s\\
&\qquad\le \sum_{k\in\mathbb{Z}^d}\sup_{\substack{s\in(0,t)\\ \xi\in\mathbb{R}^d}}(1+|\xi|)^\alpha|\mathcal{F}_{x,v}\mathcal{G}(t,s,\ell-k,\xi)|\int_0^t\frac{|k|}{(1+|k|(t-s))^\alpha}\,\mathrm{d}s,
\end{aligned}$$

and that the change of variable $\tau=|k|(t-s)$ in the last integral yields

$$\int_0^t\frac{|k|}{(1+|k|(t-s))^\alpha}\,\mathrm{d}s\le\int_0^{+\infty}\frac{\mathrm{d}\tau}{(1+\tau)^\alpha}<+\infty,$$

provided that $\alpha>1$. With this observation, we can treat the term (II) and obtain, by the Cauchy–Schwarz inequality in k,

$$\text{(II)}\le\sup_{\ell\in\mathbb{Z}^d}\sup_{t\in(0,T)}\sum_{k\in\mathbb{Z}^d}\sup_{\substack{s\in(0,t)\\ \xi\in\mathbb{R}^d}}(1+|\xi|)^{s_1}|\mathcal{F}_{x,v}\mathcal{G}(t,s,\ell-k,\xi)|\le\|\mathcal{G}\|_{T,s_1,s_2}$$

for $s_1>1$ and $s_2>d/2$. For the term (I), we use the Fubini–Tonelli theorem and get

$$\begin{aligned}
&\text{(I)}\\
&\le\sum_{\ell\in\mathbb{Z}^d}|\ell|^{2q}\int_0^T\int_0^t\sum_{k\in\mathbb{Z}^d}(t-s)^{2q}|\widehat{H}_k(s)|^2|k||\mathcal{F}_{x,v}\mathcal{G}(t,s,\ell-k,k(e^{t-s}-1))|\,\mathrm{d}s\,\mathrm{d}t\\
&=\int_0^T\sum_{k\in\mathbb{Z}^d}|\widehat{H}_k(s)|^2\int_s^T\sum_{\ell\in\mathbb{Z}^d}(t-s)^{2q}|\ell|^{2q}|k||\mathcal{F}_{x,v}\mathcal{G}(t,s,\ell-k,k(e^{t-s}-1))|\,\mathrm{d}t\,\mathrm{d}s\\
&\le\|H\|^2_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{T}^d))}\\
&\qquad\times\sup_{k\in\mathbb{Z}^d}\sup_{0\le s\le T}\int_s^T\sum_{\ell\in\mathbb{Z}^d}(t-s)^{2q}|\ell|^{2q}|k||(\mathcal{F}_{x,v}\mathcal{G})(t,s,\ell-k,k(e^{t-s}-1))|\,\mathrm{d}t.
\end{aligned}$$

Note that the last expression can be taken into account for $k\in\dot{\mathbb{Z}}^d$ only (indeed, the term corresponding to $k=0$ vanishes in (I)). We then have

$$\begin{aligned}
&\sup_{k\in\dot{\mathbb{Z}}^d}\sup_{0\le s\le T}\int_s^T(t-s)^{2q}|k|\sum_{\ell\in\mathbb{Z}^d}|\ell|^{2q}|(\mathcal{F}_{x,v}\mathcal{G})(t,s,\ell-k,k(e^{t-s}-1))|\,\mathrm{d}t\\
&\quad\le\sup_{k\in\dot{\mathbb{Z}}^d}\sup_{0\le s\le T}\int_s^T\frac{(t-s)^{2q}|k|}{(1+|k|(e^{t-s}-1))^{\alpha_1}}\\
&\qquad\quad\times\sum_{\ell\in\mathbb{Z}^d}|\ell|^{2q}(1+|k|(e^{t-s}-1))^{\alpha_1}|(\mathcal{F}_{x,v}\mathcal{G})(t,s,\ell-k,k(e^{t-s}-1))|\,\mathrm{d}t
\end{aligned}$$

$$\leq \sup_{k\in\dot{\mathbb{Z}}^d}\sup_{0\leq s\leq T}\int_s^T \frac{(t-s)^{2q}|k|}{(1+|k|(e^{t-s}-1))^{\alpha_1}}\,\mathrm{d}t$$
$$\times \sup_{0\leq s\leq T}\sup_{s\leq t\leq T}\sum_{\ell\in\mathbb{Z}^d}|\ell|^{2q}\sup_{\xi}(1+|\xi|^{\alpha_1})|(\mathcal{F}_{x,v}\mathcal{G})(t,s,\ell-k,\xi)|$$
$$\leq \sup_{k\in\dot{\mathbb{Z}}^d}\sup_{0\leq s\leq T}\int_s^T \frac{(t-s)^{2q}|k|}{(1+|k|(t-s))^{\alpha_1}}\,\mathrm{d}t$$
$$\times \sup_{0\leq s\leq T}\sup_{s\leq t\leq T}\sum_{\ell\in\mathbb{Z}^d}|\ell+k|^{2q}\sup_{\xi}(1+|\xi|^{\alpha_1})|(\mathcal{F}_{x,v}\mathcal{G})(t,s,\ell,\xi)|.$$

Therefore,

$$\sup_{k\in\dot{\mathbb{Z}}^d}\sup_{0\leq s\leq T}\int_s^T (t-s)^{2q}|k|\sum_{\ell\in\mathbb{Z}^d}|\ell|^{2q}|(\mathcal{F}_{x,v}\mathcal{G})(t,s,\ell-k,k(e^{t-s}-1))|\,\mathrm{d}t$$
$$\leq \sup_{k\in\dot{\mathbb{Z}}^d}\sup_{0\leq s\leq T}\int_s^T \frac{(t-s)^{2q}|k|}{(1+|k|(t-s))^{\alpha_1}}\,\mathrm{d}t$$
$$\times \sup_{0\leq s\leq T}\sup_{s\leq t\leq T}\sum_{\ell\in\mathbb{Z}^d}|\ell|^{2q}\sup_{\xi}(1+|\xi|^{\alpha_1})|(\mathcal{F}_{x,v}\mathcal{G})(t,s,\ell,\xi)|$$
$$+ \sup_{k\in\dot{\mathbb{Z}}^d}\sup_{0\leq s\leq T}\int_s^T \frac{(t-s)^{2q}|k|^{1+2q}}{(1+|k|(t-s))^{\alpha_1}}\,\mathrm{d}t$$
$$\times \sup_{0\leq s\leq T}\sup_{s\leq t\leq T}\sum_{\ell\in\mathbb{Z}^d}\sup_{\xi}(1+|\xi|^{\alpha_1})|(\mathcal{F}_{x,v}\mathcal{G})(t,s,\ell,\xi)|$$
$$=: \mathrm{S}_1+\mathrm{S}_2.$$

Let us treat these two terms separately.

First term. We have, by the Cauchy–Schwarz inequality,

$$\mathrm{S}_1 \leq \sup_{k\in\dot{\mathbb{Z}}^d}\sup_{0\leq s\leq T}\int_s^T \frac{(t-s)^{2q}|k|}{(1+|k|(t-s))^{\alpha_1}}\,\mathrm{d}t$$
$$\times \sup_{0\leq s\leq T}\sup_{s\leq t\leq T}\left(\sum_{\ell\in\mathbb{Z}^d}\sup_{\xi}\left\{(1+|\ell|^{2q+\alpha_2})(1+|\xi|^{\alpha_1})|(\mathcal{F}_{x,v}\mathcal{G})(t,s,\ell,\xi)|\right\}^2\right)^{\frac{1}{2}}$$

if $\alpha_2 > d/2$. For the integral term, we write, for $\alpha_1 > 3$,

$$\sup_{k\in\dot{\mathbb{Z}}^d}\sup_{0\leq s\leq T}\int_s^T \frac{(t-s)^{2q}|k|}{(1+|k|(t-s))^{\alpha_1}}\,\mathrm{d}t = \sup_{k\in\dot{\mathbb{Z}}^d}\frac{1}{|k|^{2q}}\sup_{0\leq s\leq T}\int_0^{|k|(T-s)}\frac{\tau^{2q}}{(1+\tau)^{\alpha_1}}\,\mathrm{d}\tau$$
$$\leq \sup_{k\in\dot{\mathbb{Z}}^d}\frac{1}{|k|^{2q}}\int_0^{+\infty}\frac{\tau^{2q}}{(1+\tau)^{\alpha_1}}\,\mathrm{d}\tau,$$

which is a finite constant independent of k and T (since $\alpha_1 > 1 + 2q$); therefore,

$$\mathrm{S}_1 \lesssim \sup_{0\le t\le T} \left(\sum_{\ell\in\mathbb{Z}^d} \sup_{0\le s\le t} \sup_{\xi\in\mathbb{R}^d} \left\{ (1 + |\ell|^{2q+\alpha_2})(1 + |\xi|^{\alpha_1}) |(\mathscr{F}_{x,v}\mathscr{G})(t, s, \ell, \xi)| \right\}^2 \right)^{\frac{1}{2}}.$$

Second term. We have, for $\alpha_2 > d/2$,

$$\begin{aligned}\mathrm{S}_2 \le{} & \sup_{k\in\dot{\mathbb{Z}}^d} \sup_{0\le s\le T} \int_s^T \frac{(t-s)^{2q}|k|^{1+2q}}{(1 + |k|(t-s))^{\alpha_1}}\, \mathrm{d}t \\ & \times \sup_{0\le s\le T} \sup_{s\le t\le T} \left(\sum_{\ell\in\mathbb{Z}^d} \sup_{\xi} \left\{ (1 + |m|^{\alpha_2})(1 + |\xi|^{\alpha_1}) |(\mathscr{F}_{x,v}\mathscr{G})(t, s, \ell, \xi)| \right\}^2 \right)^{\frac{1}{2}}.\end{aligned}$$

The integral term now reads, for $\alpha_1 > 3$,

$$\begin{aligned}& \sup_{k\in\dot{\mathbb{Z}}^d} \sup_{0\le s\le T} \int_s^T \frac{(t-s)^{2q}|k|^{1+2q}}{(1 + |k|(t-s))^{\alpha_1}}\, \mathrm{d}t \\ & \qquad = \sup_{k\in\dot{\mathbb{Z}}^d} \sup_{0\le s\le T} \int_0^{|k|(T-s)} \frac{\tau^{2q}}{(1+\tau)^{\alpha_1}}\, \mathrm{d}\tau \le \int_0^{+\infty} \frac{\tau^{2q}}{(1+\tau)^{\alpha_1}}\, \mathrm{d}\tau;\end{aligned}$$

therefore,

$$\mathrm{S}_2 \lesssim \sup_{0\le t\le T} \left(\sum_{\ell\in\mathbb{Z}^d} \sup_{0\le s\le t} \sup_{\xi\in\mathbb{R}^d} \{ (1 + |\ell|^{\alpha_2})(1 + |\xi|^{\alpha_1}) |(\mathscr{F}_{x,v}\mathscr{G})(t, s, \ell, \xi)| \}^2 \right)^{\frac{1}{2}}.$$

We have thus proved that

$$\text{(I)} \lesssim \|H\|^2_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{T}^d))} \big(\|\mathscr{G}\|_{T,s_1,s_2} + \|\mathscr{G}\|_{T,s_1,s_2+2q} \big)$$

for $s_1 > 1 + 2q$ and $s_2 > d/2 + 2q$. All in all, we get

$$\big\| \nabla_x^q \mathrm{K}^{\mathrm{fric}}_{G_{[q]}}[H] \big\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{T}^d))} \le C \|\mathscr{G}\|_{T,s_1,s_2} \|H\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{T}^d))},$$

which ends the proof. ∎

We then deduce the following two propositions. The first one is a direct consequence of Proposition 3.3 with $q = 0$, and states the continuity of $\mathrm{K}^{\mathrm{free}}_G$ and $\mathrm{K}^{\mathrm{fric}}_G$ on $\mathrm{L}^2_T\mathrm{L}^2_x$.

Proposition 3.4. *There exists $C > 0$ such that, for every $T > 0$, if $p > 1 + d$ and $\sigma > d/2$ then, for all $H \in \mathrm{L}^2(0, T; \mathrm{L}^2(\mathbb{T}^d))$,*

$$\begin{aligned}& \big\|\mathrm{K}^{\mathrm{free}}_G[H]\big\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{T}^d))} + \big\|\mathrm{K}^{\mathrm{fric}}_G[H]\big\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{T}^d))} \\ & \qquad \le C \sup_{0\le s,t\le T} \|G(t,s)\|_{\mathcal{H}^p_\sigma} \|H\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{T}^d))}.\end{aligned}$$

Proof. By Proposition 3.3 with $q = 0$, we have

$$\left\|\mathrm{K}_G^{\text{free}}[H]\right\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{T}^d))} + \left\|\mathrm{K}_G^{\text{fric}}[H]\right\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{T}^d))} \leq C\|G\|_{T,s_1,s_2}\|H\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{T}^d))}$$

for any $s_1 > 1$ and $s_2 > d/2$. Now, appealing to [90, Remark 5.1], one can prove that for all $p > 1 + d$ and $\sigma > d/2$, there exist $s_2 > d/2$ and $s_1 > 1$ such that

$$\|G\|_{T,s_1,s_2} \lesssim \sup_{0\leq s,t\leq T} \|G(t,s)\|_{\mathcal{H}^p_\sigma},$$

hence the result. ∎

When the kernel G vanishes along the diagonal in time $\{t = s\}$, Proposition 3.3 with $q = 1$ leads to the following additional regularizing effect of the operators K_G (as already observed in [91]). Loosely speaking, the operators $\mathrm{K}_G^{\text{free}}$ and $\mathrm{K}_G^{\text{fric}}$ are bounded from $\mathrm{L}^2_T\mathrm{L}^2_x$ to $\mathrm{L}^2_T\dot{\mathrm{H}}^1_x$ in this case.

Proposition 3.5. *There exists $C > 0$ such that, if the kernel G satisfies*

$$G(t,t,x,v) = 0$$

then the following holds. Let $T > 0$. If $p > 7 + d$ and $\sigma > d/2$, then, for $H \in \mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{T}^d))$,

$$\left\|\mathrm{K}_G^{\text{free}}[H]\right\|_{\mathrm{L}^2(0,T;\mathrm{H}^1(\mathbb{T}^d))} + \left\|\mathrm{K}_G^{\text{fric}}[H]\right\|_{\mathrm{L}^2(0,T;\mathrm{H}^1(\mathbb{T}^d))} \leq C(1+T)\sup_{0\leq s,t\leq T}\|\partial_s G(t,s)\|_{\mathcal{H}^p_\sigma}\|H\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{T}^d))}.$$

Proof. Since $G(t,t,x,v) = 0$, Taylor's formula shows that

$$G(t,s,x,v) = (t-s)\widetilde{G}(t,s,x,v),$$
$$\widetilde{G}(t,s,x,v) := -\int_0^1 \partial_s G(t, t+\tau(s-t), x, v)\,\mathrm{d}\tau.$$

By Proposition 3.3 with $q = 1$, we get, for $s_1 > 3$ and $s_2 > d/2 + 2$,

$$\left\|\nabla_x \mathrm{K}_G^{\text{free}}[H]\right\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{T}^d))} + \left\|\nabla_x \mathrm{K}_G^{\text{fric}}[H]\right\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{T}^d))} \leq C\|\widetilde{G}\|_{T,s_1,s_2}\|H\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{T}^d))}.$$

To conclude, we observe that

$$\begin{aligned}
&\left\{(1+|m|)^{s_2}(1+|\xi|)^{s_1}|(\mathcal{F}_{x,v}\widetilde{G})(t,s,m,\xi)|\right\}^2 \\
&\quad\leq \left((1+|m|)^{s_2}(1+|\xi|)^{s_1}\int_0^1 |\mathcal{F}_{x,v}(\partial_s G)(t,t+\tau(s-t),m,\xi)|\,\mathrm{d}\tau\right)^2 \\
&\quad\leq \int_0^1 \left\{(1+|m|)^{s_2}(1+|\xi|)^{s_1}|\mathcal{F}_{x,v}(\partial_s G)(t,t+\tau(s-t),m,\xi)|\right\}^2\mathrm{d}\tau;
\end{aligned}$$

therefore,

$$\left\|\nabla_x \mathrm{K}_G^{\text{free}}[H]\right\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{T}^d))} + \left\|\nabla_x \mathrm{K}_G^{\text{fric}}[H]\right\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{T}^d))} \\ \leq C\|\partial_s G\|_{T,s_1,s_2}\|H\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{T}^d))}.$$

Now, appealing to [90, Remark 5.1], one can prove that for all $p > 2\ell + s + 1 + d$ and $\sigma > d/2$ (with $\ell, s \in \mathbb{R}^+$), there exist $s_2 > \ell + d/2$ and $s_1 > s + 1$ such that

$$\|G\|_{T,s_1,s_2} \lesssim \sup_{0\leq s,t\leq T} \|G(t,s)\|_{\mathcal{H}^p_\sigma}.$$

Since $G(t,t,x,v) = 0$, we also have

$$\|G\|_{T,s_1,s_2} \lesssim T \sup_{0\leq s,t\leq T} \|\partial_s G(t,s)\|_{\mathcal{H}^p_\sigma}.$$

By Proposition 3.4, and taking $\ell = s = 2$, we end up with the desired conclusion. ■

Remark 3.6. A variant of Proposition 3.5 holds in the following form: there exists $C > 0$ such that, for $p > 7 + d$ and $\sigma > d/2$, if

$$G(t,t,x,v) = 0,$$

then we have, for all $H \in \mathrm{L}^2(0,T;\dot{\mathrm{H}}^{-1}(\mathbb{T}^d))$,

$$\left\|\mathrm{K}_G^{\text{free}}[H]\right\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{T}^d))} + \left\|\mathrm{K}_G^{\text{fric}}[H]\right\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{T}^d))} \\ \leq C(1+T) \sup_{0\leq s,t\leq T} \|\partial_s G(t,s)\|_{\mathcal{H}^p_\sigma}\|H\|_{\mathrm{L}^2(0,T;\dot{\mathrm{H}}^{-1}(\mathbb{T}^d))}.$$

We do not detail the proof, which follows the same lines as that of Proposition 3.5.

We finally investigate the smoothing properties of the difference operator $\mathrm{K}_G^{\text{free}} - \mathrm{K}_G^{\text{fric}}$. A somewhat surprising result is the fact that this operator gains one additional derivative. This is the content of the following proposition.

Proposition 3.7. *There exists $C > 0$ such that, for every $T > 0$, if $p > 8 + d$ and $\sigma > 1 + d/2$ then, for all $H \in \mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{T}^d))$,*

$$\left\|\mathrm{K}_G^{\text{free}}[H] - \mathrm{K}_G^{\text{fric}}[H]\right\|_{\mathrm{L}^2(0,T;\mathrm{H}^1(\mathbb{T}^d))} \\ \leq C\varphi(T) \sup_{0\leq s,t\leq T} \|G(t,s)\|_{\mathcal{H}^p_\sigma}\|H\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{T}^d))},$$

where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous nondecreasing function.

Proof. Following the computations performed in the proof of Proposition 3.5, we have

$$\begin{aligned}
&\mathrm{K}_G^{\mathrm{free}}[F](t,x) - \mathrm{K}_G^{\mathrm{fric}}[F](t,x) \\
&\quad = \sum_{\ell\in\mathbb{Z}^d} e^{i\ell\cdot x}\bigg\{\sum_{k\in\mathbb{Z}^d}\int_0^t \widehat{F}_k(s)(ik)\cdot\big[(\mathcal{F}_{x,v}G)(t,s,\ell-k,k(t-s)) \\
&\qquad\qquad - (\mathcal{F}_{x,v}G)(t,s,\ell-k,k(e^{t-s}-1))\big]\,\mathrm{d}s\bigg\}.
\end{aligned}$$

Therefore, if we set

$$\Theta(t,s,\ell,k) := (\mathcal{F}_{x,v}G)(t,s,\ell-k,k(t-s)) - (\mathcal{F}_{x,v}G)(t,s,\ell-k,k(e^{t-s}-1)),$$

we get

$$\begin{aligned}
&\big\|\nabla_x\big(\mathrm{K}_G^{\mathrm{free}}[H] - \mathrm{K}_G^{\mathrm{fric}}[H]\big)(t)\big\|^2_{\mathrm{L}^2(\mathbb{T}^d)} \\
&\quad \le \sum_{\ell\in\mathbb{Z}^d}|\ell|^2\bigg(\sum_{k\in\mathbb{Z}^d}\int_0^t |\widehat{H}_k(s)||k||\Theta(t,s,\ell,k)|\,\mathrm{d}s\bigg)^2.
\end{aligned}$$

We also have, by setting $\mathcal{G}^{t,s}_{\ell-k} = (\mathcal{F}_{x,v}G)(t,s,\ell-k,\cdot)$,

$$\begin{aligned}
|\Theta(t,s,\ell,k)| &= \big|\mathcal{G}^{t,s}_{\ell-k}(k(e^{t-s}-1)) - \mathcal{G}^{t,s}_{\ell-k}(k(t-s))\big| \\
&= \big|\mathcal{G}^{t,s}_{\ell-k}(k(t-s)+k(t-s)^2\varphi(t-s)) - \mathcal{G}^{t,s}_{\ell-k}(k(t-s))\big| \\
&\le \sup_{\theta\in[0,1]}\big|\nabla_\xi\mathcal{G}^{t,s}_{\ell-k}\big(\xi^{t,s}_\theta(k(t-s))\big)\big||k|(t-s)^2\varphi(t-s),
\end{aligned}$$

where $\varphi(z) = \sum_{i\ge 0}\frac{z^i}{(i+2)!}$ and $\xi^{t,s}_\theta(z) = z + \theta z(t-s)\varphi(t-s)$. By continuity, there exists $\theta^\star \in [0,1]$ (which may depend on all the other variables) such that

$$|\Theta(t,s,\ell,k)| \le \big|\nabla_\xi\mathcal{G}^{t,s}_{\ell-k}\big(\xi^{t,s}_{\theta^\star}(k(t-s))\big)\big||k|(t-s)^2\varphi(t-s).$$

This yields

$$\begin{aligned}
&\big\|\nabla_x\left(\mathrm{K}_G^{\mathrm{free}}[H] - \mathrm{K}_G^{\mathrm{fric}}[H]\right)(t)\big\|^2_{\mathrm{L}^2(\mathbb{T}^d)} \\
&\le \sum_{\ell\in\mathbb{Z}^d}|\ell|^2\bigg(\sum_{k\in\mathbb{Z}^d}\int_0^t |\widehat{H}_k(s)||k|^2(t-s)^2\varphi(t-s)\big|\nabla_\xi\mathcal{G}^{t,s}_{\ell-k}\big(\xi^{t,s}_{\theta^\star}(k(t-s))\big)\big|\,\mathrm{d}s\bigg)^2 \\
&\le \sum_{\ell\in\mathbb{Z}^d}|\ell|^2\bigg(\sum_{k\in\mathbb{Z}^d}\int_0^t |\widehat{H}_k(s)|^2|k|^2(t-s)^3\varphi(t-s)^2\big|\nabla_\xi\mathcal{G}^{t,s}_{\ell-k}\big(\xi^{t,s}_{\theta^\star}(k(t-s))\big)\big|\,\mathrm{d}s\bigg) \\
&\qquad \times\bigg(\sum_{k\in\mathbb{Z}^d}\int_0^t |k|^2(t-s)\big|\nabla_\xi\mathcal{G}^{t,s}_{\ell-k}\big(\xi^{t,s}_{\theta^\star}(k(t-s))\big)\big|\,\mathrm{d}s\bigg),
\end{aligned}$$

thanks to the Cauchy–Schwarz inequality. As in the proof of Proposition 3.5, we obtain by integrating in time that

$$\begin{aligned}
&\left\| \nabla_x \left(\mathrm{K}_G^{\mathrm{free}}[H] - \mathrm{K}_G^{\mathrm{fric}}[H] \right) \right\|^2_{\mathrm{L}^2((0,T)\times\mathbb{T}^d)} \\
&\quad \leq \Bigg(\sum_{\ell\in\mathbb{Z}^d} |\ell|^2 \int_0^T \int_0^t \sum_{k\in\mathbb{Z}^d} |\widehat{H}_k(s)|^2 |k|^2 (t-s)^3 \varphi(t-s)^2 \\
&\qquad\qquad \times \left| \nabla_\xi \mathcal{G}^{t,s}_{\ell-k} \left(\xi^{t,s}_{\theta^\star}(k(t-s)) \right) \right| \mathrm{d}s\, \mathrm{d}t \Bigg) \\
&\qquad \times \sup_{\ell\in\mathbb{Z}^d} \sup_{t\in(0,T)} \sum_{k\in\mathbb{Z}^d} \int_0^t |k|^2 (t-s) \left| \nabla_\xi \mathcal{G}^{t,s}_{\ell-k} \left(\xi^{t,s}_{\theta^\star}(k(t-s)) \right) \right| \mathrm{d}s \\
&\quad =: \ (\mathrm{A}) \times (\mathrm{B}).
\end{aligned}$$

First, we note that for all $\theta \in [0,1]$, $k \in \mathbb{Z}^d$ and $0 \leq s \leq t$,

$$\begin{aligned}
|\xi^{t,s}_\theta(k(t-s))| &= |k(t-s) + \theta k (t-s)^2 \varphi(t-s)| \\
&= |k|(t-s)[1 + \theta(t-s)\varphi(t-s)] \\
&\geq |k|(t-s).
\end{aligned}$$

For (B), we thus have

$$\begin{aligned}
(\mathrm{B}) &\leq \sup_{\ell\in\mathbb{Z}^d} \sup_{t\in(0,T)} \sum_{k\in\mathbb{Z}^d} \int_0^t \left(1 + |\xi^{t,s}_{\theta^\star}(k(t-s))|\right)^{\beta_1} \left| \nabla_\xi \mathcal{G}^{t,s}_{\ell-k} \left(\xi^{t,s}_{\theta^\star}(k(t-s)) \right) \right| \\
&\qquad\qquad \times \frac{|k|^2(t-s)}{(1 + |\xi^{t,s}_{\theta^\star}(k(t-s))|)^{\beta_1}} \mathrm{d}s \\
&\leq \sup_{\ell\in\mathbb{Z}^d} \sup_{t\in(0,T)} \sum_{k\in\mathbb{Z}^d} \sup_{s,\xi} \left\{ (1+|\xi|)^{\beta_1} |\nabla_\xi \mathcal{G}^{t,s}_{\ell-k}(\xi)| \right\} \int_0^t \frac{|k|^2(t-s)}{(1+|k|(t-s))^{\beta_1}} \mathrm{d}s.
\end{aligned}$$

Since

$$\int_0^t \frac{|k|^2(t-s)}{(1+|k|(t-s))^{\beta_1}} \mathrm{d}s = \int_0^{|k|t} \frac{\tau}{(1+\tau)^{\beta_1}} \mathrm{d}\tau \leq \int_0^{+\infty} \frac{\tau}{(1+\tau)^{\beta_1}} \mathrm{d}\tau < +\infty$$

if $\beta_1 > 2$, we get

$$(\mathrm{B}) \lesssim \sup_{\ell\in\mathbb{Z}^d} \sup_{t\in(0,T)} \sum_{k\in\mathbb{Z}^d} \sup_{s,\xi} \left\{ (1+|\xi|)^{\beta_1} |\nabla_\xi \mathcal{G}^{t,s}_{\ell-k}(\xi)| \right\}.$$

By choosing $\beta_2 > d/2$ and by the Cauchy–Schwarz inequality, this yields

$$(\mathrm{B}) \lesssim \sup_{t\in(0,T)} \left(\sum_{k\in\mathbb{Z}^d} \sup_{s,\xi} \left\{ (1+|k|)^{\beta_2} (1+|\xi|)^{\beta_1} |\nabla_\xi \mathcal{G}^{t,s}_k(\xi)| \right\}^2 \right)^{\frac{1}{2}}.$$

Let us estimate the other term, (A). By the Fubini–Tonelli theorem, we have

$$
\begin{aligned}
\text{(A)} &= \sum_{\ell\in\mathbb{Z}^d} |\ell|^2 \int_0^T \int_0^t \sum_{k\in\mathbb{Z}^d} |\widehat{H}_k(s)|^2 |k|^2 (t-s)^3 \varphi(t-s)^2 \\
&\qquad\qquad \times \left|\nabla_\xi \mathcal{G}^{t,s}_{\ell-k}\big(\xi^{t,s}_{\theta^\star}(k(t-s))\big)\right| \mathrm{d}s\, \mathrm{d}t \\
&= \int_0^T \sum_{k\in\mathbb{Z}^d} |\widehat{H}_k(s)|^2 \int_s^T \sum_{\ell\in\mathbb{Z}^d} |\ell|^2 |k|^2 (t-s)^3 \varphi(t-s)^2 \\
&\qquad\qquad \times \left|\nabla_\xi \mathcal{G}^{t,s}_{\ell-k}\big(\xi^{t,s}_{\theta^\star}(k(t-s))\big)\right| \mathrm{d}t\, \mathrm{d}s \\
&\le \|H\|^2_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{T}^d))} \\
&\quad \times \sup_{k\in\mathbb{Z}^d} \sup_{0\le s\le T} \int_s^T \sum_{\ell\in\mathbb{Z}^d} |\ell|^2 |k|^2 (t-s)^3 \varphi(t-s)^2 \left|\nabla_\xi \mathcal{G}^{t,s}_{\ell-k}\big(\xi^{t,s}_{\theta^\star}(k(t-s))\big)\right| \mathrm{d}t.
\end{aligned}
$$

As in the proof of Proposition 3.3, we have

$$
\begin{aligned}
&\sup_{k\in\mathbb{Z}^d} \sup_{0\le s\le T} \int_s^T |k|^2 (t-s)^3 \varphi(t-s)^2 \sum_{\ell\in\mathbb{Z}^d} |\ell|^2 \left|\nabla_\xi \mathcal{G}^{t,s}_{\ell-k}\big(\xi^{t,s}_{\theta^\star}(k(t-s))\big)\right| \mathrm{d}t \\
&\quad \le \sup_{k\in\dot{\mathbb{Z}}^d} \sup_{0\le s\le T} \int_s^T \frac{|k|^2 (t-s)^3 \varphi(t-s)^2}{(1+|k|(t-s))^{\alpha_1}} \,\mathrm{d}t \\
&\qquad\quad \times \sup_{0\le s\le T} \sup_{s\le t\le T} \sum_{\ell\in\mathbb{Z}^d} |\ell|^2 \sup_\xi (1+|\xi|^{\alpha_1}) |\nabla_\xi \mathcal{G}^{t,s}_\ell(\xi)| \\
&\qquad + \sup_{k\in\dot{\mathbb{Z}}^d} \sup_{0\le s\le T} \int_s^T \frac{|k|^4 (t-s)^3 \varphi(t-s)^2}{(1+|k|(t-s))^{\alpha_1}} \,\mathrm{d}t \\
&\qquad\quad \times \sup_{0\le s\le T} \sup_{s\le t\le T} \sum_{\ell\in\mathbb{Z}^d} \sup_\xi (1+|\xi|^{\alpha_1}) |\nabla_\xi \mathcal{G}^{t,s}_\ell(\xi)| \\
&\quad =: \mathrm{T}_1 + \mathrm{T}_2.
\end{aligned}
$$

We treat these two terms separately.

First term. In T_1, the integral term can be bounded via

$$
\begin{aligned}
&\sup_{k\in\dot{\mathbb{Z}}^d} \sup_{0\le s\le T} \int_s^T \frac{|k|^2 (t-s)^3 \varphi(t-s)^2}{(1+|k|(t-s))^{\alpha_1}} \,\mathrm{d}t \\
&\quad \le \sup_{k\in\dot{\mathbb{Z}}^d} \frac{1}{|k|^2} \sup_{0\le s\le T} \varphi(T-s)^2 \int_0^{+\infty} \frac{\tau^3}{(1+\tau)^{\alpha_1}} \,\mathrm{d}\tau. \\
&\quad \le \varphi(T)^2 \sup_{k\in\dot{\mathbb{Z}}^d} \frac{1}{|k|^2} \int_0^{+\infty} \frac{\tau^3}{(1+\tau)^{\alpha_1}} \,\mathrm{d}\tau \\
&\quad \lesssim \varphi(T)^2,
\end{aligned}
$$

provided that $\alpha_1 > 4$. This implies that for any $\alpha_2 > d/2$,

$$\mathrm{T}_1 \lesssim \varphi(T)^2 \sup_{0\le s\le T} \sup_{s\le t\le T} \left(\sum_{m\in\mathbb{Z}^d} \sup_{\xi} \left\{ (1+|m|^{2+\alpha_2})(1+|\xi|^{\alpha_1})|\nabla_\xi \mathcal{G}_m^{t,s}(\xi)| \right\}^2 \right)^{\frac{1}{2}}.$$

Second term. In T_2, the integral term can be bounded in a similar way via

$$\begin{aligned}\sup_{k\in\dot{\mathbb{Z}}^d} \sup_{0\le s\le T} \int_s^T \frac{|k|^4 (t-s)^3 \varphi(t-s)^2}{(1+|k|(t-s))^{\alpha_1}}\,\mathrm{d}t &\le \sup_{0\le s\le T} \varphi(T-s)^2 \int_0^{+\infty} \frac{\tau^3}{(1+\tau)^{\alpha_1}}\,\mathrm{d}\tau \\ &\le \varphi(T)^2 \int_0^{+\infty} \frac{\tau^3}{(1+\tau)^{\alpha_1}}\,\mathrm{d}\tau \\ &\lesssim \varphi(T)^2,\end{aligned}$$

provided that $\alpha_1 > 4$. This implies that for any $\alpha_2 > d/2$,

$$\mathrm{T}_2 \lesssim \varphi(T) \sup_{0\le s\le T} \sup_{s\le t\le T} \left(\sum_{m\in\mathbb{Z}^d} \sup_{\xi} \left\{ (1+|m|^{\alpha_2})(1+|\xi|^{\alpha_1})|\nabla_\xi \mathcal{G}_m^{t,s}(\xi)| \right\}^2 \right)^{\frac{1}{2}}.$$

All in all, we get, for $\alpha_1 > 4$ and $\alpha_2 > 2 + d/2$,

$$\begin{aligned}(\mathrm{A}) &\lesssim \varphi(T)^2 \|H\|^2_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{T}^d))} \\ &\quad\times \sup_{0\le s\le T} \sup_{s\le t\le T} \left(\sum_{m\in\mathbb{Z}^d} \sup_{\xi} \left\{ (1+|m|^{\alpha_2})(1+|\xi|^{\alpha_1})|\nabla_\xi \mathcal{G}_m^{t,s}(\xi)| \right\}^2 \right)^{\frac{1}{2}} \\ &\lesssim \varphi(T)^2 \|H\|^2_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{T}^d))} \\ &\quad\times \sup_{0\le t\le T} \left(\sum_{m\in\mathbb{Z}^d} \sup_{0\le s\le t} \sup_{\xi} \left\{ (1+|m|^{\alpha_2})(1+|\xi|^{\alpha_1})|\nabla_\xi \mathcal{G}_m^{t,s}(\xi)| \right\}^2 \right)^{\frac{1}{2}}.\end{aligned}$$

We have thus proved that for $s_1 > 4$ and $s_2 > 2 + d/2$,

$$\left\| \nabla_x \left(\mathrm{K}_G^{\mathrm{free}}[H] - \mathrm{K}_G^{\mathrm{fric}}[H] \right) \right\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{T}^d))} \lesssim \varphi(T) \|v\otimes G\|_{T,s_1,s_2} \|H\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{T}^d))},$$

where we have used the seminorm $\|\cdot\|_{T,s_1,s_2}$ from Proposition 3.3. We can now conclude as in the proof of Proposition 3.5. We observe that for all $p > 2\ell + s + 1 + d$ and $\sigma > 1 + d/2$ (with $\ell, s \in \mathbb{R}^+$), there exist $s_1 > s + 1$ and $s_2 > \ell + d/2$ such that

$$\|v\otimes G\|_{T,s_1,s_2} \lesssim \sup_{0\le s,t\le T} \|G(t,s)\|_{\mathcal{H}^p_\sigma}.$$

By taking $\ell = 2$ and $s = 3$, and by finally using Proposition 3.3, we reach the desired conclusion. ∎