Chapter 4

Analysis of the kinetic moments

Following the bootstrap procedure initiated in Section 2.2, we aim at controlling the term $\|\varrho_\varepsilon\|_{L^2(0,T;H^m)}$ uniformly in ε and for $T < T_\varepsilon$. In view of the transport equation bearing on ϱ_ε (see Lemma 2.2), we will relate the kinetic moments ρ_{f_ε} and j_{f_ε} to the fluid density ϱ_ε itself.

In this section, to ease readability, we drop out the subscript ε when we refer to the solution. Recall that Λ will always stand for a nonnegative continuous function which is independent of ε, nondecreasing with respect to each of its arguments, that may depend implicitly on the initial data and that may change from line to line.

For all $T \in [0, T_\varepsilon)$ small enough, the goal of this chapter is to prove the following result.

Proposition 4.1. *Let $T \in (0, \min(T_\varepsilon(R), \overline{T}(R)))$. For all $|I| \leq m$, we have, for any $t \in (0, T)$,*

$$\partial_x^I \rho_f(t,x)$$
$$= p'(\varrho(t,x)) \int_0^t \int_{\mathbb{R}^d} \nabla_x [\mathrm{J}_\varepsilon \partial_x^I \varrho](s, x-(t-s)v) \cdot \nabla_v f(t,x,v) \,\mathrm{d}v \,\mathrm{d}s + R^I[\rho_f](t,x),$$

$$\partial_x^I j_f(t,x)$$
$$= p'(\varrho(t,x)) \int_0^t \int_{\mathbb{R}^d} v \nabla_x [\mathrm{J}_\varepsilon \partial_x^I \varrho](s, x-(t-s)v) \cdot \nabla_v f(t,x,v) \,\mathrm{d}v \,\mathrm{d}s + R^I[j_f](t,x),$$

where the remainders $R^I[\rho_f]$ and $R^I[j_f]$ satisfy

$$\left\| R^I[\rho_f] \right\|_{L^2(0,T,H^1_x)} \leq \Lambda(T,R), \quad \left\| R^I[j_f] \right\|_{L^2(0,T,H^1_x)} \leq \Lambda(T,R).$$

We recall the definition of the time $T_\varepsilon(R)$ from our bootstrap procedure presented in Section 2.2, as well as the definition of the time $\overline{T}(R)$ from Lemma 2.26 in Section 2.3. Note that the latter is independent of ε. In the rest of this section, we will always implicitly consider times $T > 0$ such that

$$T < \min(T_\varepsilon(R), \overline{T}(R)).$$

From Proposition 4.1, thanks to the analysis of Chapter 3, we can immediately infer the following corollary.

Corollary 4.2. *For $m > 2 + d$, $\sigma > 1 + d/2$ and $|I| \leq m$, we have*

$$\left\| \partial_x^I \rho_f \right\|_{L^2(0,T;L^2)} \leq \Lambda(T,R),$$
$$\left\| \partial_x^I j_f \right\|_{L^2(0,T;L^2)} \leq \Lambda(T,R).$$

Proof. By Proposition 4.1, we can write

$$\partial_x^I \rho_f = p'(\varrho)\mathrm{K}_G^{\mathrm{free}}\big[\mathrm{J}_\varepsilon \partial_x^I \varrho\big] + R^I[\rho_f],$$

with $G(t,x,v) = \nabla_v f(t,x,v)$ and $\big\| R^I[\rho_f]\big\|_{\mathrm{L}^2(0,T,\mathrm{H}^1_x)} \leq \Lambda(T,R)$. Since the kernel G satisfies, for $p > 1+d$,

$$\sup_{0\leq t\leq T} \|G\|_{\mathcal{H}^p_\sigma} \leq \|f\|_{\mathrm{L}^\infty(0,T;\mathcal{H}^{m-1}_\sigma)} \leq \Lambda(T,R),$$

we can use the estimate from Proposition 3.1 to get

$$\begin{aligned}\big\|\partial_x^I \rho_f\big\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)} &\lesssim \|p'(\varrho)\|_{\mathrm{L}^2(0,T;\mathrm{L}^\infty)}\Lambda(T,R)\big\|\partial_x^I\varrho\big\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)} \\ &\quad + \big\|R^I[\rho_f]\big\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)} \\ &\lesssim \mathrm{C}\big(\|\varrho\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^{m-2})}\big)\Lambda(T,R) + \Lambda(T,R) \\ &\leq \Lambda(T,R),\end{aligned}$$

by Sobolev embedding, Proposition A.3 and Lemma 2.20. The same argument applies for $\big\|\partial_x^I j_f\big\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)}$. ■

Our strategy to prove Proposition 4.1 goes as follows:

- first, we take derivatives in the Vlasov equation to obtain a system of coupled kinetic equations satisfied by the augmented unknown $\big(\partial_x^I\partial_v^J f_\varepsilon\big)_{|I|+|J|=m-1,m}$;
- next, we study the average in velocity of $\mathcal{F}$ by relying on the Duhamel formula and the Lagrangian point of view of Section 2.3. We isolate the leading terms and prove estimates for the remainders, using crucially the techniques developed in Section 2.3 and Chapter 3.

4.1 The integro-differential system for derivatives of moments

4.1.1 Applying derivatives

We start with the following algebraic lemma, where we apply $\partial_x^I\partial_v^J$ to the Vlasov equation. Let us recall the notation $\widehat{\alpha}^k$ and $\overline{\alpha}^k$ for shifted indices (see Notation 2.8).

Lemma 4.3. *For any $I = (i_1,\dots,i_d)$, $J = (j_1,\dots,j_d) \in \mathbb{N}^d$ such that $|I|+|J| \in \{m-1,m\}$ and for any smooth function $f(t,x,v)$, we have*

$$\big[\partial_x^I\partial_v^J, \mathcal{T}^{u,\varrho}_{\mathrm{reg},\varepsilon}\big]f = \partial_x^I\partial_v^J E^{u,\varrho}_{\mathrm{reg},\varepsilon}\cdot\nabla_v f + \mathcal{M}^{I,J}\mathcal{F} + \mathcal{R}_1^{I,J} + \mathcal{R}_0^{I,J},$$

where

$$\mathcal{F} := \big(\partial_x^I\partial_v^J f\big)_{\substack{I,J\in\mathbb{N}^d,\\ |I|+|J|\in\{m-1,m\}}},$$

$$\mathcal{M}^{I,J}\mathcal{F} := \sum_{p=1}^{d} \mathbf{1}_{j_p\neq 0}\left(\partial_x^{\widehat{I}^p}\partial_v^{\overline{J}^p} f - \partial_x^I\partial_v^J f\right) + \mathbf{1}_{|I|>2}\sum_{\substack{0<\alpha<I\\ |\alpha|\in\{1,2\}}}\binom{I}{\alpha}\partial_x^\alpha E^{u,\varrho}_{\mathrm{reg},\varepsilon}\cdot\nabla_v\partial_x^{I-\alpha}\partial_v^J f + \mathbf{1}_{|I|=1,2}\partial_x^I E^{u,\varrho}_{\mathrm{reg},\varepsilon}\cdot\partial_v^J\nabla_v f,$$

$$\mathcal{R}_1^{I,J} := \mathbf{1}_{\substack{|I|>2\\ |J|\neq 0}}\partial_x^I E^{u,\varrho}_{\mathrm{reg},\varepsilon}\cdot\nabla_v\partial_v^J f + \mathbf{1}_{|I|>1}\sum_{\substack{0<\alpha<I\\ |\alpha|=m-1}}\binom{I}{\alpha}\partial_x^\alpha E^{u,\varrho}_{\mathrm{reg},\varepsilon}\cdot\nabla_v\partial_x^{I-\alpha}\partial_v^J f,$$

where

$$\left\|\mathcal{R}_1^{I,J}\right\|_{\mathrm{L}^2(0,T;\mathcal{H}^0_r)} \leq \Lambda(T,R), \tag{4.1}$$

and $\mathcal{R}_0^{I,J}$ *is a remainder satisfying*

$$\left\|\mathcal{R}_0^{I,J}\right\|_{\mathrm{L}^2(0,T;\mathcal{H}^1_r)} \leq \Lambda(T,R). \tag{4.2}$$

Proof. Using Lemma 2.9, we have

$$\partial_x^I\partial_v^J\left(\mathcal{T}^{u,\varrho}_{\mathrm{reg},\varepsilon} f\right) = \mathcal{T}^{u,\varrho}_{\mathrm{reg},\varepsilon}\left(\partial_x^I\partial_v^J f\right) + \sum_{p=1}^{d}\mathbf{1}_{j_p\neq 0}\left(\partial_x^{\widehat{I}^p}\partial_v^{\overline{J}^p} f - \partial_x^I\partial_v^J f\right) + \left[\partial_x^I\partial_v^J, E^{u,\varrho}_{\mathrm{reg},\varepsilon}\cdot\nabla_v\right]f.$$

Since the force $E^{u,\varrho}_{\mathrm{reg},\varepsilon}(t,x)$ does not depend on v, we expand the commutator as

$$\left[\partial_x^I\partial_v^J, E(t,x)\cdot\nabla_v\right]f = \mathbf{1}_{I\neq 0}\partial_x^I E^{u,\varrho}_{\mathrm{reg},\varepsilon}\cdot\partial_v^J\nabla_v f + \mathbf{1}_{|I|>1}\sum_{0<\alpha<I}\binom{I}{\alpha}\partial_x^\alpha E^{u,\varrho}_{\mathrm{reg},\varepsilon}\cdot\partial_x^{I-\alpha}\partial_v^J\nabla_v f.$$

Note that if $J\neq 0$ then $|I|\leq m-1$, and if $|I|=1,2$ then $J\neq 0$. The terms that cannot be considered as remainders in the last sum are those for which $|I|-|\alpha|+|J|+1\in\{m-1,m\}$, that is, those with $|\alpha|\in\{1,2\}$. We thus have

$$\left[\partial_x^I\partial_v^J, E^{u,\varrho}_{\mathrm{reg},\varepsilon}\cdot\nabla_v\right]f = \mathbf{1}_{\substack{|I|\neq 0\\ J=0}}\partial_x^I\partial_v^J E^{u,\varrho}_{\mathrm{reg},\varepsilon}\cdot\nabla_v f + \mathbf{1}_{|I|=1,2}\partial_x^I E^{u,\varrho}_{\mathrm{reg},\varepsilon}\cdot\partial_v^J\nabla_v f + \mathbf{1}_{|I|>2}\sum_{\substack{0<\alpha<I\\ |\alpha|\in\{1,2\}}}\binom{I}{\alpha}\partial_x^\alpha E^{u,\varrho}_{\mathrm{reg},\varepsilon}\cdot\nabla_v\partial_x^{I-\alpha}\partial_v^J f + \mathcal{R}_1^{I,J} + \mathcal{R}_0^{I,J},$$

where

$$\mathcal{R}_1^{I,J} := \mathbf{1}_{\substack{|I|>2\\ |J|\neq 0}}\partial_x^I E^{u,\varrho}_{\mathrm{reg},\varepsilon}\cdot\partial_v^J\nabla_v f + \mathbf{1}_{|I|>1}\sum_{\substack{0<\alpha<I\\ |\alpha|=m-1}}\binom{I}{\alpha}\partial_x^\alpha E^{u,\varrho}_{\mathrm{reg},\varepsilon}\cdot\nabla_v\partial_x^{I-\alpha}\partial_v^J f,$$

$$\mathcal{R}_0^{I,J} := \mathbf{1}_{|I|>1} \sum_{\substack{0<\alpha<I \\ 3\leq|\alpha|\leq m-2}} \binom{I}{\alpha} \partial_x^\alpha E^{u,\varrho}_{\mathrm{reg},\varepsilon} \cdot \nabla_v \partial_x^{I-\alpha} \partial_v^J f.$$

Let us estimate the remainder $\mathcal{R}_0^{I,J}$ in $\mathcal{H}^1_r$: setting $\chi(v) = (1+|v|^2)^{r/2}$, we have

$$\begin{aligned}
\|\mathcal{R}_0^{I,J}\|_{\mathcal{H}^1_r} \lesssim & \sum_{\substack{0<\alpha<I \\ 3\leq|\alpha|\leq m-2}} \|\chi \partial_x^\alpha E^{u,\varrho}_{\mathrm{reg},\varepsilon} \cdot \nabla_v \partial_x^{I-\alpha} \partial_v^J f\|_{\mathrm{L}^2_{x,v}} \\
& + \sum_{\substack{0<\alpha<I \\ 3\leq|\alpha|\leq m-2}} \sum_{k=1}^d \Big(\|\chi \partial_x^{\widehat{\alpha}^k} E^{u,\varrho}_{\mathrm{reg},\varepsilon} \cdot \nabla_v \partial_x^{I-\alpha} \partial_v^J f\|_{\mathrm{L}^2_{x,v}} \\
& \qquad + \|\chi \partial_x^\alpha E^{u,\varrho}_{\mathrm{reg},\varepsilon} \cdot \partial_{x_k} \nabla_v \partial_x^{I-\alpha} \partial_v^J f\|_{\mathrm{L}^2_{x,v}} \\
& \qquad + \|\chi \partial_x^\alpha E^{u,\varrho}_{\mathrm{reg},\varepsilon} \cdot \partial_{v_k} \nabla_v \partial_x^{I-\alpha} \partial_v^J f\|_{\mathrm{L}^2_{x,v}} \Big).
\end{aligned}$$

First case: $\frac{m+1}{2} \leq |\alpha| \leq m-2$. We have, for all k,

$$\begin{aligned}
&\|\chi \partial_x^\alpha E^{u,\varrho}_{\mathrm{reg},\varepsilon} \cdot \nabla_v \partial_x^{I-\alpha} \partial_v^J f\|^2_{\mathrm{L}^2_{x,v}} + \|\chi \partial_x^{\widehat{\alpha}^k} E^{u,\varrho}_{\mathrm{reg},\varepsilon} \cdot \nabla_v \partial_x^{I-\alpha} \partial_v^J f\|^2_{\mathrm{L}^2_{x,v}} \\
&\quad \leq \Big(\|\partial_x^\alpha E^{u,\varrho}_{\mathrm{reg},\varepsilon}\|^2_{\mathrm{L}^2} + \|\partial_x^{\widehat{\alpha}^k} E^{u,\varrho}_{\mathrm{reg},\varepsilon}\|^2_{\mathrm{L}^2} \Big) \int_{\mathbb{R}^d} \chi(v)^2 \|\nabla_v \partial_x^{I-\alpha} \partial_v^J f\|^2_{\mathrm{L}^\infty} \,\mathrm{d}v \\
&\quad \lesssim \|E^{u,\varrho}_{\mathrm{reg},\varepsilon}\|^2_{\mathrm{H}^{m-1}} \int_{\mathbb{R}^d} \chi(v)^2 \|\nabla_v \partial_v^J f\|^2_{\mathrm{H}^\sigma} \,\mathrm{d}v,
\end{aligned}$$

provided that $\sigma > |I-\alpha| + d/2$. Since $|J| + 1 + |I| - |\alpha| + d/2 \leq m + 1 - |\alpha| + d/2 \leq \frac{m+1+d}{2}$, and since $m > 4 + d$, we can find such a σ so that

$$\begin{aligned}
&\|\chi \partial_x^\alpha E^{u,\varrho}_{\mathrm{reg},\varepsilon} \cdot \nabla_v \partial_x^{I-\alpha} \partial_v^J f\|_{\mathrm{L}^2_{x,v}} + \|\chi \partial_x^{\widehat{\alpha}^k} E^{u,\varrho}_{\mathrm{reg},\varepsilon} \cdot \nabla_v \partial_x^{I-\alpha} \partial_v^J f\|_{\mathrm{L}^2_{x,v}} \\
&\quad \lesssim \|E^{u,\varrho}_{\mathrm{reg},\varepsilon}\|_{\mathrm{H}^{m-1}} \|f\|_{\mathcal{H}^{m-1}_r}.
\end{aligned}$$

Likewise, we have, for all k,

$$\|\chi \partial_x^\alpha E^{u,\varrho}_{\mathrm{reg},\varepsilon} \cdot \partial_{x_k} \nabla_v \partial_x^{I-\alpha} \partial_v^J f\|^2_{\mathrm{L}^2_{x,v}} \leq \|E^{u,\varrho}_{\mathrm{reg},\varepsilon}\|^2_{\mathrm{H}^{m-1}} \int_{\mathbb{R}^d} \chi(v)^2 \|\nabla_v \partial_v^J f\|^2_{\mathrm{H}^\sigma} \,\mathrm{d}v,$$

provided that $\sigma > 1 + |I-\alpha| + d/2$. Since $|J| + 1 + 1 + |I| - |\alpha| + d/2 \leq m + 2 - |\alpha| + d/2 \leq \frac{m+3+d}{2}$, and since $m > 6 + d$, there exists such a σ so that

$$\|\chi \partial_x^\alpha E \cdot \partial_{x_k} \nabla_v \partial_x^{I-\alpha} \partial_v^J f\|_{\mathrm{L}^2_{x,v}} \lesssim \|E\|_{\mathrm{H}^{m-1}} \|f\|_{\mathcal{H}^{m-1}_r}.$$

The same procedure can be applied for the terms $\|\chi \partial_x^\alpha E^{u,\varrho}_{\mathrm{reg},\varepsilon} \cdot \partial_{v_k} \nabla_v \partial_x^{I-\alpha} \partial_v^J f\|_{\mathrm{L}^2_{x,v}}$.

Second case: $3 \leq |\alpha| < \frac{m+1}{2}$. We have

$$\begin{aligned}
&\big\|\chi \partial_x^\alpha E^{u,\varrho}_{\mathrm{reg},\varepsilon} \cdot \nabla_v \partial_x^{I-\alpha} \partial_v^J f\big\|_{\mathrm{L}^2_{x,v}} + \big\|\chi \partial_x^{\widehat{\alpha}^k} E^{u,\varrho}_{\mathrm{reg},\varepsilon} \cdot \nabla_v \partial_x^{I-\alpha} \partial_v^J f\big\|_{\mathrm{L}^2_{x,v}} \\
&\quad \leq \Big(\big\|\partial_x^\alpha E^{u,\varrho}_{\mathrm{reg},\varepsilon}\big\|_{\mathrm{L}^\infty} + \big\|\partial_x^{\widehat{\alpha}^k} E^{u,\varrho}_{\mathrm{reg},\varepsilon}\big\|_{\mathrm{L}^\infty}\Big) \big\|\chi \nabla_v \partial_x^{I-\alpha} \partial_v^J f\big\|_{\mathrm{L}^2_{x,v}} \\
&\quad \lesssim \big\|E^{u,\varrho}_{\mathrm{reg},\varepsilon}\big\|_{\mathrm{H}^\sigma} \|f\|_{\mathcal{H}^{m-1}_r},
\end{aligned}$$

provided that $\sigma > 1 + |\alpha| + d/2$. Since $1 + |\alpha| + d/2 \leq \frac{m+3+d}{2}$ and $m > 5 + d$, we can find such a σ so that

$$\big\|\chi \partial_x^{\widehat{\alpha}^k} E^{u,\varrho}_{\mathrm{reg},\varepsilon} \cdot \nabla_v \partial_x^{I-\alpha} \partial_v^J f\big\|_{\mathrm{L}^2_{x,v}} \lesssim \big\|E^{u,\varrho}_{\mathrm{reg},\varepsilon}\big\|_{\mathrm{H}^{m-1}} \|f\|_{\mathcal{H}^{m-1}_r}.$$

Likewise, we have, for all k,

$$\begin{aligned}
\big\|\chi \partial_x^\alpha E^{u,\varrho}_{\mathrm{reg},\varepsilon} \cdot \partial_{x_k} \nabla_v \partial_x^{I-\alpha} \partial_v^J f\big\|^2_{\mathrm{L}^2_{x,v}} &\leq \big\|\partial_x^\alpha E^{u,\varrho}_{\mathrm{reg},\varepsilon}\big\|_{\mathrm{L}^\infty} \big\|\chi \partial_x \nabla_v \partial_x^{I-\alpha} \partial_v^J f\big\|_{\mathrm{L}^2_{x,v}} \\
&\lesssim \big\|E^{u,\varrho}_{\mathrm{reg},\varepsilon}\big\|_{\mathrm{H}^\sigma} \|f\|_{\mathcal{H}^{m-1}_r},
\end{aligned}$$

provided that $\sigma > |I - \alpha| + d/2$. Since $|\alpha| + d/2 \leq \frac{m+2+d}{2}$ and $m > 4 + d$, there exists such a σ so that

$$\big\|\chi \partial_x^\alpha E^{u,\varrho}_{\mathrm{reg},\varepsilon} \cdot \nabla_v \partial_x \partial_x^{I-\alpha} \partial_v^J f\big\|_{\mathrm{L}^2_{x,v}} \lesssim \big\|E^{u,\varrho}_{\mathrm{reg},\varepsilon}\big\|_{\mathrm{H}^{m-1}} \|f\|_{\mathcal{H}^{m-1}_r}.$$

The same procedure can be applied for the terms $\big\|\chi \partial_x^\alpha E^{u,\varrho}_{\mathrm{reg},\varepsilon} \cdot \partial_{v_k} \nabla_v \partial_x^{I-\alpha} \partial_v^J f\big\|_{\mathrm{L}^2_{x,v}}$.

All in all, we have proved that for all $t \in [0, T]$,

$$\begin{aligned}
\big\|\mathcal{R}_0^{I,J}(t)\big\|_{\mathcal{H}^1_r} &\lesssim \big\|E^{u,\varrho}_{\mathrm{reg},\varepsilon}(t)\big\|_{\mathrm{H}^{m-1}} \|f(t)\|_{\mathcal{H}^{m-1}_r} \\
&\leq \|f\|_{\mathrm{L}^\infty(0,T;\mathcal{H}^{m-1}_r)} \big\|E^{u,\varrho}_{\mathrm{reg},\varepsilon}(t)\big\|_{\mathrm{H}^{m-1}} \\
&\leq R \|E(t)\|_{\mathrm{H}^{m-1}}.
\end{aligned}$$

By the estimate (2.2) from Lemma 2.7 and by Lemma 2.20, we finally have

$$\big\|\mathcal{R}_0^{I,J}\big\|_{\mathrm{L}^2(0,T;\mathcal{H}^1_r)} \leq \Lambda(T, R).$$

With the same exact arguments, we easily obtain

$$\big\|\mathcal{R}_1^{I,J}\big\|_{\mathrm{L}^2(0,T;\mathcal{H}^0_r)} \leq \Lambda(T, R),$$

and this concludes the proof. ∎

Remark 4.4. We will actually obtain an improved $\mathrm{L}^2(0, T; \mathcal{H}^1_r)$ estimate for the term $\mathcal{R}_1^{I,J}$ (or, more precisely, for related terms) at the end of the current section.

We can see $\mathcal{M}^{I,J}\mathcal{F}$ appearing in Lemma 4.3 as a linear combination of $\mathcal{F}^{K,L} = \partial_x^K \partial_v^L f$. More precisely, we can write, for all (I, J),

$$\mathcal{M}^{I,J}\mathcal{F} = \sum_{K,L} \mathcal{M}_{(I,J),(K,L)} \mathcal{F}^{K,L},$$

$$\begin{aligned}\mathcal{M}_{(I,J),(K,L)} := \sum_{p=1}^{d} &\mathbf{1}_{j_p \neq 0} \big(\mathbf{1}_{(K,L)=(\hat{I}^p, \bar{J}^p)} - \mathbf{1}_{(K,L)=(I,J)}\big) \\ &+ \mathbf{1}_{|I|=1,2} \sum_{p=1}^{d} \mathbf{1}_{(K,L)=(0,\hat{J}^p)} \partial_x^I (E^{u,\varrho}_{\mathrm{reg},\varepsilon})_p \\ &+ \sum_{p=1}^{d} \sum_{\substack{0<\alpha<I \\ |\alpha|\in\{1,2\}}} \binom{I}{\alpha} \mathbf{1}_{(K,L)=(I-\alpha,\hat{J}^p)} \partial_x^\alpha (E^{u,\varrho}_{\mathrm{reg},\varepsilon})_p.\end{aligned}$$

Let us observe that the coefficients of the operator $\mathcal{M}$ involve only 0, 1 or 2 derivatives of the force field $E^{u,\varrho}_{\mathrm{reg},\varepsilon}(t, x)$, but nothing coming from f.

We finally introduce the following additional notation which will allow us to reformulate Lemma 4.3 in a compact way.

Notation 4.5. We consider the following quantities:

(1) $\mathcal{R}_0$ and $\mathcal{R}_1$ are the vectors defined by

$$\mathcal{R}_0 = \big(\mathcal{R}_0^{I,J}\big)_{\substack{I,J\in\mathbb{N}^d, \\ |I|+|J|\in\{m-1,m\}}}, \quad \mathcal{R}_1 = \big(\mathcal{R}_0^{I,J}\big)_{\substack{I,J\in\mathbb{N}^d, \\ |I|+|J|\in\{m-1,m\}}};$$

(2) $\mathcal{M}$ is the linear map defined by

$$\mathcal{M} = \big(\mathcal{M}_{(I,J),(K,L)}\big)_{\substack{(I,J),(K,L) \\ |I|+|J|,|K|+|L|\in\{m-1,m\}}};$$

(3) $\mathcal{L}$ is the vector defined by

$$\mathcal{L} = \big(\partial_x^I \partial_v^J E^{u,\varrho}_{\mathrm{reg},\varepsilon} \cdot \nabla_v f\big)_{\substack{I,J\in\mathbb{N}^d, \\ |I|+|J|\in\{m-1,m\}}}.$$

4.1.2 The semi-Lagrangian approach

If f satisfies the Vlasov equation (in a strong sense), we have $\partial_x^I \partial_v^J (\mathcal{T}^{u,\varrho}_{\mathrm{reg},\varepsilon} f) = 0$ for any I, J. We can apply Lemma 4.3 to obtain the following coupled system of equations satisfied by the family $\mathcal{F} = (\partial_x^I \partial_v^J f)_{I,J}$:

$$\mathcal{T}^{u,\varrho}_{\mathrm{reg},\varepsilon}\mathcal{F} + \mathcal{M}\mathcal{F} + \mathcal{L} = -\mathcal{R}_0 - \mathcal{R}_1. \tag{4.3}$$

For any function $g(t, x, v)$, we set

$$\widetilde{g}(t, x, v) = g(t, \mathrm{X}^{t;0}(x, v), \mathrm{V}^{t;0}(x, v)),$$

where

$$s \mapsto \mathrm{Z}^{s;t}(x,v) = \big(\mathrm{X}^{s;t}(x,v), \mathrm{V}^{s;t}(x,v)\big)$$

is the solution to

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}s}\mathrm{X}^{s;t}(x,v) = \mathrm{V}^{s;t}(x,v), \;\; \mathrm{X}^{t;t}(x,v) = x \in \mathbb{T}^d, \\ \dfrac{\mathrm{d}}{\mathrm{d}s}\mathrm{V}^{s;t}(x,v) = -\mathrm{V}^{s;t}(x,v) + E^{u,\varrho}_{\mathrm{reg},\varepsilon}(s, \mathrm{X}^{s;t}(x,v)), \;\; \mathrm{V}^{t;t}(x,v) = v \in \mathbb{R}^d. \end{cases} \tag{4.4}$$

After the composition by $(t,x,v) \mapsto (t, \mathrm{X}^{t;0}(x,v), \mathrm{V}^{t;0}(x,v))$, we thus obtain, by the method of characteristics,

$$\partial_t \widetilde{\mathcal{F}} + \widetilde{\mathcal{M}}\widetilde{\mathcal{F}} + \widetilde{\mathcal{L}} = d\widetilde{\mathcal{F}} - \widetilde{\mathcal{R}}_0 - \widetilde{\mathcal{R}}_1. \tag{4.5}$$

To deal with the coupling matrix $\mathcal{M}$, we introduce the following resolvent.

Definition 4.6. For all (x,v) and $s,t \geq 0$, we define the *resolvent operator* $\mathfrak{N}^{s,t}(x,v)$ as the solution $s \mapsto \mathfrak{N}^{s,t}(x,v)$ of

$$\begin{cases} \partial_s \mathfrak{N}^{s;t} + \big[\mathcal{M} \circ \mathrm{Z}^{s;0} - d\,\mathrm{Id}\big]\mathfrak{N}^{s;t} = 0, \\ \mathfrak{N}^{t;t} = \mathrm{Id}. \end{cases} \tag{4.6}$$

The resolvent is well defined thanks to the Cauchy–Lipschitz theorem. We also have

$$\mathfrak{N}^{s;t}(x,v) = e^{d(s-t)}\mathrm{N}^{s;t}(x,v),$$

where

$$\begin{cases} \partial_s \mathrm{N}^{s;t} + \mathcal{M} \circ \mathrm{Z}^{s;0}\mathrm{N}^{s;t} = 0, \\ \mathrm{N}^{t;t} = \mathrm{Id}. \end{cases} \tag{4.7}$$

For the upcoming analysis, we need the following bounds on the resolvent.

Lemma 4.7. *For all* $0 \leq k < m - 3 - d/2$*, we have*

$$\sup_{0\leq s,t\leq T} \|\mathrm{N}^{s;t}\|_{\mathrm{W}^{k,\infty}_{x,v}} + \sup_{0\leq s,t\leq T} \|\partial_s \mathrm{N}^{s;t}\|_{\mathrm{W}^{k,\infty}_{x,v}} + \sup_{0\leq s,t\leq T} \|\partial_t \mathrm{N}^{s;t}\|_{\mathrm{W}^{k,\infty}_{x,v}} \leq \Lambda(T,R).$$

Proof. We have

$$\mathrm{N}^{s;t} = \mathrm{Id} - \int_t^s \big[\mathcal{M} \circ \mathrm{Z}^{\tau;0}\big]\mathrm{N}^{\tau;t}\,\mathrm{d}\tau.$$

By the definition of coefficients of the matrix $\mathcal{M}$, we also have

$$\|\mathcal{M}\|_{\mathrm{L}^2(0,T;\mathrm{L}^\infty)} \lesssim \sup_{|\alpha|\leq 2} \|\partial_x^\alpha E\|_{\mathrm{L}^2(0,T;\mathrm{L}^\infty)} \leq \Lambda(T,R),$$

thanks to estimate (2.2) from Lemma 2.7, and by Sobolev embedding (with $m - 1 > 2 + d/2$). Grönwall's lemma leads to the conclusion. ∎

We first obtain the following decomposition for the kinetic moments of the vector $\mathcal{F}$.

Lemma 4.8. *We have*

$$\begin{aligned}\int_{\mathbb{R}^d} &\mathcal{F}(t,x,v)\,\mathrm{d}v\\ &= \mathcal{I}^0_{\mathrm{in}}(t,x) + \mathcal{I}^0_{\mathcal{R}_0}(t,x) + \mathcal{I}^0_{\mathcal{R}_1}(t,x)\\ &\quad - \int_0^t e^{d(t-s)} \int_{\mathbb{R}^d} \mathrm{N}^{t;s}(\mathrm{Z}^{0;t}(x,v))\mathcal{L}(s,\mathrm{Z}^{s;t}(x,v))\,\mathrm{d}v\,\mathrm{d}s,\end{aligned}$$

$$\begin{aligned}\int_{\mathbb{R}^d} &v\otimes\mathcal{F}(t,x,v)\,\mathrm{d}v\\ &= \mathcal{I}^1_{\mathrm{in}}(t,x) + \mathcal{I}^1_{\mathcal{R}_0}(t,x) + \mathcal{I}^1_{\mathcal{R}_1}(t,x)\\ &\quad - \int_0^t e^{d(t-s)} \int_{\mathbb{R}^d} v\otimes\mathrm{N}^{t;s}(\mathrm{Z}^{0;t}(x,v))\mathcal{L}(s,\mathrm{Z}^{s;t}(x,v))\,\mathrm{d}v\,\mathrm{d}s,\end{aligned}$$

where

$$\mathcal{I}^0_{\mathrm{in}}(t,x) := e^{dt}\int_{\mathbb{R}^d} \mathrm{N}^{t;0}(\mathrm{Z}^{0;t}(x,v))\mathcal{F}_{|t=0}(\mathrm{Z}^{0;t}(x,v))\,\mathrm{d}v,$$

$$\mathcal{I}^0_{\mathcal{R}_j}(t,x) := -\int_0^t e^{d(t-s)}\int_{\mathbb{R}^d} \mathrm{N}^{t;s}(\mathrm{Z}^{0;t}(x,v))\mathcal{R}_j(s,\mathrm{Z}^{s;t}(x,v))\,\mathrm{d}v\,\mathrm{d}s, \quad j=0,1,$$

$$\mathcal{I}^1_{\mathrm{in}}(t,x) := e^{dt}\int_{\mathbb{R}^d} v\otimes\mathrm{N}^{t;0}(\mathrm{Z}^{0;t}(x,v))\mathcal{F}_{|t=0}(\mathrm{Z}^{0;t}(x,v))\,\mathrm{d}v,$$

$$\mathcal{I}^1_{\mathcal{R}_j}(t,x) := -\int_0^t e^{d(t-s)}\int_{\mathbb{R}^d} v\otimes\mathrm{N}^{t;s}(\mathrm{Z}^{0;t}(x,v))\mathcal{R}_j(s,\mathrm{Z}^{s;t}(x,v))\,\mathrm{d}v\,\mathrm{d}s, \quad j=0,1.$$

Proof. We only explain the case of the moment of order 0, the other being similar. Starting from the equation (4.5) and using the resolvent operators $\mathfrak{N}$ and N defined in (4.6) and (4.7), we have

$$\begin{aligned}\widetilde{\mathcal{F}}(t) &= \mathfrak{N}^{t,0}\widetilde{\mathcal{F}}_{|t=0} - \int_0^t \mathfrak{N}^{t,s}\big[\widetilde{\mathcal{L}}(s) + \widetilde{\mathcal{R}}_0(s) + \widetilde{\mathcal{R}}_1(s)\big]\,\mathrm{d}s\\ &= e^{dt}\mathrm{N}^{t;0}\widetilde{\mathcal{F}}_{|t=0} - \int_0^t e^{d(t-s)}\mathrm{N}^{t;s}\big[\widetilde{\mathcal{L}}(s) + \widetilde{\mathcal{R}}_0(s) + \widetilde{\mathcal{R}}_1(s)\big]\,\mathrm{d}s.\end{aligned}$$

After a composition by the map $(t,x,v)\mapsto(t,\mathrm{X}^{0;t}(x,v),\mathrm{V}^{0;t}(x,v))$, we obtain

$$\begin{aligned}\mathcal{F}(t) = e^{dt}&\big[\mathrm{N}^{t;0}\circ\mathrm{Z}^{0;t}\big]\mathcal{F}_{|t=0}\circ\mathrm{Z}^{0;t}\\ &- \int_0^t e^{d(t-s)}\big[\mathrm{N}^{t;s}\circ\mathrm{Z}^{0;t}\big]\big(\mathcal{R}_0(s,\mathrm{Z}^{s;t}) + \mathcal{R}_1(s,\mathrm{Z}^{s;t})\big)\,\mathrm{d}s\\ &- \int_0^t e^{d(t-s)}\big[\mathrm{N}^{t;s}\circ\mathrm{Z}^{0;t}\big]\mathcal{L}(s,\mathrm{Z}^{s;t})\,\mathrm{d}s.\end{aligned}$$

We reach the desired conclusion by integrating in velocity. ∎

4.2 First remainders

Let us show straightaway that some of the previous terms can be considered as remainders.

Lemma 4.9. *We have*

$$\begin{aligned} \|\mathcal{I}^0_{\mathrm{in}}\|_{\mathrm{L}^2(0,T;\mathrm{H}^1)}\| + \|\mathcal{I}^1_{\mathrm{in}}\|_{\mathrm{L}^2(0,T;\mathrm{H}^1)} &\leq \Lambda(T,R)\|f^{\mathrm{in}}\|_{\mathcal{H}^{m+1}_r}, \\ \|\mathcal{I}^0_{\mathcal{R}_0}\|_{\mathrm{L}^2(0,T;\mathrm{H}^1)} + \|\mathcal{I}^1_{\mathcal{R}_0}\|_{\mathrm{L}^2(0,T;\mathrm{H}^1)} &\leq \Lambda(T,R), \\ \|\mathcal{I}^0_{\mathcal{R}_1}\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)} + \|\mathcal{I}^1_{\mathcal{R}_1}\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)} &\leq \Lambda(T,R). \end{aligned} \tag{4.8}$$

Proof. We shall first estimate the term $\mathcal{I}^0_{\mathrm{in}}$. We have

$$\begin{aligned} |\mathcal{I}^0_{\mathrm{in}}(t,x)| &\lesssim e^{dT} \sup_{0\leq t\leq T} \|\mathrm{N}^{t;s}\|_{\mathrm{L}^\infty_{x,v}} \int_{\mathbb{R}^d} |\mathcal{F}_0(\mathrm{Z}^{0;t}(x,v))|\,\mathrm{d}v \\ &\lesssim \Lambda(T,R) \sum_{I,J} \int_{\mathbb{R}^d} \left|\partial^I_x \partial^J_v f^{\mathrm{in}}(\mathrm{Z}^{0;t}(x,v))\right| \mathrm{d}v, \end{aligned}$$

thanks to Lemma 4.7 with $k = 0$. Using the generalized Minkowski inequality and the Cauchy–Schwarz inequality, we get

$$\begin{aligned} &\|\mathcal{I}^0_{\mathrm{in}}\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)} \\ &\lesssim \Lambda(T,R) \sum_{I,J} \left\| \int_{\mathbb{R}^d} \left\|\partial^I_x \partial^J_v f^{\mathrm{in}}(\mathrm{Z}^{0;t}(\cdot,v))\right\|_{\mathrm{L}^2} \mathrm{d}v \right\|_{\mathrm{L}^2(0,T)} \\ &\lesssim \Lambda(T,R) \sum_{I,J} \left\| \left(\int_{\mathbb{T}^d\times\mathbb{R}^d} \left(1+|v|^2\right)^r \left|\partial^I_x \partial^J_v f^{\mathrm{in}}(\mathrm{Z}^{0;t}(x,v))\right|^2 \mathrm{d}x\,\mathrm{d}v \right)^{1/2} \right\|_{\mathrm{L}^2(0,T)}, \end{aligned}$$

since $2r > d$. We then perform the change of variable $(x,v) \mapsto \mathrm{Z}^{0;t}(x,v)$, which yields

$$\begin{aligned} &\int_{\mathbb{T}^d\times\mathbb{R}^d} \left(1+|v|^2\right)^r \left|\partial^I_x \partial^J_v f^{\mathrm{in}}(\mathrm{Z}^{0;t}(x,v))\right|^2 \mathrm{d}x\,\mathrm{d}v \\ &\qquad \lesssim \Lambda(T,R) \int_{\mathbb{T}^d\times\mathbb{R}^d} \left(1+|\mathrm{V}^{t;0}(x,v)|^2\right)^r \left|\partial^I_x \partial^J_v f^{\mathrm{in}}(x,v)\right|^2 \mathrm{d}x\,\mathrm{d}v. \end{aligned}$$

Since

$$\mathrm{V}^{t;0}(x,v) = e^{-t}v + \int_0^t e^{\tau-t} E^{u,\varrho}_{\mathrm{reg},\varepsilon}(\tau, \mathrm{X}^{\tau;t}(x,v))\,\mathrm{d}\tau,$$

we have by Sobolev embedding

$$|\mathrm{V}^{t;0}(x,v)|^2 \lesssim |v|^2 + \left|\int_0^t \|E^{u,\varrho}_{\mathrm{reg},\varepsilon}(\tau)\|_{\mathrm{L}^\infty}\,\mathrm{d}\tau\right|^2 \lesssim |v|^2 + T\|E^{u,\varrho}_{\mathrm{reg},\varepsilon}\|^2_{\mathrm{L}^2(0,T;\mathrm{H}^{m-1})}.$$

By the estimate (2.2) from Lemma 2.7, we get

$$|\mathrm{V}^{t;0}(x,v)|^2 \leq |v|^2 \Lambda(T,R),$$

and then

$$\begin{aligned}\int_{\mathbb{T}^d\times\mathbb{R}^d} &\big(1+|\mathrm{V}^{t;0}(x,v)|^2\big)^r \big|\partial_x^I \partial_v^J f^{\mathrm{in}}(x,v)\big|^2 \,\mathrm{d}x\,\mathrm{d}v \\ &\lesssim \Lambda(T,R) \int_{\mathbb{T}^d\times\mathbb{R}^d} \big(1+|v|^2\big)^r \big|\partial_x^I \partial_v^J f^{\mathrm{in}}(x,v)\big|^2 \,\mathrm{d}x\,\mathrm{d}v \\ &\lesssim \Lambda(T,R)\|f^{\mathrm{in}}\|^2_{\mathcal{H}^m_r}.\end{aligned}$$

This implies

$$\|\mathcal{I}^0_{\mathrm{in}}\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)} \leq \Lambda(T,R)\|f^{\mathrm{in}}\|_{\mathcal{H}^m_r}.$$

Likewise, using $2r > d+1$, we have

$$\|\mathcal{I}^1_{\mathrm{in}}\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)} \lesssim \Lambda(T,R)\|f^{\mathrm{in}}\|_{\mathcal{H}^m_r}.$$

We also have

$$\big[\mathcal{I}^0_{\mathrm{in}}\big]_{(I,J)}(t,x) = e^{dt}\int_{\mathbb{R}^d} \sum_{(K,L)} \mathrm{N}^{t;0}_{(I,J),(K,L)}(\mathrm{Z}^{0;t}(x,v))\big[\mathcal{F}_{|t=0}\big]_{(K,L)}(\mathrm{Z}^{0;t}(x,v))\,\mathrm{d}v,$$

and

$$\begin{aligned}&\nabla_x\big[\mathcal{I}^0_{\mathrm{in}}\big]_{(I,J)}(t,x) \\ &= e^{dt}\int_{\mathbb{R}^d} \sum_{(K,L)} \nabla_x \mathrm{Z}^{0;t}(x,v)\nabla_x \mathrm{N}^{t;0}_{(I,J),(K,L)}(\mathrm{Z}^{0;t}(x,v))\big[\mathcal{F}_{|t=0}\big]_{(K,L)}(\mathrm{Z}^{0;t}(x,v))\,\mathrm{d}v \\ &\quad + e^{dt}\int_{\mathbb{R}^d} \sum_{(K,L)} \mathrm{N}^{t;0}_{(I,J),(K,L)}(\mathrm{Z}^{0;t}(x,v))\nabla_x \mathrm{Z}^{0;t}(x,v)\nabla_x\big[\mathcal{F}_{|t=0}\big]_{(K,L)}(\mathrm{Z}^{0;t}(x,v))\,\mathrm{d}v.\end{aligned}$$

The same procedure as before, using Lemma 4.7 with $k=1$ and the pointwise bounds (2.16) and (2.17) from Remark 2.27, gives

$$\|\nabla_x \mathcal{I}^0_{\mathrm{in}}\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)} + \|\nabla_x \mathcal{I}^1_{\mathrm{in}}\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)} \lesssim \Lambda(T,R)\|f^{\mathrm{in}}\|_{\mathcal{H}^{m+1}_r}.$$

We now estimate the terms $\mathcal{I}^0_{\mathcal{R}_0}$ and $\mathcal{I}^1_{\mathcal{R}_0}$. We rely on the same arguments as before, with an additional Cauchy–Schwarz inequality in time leading to

$$\begin{aligned}\|\mathcal{I}^0_{\mathcal{R}_0}\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)} &\leq \Lambda(T,R)\sum_{I,J}\left\| \int_0^t \int_{\mathbb{R}^d} \big\|\mathcal{R}_0^{I,J}(s,\mathrm{Z}^{s;t}(\cdot,v))\big\|_{\mathrm{L}^2}\,\mathrm{d}s\,\mathrm{d}v\right\|_{\mathrm{L}^2(0,T)} \\ &\leq \Lambda(T,R)\left\|\int_0^t \|\mathcal{R}_0(s)\|_{\mathcal{H}^0_r}\,\mathrm{d}s\right\|_{\mathrm{L}^2(0,T)}\end{aligned}$$

$$\leq \Lambda(T,R)T\,\|\mathcal{R}_0\|_{\mathrm{L}^2(0,T;\mathcal{H}^0_r)}$$
$$\leq \Lambda(T,R),$$

thanks to (4.2) in Lemma 4.3. We obtain the same result for $\mathcal{I}^1_{\mathcal{R}_0}$. Using again (4.2) for the first order derivative, we also have

$$\|\mathcal{I}_{\mathcal{R}_0}\|_{\mathrm{L}^2(0,T;\mathrm{H}^1)} + \|\mathcal{I}^1_{\mathcal{R}_0}\|_{\mathrm{L}^2(0,T;\mathrm{H}^1)} \lesssim \Lambda(T,R).$$

For the estimate in $\mathrm{L}^2(0,T;\mathrm{L}^2)$ of the last term $\mathcal{I}^0_{\mathcal{R}_1}$, we end up with the conclusion thanks to the inequality (4.1) in Lemma 4.3, combined with the same arguments as before. ∎

Note that the previous lemma does not give any control on $\|\nabla_x \mathcal{I}^0_{\mathcal{R}_1}\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)} + \|\nabla_x \mathcal{I}^1_{\mathcal{R}_1}\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)}$. The treatment of these terms requires additional arguments that we will develop in the next sections. For now, we merely state the result and postpone the proof to the end of Section 4.4.

Lemma 4.10. *We have*

$$\mathcal{I}^0_{\mathcal{R}_1}\|_{\mathrm{L}^2(0,T;\mathrm{H}^1)} + \|\mathcal{I}^1_{\mathcal{R}_1}\|_{\mathrm{L}^2(0,T;\mathrm{H}^1)} \leq \Lambda(T,R).$$

4.3 Leading terms and conclusion

In this section, we focus on the following two terms:

$$\mathfrak{L}^0(t,x) := -\int_0^t e^{d(t-s)} \int_{\mathbb{R}^d} \mathrm{N}^{t;s}(\mathrm{Z}^{0;t}(x,v))\mathcal{L}(s,\mathrm{Z}^{s;t}(x,v))\,\mathrm{d}v\,\mathrm{d}s$$

and

$$\mathfrak{L}^1(t,x) := -\int_0^t e^{d(t-s)} \int_{\mathbb{R}^d} v \otimes \mathrm{N}^{t;s}(\mathrm{Z}^{0;t}(x,v))\mathcal{L}(s,\mathrm{Z}^{s;t}(x,v))\,\mathrm{d}v\,\mathrm{d}s.$$

The goal is to prove that $\mathfrak{L}^0$ and $\mathfrak{L}^1$ can be decomposed as a sum of a leading term and a remainder in $\mathrm{L}^2_T\mathrm{H}^1$. This will imply the result stated in Proposition 4.1. Since the treatment of $\mathfrak{L}^1$ is similar, we focus on $\mathfrak{L}^0$.

First, we have the following decomposition, which introduces several remainder terms that we shall estimate later on. Recall the definition of the straightening diffeomorphism $\psi_{s,t}(x,v)$ from Lemma 2.26.

Lemma 4.11. *For $|I| \leq m$, we have*

$$[\mathfrak{L}^0]_{(I,0)}(t,x) = -\int_0^t \int_{\mathbb{R}^d} \partial_x^I E^{u,\varrho}_{\mathrm{reg},\varepsilon}(s,x-(t-s)v)\cdot\nabla_v f(t,x,v)\,\mathrm{d}v\,\mathrm{d}s$$
$$+ \mathfrak{R}^{\mathrm{Diff}}_I(t,x) + \mathfrak{R}^{\mathrm{Duha}}_{I,1}(t,x) + \mathfrak{R}^{\mathrm{Duha}}_{I,2}(t,x),$$

with

$$\mathcal{R}_I^{\mathrm{Diff}}(t,x) := \int_0^t \int_{\mathbb{R}^d} \big[\partial_x^I E^{u,\varrho}_{\mathrm{reg},\varepsilon}(s, x-(t-s)v) - \partial_x^I E^{u,\varrho}_{\mathrm{reg},\varepsilon}(s, x+(1-e^{t-s})v)\big] \cdot \nabla_v f(t,x,v)\,\mathrm{d}v\,\mathrm{d}s,$$

$$\mathcal{R}_{I,1}^{\mathrm{Duha}}(t,x) := -\sum_K \int_0^t \int_{\mathbb{R}^d} \partial_x^K E^{u,\varrho}_{\mathrm{reg},\varepsilon}(s, x+(1-e^{t-s})v) \cdot \mathrm{H}^{K,I}(s,t,x,v)\,\mathrm{d}v\,\mathrm{d}s,$$

$$\mathcal{R}_{I,2}^{\mathrm{Duha}}(t,x) := -\sum_K \int_0^t \int_{\mathbb{R}^d} \partial_x^K E^{u,\varrho}_{\mathrm{reg},\varepsilon}(s, x+(1-e^{t-s}v)) \cdot \mathfrak{H}^{K,I}(t,s,x,v)\,\mathrm{d}v\,\mathrm{d}s,$$

where $\mathrm{H}^{K,I}$ *and* $\mathfrak{H}^{K,I}$ *are vector fields defined by*

$$\mathrm{H}^{K,I}(t,s,x,v) := \mathrm{N}^{t;s}(\mathrm{Z}^{0;t}(x,\psi_{s,t}(x,v)))_{(I,0),(K,0)}\mathrm{J}^{s,t}(x,v)\nabla_v f(t,x,\psi_{s,t}(x,v)) - \nabla_v f(t,x,v), \tag{4.9}$$

and

$$\mathfrak{H}^{K,I}(t,s,x,v) := \int_s^t e^{d(t-\tau)}\mathrm{N}^{t;s}\big(\mathrm{Z}^{0;t}(x,\psi_{s,t}(x,v))\big)_{(I,0),(K,0)} \times \big(\nabla_x f(\tau,\mathrm{Z}^{\tau;t}(x,\psi_{s,t}(x,v))) - \nabla_v f(\tau,\mathrm{Z}^{\tau;t}(x,\psi_{s,t}(x,v)))\big) \times \mathrm{J}^{s,t}(x,v)\,\mathrm{d}\tau, \tag{4.10}$$

with

$$\mathrm{J}^{s,t}(x,w) = |\det(\mathrm{D}_w\psi_{s,t}(x,w))|.$$

Proof. Let $T \in (0, \min(T_\varepsilon(R), \overline{T}(R)))$. We have, for $|I| \le m$,

$$\begin{aligned}[\mathfrak{L}^0]_{(I,0)}(t,x) &= -\sum_{(K,L)} \int_0^t e^{d(t-s)} \int_{\mathbb{R}^d} \mathrm{N}^{t;s}(\mathrm{Z}^{0;t}(x,v))_{(I,0),(K,L)}\partial_x^K\partial_v^L E^{u,\varrho}_{\mathrm{reg},\varepsilon}(s,\mathrm{X}^{s;t}(x,v)) \cdot \nabla_v f(s,\mathrm{Z}^{s;t}(x,v))\,\mathrm{d}v\,\mathrm{d}s \\ &= -\sum_K \int_0^t e^{d(t-s)} \int_{\mathbb{R}^d} \mathrm{N}^{t;s}(\mathrm{Z}^{0;t}(x,v))_{(I,0),(K,0)}\partial_x^K E^{u,\varrho}_{\mathrm{reg},\varepsilon}(s,\mathrm{X}^{s;t}(x,v)) \cdot \nabla_v f(s,\mathrm{Z}^{s;t}(x,v))\,\mathrm{d}v\,\mathrm{d}s,\end{aligned} \tag{4.11}$$

since $E^{u,\varrho}_{\mathrm{reg},\varepsilon}$ does not depend on the v variable. By Lemma 2.9, we know that

$$\mathcal{T}^{u,\varrho}_{\mathrm{reg},\varepsilon}(\nabla_v f) = \nabla_v f - \nabla_x f.$$

Invoking the Duhamel formula, we get

$$\nabla_v f(t,x,v) = e^{d(t-s)}\nabla_v f(s,\mathrm{Z}^{s;t}(x,v)) + \int_s^t e^{d(t-\tau)}\big(\nabla_v f(\tau,\mathrm{Z}^{\tau;t}(x,v)) - \nabla_x f(\tau,\mathrm{Z}^{\tau;t}(x,v))\big)\,\mathrm{d}\tau,$$

and therefore

$$e^{d(t-s)}\nabla_v f(s,\mathrm{Z}^{s;t}(x,v)) = \nabla_v f(t,x,v) + \int_s^t e^{d(t-\tau)}\big(\nabla_x f(\tau,\mathrm{Z}^{\tau;t}(x,v)) - \nabla_v f(\tau,\mathrm{Z}^{\tau;t}(x,v))\big)\,\mathrm{d}\tau.$$

Inserting this expression in (4.11) yields

$$[\mathfrak{L}^0]_{(I,0)}(t,x) = \mathcal{L}_I^1(t,x) + \mathcal{L}_I^2(t,x),$$

where

$$\mathcal{L}_I^1(t,x) := -\sum_K \int_0^t \int_{\mathbb{R}^d} \mathrm{N}^{t;s}(\mathrm{Z}^{0;t}(x,v))_{(I,0),(K,0)}\partial_x^K E^{u,\varrho}_{\mathrm{reg},\varepsilon}(s,\mathrm{X}^{s;t}(x,v)) \cdot \nabla_v f(t,x,v)\,\mathrm{d}v\,\mathrm{d}s,$$

$$\mathcal{L}_I^2(t,x) := -\sum_K \int_0^t \int_{\mathbb{R}^d} \int_s^t e^{d(t-\tau)}\mathrm{N}^{t;s}(\mathrm{Z}^{0;t}(x,v))_{(I,0),(K,0)}\partial_x^K E^{u,\varrho}_{\mathrm{reg},\varepsilon}(s,\mathrm{X}^{s;t}(x,v)) \cdot \big(\nabla_x f(\tau,\mathrm{Z}^{\tau;t}(x,v)) - \nabla_v f(\tau,\mathrm{Z}^{\tau;t}(x,v))\big)\,\mathrm{d}\tau\,\mathrm{d}v\,\mathrm{d}s.$$

Let us transform these two expressions in order to make the terms $\mathcal{R}_I^1(t,x)$, $\mathcal{R}_I^2(t,x)$ and $\mathcal{R}_I^3(t,x)$ appear.

First term. We first focus on $\mathcal{L}_I^1$, which will produce the leading term in the result. Using the change of variable $v = \psi_{s,t}(x,w)$ coming from Lemma 2.26, we have (since $t \le \overline{T}(R)$)

$$\mathcal{L}_I^1(t,x) = -\sum_K \int_0^t \int_{\mathbb{R}^d} \mathrm{N}^{t;s}(\mathrm{Z}^{0;t}(x,\psi_{s,t}(x,v)))_{(I,0),(K,0)} \times \partial_x^K E^{u,\varrho}_{\mathrm{reg},\varepsilon}(s,x+(1-e^{t-s})v) \cdot \nabla_v f(t,x,\psi_{s,t}(x,v))\mathrm{J}^{s,t}(x,v)\,\mathrm{d}v\,\mathrm{d}s,$$

where $\mathrm{J}^{s,t}(x,w) = |\det(\mathrm{D}_w\psi_{s,t}(x,w))|$. We obtain

$$\begin{aligned}\mathcal{L}_I^1(t,x) = &-\sum_K \int_0^t \int_{\mathbb{R}^d} \partial_x^K E^{u,\varrho}_{\mathrm{reg},\varepsilon}(s,x+(1-e^{t-s})v) \cdot \nabla_v f(t,x,\psi_{t,t}(x,v)) \\ &\qquad \times \mathrm{N}^{t;t}(\mathrm{Z}^{0;t}(x,\psi_{t,t}(x,v)))_{(I,0),(K,0)}\mathrm{J}^{t,t}(x,v)\,\mathrm{d}v\,\mathrm{d}s \\ &-\sum_K \int_0^t \int_{\mathbb{R}^d} \partial_x^K E^{u,\varrho}_{\mathrm{reg},\varepsilon}(s,x+(1-e^{t-s})v) \cdot \mathrm{H}^{K,I}(t,s,x,v)\,\mathrm{d}v\,\mathrm{d}s,\end{aligned}$$

with

$$\begin{aligned}\mathrm{H}^{K,I}(t,s,x,v) := {}&\mathrm{N}^{t;s}(\mathrm{Z}^{0;t}(x,\psi_{s,t}(x,v)))_{(I,0),(K,0)}\mathrm{J}^{s,t}(x,v)\nabla_v f(t,x,\psi_{s,t}(x,v)) \\ &- \mathrm{N}^{t;t}(\mathrm{Z}^{0;t}(x,\psi_{t,t}(x,v)))_{(I,0),(K,0)}\mathrm{J}^{t,t}(x,v)\nabla_v f(t,x,v).\end{aligned}$$

Now observe that, since $\mathrm{N}^{t;t} = \mathrm{Id}$ and $\psi_{t,t} = \mathrm{Id}$, we have

$$\nabla_v f(t,x,\psi_{t,t}(x,v))\mathrm{N}^{t;t}(\mathrm{Z}^{0;t}(x,\psi_{t,t}(x,v)))_{(I,0),(K,0)}\mathrm{J}^{t,t}(x,v) \\ = \mathbf{1}_{I=K}\nabla_v f(t,x,v);$$

therefore,

$$\begin{aligned}\mathcal{L}_I^1(t,x) &= -\int_0^t \int_{\mathbb{R}^d} \partial_x^I E^{u,\varrho}_{\mathrm{reg},\varepsilon}(s, x+(1-e^{t-s})v)\cdot\nabla_v f(t,x,v)\,\mathrm{d}v\,\mathrm{d}s \\ &\quad -\sum_K \int_0^t \int_{\mathbb{R}^d} \partial_x^K E^{u,\varrho}_{\mathrm{reg},\varepsilon}(s, x+(1-e^{t-s})v)\cdot \mathrm{H}^{K,I}(s,t,x,v)\,\mathrm{d}v\,\mathrm{d}s \\ &= -\int_0^t \int_{\mathbb{R}^d} \partial_x^I E^{u,\varrho}_{\mathrm{reg},\varepsilon}(s, x-(t-s)v)\cdot\nabla_v f(t,x,v)\,\mathrm{d}v\,\mathrm{d}s \\ &\quad + \mathcal{R}_I^{\mathrm{Diff}}(t,x) + \mathcal{R}_{I,1}^{\mathrm{Duha}}(t,x).\end{aligned}$$

Second term. To deal with the term $\mathcal{L}_I^2$, we apply again the change of variable $v = \psi_{s,t}(x,w)$ from Lemma 2.26 and get (since $T \leq \overline{T}(R)$)

$$\mathcal{L}_I^2(t,x) = -\sum_K \int_0^t \int_{\mathbb{R}^d} \partial_x^K E^{u,\varrho}_{\mathrm{reg},\varepsilon}(s, x+(1-e^{t-s}v))\cdot \mathfrak{H}^{K,I}(t,s,x,v)\,\mathrm{d}v\,\mathrm{d}s,$$

where

$$\begin{aligned}&\mathfrak{H}^{K,I}(t,s,x,v) \\ &\quad := \int_s^t e^{d(t-\tau)}\mathrm{N}^{t;s}\big(\mathrm{Z}^{0;t}(x,\psi_{s,t}(x,v))\big)_{(I,0),(K,0)} \\ &\qquad\qquad \times \big(\nabla_x f(\tau,\mathrm{Z}^{\tau;t}(x,\psi_{s,t}(x,v))) - \nabla_v f(\tau,\mathrm{Z}^{\tau;t}(x,\psi_{s,t}(x,v)))\big)\,\mathrm{J}^{s,t}(x,v)\,\mathrm{d}\tau,\end{aligned}$$

which means that $\mathcal{L}_I^2(t,x) = \mathcal{R}_{I,2}^{\mathrm{Duha}}(t,x)$. Combining the previous decompositions finally yields the conclusion. ■

We now have the following lemma, which is the continuation of Lemma 4.11, in which we express $[\mathfrak{L}^0]_{(I,0)}$ as a sum of a leading term and a remainder that is controlled in $\mathrm{L}^2_T\mathrm{H}^1$. The proof, which is rather lengthy and technical, is based on the smoothing estimates of Chapter 3; we postpone it to Section 4.4.

Lemma 4.12. *We have, for all $|I| = m$,*

$$\begin{aligned}&[\mathfrak{L}^0]_{(I,0)}(t,x) \\ &\quad = p'(\varrho(t,x))\int_0^t \int_{\mathbb{R}^d} \nabla_x[\mathrm{J}_\varepsilon \partial_x^I \varrho](s,x-(t-s)v)\cdot\nabla_v f(t,x,v)\,\mathrm{d}v\,\mathrm{d}s + \mathcal{R}_I(t,x),\end{aligned}$$

with

$$\|\mathcal{R}_I\|_{\mathrm{L}^2(0,T,\mathrm{H}^1)} \leq \Lambda(T,R).$$

We can finally proceed with the proof of Proposition 4.1.

Proof of Proposition 4.1. We only treat the case of $\partial_x^I \rho_f$, that of $\partial_x^I j_f$ being similar. First invoking Lemma 4.8, we have

$$\partial_x^I \rho_f = [\mathcal{I}_{\mathrm{in}}^0]_{(I,0)} + [\mathcal{I}_{\mathcal{R}_0}^0]_{(I,0)} + [\mathcal{I}_{\mathcal{R}_1}^0]_{(I,0)} + [\mathfrak{L}^0]_{(I,0)}.$$

Thanks to Lemmas 4.9, 4.10 and 4.12, we infer that

$$\partial_x^I \rho_f(t,x) = p'(\varrho(t,x)) \int_0^t \int_{\mathbb{R}^d} \nabla_x [\mathrm{J}_\varepsilon \partial_x^I \varrho](s, x-(t-s)v) \cdot \nabla_v f(t,x,v)\, \mathrm{d}v\, \mathrm{d}s$$
$$+ [\mathcal{I}_0^0]_{(I,0)}(t,x) + [\mathcal{I}_{\mathcal{R}_0}^0]_{(I,0)}(t,x) + [\mathcal{I}_{\mathcal{R}_1}^0]_{(I,0)}(t,x) + \mathcal{R}_I(t,x),$$

where the previous remainders are estimated as

$$\begin{aligned} \|\mathcal{I}_{\mathrm{in}}^0\|_{\mathrm{L}^2(0,T;\mathrm{H}^1)}\| &\leq \Lambda(T,R)\|f^{\mathrm{in}}\|_{\mathcal{H}_r^{m+1}}, \\ \|\mathcal{I}_{\mathcal{R}_0}^0\|_{\mathrm{L}^2(0,T;\mathrm{H}^1)} + \|\mathcal{I}_{\mathcal{R}_1}^0\|_{\mathrm{L}^2(0,T;\mathrm{H}^1)} &\leq \Lambda(T,R), \\ \|\mathcal{R}_I\|_{\mathrm{L}^2(0,T,\mathrm{H}^1)} &\leq \Lambda(T,R). \end{aligned}$$

This concludes the proof. ∎

4.4 Estimates of last remainders

In this section, we mainly aim at giving a proof for Lemma 4.12 and Lemma 4.10, that we have previously stated. We shall rely on the crucial smoothing estimates derived in Chapter 3 to treat the different remainders. Broadly speaking, there are three types of terms requiring a gain of regularity:

Type I. Terms that will be treated thanks to the continuity estimate of Proposition 3.1, as in [90], and of Proposition 3.4. Such terms will be useful to treat $\nabla_x \mathcal{I}_{\mathcal{R}_1}^0$ and $\nabla_x \mathcal{I}_{\mathcal{R}_1}^1$ in the proof of Lemma 4.10.

Type II. Terms involving a kernel vanishing on the diagonal in time. They will be treated by the regularization estimate of Proposition 3.5. Such terms will appear in the proof of Lemma 4.12, as well as for the remainders $\mathcal{R}_I^{\mathrm{Diff}}$, $\mathcal{R}_{I,1}^{\mathrm{Duha}}$ and $\mathcal{R}_{I,2}^{\mathrm{Duha}}$.

Type III. Terms involving the difference between the integral operators $\mathrm{K}^{\mathrm{free}}$ and $\mathrm{K}^{\mathrm{fric}}$ (see Chapter 3). They will be handled thanks to the regularization estimate of Proposition 3.7. They will also appear in the treatment of the different remainders.

Let us mention that the previous gains require to control a fixed number of derivatives of the kernels that are involved (see again Chapter 3).

Recall also the expression

$$E_{\mathrm{reg},\varepsilon}^{u,\varrho}(t,x) = u(t,x) - p'(\varrho)\nabla_x[\mathrm{J}_\varepsilon \varrho](t,x),$$

as well as Notation 2.8 for shifted indices. In order to check that the assumptions of the smoothing estimates of Chapter 3 are satisfied, it is convenient to have the following result.

Lemma 4.13. *For any* $K \in \mathbb{N}^d$ *such that* $|K| > 0$, *we have*

$$\begin{aligned}\partial_x^K E^{u,\varrho}_{\mathrm{reg},\varepsilon} &= -p'(\varrho)\nabla_x[\partial_x^K \mathrm{J}_\varepsilon \varrho] + \partial_x^K u \\ &\quad - \sum_{\ell=0}^{\lfloor \frac{|K|-1}{2} \rfloor} \sum_{\substack{i=1,\dots,d \\ \beta \in \mathbb{B}_K(i,\ell)}} \binom{K}{\widehat{\beta}^i} \nabla_x\big(\partial_x^{\overline{K}^i - \beta} \mathrm{J}_\varepsilon \varrho\big) \nabla_x\big(\partial_x^\beta p'(\varrho)\big) \cdot e_i \\ &\quad - \sum_{\ell=\lfloor \frac{|K|-1}{2} \rfloor + 1}^{|K|-1} \sum_{\substack{i=1,\dots,d \\ \beta \in \mathbb{B}_K(i,\ell)}} \binom{K}{\widehat{\beta}^i} \nabla_x\big(\partial_x^\beta p'(\varrho)\big) \cdot e_i \nabla_x\big(\partial_x^{\overline{K}^i - \beta} \mathrm{J}_\varepsilon \varrho\big),\end{aligned}$$

where

$$\mathbb{B}_K(i,\ell) := \big\{\beta \in \mathbb{N}^d \mid |\beta| = \ell,\, 0 < \widehat{\beta}^i \le K\big\},\ i = 1,\dots,d,\ \ell = 0,\dots,|K|-1.$$

Proof. The proof follows directly from the Leibniz formula, which provides

$$\begin{aligned}\partial_x^K E^{u,\varrho}_{\mathrm{reg},\varepsilon} &= -p'(\varrho)\nabla_x \partial_x^K \varrho + \partial_x^K u \\ &\quad - \sum_{\ell=0}^{|K|-1} \sum_{\substack{i=1,\dots,d \\ \beta \in \mathbb{B}_K(i,\ell)}} \binom{K}{\widehat{\beta}^i} \nabla_x\big(\partial_x^\beta p'(\varrho)\big) \cdot e_i \nabla_x\big(\partial_x^{\overline{K}^i - \beta} \mathrm{J}_\varepsilon \varrho\big),\end{aligned}$$

and yields the desired result. ∎

In the next lemma, we show how to obtain the leading term of Lemma 4.12 (up to some good remainder) from the integral term of Lemma 4.11.

Lemma 4.14. *We have*

$$\begin{aligned}&- \int_0^t \int_{\mathbb{R}^d} \partial_x^I E^{u,\varrho}_{\mathrm{reg},\varepsilon}(s, x-(t-s)v) \cdot \nabla_v f(t,x,v)\, \mathrm{d}v\, \mathrm{d}s \\ &\quad = p'(\varrho(t,x)) \int_0^t \int_{\mathbb{R}^d} \nabla_x[\mathrm{J}_\varepsilon \partial_x^I \varrho](s, x-(t-s)v) \cdot \nabla_v f(t,x,v)\, \mathrm{d}v\, \mathrm{d}s + \widetilde{\mathcal{R}}_I(t,x),\end{aligned}$$

where the remainder $\widetilde{\mathcal{R}}_I$ *satisfies*

$$\|\widetilde{\mathcal{R}}_I\|_{\mathrm{L}^2(0,T;\mathrm{H}^1)} \le \Lambda(T,R).$$

Proof. Let us introduce the following vector fields:

$$G_1(s,t,x,v) = [p'(\varrho(s, x-(t-s)v)) - p'(\varrho(t,x))]\nabla_v f(t,x,v),$$

and

$$G_{3,i}^{\beta}(s,t,x,v) := \big(\nabla_x\big(\partial_x^{\beta} p'(\varrho)\big)(s,x-(t-s)v)\cdot e_i\big)\nabla_v f(t,x,v),$$
$$G_{4,i}^{K,\beta}(s,t,x,v) := \big(\nabla_x\big(\partial_x^{\overline{K}^i-\beta}\mathrm{J}_\varepsilon\varrho\big)(s,x-(t-s)v)\cdot\nabla_v f(t,x,v)\big)e_i$$

for

$$\beta\in\mathbb{B}_K(i,\ell)=\big\{\beta\in\mathbb{N}^d\mid|\beta|=\ell,\ 0<\widehat{\beta}^i\leq K\big\},\ i=1,\dots,d,\ \ell=0,\dots,|K|-1.$$

Thanks to Lemma 4.13, we can write

$$\begin{aligned}-\int_0^t\int_{\mathbb{R}^d}&\partial_x^I E_{\mathrm{reg},\varepsilon}^{u,\varrho}(s,x-(t-s)v)\cdot\nabla_v f(t,x,v)\,\mathrm{d}v\,\mathrm{d}s\\ &=p'(\varrho(t,x))\int_0^t\int_{\mathbb{R}^d}\nabla_x[\mathrm{J}_\varepsilon\partial_x^I\varrho](s,x-(t-s)v)\cdot\nabla_v f(t,x,v)\,\mathrm{d}v\,\mathrm{d}s\\ &\quad+\mathbf{S}_1(t,x)+\mathbf{S}_2(t,x)+\mathbf{S}_3(t,x)+\mathbf{S}_4(t,x),\end{aligned}$$

where

$$\begin{aligned}\mathbf{S}_1(t,x)&:=\mathrm{K}_{G_1}^{\mathrm{free}}[\partial_x^I\mathrm{J}_\varepsilon\varrho],\\ \mathbf{S}_2(t,x)&:=-\int_0^t\int_{\mathbb{R}^d}\partial_x^I u(s,x-(t-s)v)\cdot\nabla_v f(t,x,v)\,\mathrm{d}v\,\mathrm{d}s,\\ \mathbf{S}_3(t,x)&:=\sum_{\ell=0}^{\lfloor\frac{|I|-1}{2}\rfloor}\sum_{\substack{i=1,\dots,d\\ \beta\in\mathbb{B}_I(i,\ell)}}\binom{I}{\widehat{\beta}^i}\mathrm{K}_{G_{3,i}^{\beta}}^{\mathrm{free}}\big[\partial_x^{\overline{I}^i-\beta}\mathrm{J}_\varepsilon\varrho\big],\\ \mathbf{S}_4(t,x)&:=\sum_{\ell=\lfloor\frac{|I|-1}{2}\rfloor+1}^{|I|-1}\sum_{\substack{i=1,\dots,d\\ \beta\in\mathbb{B}_I(i,\ell)}}\binom{I}{\widehat{\beta}^i}\mathrm{K}_{G_{4,i}^{\overline{I}^i,\beta}}^{\mathrm{free}}\big[\partial_x^{\beta}p'(\varrho)\big].\end{aligned}$$

The treatment of the term $\mathbf{S}_2$ will follow from a straightforward estimate. The terms $\mathbf{S}_3$ and $\mathbf{S}_4$ are terms of Type I and we will use the continuity estimates provided by Proposition 3.1. The term $\mathbf{S}_1$, which already contains I derivatives of ϱ, is a term of Type II (since $G_1(t,t,x,v)=0$) and we will rely on the regularization estimate of Proposition 3.5.

Estimate of $\mathbf{S}_1$. Since $G_1(t,t,x,v)=0$, we apply Proposition 3.5 to get

$$\begin{aligned}\|\mathbf{S}_1\|_{\mathrm{L}^2(0,T;\mathrm{H}^1)}&\lesssim(1+T)\sup_{0\leq s,t\leq T}\|\partial_s G_1(s,t)\|_{\mathcal{H}_\sigma^\ell}\|\partial_x^I\varrho\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)}\\ &\lesssim(1+T)\sup_{0\leq s,t\leq T}\|\partial_s G_1(s,t)\|_{\mathcal{H}_\sigma^\ell}\|\varrho\|_{\mathrm{L}^2(0,T;\mathrm{H}^m)}\end{aligned}$$

for $\ell>7+d$ and $\sigma>d/2$. A direct computation gives

$$\partial_s G_1(s,t,x,v)=p''(\varrho)[\partial_s\varrho+v\cdot\nabla_x\varrho](s,x-(t-s)v)\nabla_v f(t,x,v).$$

We thus have, for all $0 \leq s, t \leq T$,

$$
\begin{aligned}
&\|\partial_s G_1(s,t,x,v)\|^2_{\mathcal{H}^\ell_\sigma} \\
&\quad \lesssim C_T \sum_{|\mu|+|\nu|\leq \ell} \sum_{\gamma=0}^{\mu+\nu} \big(\|\partial_x^\gamma (p''(\varrho)\partial_s \varrho(s))\|^2_{\mathrm{L}^\infty} + \|\partial_x^\gamma (p''(\varrho)\nabla_x \varrho(s))\|^2_{\mathrm{L}^\infty}\big) \\
&\qquad\qquad \times \int_{\mathbb{T}^d\times\mathbb{R}^d} \langle v\rangle^{2\sigma+2} |\partial_{x,v}^{\mu+\nu-\gamma} \nabla_v f(t,x,v)|^2 \,\mathrm{d}x\,\mathrm{d}v \\
&\quad \lesssim C_T \big(\|p''(\varrho)\partial_s\varrho(s)\|^2_{\mathrm{H}^k} + \|p''(\varrho)\nabla_x\varrho(s)\|^2_{\mathrm{H}^k}\big)\|f(t)\|^2_{\mathcal{H}^{m-1}_{\sigma+1}} \\
&\quad \lesssim C_T \|p''(\varrho(s))\|^2_{\mathrm{H}^k} \big(\|\partial_s\varrho(s)\|^2_{\mathrm{H}^k} + \|\nabla_x\varrho(s)\|^2_{\mathrm{H}^k}\big)\|f(t)\|^2_{\mathcal{H}^{m-1}_{\sigma+1}}
\end{aligned}
$$

for $k > \frac{d}{2} + \ell > \frac{d}{2} + 1 + d$ and $m - 1 > \ell > 1 + d$. Using the equation satisfied by ϱ, we get

$$\|p''(\varrho(s))\|_{\mathrm{H}^k} \leq \Lambda(\|\varrho(s)\|_{\mathrm{H}^k})\|\varrho(s)\|_{\mathrm{H}^k}$$

and

$$
\begin{aligned}
&\|\partial_s\varrho(s)\|_{\mathrm{H}^k} + \|\nabla_x\varrho(s)\|_{\mathrm{H}^k} \\
&\quad \lesssim \|u\|_{\mathrm{H}^k}\|\nabla_x\varrho\|_{\mathrm{H}^k} + \left\|\frac{\varrho}{1-\rho_f}\right\|_{\mathrm{H}^k} \big\|\mathrm{div}_x\big(j_f - \rho_f u + u\big)\big\|_{\mathrm{H}^k} + \|\varrho\|_{\mathrm{H}^{k+1}},
\end{aligned}
$$

thanks to the algebra property of H^k. Taking $k+1 < m-3$ and using the bootstrap assumption combined with (the proof of) Lemma 2.20, we obtain

$$\sup_{0\leq s,t\leq T} \|\partial_s G_1(s,t)\|_{\mathcal{H}^\ell_\sigma} \lesssim \Lambda(T,R).$$

Estimate of $\mathbf{S}_2$. Using the generalized Minkowski inequality followed by the Cauchy–Schwarz inequality, we have, for $r > d/2$,

$$
\begin{aligned}
&\|\mathbf{S}_2\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)} \\
&\quad \leq \left\| \int_0^t \left(\int_{\mathbb{T}^d\times\mathbb{R}^d} (1+|v|^2)^r \left|\partial_x^I u(s, x-(t-s)v)\right|^2 \right.\right. \\
&\qquad\qquad \left.\left. \times |\nabla_v f(t,x,v)|^2 \,\mathrm{d}x\,\mathrm{d}v \right)^{1/2} \mathrm{d}s \right\|_{\mathrm{L}^2(0,T)} \\
&\quad \leq \left\| \int_0^t \left(\int_{\mathbb{T}^d\times\mathbb{R}^d} (1+|v|^2)^r \left|\partial_x^I u(s,y)\right|^2 \right.\right. \\
&\qquad\qquad \left.\left. \times |\nabla_v f(t, y+(t-s)v, v)|^2 \,\mathrm{d}y\,\mathrm{d}v \right)^{1/2} \mathrm{d}s \right\|_{\mathrm{L}^2(0,T)} \\
&\quad \leq \sup_{t\in[0,T]} \left(\int_{\mathbb{R}^d} (1+|v|^2)^r \|\nabla_v f(t,\cdot,v)\|^2_{\mathrm{L}^\infty} \,\mathrm{d}v \right)^{1/2} \left\| \int_0^t \left\|\partial_x^I u(s)\right\|_{\mathrm{L}^2} \mathrm{d}s \right\|_{\mathrm{L}^2(0,T)} \\
&\quad \lesssim \|f\|_{\mathrm{L}^\infty(0,T;\mathcal{H}^{m-1}_r)} \|u\|_{\mathrm{L}^2(0,T;\mathrm{H}^m)},
\end{aligned}
$$

by Sobolev embedding in the last line, since $m-1>1+d/2$. This yields

$$\|\mathbf{S}_2\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)} \lesssim \Lambda(T,R).$$

Likewise, since $m-1>2+d/2$, we have

$$\begin{aligned}\|\nabla_x \mathbf{S}_2\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)} &\lesssim \|f\|_{\mathrm{L}^\infty(0,T;\mathcal{H}^{m-1}_r)}\|u\|_{\mathrm{L}^2(0,T;\mathrm{H}^m)} \\ &\quad + \|f\|_{\mathrm{L}^\infty(0,T;\mathcal{H}^{m-1}_r)}\|u\|_{\mathrm{L}^2(0,T;\mathrm{H}^{m+1})} \\ &\leq \Lambda(T,R),\end{aligned}$$

which gives the conclusion.

Estimate of $\mathbf{S}_3$ *and* $\mathbf{S}_4$. For all $0\leq|\beta|\leq\left\lfloor\frac{|I|-1}{2}\right\rfloor$ and $i=1,\ldots,d$, we use Proposition 3.1 (after taking one derivative in space) to get

$$\left\|\mathrm{K}^{\mathrm{free}}_{G^\beta_{3,i}}\left[\partial_x^{\overline{I}^i-\beta}\mathrm{J}_\varepsilon\varrho\right]\right\|_{\mathrm{L}^2(0,T;\mathrm{H}^1)} \lesssim \sup_{0\leq s,t\leq T}\left\|G^\beta_{3,i}(t,s)\right\|_{\mathcal{H}^{\ell+1}_\sigma}\left\|\partial_x^{\overline{I}^i-\beta}\varrho\right\|_{\mathrm{L}^2(0,T;\mathrm{H}^1)}$$

for $\ell>1+d$ and $\sigma>d/2$; therefore

$$\left\|\mathrm{K}^{\mathrm{free}}_{G^\beta_{3,i}}\left[\partial_x^{\overline{I}^i-\beta}\mathrm{J}_\varepsilon\varrho\right]\right\|_{\mathrm{L}^2(0,T;\mathrm{H}^1)} \lesssim \sup_{0\leq s,t\leq T}\left\|G^\beta_{3,i}(t,s)\right\|_{\mathcal{H}^{\ell+1}_\sigma}\|\varrho\|_{\mathrm{L}^2(0,T;\mathrm{H}^m)}\,.$$

We have, for all $0\leq s,t\leq T$,

$$\begin{aligned}&\left\|G^\beta_{3,i}(t,s)\right\|^2_{\mathcal{H}^{\ell+1}_\sigma} \\ &\quad\leq C_T\sum_{|\mu|+|\nu|\leq\ell+1}\sum_{\gamma=0}^{\mu+\nu}\left\|\partial_x^\gamma\nabla_x(\partial_x^\beta p'(\varrho))(s)\right\|^2_{\mathrm{L}^\infty} \\ &\qquad\qquad\times\int_{\mathbb{T}^d\times\mathbb{R}^d}\langle v\rangle^{2\sigma}\left|\partial_{x,v}^{\mu+\nu-\gamma}\nabla_v f(t,x,v)\right|^2\mathrm{d}x\,\mathrm{d}v \\ &\quad\lesssim C_T\|p'(\varrho)\|^2_{\mathrm{L}^\infty(0,T;\mathrm{H}^k)}\|f(t)\|^2_{\mathcal{H}^{m-1}_\sigma},\end{aligned}$$

provided that $m-1\geq\ell+2$ and $k>\frac{d}{2}+|\gamma|+1+|\beta|$. Since $\ell>1+d$ and $\frac{d}{2}+|\gamma|+1+|\beta|\leq\frac{d}{2}+\ell+2+\frac{m-1}{2}$, a condition such as $3d+9<m$ ensures that

$$\left\|G^\beta_{3,i}(t,s)\right\|_{\mathcal{H}^{\ell+1}_\sigma}\lesssim C_T\|p'(\varrho)\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^{m-2})}\|f\|_{\mathrm{L}^\infty(0,T;\mathcal{H}^{m-1}_\sigma)}\leq\Lambda(T,R),$$

thanks to (2.1) from Lemma 2.7, Sobolev embedding and Lemma 2.20. We thus obtain

$$\left\|\mathrm{K}^{\mathrm{free}}_{G^\beta_{3,i}}\left[\partial_x^{\overline{I}^i-\beta}\mathrm{J}_\varepsilon\varrho\right]\right\|_{\mathrm{L}^2(0,T;\mathrm{H}^1)}\leq\Lambda(T,R).$$

We proceed in the same way for $\mathbf{S}_4$: for all $\left\lfloor\frac{|I|-1}{2}\right\rfloor+1\leq|\beta|\leq|I|-1$ and all $i=1,\ldots,d$, we take one derivative in space and apply Proposition 3.1 to write, for

$\ell > 1+d$ and $\sigma > d/2$,

$$\left\|\mathrm{K}^{\mathrm{free}}_{G^{\overline{I}^i,\beta}_{4,i}}\left[\partial^\beta_x p'(\varrho)\right]\right\|_{\mathrm{L}^2(0,T;\mathrm{H}^1)} \leq \sup_{0\leq s,t\leq T} \left\|G^{\overline{I}^i,\beta}_{4,i}(t,s)\right\|_{\mathcal{H}^{\ell+1}_\sigma} \|p'(\varrho)\|_{\mathrm{L}^2(0,T;\mathrm{H}^m)}.$$

The kernel $G^{\overline{I}^i,\beta}_{4,i}$ is estimated as before: using the Leibniz rule, we have, for all $0 \leq s,t \leq T$,

$$\left\|G^{\overline{I}^i,\beta}_{4,i}(t,s)\right\|^2_{\mathcal{H}^{\ell+1}_\sigma} \lesssim C_T \|\varrho\|^2_{\mathrm{L}^\infty(0,T;\mathrm{H}^k)} \|f(t)\|^2_{\mathcal{H}^{m-1}_\sigma},$$

provided that $m-1 > 2+\ell$ and $k > \frac{d}{2} + |\gamma| + 1 + |\overline{I}^i| - |\beta|$. Since $\ell > 1+d$ and

$$\frac{d}{2} + |\gamma| + 1 + |\overline{I}^i| - |\beta| \leq \frac{d+2\ell+m+1}{2},$$

a condition such as $3d+9 < m$ ensures that

$$\left\|G^{\overline{I}^i,\beta}_{4,i}(t,s)\right\|_{\mathcal{H}^{\ell+1}_\sigma} \lesssim C_T \|\varrho\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^{m-2})} \|f\|_{\mathrm{L}^\infty(0,T;\mathcal{H}^{m-1}_\sigma)}.$$

Therefore, by Sobolev embedding, the bound (2.1) in Lemma 2.7 and Lemma 2.20, we obtain

$$\left\|\mathrm{K}^{\mathrm{free}}_{G^{\overline{I}^i,\beta}_{4,i}}\left[\partial^\beta_x p'(\varrho)\right]\right\|_{\mathrm{L}^2(0,T;\mathrm{H}^1)} \leq \Lambda(T,R).$$ ■

The remaining task is to show that the remainder terms $\mathcal{R}^{\mathrm{Diff}}_I$, $\mathcal{R}^{\mathrm{Duha}}_{I,1}$ and $\mathcal{R}^{\mathrm{Duha}}_{I,2}$ introduced in Lemma 4.11 are well controlled in $\mathrm{L}^2(0,T;\mathrm{H}^1)$. For the first one, we have the following lemma.

Lemma 4.15. *We have*

$$\left\|\mathcal{R}^{\mathrm{Diff}}_I\right\|_{\mathrm{L}^2(0,T;\mathrm{H}^1)} \leq \Lambda(T,R).$$

Proof. In view of Lemma 4.13, let us write, for $0 \leq |\mu| \leq \left\lfloor\frac{|K|-1}{2}\right\rfloor$ and $\left\lfloor\frac{|K|-1}{2}\right\rfloor + 1 \leq |\nu| \leq |K|-1$,

$$\begin{aligned}
&\Big(\nabla_x\big[\partial^{\overline{K}^i-\mu}_x \mathrm{J}_\varepsilon\varrho\big]\nabla_x[\partial^\mu_x p'(\varrho)]\cdot e_i + \nabla_x[\partial^\mu_x p'(\varrho)]\cdot e_i \nabla_x\big[\partial^{\overline{K}^i-\nu}_x \mathrm{J}_\varepsilon\varrho\big]\Big)(s,x-(t-s)v) \\
&\qquad \cdot \nabla_v f(t,x,v) \\
&-\Big(\nabla_x\big[\partial^{\overline{K}^i-\mu}_x \mathrm{J}_\varepsilon\varrho\big]\nabla_x[\partial^\mu_x p'(\varrho)]\cdot e_i + \nabla_x[\partial^\mu_x p'(\varrho)]\cdot e_i \nabla_x\big[\partial^{\overline{K}^i-\nu}_x \mathrm{J}_\varepsilon\varrho\big]\Big)(s,x+(1-e^{t-s})v) \\
&\qquad \cdot \nabla_v f(t,x,v) \\
&=\Big(\nabla_x\big[\partial^{\overline{K}^i-\mu}_x \mathrm{J}_\varepsilon\varrho\big](s,x-(t-s)v) - \nabla_x\big[\partial^{\overline{K}^i-\mu}_x \mathrm{J}_\varepsilon\varrho\big](s,x+(1-e^{t-s})v)\Big) \\
&\qquad\quad \times G_{8,\mu,i}(s,t,x,v) \\
&\qquad + \nabla_x\big[\partial^{\overline{K}^i-\mu}_x \mathrm{J}_\varepsilon\varrho\big](s,x+(1-e^{t-s})v)\cdot G_{9,\mu,i}(s,t,x,v)
\end{aligned}$$

$$+ \nabla_x[\partial_x^\nu p'(\varrho)](s, x - (t-s)v) \cdot G^K_{10,\nu,i}(s,t,x,v)$$
$$+ \Big(\nabla_x[\partial_x^\nu p'(\varrho)](s, x - (t-s)v) - \nabla_x[\partial_x^\nu p'(\varrho)](s, x + (1-e^{t-s})v)\Big)$$
$$\times G^K_{11,\nu,i}(s,t,x,v),$$

where

$$G_{8,\mu,i}(s,t,x,v) := \nabla_x[\partial_x^\mu p'(\varrho)](s, x-(t-s)v) \cdot e_i \nabla_v f(t,x,v),$$
$$G_{9,\mu,i}(s,t,x,v)$$
$$:= \big(\nabla_x[\partial_x^\mu p'(\varrho)] \cdot e_i - \nabla_x[\partial_x^\mu p'(\varrho)](s, x+(1-e^{t-s})v) \cdot e_i\big)\nabla_v f(t,x,v),$$
$$G^K_{10,\nu,i}(s,t,x,v)$$
$$:= \big(\nabla_x\big[\partial_x^{\overline{K}^i-\nu}\mathrm{J}_\varepsilon\varrho\big](s, x-(t-s)v) - \nabla_x\big[\partial_x^{\overline{K}^i-\nu}\mathrm{J}_\varepsilon\varrho\big](s, x+(1-e^{t-s})v)\big)$$
$$\cdot \nabla_v f(t,x,v) e_i,$$
$$G^K_{11,\nu,i}(s,t,x,v) := \nabla_x\big[\partial_x^{\overline{K}^i-\nu}\mathrm{J}_\varepsilon\varrho\big](s, x+(1-e^{t-s})v) \cdot \nabla_x f(t,x,v) e_i.$$

By Lemma 4.13, we can thus rewrite

$$\mathcal{R}_I^{\mathrm{Diff}} = \mathbf{S}_5 + \mathbf{S}_6 + \mathbf{S}_7 + \mathbf{S}_8 + \mathbf{S}_9 + \mathbf{S}_{10} + \mathbf{S}_{11},$$

where

$$\mathbf{S}_5(t,x) := -\int_0^t \int_{\mathbb{R}^d} \big[\partial_x^I u(s, x+(1-e^{t-s})v) - \partial_x^I u(s, x-(t-s)v)\big]$$
$$\cdot \nabla_v f(t,x,v)\,\mathrm{d}v\,\mathrm{d}s,$$

and

$\mathbf{S}_6 := \mathrm{K}^{\mathrm{free}}_{G_6}\big[\partial_x^I \mathrm{J}_\varepsilon\varrho\big] - \mathrm{K}^{\mathrm{fric}}_{G_6}\big[\partial_x^I \mathrm{J}_\varepsilon\varrho\big]$, with

$G_6(t,s,x,v) := p'(\varrho(s, x-(t-s)v))\nabla_v f(t,x,v),$

$\mathbf{S}_7 := \mathrm{K}^{\mathrm{fric}}_{G_7}\big[\partial_x^I \mathrm{J}_\varepsilon\varrho\big]$, with

$G_7(t,s,x,v) := \big[(p'(\varrho)(s, x-(t-s)v) - p'(\varrho)(s, x+(1-e^{t-s})v))\nabla_v f(t,x,v)\big],$

$$\mathbf{S}_8 := \sum_{\ell=0}^{\lfloor\frac{|I|-1}{2}\rfloor} \sum_{\substack{i=1,\dots,d\\ \beta\in\mathbb{B}_I(i,\ell)}} \binom{I}{\widehat{\beta}^i}\Big(\mathrm{K}^{\mathrm{free}}_{G_{8,\beta,i}}\big[\partial_x^{\overline{I}^i-\beta}\mathrm{J}_\varepsilon\varrho\big] - \mathrm{K}^{\mathrm{fric}}_{G_{8,\beta,i}}\big[\partial_x^{\overline{I}^i-\beta}\mathrm{J}_\varepsilon\varrho\big]\Big),$$

$$\mathbf{S}_9 := \sum_{\ell=0}^{\lfloor\frac{|I|-1}{2}\rfloor} \sum_{\substack{i=1,\dots,d\\ \beta\in\mathbb{B}_I(i,\ell)}} \binom{I}{\widehat{\beta}^i}\mathrm{K}^{\mathrm{fric}}_{G_{9,\beta,i}}\big[\partial_x^{\overline{I}^i-\beta}\mathrm{J}_\varepsilon\varrho\big],$$

$$\mathbf{S}_{10} := \sum_{\ell=\lfloor\frac{|I|-1}{2}\rfloor+1}^{|I|-1} \sum_{\substack{i=1,\dots,d\\ \beta\in\mathbb{B}_I(i,\ell)}} \binom{I}{\widehat{\beta}^i}\mathrm{K}^{\mathrm{free}}_{G^I_{10,\beta,i}}\big[\partial_x^\beta p'(\varrho)\big],$$

$$\mathbf{S}_{11} := \sum_{\ell=\lfloor\frac{|I|-1}{2}\rfloor+1}^{|I|-1} \sum_{\substack{i=1,\dots,d\\ \beta\in\mathbb{B}_I(i,\ell)}} \binom{I}{\widehat{\beta}^i}\Big(\mathrm{K}^{\mathrm{free}}_{G^I_{11,\beta,i}}\big[\partial_x^\beta p'(\varrho)\big] - \mathrm{K}^{\mathrm{fric}}_{G^I_{11,\beta,i}}\big[\partial_x^\beta p'(\varrho)\big]\Big).$$

Let us explain how to estimate each of these terms. The term $\mathbf{S}_5$ will be estimated by a direct proof. All the other terms actually require the smoothing estimates coming either from Proposition 3.5 or Proposition 3.7:

- for $\mathbf{S}_6, \mathbf{S}_8$ and $\mathbf{S}_{11}$, the difference of the operators $\mathrm{K}^{\mathrm{free}}$ and $\mathrm{K}^{\mathrm{fric}}$ appears; these terms are therefore of Type III, and we will apply Proposition 3.7;
- since the kernels G_7, G_9 and G_{10} vanish on the diagonal $\{s=t\}$, the terms $\mathbf{S}_7, \mathbf{S}_9$ and $\mathbf{S}_{10}$ are of Type II, and we will appeal to Proposition 3.5.

Let us now turn to the estimates.

Estimate of $\mathbf{S}_5$. The argument is the same as for $\mathbf{S}_2$ above. We obtain

$$\|\mathbf{S}_5\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)} \lesssim \Lambda(T,R).$$

Estimate of $\mathbf{S}_6, \mathbf{S}_8$ *and* $\mathbf{S}_{11}$. By Proposition 3.7, we have, for all $\ell > 8+d$ and $\sigma > 1+d/2$,

$$\|\mathbf{S}_6\|_{\mathrm{L}^2(0,T;\mathrm{H}^1(\mathbb{T}^d))} \lesssim \sup_{0\le s,t\le T} \|G_6(t,s)\|_{\mathcal{H}^\ell_\sigma} \|\partial_x^I \varrho\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{T}^d))} \le \Lambda(T,R),$$

thanks to the bound (2.1) in Lemma 2.7, Lemma 2.20 and provided that we can take $m-2 \ge \frac{d}{2}+8+d$.

Likewise, for $\mathbf{S}_8$, we use Proposition 3.7 to get

$$\|\mathbf{S}_8\|_{\mathrm{L}^2(0,T;\mathrm{H}^1)} \lesssim \sup_{\substack{0\le s,t\le T\\ 0\le|\beta|\le\lfloor\frac{|I|-1}{2}\rfloor\\ i=1,\dots,d}} \|G_{8,\beta,i}(s,t)\|_{\mathcal{H}^\ell_\sigma} \|\varrho\|_{\mathrm{L}^2(0,T;\mathrm{H}^{m-1})}$$

for all $\ell > 8+d$ and $\sigma > 1+d/2$. As in the treatment of $\mathbf{S}_3$ above, we deduce that

$$\begin{aligned}\|\mathbf{S}_8\|_{\mathrm{L}^2(0,T;\mathrm{H}^1)} &\lesssim C_T \|p'(\varrho)\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^{m-2})} \|f\|_{\mathrm{L}^\infty(0,T;\mathcal{H}^{m-1}_\sigma)} \|\varrho\|_{\mathrm{L}^2(0,T;\mathrm{H}^{m-1})}\\ &\le \Lambda(T,R),\end{aligned}$$

since $m > 3d+21$. We argue in the same way for $\mathbf{S}_{11}$ (see the treatment of $\mathbf{S}_4$ above) and obtain

$$\|\mathbf{S}_{11}\|_{\mathrm{L}^2(0,T;\mathrm{H}^1)} \le \Lambda(T,R).$$

Estimate of $\mathbf{S}_7, \mathbf{S}_9$ *and* $\mathbf{S}_{10}$. We proceed exactly as in the estimate of $\mathbf{S}_1$ above, since $G_7(t,t,x,v)=0$. Here, we have

$$\begin{aligned}\partial_s G_7(s,t,x,v) = \big[&p''(\varrho)[\partial_s\varrho + v\cdot\nabla_x\varrho](s,x-(t-s)v)\\ &- p''(\varrho)[\partial_s\varrho + e^{t-s}v\cdot\nabla_x\varrho](s,x+(1-e^{t-s})v)\big]\nabla_v f(t,x,v).\end{aligned}$$

Using the triangle inequality with Proposition 3.5, we end up with

$$\|\mathbf{S}_7\|_{\mathrm{L}^2(0,T;\mathrm{H}^1)} \leq \Lambda(T, R).$$

For $\mathbf{S}_9$, using $G_{9,\beta,i}(t,t,x,v) = 0$ for $0 \leq |\beta| \leq \lfloor \frac{|I|-1}{2} \rfloor$, we also have, by Proposition 3.5,

$$\begin{aligned}&\Big\|\mathrm{K}^{\mathrm{fric}}_{G_{9,\beta,i}}\big[\partial_x^{\overline{I}^i-\beta}\mathrm{J}_\varepsilon\varrho\big]\Big\|_{\mathrm{L}^2(0,T;\mathrm{H}^1)}\\ &\qquad \lesssim (1+T)\sup_{0\leq s,t\leq T}\|\partial_s G_{9,\beta,i}(s,t)\|_{\mathcal{H}^\ell_\sigma}\|\varrho\|_{\mathrm{L}^2(0,T;\mathrm{H}^{m-1})}\end{aligned}$$

for all $\ell > 7+d$ and $\sigma > d/2$. We then proceed, as in the estimate of $\mathbf{S}_1$, to deal with the time derivative, combined with what we have done for the estimate of $\mathbf{S}_3$ (since $m > 3d+19$, for instance) and get

$$\|\mathbf{S}_9\|_{\mathrm{L}^2(0,T;\mathrm{H}^1)} \leq \Lambda(T,R)\|f\|_{\mathrm{L}^\infty(0,T;\mathcal{H}^{m-1}_\sigma)}\|\varrho\|_{\mathrm{L}^2(0,T;\mathrm{H}^{m-1})} \leq \Lambda(T,R).$$

Finally, we apply the same exact arguments as before for $\mathbf{S}_{10}$ (see the estimate of $\mathbf{S}_4$ above, for instance) to get

$$\|\mathbf{S}_{10}\|_{\mathrm{L}^2(0,T;\mathrm{H}^1)} \leq \Lambda(T,R)\|\varrho\|_{\mathrm{L}^2(0,T;\mathrm{H}^{m-1})} \leq \Lambda(T,R). \qquad \blacksquare$$

The second term $\mathcal{R}^{\mathrm{Duha}}_{I,1}$ from Lemma 4.11 is estimated thanks to the following lemma.

Lemma 4.16. *We have*

$$\|\mathcal{R}^{\mathrm{Duha}}_{I,1}\|_{\mathrm{L}^2(0,T;\mathrm{H}^1)} \leq \Lambda(T,R).$$

Proof. Let us introduce the following vector fields:

$$\begin{aligned}G^{K,I}_{14,i,\beta}(s,t,x,v) &:= \Big(\nabla_x\big(\partial_x^\beta p'(\varrho)\big)(s, x+(1-e^{t-s}v))\cdot e_i\Big)\mathrm{H}^{K,I}(s,t,x,v),\\ G^{K,I}_{15,i,\beta}(s,t,x,v) &:= \Big(\nabla_x\big(\partial_x^{\overline{K}^i-\beta}\mathrm{J}_\varepsilon\varrho\big)(s,x+(1-e^{t-s}v))\cdot\mathrm{H}^{K,I}(s,t,x,v)\Big)e_i\end{aligned}$$

for

$$\beta \in \mathbb{B}_K(i,\ell) = \big\{\beta\in\mathbb{N}^d \mid |\beta| = \ell,\, 0 < \widehat{\beta}^i \leq K\big\},\ i = 1,\dots,d,\ \ell = 0,\dots,|K|-1,$$

where we recall the expression of the kernel H defined in (4.9) by

$$\begin{aligned}&\mathrm{H}^{K,I}(t,s,x,v)\\ &\quad := \mathrm{N}^{t;s}\big(\mathrm{Z}^{0;t}(x,\psi_{s,t}(x,v))\big)_{(I,0),(K,0)}\mathrm{J}^{s,t}(x,v)\nabla_v f(t,x,\psi_{s,t}(x,v)) - \nabla_v f(t,x,v).\end{aligned}$$

By Lemma 4.13, we now decompose $\mathcal{R}^{\mathrm{Duha}}_{I,1}$ as

$$\mathcal{R}^{\mathrm{Duha}}_{I,1} = \mathbf{S}_{12}+\mathbf{S}_{13}+\mathbf{S}_{14}+\mathbf{S}_{15},$$

where

$$\mathbf{S}_{12}(t,x) := -\sum_K \int_0^t \int_{\mathbb{R}^d} \partial_x^K u(s, x + (1-e^{t-s})v) \cdot \mathrm{H}^{K,I}(t,s,x,v)\,\mathrm{d}v\,\mathrm{d}s,$$

and

$$\mathbf{S}_{13} := \sum_K \mathrm{K}^{\mathrm{fric}}_{G_{13}}[\partial_x^K \mathrm{J}_\varepsilon \varrho], \text{ with}$$

$$G_{13}(t,s,x,v) := p'(\varrho)(s, x + (1-e^{t-s})v)\mathrm{H}^{K,I}(t,s,x,v),$$

$$\mathbf{S}_{14} := \sum_K \sum_{\ell=0}^{\lfloor \frac{|K|-1}{2} \rfloor} \sum_{\substack{i=1,\dots,d \\ \beta \in \mathbb{B}_K(i,\ell)}} \binom{I}{\widehat{\beta}^i} \mathrm{K}^{\mathrm{free}}_{G^{K,I}_{14,i,\beta}}[\partial_x^{\bar{K}^i-\beta} \mathrm{J}_\varepsilon \varrho],$$

$$\mathbf{S}_{15} := \sum_K \sum_{\ell=\lfloor \frac{|K|-1}{2} \rfloor+1}^{|K|-1} \sum_{\substack{i=1,\dots,d \\ \beta \in \mathbb{B}_K(i,\ell)}} \binom{I}{\widehat{\beta}^i} \mathrm{K}^{\mathrm{free}}_{G^{K,I}_{15,i,\beta}}[\partial_x^{\beta} p'(\varrho)].$$

Let us estimate each of these terms. Note that $\mathrm{H}^{K,I}(t,t,x,v) = 0$, so the kernels appearing in $\mathbf{S}_{13}, \mathbf{S}_{14}$ and $\mathbf{S}_{15}$ vanish in the diagonal in time: these terms are therefore of Type II and we will rely on the regularization property from Proposition 3.5 to handle them.

Estimate of $\mathbf{S}_{12}$. We proceed as in the estimate for $\mathbf{S}_2$ above and first get, for $k > 1 + \frac{d}{2}$,

$$\|J_8\|_{\mathrm{L}^2(0,T;\mathrm{H}^1)} \lesssim \sum_K \|\mathrm{H}^{K,I}\|_{\mathrm{L}^\infty(0,T;\mathcal{H}^k_r)} \|u\|_{\mathrm{L}^2(0,T;\mathrm{H}^{m+1})}.$$

Now observe that for s,t we have

$$\begin{aligned}
\|\mathrm{H}^{K,I}(s,t)\|_{\mathcal{H}^k_r} &\lesssim \|J^{s,t}\|_{\mathrm{W}^{k,\infty}_{x,v}} \|\mathrm{N}^{s;t}(\mathrm{Z}^{0;t}(\cdot,\psi_{s,t}))\nabla_v f(t,\cdot,\psi_{s,t})\|_{\mathcal{H}^k_r} + \|f(t)\|_{\mathcal{H}^{k+1}_r} \\
&\lesssim \|J^{s,t}\|_{\mathrm{W}^{k,\infty}_{x,v}} \Big(1 + \|\mathrm{Z}^{0;t}\|_{\mathrm{W}^{k,\infty}_{x,v}}\Big)\Big(1 + \|\psi_{s,t}\|_{\dot{\mathrm{W}}^{k,\infty}_{x,v}}\Big) \\
&\qquad \times \|\mathrm{N}^{t;s}\|_{\mathrm{W}^{k,\infty}_{x,v}} \sum_{|\gamma|\leq k} \|\partial^\gamma_{x,v}(\nabla_v f)(t,\cdot,\psi_{s,t})\|_{\mathcal{H}^0_r} + \|f(t)\|_{\mathcal{H}^{k+1}_r} \\
&\lesssim \Lambda(T,R) \sum_{|\gamma|\leq k} \|\partial^\gamma_{x,v}(\nabla_v f)(t,\cdot,\psi_{s,t})\|_{\mathcal{H}^0_r} + \|f(t)\|_{\mathcal{H}^{k+1}_r},
\end{aligned}$$

thanks to the estimate (2.9) of Lemma 2.26, Remark 2.27 and Lemma 4.7, and since $m > 4 + d$. To handle the last sum, we write

$$\begin{aligned}
&\int_{\mathbb{T}^d\times\mathbb{R}^d} |\partial^\gamma_{x,v}\nabla_v f(t,x,\psi_{s,t}(x,v))|^2 (1+|v|^2)^r\,\mathrm{d}x\,\mathrm{d}v \\
&\quad \leq \int_{\mathbb{T}^d\times\mathbb{R}^d} |\partial^\gamma_{x,v}\nabla_v f(t,x,\psi_{s,t}(x,v))|^2 \\
&\qquad\qquad \times \big(1 + |v-\psi_{s,t}(x,v)|^2 + |\psi_{s,t}(x,v)|^2\big)^r\,\mathrm{d}x\,\mathrm{d}v,
\end{aligned}$$

and apply the change of variable $v \mapsto w = \psi_{s,t}(x,v)$ from Lemma 2.26, combined with the bounds (2.8) and (2.7) to get (choosing k such that $k+1 \leq m-1$)

$$\left\| \mathrm{H}^{K,I}(s,t) \right\|_{\mathcal{H}^k_r} \leq \Lambda(T,R).$$

Taking a supremum in time we obtain

$$\|\mathbf{S}_{12}\|_{\mathrm{L}^2(0,T;\mathrm{H}^1)} \leq \Lambda(T,R),$$

which yields the result.

Estimate of $\mathbf{S}_{13}, \mathbf{S}_{14}$ *and* $\mathbf{S}_{15}$. Since $\mathrm{H}^{K,I}(t,t,x,v) = 0$, we first observe the cancellation $G_{13}(t,t,x,v) = 0$. By Proposition 3.5, we therefore have, for $\ell > 7+d$ and $\sigma > d/2$,

$$\begin{aligned}
&\|\mathbf{S}_{13}\|_{\mathrm{L}^2(0,T;\mathrm{H}^1)} \\
&\quad \lesssim (1+T) \sum_K \sup_{0 \leq s,t \leq T} \left\| \partial_s \left[p'(\varrho(s, x + (1-e^{t-s})v)) \mathrm{H}^{K,I}(s,t) \right] \right\|_{\mathcal{H}^\ell_\sigma} \\
&\qquad\qquad \times \left\| \partial_x^K \varrho \right\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)} \\
&\quad \lesssim C_T \sum_K \sup_{0 \leq s,t \leq T} \left\| p''(\varrho) \partial_s \varrho(s, x + (1-e^{t-s})v) \mathrm{H}^{K,I}(s,t) \right\|_{\mathcal{H}^\ell_\sigma} \\
&\qquad\qquad \times \left\| \partial_x^K \varrho \right\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)} \\
&\qquad + C_T \sum_K \sup_{0 \leq s,t \leq T} \left\| p''(\varrho) \nabla_x \varrho(s, x + (1-e^{t-s})v) \mathrm{H}^{K,I}(s,t) \right\|_{\mathcal{H}^\ell_{\sigma+1}} \\
&\qquad\qquad \times \left\| \partial_x^K \varrho \right\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)} \\
&\qquad + C_T \sum_K \sup_{0 \leq s,t \leq T} \left\| p'(\varrho(s, x + (1-e^{t-s})v)) \partial_s \mathrm{H}^{K,I}(s,t) \right\|_{\mathcal{H}^\ell_\sigma} \\
&\qquad\qquad \times \left\| \partial_x^K \varrho \right\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)}.
\end{aligned}$$

The first two terms can be handled by arguments similar to those used for $\mathbf{S}_1$ and $\mathbf{S}_{12}$, a fixed number of derivatives being involved. For the last one, we proceed as for the other terms, combined with the arguments used for $\mathbf{S}_{12}$, and write, for all t, s,

$$\begin{aligned}
&\| p'(\varrho(s, x + (1-e^{t-s})v)) \partial_s \mathrm{H}^{K,I}(s,t) \|_{\mathcal{H}^\ell_\sigma} \\
&\quad \leq C_T \| p'(\varrho) \|_{\mathrm{L}^\infty(0,T;\mathrm{H}^{m-2})} \| \partial_s \mathrm{H}^{K,I}(s,t) \|_{\mathcal{H}^\ell_\sigma} \\
&\quad \leq \Lambda(T,R) \big(\| J^{s,t} \|_{\mathrm{W}^{\ell,\infty}_{x,v}} \| \nabla_v f(t) \|_{\mathcal{H}^\ell_\sigma} \\
&\qquad\qquad + \| \partial_s J^{s,t} \|_{\mathrm{W}^{\ell,\infty}_{x,v}} \| \nabla_v f(t, \cdot, \psi_{s,t}) \|_{\mathcal{H}^\ell_\sigma} + \| \nabla_v f(t) \|_{\mathcal{H}^{\ell+1}_\sigma} \big) \\
&\quad \leq \Lambda(T,R),
\end{aligned}$$

since $m > 3d/2 + 11$. This yields

$$\|\mathbf{S}_{13}\|_{\mathrm{L}^2(0,T;\mathrm{H}^1)} \leq \Lambda(T,R).$$

Next, for $\mathbf{S}_{14}$, we use the fact that $G^{K,I}_{14,i,\beta}(t,t,x,v) = 0$ for $0 \leq |\beta| \leq \lfloor \frac{|K|-1}{2} \rfloor$ and $i = 1, \dots, d$ so that by Proposition 3.5, we have

$$\left\| \mathrm{K}_{G^{K,I}_{14,i,\beta}}[\partial_x^{\overline{K}^i-\beta} \mathrm{J}_\varepsilon \varrho] \right\|_{\mathrm{L}^2(0,T;\mathrm{H}^1)} \lesssim (1+T) \sup_{0 \leq t,s \leq T} \left\| \partial_s G^{K,I}_{14,i,\beta}(s,t) \right\|_{\mathcal{H}^\ell_\sigma} \|\varrho\|_{\mathrm{L}^2(0,T;\mathrm{H}^m)}$$

for $\ell > 7+d$ and $\sigma > d/2$. Next, we have, for all t, s,

$$\left\| \partial_s G^{K,I}_{14,i,\beta}(s,t) \right\|_{\mathcal{H}^\ell_\sigma} \lesssim \left\| \partial_s \left[\nabla_x \left(\partial_x^\beta p'(\varrho) \right)(s, x + (1-e^{t-s})v) \right] \mathrm{H}^{K,I}(s,t) \right\|_{\mathcal{H}^\ell_\sigma} + \left\| \nabla_x \left(\partial_x^\beta p'(\varrho) \right)(s, x + (1-e^{t-s})v) \partial_s \mathrm{H}^{K,I}(s,t) \right\|_{\mathcal{H}^\ell_\sigma}.$$

The first term can be handled by arguments similar to those used for $\mathbf{S}_9$ and $\mathbf{S}_{12}$. The second can be addressed by the same procedure where one relies on the arguments for handling $\mathbf{S}_{13}$. Likewise, we obtain

$$\|\mathbf{S}_{14}\|_{\mathrm{L}^2(0,T;\mathrm{H}^1)} \leq \Lambda(T,R). \qquad \blacksquare$$

We finally estimate the third term $\mathcal{R}^{\mathrm{Duha}}_{I,2}$ from Lemma 4.11.

Lemma 4.17. *We have*

$$\left\| \mathcal{R}^{\mathrm{Duha}}_{I,2} \right\|_{\mathrm{L}^2(0,T;\mathrm{H}^1)} \leq \Lambda(T,R).$$

Proof. We proceed as before by introducing the vector fields

$$G^{K,I}_{18,i,\beta}(t,s,x,v) := \left(\nabla_x \left(\partial_x^\beta p'(\varrho) \right)(s, x + (1-e^{t-s}v)) \cdot e_i \right) \mathfrak{H}^{K,I}(s,t,x,v),$$

$$G^{K,I}_{19,i,\beta}(t,s,x,v) := \left(\nabla_x \left(\partial_x^{\overline{K}^i-\beta} \mathrm{J}_\varepsilon \varrho \right)(s, x + (1-e^{t-s}v)) \cdot \mathfrak{H}^{K,I}(s,t,x,v) \right) e_i,$$

where

$$\beta \in \mathbb{B}_K(i,\ell) = \left\{ \beta \in \mathbb{N}^d \mid |\beta| = \ell,\ 0 < \widehat{\beta}^i \leq K \right\},\ i = 1, \dots, d,\ \ell = 0, \dots, |K|-1.$$

Let us also recall the expression of the kernel $\mathfrak{H}$ defined in (4.10) by

$$\mathfrak{H}^{K,I}(t,s,x,v) := \int_s^t e^{d(t-\tau)} \mathrm{N}^{t;s}(\mathrm{Z}^{0;t}(x, \psi_{s,t}(x,v)))_{(I,0),(K,0)} \times \left(\nabla_x f\left(\tau, \mathrm{Z}^{\tau;t}(x, \psi_{s,t}(x,v))\right) - \nabla_v f\left(\tau, \mathrm{Z}^{\tau;t}(x, \psi_{s,t}(x,v))\right) \right) \mathrm{J}^{s,t}(x,v)\, \mathrm{d}\tau.$$

Thanks to Lemma 4.13, we can write

$$\mathcal{R}^{\mathrm{Duha}}_{I,2} = \mathbf{S}_{16} + \mathbf{S}_{17} + \mathbf{S}_{18} + \mathbf{S}_{19},$$

where

$$\mathbf{S}_{16}(t,x) := -\sum_K \int_0^t \int_{\mathbb{R}^d} \partial_x^K u(s, x + (1-e^{t-s}v)) \cdot \mathfrak{H}^{K,I}(t,s,x,v)\,\mathrm{d}v\,\mathrm{d}s,$$

and

$$\begin{aligned}
&\mathbf{S}_{17} := \mathrm{K}^{\mathrm{fric}}_{G_{17}}\big[\partial_x^K \mathrm{J}_\varepsilon \varrho\big], \text{ with}\\
&G_{17}(t,s,x,v) := p'(\varrho(s, x + (1-e^{t-s})v))\mathfrak{H}^{K,I}(t,s,x,v),\\
&\mathbf{S}_{18} := \sum_K \sum_{\ell=0}^{\lfloor \frac{|K|-1}{2} \rfloor} \sum_{\substack{i=1,\dots,d\\ \beta \in \mathbb{B}_K(i,\ell)}} \binom{I}{\widehat{\beta}^i} \mathrm{K}_{G^{K,I}_{18,i,\beta}}\big[\partial_x^{\bar{K}^i-\beta} \mathrm{J}_\varepsilon \varrho\big],\\
&\mathbf{S}_{19} := \sum_K \sum_{\ell=\lfloor \frac{|K|-1}{2} \rfloor+1}^{|K|-1} \sum_{\substack{i=1,\dots,d\\ \beta \in \mathbb{B}_K(i,\ell)}} \binom{I}{\widehat{\beta}^i} \mathrm{K}_{G^{K,I}_{19,i,\beta}}\big[\partial_x^{\beta} p'(\varrho)\big].
\end{aligned}$$

Let us estimate these terms, as we have done previously. Let us observe the cancellation $\mathfrak{H}^{K,I}(t,t,x,v) = 0$; therefore, the terms $\mathbf{S}_{17}, \mathbf{S}_{18}$ and $\mathbf{S}_{19}$ are of Type II and we can rely on Proposition 3.5.

Estimate of $\mathbf{S}_{16}$. We mainly proceed as for $\mathbf{S}_{12}$, the kernel being changed from H to $\mathfrak{H}$. Hence, we only have to give an estimate for $\|\mathfrak{H}^{K,I}(s,t)\|_{\mathcal{H}^k_r}$ (for k and r large enough). As in the estimate of $\mathbf{S}_{12}$, we use $\mathrm{W}^{k,\infty}_{x,v}$ bounds on $\psi_{s,t}, \mathrm{Z}^{s;t}$ and $\mathrm{N}^{s;t}$ from Lemma 2.26, Remark 2.27 and Lemma 4.7 to write

$$\begin{aligned}
&\big\|\mathfrak{H}^{K,I}(s,t)\big\|^2_{\mathcal{H}^k_r}\\
&\quad\lesssim \Lambda(T,R)\|J^{s,t}\|^2_{\mathrm{W}^{k,\infty}_{x,v}}\Big(1 + \|\mathrm{N}^{s;t}\|^2_{\mathrm{W}^{k,\infty}_{x,v}}\Big)\\
&\qquad\times \left\| \int_s^t \big[\nabla_x f(\tau, \mathrm{Z}^{\tau;t}(\cdot,\psi_{s,t})) - \nabla_v f(\tau, \mathrm{Z}^{\tau;t}(\cdot,\psi_{s,t}))\big]\,\mathrm{d}\tau \right\|^2_{\mathcal{H}^k_r}\\
&\quad\lesssim \Lambda(T,R) \sum_{|\gamma|\le k} \int_{\mathbb{T}^d\times\mathbb{R}^d} \langle v\rangle^{2r} \bigg| \int_s^t \big[\partial^\gamma_{x,v}(\nabla_x f)(\tau, \mathrm{Z}^{\tau;t}(\cdot,\psi_{s,t}))\\
&\qquad\qquad - \partial^\gamma_{x,v}(\nabla_v f)(\tau, \mathrm{Z}^{\tau;t}(\cdot,\psi_{s,t}))\big]\,\mathrm{d}\tau \bigg|^2 \mathrm{d}x\,\mathrm{d}v.
\end{aligned}$$

By the generalized Minkowski inequality, and performing the change of variable $v \mapsto w = \psi_{s,t}(x,v)$ from Lemma 2.26 followed by $(x,w) \mapsto \mathrm{Z}^{0;t}(x,w)$, the last expression is bounded by

$$\sum_{|\gamma|\le k}\Big(\int_s^t\Big(\int_{\mathbb{T}^d\times\mathbb{R}^d}\langle v\rangle^{2r}\big|\partial^\gamma_{x,v}(\nabla_x f)(\tau,\mathrm{Z}^{\tau;t}(\cdot,\psi_{s,t})) - \partial^\gamma_{x,v}(\nabla_v f)(\tau,\mathrm{Z}^{\tau;t}(\cdot,\psi_{s,t}))\big|^2\,\mathrm{d}x\,\mathrm{d}v\Big)^{1/2}\mathrm{d}\tau\Big)^2$$

$$\le \Lambda(T,R)\sum_{|\gamma|\le k}\Big(\int_s^t\Big(\int_{\mathbb{T}^d\times\mathbb{R}^d}\langle v\rangle^{2r}\big|\partial^\gamma_{x,v}(\nabla_x f)(\tau,x,v) - \partial^\gamma_{x,v}(\nabla_v f)(\tau,x,v)\big|^2\,\mathrm{d}x\,\mathrm{d}v\Big)^{1/2}\mathrm{d}\tau\Big)^2.$$

Here, we have used the bounds on the Jacobian from (2.7), as well as the bound on $|v-\psi_{s,t}(x,v)|$ via (2.8). It follows that

$$\big\|\mathfrak{H}^{K,I}(s,t)\big\|^2_{\mathcal{H}^k_r}\lesssim \Lambda(T,R)|t-s|^2\sup_{0\le\tau\le T}\Big\{\|\nabla_x f(\tau)\|^2_{\mathcal{H}^k_r}+\|\nabla_v f(\tau)\|^2_{\mathcal{H}^k_r}\Big\},$$

and therefore

$$\|\mathbf{S}_{16}\|_{\mathrm{L}^2(0,T;\mathrm{H}^1)}\le\Lambda(T,R).$$

Estimate of $\mathbf{S}_{17},\mathbf{S}_{18}$ *and* $\mathbf{S}_{19}$. We mainly proceed as for $\mathbf{S}_{13},\mathbf{S}_{14}$ and $\mathbf{S}_{15}$, using Proposition 3.5. As before, the kernel has just been changed from H to $\mathfrak{H}$. Hence, we only have to provide an estimate for $\|\partial_s\mathfrak{H}^{K,I}(s,t)\|_{\mathcal{H}^k_r}$ (for k and r large enough). We have

$$\begin{aligned}
&\partial_s\mathfrak{H}^{K,I}(s,t,x,v)\\
&=\partial_s J^{s,t}\mathrm{N}^{t;s}(\mathrm{Z}^{0;t}(x,\psi_{s,t}(x,v)))_{(I,0),(K,0)}\\
&\qquad\times\int_s^t e^{d(t-\tau)}\big[\nabla_x f(\tau,\mathrm{Z}^{\tau;t}(x,\psi_{s,t}(x,v)))-\nabla_v f(\tau,\mathrm{Z}^{\tau;t}(x,\psi_{s,t}(x,v)))\big]\,\mathrm{d}\tau\\
&\quad+J^{s,t}\partial_s\big\{\mathrm{N}^{t;s}(\mathrm{Z}^{0;t}(x,\psi_{s,t}(x,v)))_{(I,0),(K,0)}\big\}\\
&\qquad\times\int_s^t e^{d(t-\tau)}\big[\nabla_x f(\tau,\mathrm{Z}^{\tau;t}(x,\psi_{s,t}(x,v)))-\nabla_v f(\tau,\mathrm{Z}^{\tau;t}(x,\psi_{s,t}(x,v)))\big]\,\mathrm{d}\tau\\
&\quad+J^{s,t}\mathrm{N}^{t;s}(\mathrm{Z}^{0;t}(x,\psi_{s,t}(x,v)))_{(I,0),(K,0)}\\
&\qquad\times\int_s^t e^{d(t-\tau)}\partial_s\big[\nabla_x f(\tau,\mathrm{Z}^{\tau;t}(x,\psi_{s,t}(x,v)))-\nabla_v f(\tau,\mathrm{Z}^{\tau;t}(x,\psi_{s,t}(x,v)))\big]\,\mathrm{d}\tau\\
&\quad-e^{d(t-s)}\mathrm{J}^{s,t}\mathrm{N}^{t;s}(\mathrm{Z}^{0;t}(x,\psi_{s,t}(x,v)))_{(I,0),(K,0)}\\
&\qquad\times\big[\nabla_x f(s,\mathrm{Z}^{s;t}(x,\psi_{s,t}(x,v)))-\nabla_v f(s,\mathrm{Z}^{s;t}(x,\psi_{s,t}(x,v)))\big].
\end{aligned}$$

Using the same $\mathrm{W}^{k,\infty}_{x,v}$ bounds on $J^{s,t}$, $\mathrm{N}^{s;t}$ and $\dot{\mathrm{W}}^{k,\infty}_{x,v}$ bounds on $\psi_{s,t}$ as before, as well as on their time derivatives, we obtain

$$\|\partial_s\mathfrak{H}^{K,I}(s,t)\|_{\mathcal{H}^k_r}\le\Lambda(T,R).$$

This finally yields

$$\|\mathbf{S}_{17}\|_{\mathrm{L}^2(0,T;\mathrm{H}^1)} + \|\mathbf{S}_{18}\|_{\mathrm{L}^2(0,T;\mathrm{H}^1)} + \|\mathbf{S}_{19}\|_{\mathrm{L}^2(0,T;\mathrm{H}^1)} \leq \Lambda(T, R),$$

which ends the proof. ■

We end this section by giving a proof of Lemma 4.10. Let us mention that it only requires the continuity estimate coming from Proposition 3.4 and not those of Propositions 3.5 and 3.7.

Proof of Lemma 4.10. In view of the estimate (4.8) of Lemma 4.9, we only have to prove that

$$\|\nabla_x \mathcal{I}^0_{\mathcal{R}_1}\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)} + \|\nabla_x \mathcal{I}^1_{\mathcal{R}_1}\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)} \leq \Lambda(T, R).$$

We only write the proof for $\nabla_x \mathcal{I}^0_{\mathcal{R}_1}$. Let us recall that

$$\mathcal{I}^0_{\mathcal{R}_1}(t,x) = -\int_0^t e^{d(t-s)} \int_{\mathbb{R}^d} \mathrm{N}^{t;s}(\mathrm{Z}^{0;t}(x,v))\mathcal{R}_1(s, \mathrm{Z}^{s;t}(x,v))\, \mathrm{d}v\, \mathrm{d}s,$$

where

$$\mathcal{R}_1 = \left(\mathcal{R}_1^{K,L}\right)_{|K|+|L|\in\{m-1,m\}},$$

with

$$\mathcal{R}_1^{K,L} = \mathbf{1}_{\substack{|K|>2\\ L\neq 0}} \partial_x^K E^{u,\varrho}_{\mathrm{reg},\varepsilon} \cdot \nabla_v \partial_v^L f + \mathbf{1}_{|K|>1} \sum_{\substack{0<\alpha<K\\ |\alpha|=m-1}} \binom{L}{\alpha} \partial_x^\alpha E^{u,\varrho}_{\mathrm{reg},\varepsilon} \cdot \nabla_v \partial_x^{L-\alpha} \partial_v^K f.$$

We have, for all (I, J),

$$\begin{aligned}
&[\mathcal{I}^0_{\mathcal{R}_1}]_{(I,J)}(t,x) \\
&\quad = -\int_0^t e^{d(t-s)} \int_{\mathbb{R}^d} \sum_{(K,L)} \mathrm{N}^{t;s}_{(I,J),(K,L)}(\mathrm{Z}^{0;t}(x,v))[\mathcal{R}_1]_{(K,L)}(s, \mathrm{Z}^{s;t}(x,v))\, \mathrm{d}v\, \mathrm{d}s;
\end{aligned}$$

therefore,

$$\begin{aligned}
\nabla_x&[\mathcal{I}^0_{\mathcal{R}_1}]_{(I,J)}(t,x) \\
&= -\int_0^t e^{d(t-s)} \int_{\mathbb{R}^d} \sum_{(K,L)} \nabla_x \mathrm{Z}^{0;t}(x,v) \nabla_x \mathrm{N}^{t;s}_{(I,J),(K,L)}(\mathrm{Z}^{0;t}(x,v)) \\
&\qquad\qquad \times [\mathcal{R}_1]_{(K,L)}(s, \mathrm{Z}^{s;t}(x,v))\, \mathrm{d}v\, \mathrm{d}s \\
&\quad - \int_0^t e^{d(t-s)} \int_{\mathbb{R}^d} \sum_{(K,L)} \mathrm{N}^{t;s}_{(I,J),(K,L)}(\mathrm{Z}^{0;t}(x,v)) \nabla_x \mathrm{Z}^{0;t}(x,v) \\
&\qquad\qquad \times \nabla_x [\mathcal{R}_1]_{(K,L)}(s, \mathrm{Z}^{s;t}(x,v))\, \mathrm{d}v\, \mathrm{d}s.
\end{aligned}$$

For the first term, there is no derivative on $\mathcal{R}_1$. We can proceed exactly as for $\mathcal{R}_0$ in Lemma 4.9, relying on the estimate (4.8).

The second term is more involved, since $\nabla_x[\mathcal{R}_1]_{(K,L)}$ contains several types of terms:

- Some are of the form
$$\mathbf{1}_{\substack{|K|>2\\ L\neq 0}}\nabla_x\big[\nabla_v\partial_v^L f\big]\partial_x^K E^{u,\varrho}_{\mathrm{reg},\varepsilon}.$$
They involve at most m derivatives of ϱ and at most $2+(m-3)=m-1$ derivatives of f. We can bound this term in $\mathrm{L}^2(0,T;\mathcal{H}^0_r)$, following the argument used for $\mathcal{R}_0$ in the proof of Lemma 4.3. Its contribution to $\nabla_x \mathcal{I}^0_{\mathcal{R}_1}$ is handled as in the proof of Lemma 4.9 (for the term $\nabla_x \mathcal{I}^0_{\mathcal{R}_0}$).
- Some are of the form
$$\nabla_x\big[\nabla_v\partial_x^{L-\alpha}\partial_v^K f\big]\partial_x^\alpha E^{u,\varrho}_{\mathrm{reg},\varepsilon},$$
with $|K|>1$ and $|\alpha|=m-1$. They involve at most m derivatives of ϱ and at most three derivatives of f. We can also bound this term in $\mathrm{L}^2(0,T;\mathcal{H}^0_r)$, following the argument used for $\mathcal{R}_0$ in the proof of Lemma 4.3. As before, we can follow the proof of Lemma 4.9 to control its contribution to $\nabla_x \mathcal{I}^0_{\mathcal{R}_1}$.
- Some are of the form
$$\mathbf{1}_{\substack{|K|>2\\ L\neq 0}}\nabla_x\big[\partial_x^K E^{u,\varrho}_{\mathrm{reg},\varepsilon}\big]\nabla_v\partial_v^L f,$$
and of the form
$$\nabla_x\big[\partial_x^\alpha E^{u,\varrho}_{\mathrm{reg},\varepsilon}\big]\nabla_v\partial_x^{L-\alpha}\partial_v^K f,$$
with $|K|>1$ and $|\alpha|=m-1$. These terms involve at most $m+1$ derivatives of ϱ and cannot be directly treated as the previous ones. We need to rely on the smoothing estimates from Chapter 3 (by making some terms of Type I appear).

We then focus on the last two types of terms, and we have to deal with

$$\mathbb{W}_1^{K,L}(t,x) := -\int_0^t e^{d(t-s)}\int_{\mathbb{R}^d} \mathrm{N}^{t;s}_{(I,J),(K,L)}(\mathrm{Z}^{0;t}(x,v)) \times \nabla_x \mathrm{Z}^{0;t}(x,v)\mathbf{1}_{\substack{|K|>2\\ L\neq 0}}\nabla_x\big[\partial_x^K E^{u,\varrho}_{\mathrm{reg},\varepsilon}\big]\nabla_v\partial_v^L f(s,\mathrm{Z}^{s;t}(x,v))\,\mathrm{d}v\,\mathrm{d}s,$$

and

$$\mathbb{W}_2^{K,L}(t,x) := -\int_0^t e^{d(t-s)}\int_{\mathbb{R}^d} \mathrm{N}^{t;s}_{(I,J),(K,L)}(\mathrm{Z}^{0;t}(x,v)) \times \nabla_x \mathrm{Z}^{0;t}(x,v)\mathbf{1}_{|K|>1} \times \sum_{\substack{0<\alpha<K\\ |\alpha|=m-1}}\binom{L}{\alpha}\nabla_x\big[\partial_x^\alpha E^{u,\varrho}_{\mathrm{reg},\varepsilon}\big]\nabla_v\partial_x^{L-\alpha}\partial_v^K f(s,\mathrm{Z}^{s;t}(x,v))\,\mathrm{d}v\,\mathrm{d}s$$

for $|K|+|L|\leq m$. Let us turn to the estimates of these terms in $\mathrm{L}^2(0,T;\mathrm{L}^2)$.

Estimate of $\mathbb{W}_1^{K,L}$ *when* $\frac{m-2}{2} \leq |L| < m-2$. Since $|K| + |L| \leq m$, this implies that $|K| \leq 1 + m/2$. As in the proof of Lemma 4.9, we have, by the Cauchy–Schwarz inequality,

$$\big\| \mathbb{W}_1^{K,L} \big\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)} \leq \Lambda(T,R) \big\| \nabla_x \partial_x^K E_{\mathrm{reg},\varepsilon}^{u,\varrho} \cdot \nabla_v \partial_v^L f \big\|_{\mathrm{L}^2(0,T;\mathcal{H}_r^0)},$$

and, by setting $\chi(v) = (1+|v|^2)^{r/2}$, we have

$$\begin{aligned}\big\| \chi \nabla_x \partial_x^K E_{\mathrm{reg},\varepsilon}^{u,\varrho} \cdot \nabla_v \partial_v^L f \big\|_{\mathrm{L}^2_{x,v}} &\lesssim \big\| \nabla_x \partial_x^K E_{\mathrm{reg},\varepsilon}^{u,\varrho} \big\|_{\mathrm{L}^\infty_x} \big\| \chi \nabla_v \partial_v^L f \big\|_{\mathrm{L}^2_{x,v}} \\ &\lesssim \|E\|_{\mathrm{H}^k} \|f\|_{\mathcal{H}_r^{m-1}},\end{aligned}$$

with $k > \frac{d}{2} + 1 + |K|$. Since $d + 6 \leq m$, we can choose k such that

$$\big\| \mathbb{W}_1^{K,L} \big\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)} \leq \Lambda(T,R) \big\| E_{\mathrm{reg},\varepsilon}^{u,\varrho} \big\|_{\mathrm{L}^2(0,T;\mathrm{H}^{m-1})} \|f\|_{\mathrm{L}^\infty(0,T;\mathcal{H}_r^{m-1})} \leq \Lambda(T,R),$$

and conclude as in the proof of Lemma 4.3. We obtain in that case

$$\big\| \mathbb{W}_1^{K,L} \big\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)} \leq \Lambda(T,R).$$

Estimate of $\mathbb{W}_1^{K,L}$ *when* $1 \leq |L| \leq \frac{m-2}{2} - 1$. In that case, we only have $|K| \leq m-1$. We shall rely on the smoothing estimate of Proposition 3.4 (to treat terms of Type I) and to recover the loss of the extra derivative on ϱ. To do so, we first write

$$\begin{aligned}\big|\mathbb{W}_1^{K,L}(t,x)\big| \leq \Lambda(T,R) \big\| \nabla_x \mathrm{Z}^{0;t} \big\|_{\mathrm{L}^\infty_{x,v}} &\sup_{0 \leq s \leq T} \big\| \mathrm{N}^{t;s} \big\|_{\mathrm{L}^\infty_{x,v}} \\ &\times \mathbf{1}_{\substack{|K|>2 \\ L \neq 0}} \left| \int_0^t \int_{\mathbb{R}^d} \nabla_x \big[\partial_x^K E_{\mathrm{reg},\varepsilon}^{u,\varrho}\big] \nabla_v \partial_v^L f(s, \mathrm{Z}^{s;t}(x,v))\, \mathrm{d}v\, \mathrm{d}s \right|,\end{aligned}$$

and we perform the change of variable $v \mapsto \psi_{s,t}(x,w)$ from Lemma 2.26 in the last integral (since $t \leq \overline{T}(R)$) to get

$$\begin{aligned}\big|\mathbb{W}_1^{K,L}(t,x)\big| \leq \Lambda(T,R) \mathbf{1}_{\substack{|K|>2 \\ L \neq 0}} \bigg| \int_0^t \int_{\mathbb{R}^d} &\nabla_x \big[\partial_x^K E_{\mathrm{reg},\varepsilon}^{u,\varrho}\big](s, x + (1-e^{t-s})w) \\ &\times \nabla_v \partial_v^L f(s, \mathrm{Z}^{s;t}(x, \psi_{s,t}(x,w)))\, \mathrm{d}w\, \mathrm{d}s \bigg|,\end{aligned}$$

thanks to the bounds (2.7) of Lemma 2.26, Remark 2.27 and Lemma 4.7.

Relying on the decomposition of Lemma 4.13, we can make the operator $\mathrm{K}^{\mathrm{fric}}$ appear, hence dealing with terms of Type I, and apply Proposition 3.4 (combined with the bounds on $\mathrm{Z}^{s;t}$ and $\psi_{s,t}$) to estimate the last integral in $\mathrm{L}^2(0,T;\mathrm{L}^2)$. The proof follows the same lines as that of Lemma 4.14. Note that Proposition 3.4 only requires an estimate of $\nabla_v \partial_v^L f$ in $\mathrm{L}^\infty(0,T;\mathcal{H}_r^s)$ with $s > 1 + d$. Since $m > 2d + 2$, we obtain in that case

$$\big\| \mathbb{W}_1^{K,L} \big\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)} \leq \Lambda(T,R).$$

Estimate of $\mathbb{W}_2^{K,L}$. To treat this term, we observe that it involves at most two derivatives of f and the gradient of $m-1$ derivatives of the force field $E^{u,\varrho}_{\mathrm{reg},\varepsilon}$. Hence, we can proceed exactly as we did previously for $\mathbb{W}_1^{K,L}$. We likewise obtain

$$\| \mathbb{W}_2^{K,L} \|_{\mathrm{L}^2(0,T;\mathrm{L}^2)} \leq \Lambda(T, R).$$

This concludes the proof of Lemma 4.10. ∎