Chapter 5

Analysis of the fluid density

Our goal in this chapter is to obtain a uniform control on $\|\varrho\|_{\mathrm{L}^2(0,T;\mathrm{H}^m)}$. In the previous section, we have related the kinetic moments ρ_f and j_f to the fluid density ϱ, up to some well-controlled remainders. In this section, we build on those relations to further analyze ϱ.

We start by taking derivatives in the transport equation satisfied by ϱ. By using the key Proposition 4.1 (which was precisely the main outcome of Chapter 4), we obtain a factorization of the equation for the derivatives between

- a purely hyperbolic part, which is the transport operator $\partial_t + u \cdot \nabla_x$;
- an integro-differential operator part.

This is where the crucial Penrose condition (P), assumed to be satisfied by the initial data, steps in and allows us to justify that this last operator is actually elliptic in space-time and therefore can provide $\mathrm{L}^2_T\mathrm{L}^2_x$ estimates without loss. This relies on a semiclassical pseudodifferential analysis, in the spirit of [90].

In this chapter, we will use the notation M_{in}, which stands for a positive constant depending only on the initial data.

5.1 Equation for the derivatives of the fluid density

For $T \in [0, \min(T_\varepsilon(R), \overline{T}(R)))$, the aim of this section is to prove the following proposition.

Proposition 5.1. *Setting $h = \partial_x^\alpha \varrho$ for $|\alpha| \le m$, one has*

$$\Big(\mathrm{Id} - \frac{\varrho}{1-\rho_f}\mathrm{K}_G^{\mathrm{free}} \circ \mathrm{J}_\varepsilon\Big)\big[\partial_t h + u \cdot \nabla_x h\big] = \mathcal{R}, \quad t \in (0,T), \tag{5.1}$$

with $G(t,x,v) = p'(\varrho(t,x))\nabla_v f(t,x,v)$ and

$$\|\mathcal{R}\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{T}^d))} \le \Lambda\big(T, R, \|h(0)\|_{\mathrm{H}^1(\mathbb{T}^d)}\big).$$

We start with a commutation result for the derivatives in the equation for ϱ.

Lemma 5.2. *For all $|\alpha| \le m$, $\partial_x^\alpha \varrho$ satisfies the equation*

$$\partial_t(\partial_x^\alpha \varrho) + u \cdot \nabla_x(\partial_x^\alpha \varrho) + \frac{\varrho}{1-\rho_f}\mathrm{div}_x\big[j_{\partial_x^\alpha f} - \rho_{\partial_x^\alpha f} u\big] = R^\alpha,$$

with

$$\|R^\alpha\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)} \le \Lambda(T,R).$$

Proof. Recall that by Lemma 2.2, the transport equation for ϱ reads

$$\partial_t \varrho + u \cdot \nabla_x \varrho + \frac{\varrho}{1-\rho_f} \operatorname{div}_x [j_f - \rho_f u] = S, \quad S = -\frac{\varrho}{1-\rho_f} \operatorname{div}_x u.$$

We get, for all $\alpha \in \mathbb{N}^d$,

$$\begin{aligned} &\partial_t (\partial_x^\alpha \varrho) + u \cdot \nabla_x (\partial_x^\alpha \varrho) + [\partial_x^\alpha, u \cdot \nabla_x] \varrho \\ &\quad + \frac{\varrho}{1-\rho_f} \operatorname{div}_x \partial_x^\alpha (j_f - \rho_f u) + \left[\partial_x^\alpha, \frac{\varrho \operatorname{div}_x}{1-\rho_f}\right](j_f - \rho_f u) = \partial_x^\alpha S, \end{aligned}$$

and therefore $\partial_x^\alpha \varrho$ satisfies the equation

$$\partial_t (\partial_x^\alpha \varrho) + u \cdot \nabla_x (\partial_x^\alpha \varrho) + \frac{\varrho}{1-\rho_f} \operatorname{div}_x [j_{\partial_x^\alpha f} - \rho_{\partial_x^\alpha f} u] = R^\alpha,$$

where R^α is a remainder defined by

$$\begin{aligned} R^\alpha &:= \partial_x^\alpha S - [\partial_x^\alpha, u \cdot \nabla_x] \varrho - \left[\partial_x^\alpha, \frac{\varrho \operatorname{div}_x}{1-\rho_f}\right](j_f - \rho_f u) + \frac{\varrho}{1-\rho_f} \operatorname{div}_x \big([\partial_x^\alpha, u] \rho_f\big) \\ &=: \partial_x^\alpha S + \mathscr{C}_1 + \mathscr{C}_2 + \mathscr{C}_3. \end{aligned}$$

Let us estimate each of these terms in $\mathrm{L}^2(0,T;\mathrm{L}^2)$, for all $|\alpha| \leq m$.

Estimate of $\partial_x^\alpha S$. We rely on the tame estimate from Proposition A.2 to write

$$\|\partial_x^\alpha S\|_{\mathrm{L}^2} \lesssim \left\| \frac{\varrho}{1-\rho_f} \right\|_{\mathrm{L}^\infty} \|\operatorname{div}_x u\|_{\mathrm{H}^m} + \left\| \frac{\varrho}{1-\rho_f} \right\|_{\mathrm{H}^m} \|\operatorname{div}_x u\|_{\mathrm{L}^\infty} = \mathfrak{S}_1 + \mathfrak{S}_2.$$

Since $m - 2 > d/2$, we have, by Lemma 2.20 and (2.5),

$$\|\mathfrak{S}_1\|_{\mathrm{L}^2(0,T)} \lesssim \left\| \frac{1}{1-\rho_f} \right\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty)} \|\varrho\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^{m-2})} \|u\|_{\mathrm{L}^2(0,T;\mathrm{H}^{m+1})} \leq \Lambda(T,R).$$

For $\mathfrak{S}_2$, we combine the tame estimate from Proposition A.2 with Lemma A.5, which provides

$$\begin{aligned} \|\mathfrak{S}_2\|_{\mathrm{L}^2(0,T)} &\lesssim \|\varrho\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^{m-2})} \|u\|_{\mathrm{L}^2(0,T;\mathrm{H}^m)} \\ &\quad + \|\varrho\|_{\mathrm{L}^2(0,T;\mathrm{H}^m)} \left\| \frac{1}{1-\rho_f} \right\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty)} \|u\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^m)} \\ &\quad + \|\varrho\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^{m-2})} \big(\|\rho_f\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty)}\big) \|\rho_f\|_{\mathrm{L}^2(0,T;\mathrm{H}^m)} \|u\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^m)} \\ &\lesssim \Lambda(T,R) + \Lambda(T,R) \|\rho_f\|_{\mathrm{L}^2(0,T;\mathrm{H}^m)}, \end{aligned}$$

since $m - 2 > d/2$ and again thanks to Lemma 2.20 and (2.5). We obtain

$$\|\partial_x^\alpha S\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)} \leq \Lambda(T,R),$$

by using Corollary 4.2.

Estimate of $\mathcal{C}_1$. We have $-\mathcal{C}_1 = [\partial_x^\alpha, u\cdot](\nabla_x \varrho)$. Therefore, the commutator estimate from Proposition A.1 yields

$$\|\mathcal{C}_1\|_{\mathrm{L}^2} \lesssim \|\nabla_x u\|_{\mathrm{L}^\infty} \|\nabla_x \varrho\|_{\mathrm{H}^{m-1}} + \|u\|_{\mathrm{H}^m} \|\nabla_x \varrho\|_{\mathrm{L}^\infty} \lesssim \|u\|_{\mathrm{H}^m} \|\varrho\|_{\mathrm{H}^m},$$

since $m > 1 + d/2$. We then obtain, by Corollary 4.2,

$$\|\mathcal{C}_1\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)} \lesssim \|u\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^m)} \|\varrho\|_{\mathrm{L}^2(0,T;\mathrm{H}^m)} \leq \Lambda(T, R).$$

Estimate of $\mathcal{C}_2$. We have

$$\mathcal{C}_2 = \left[\partial_x^\alpha, \frac{\varrho}{1-\rho_f}\right](\mathrm{div}_x(j_f - \rho_f u)).$$

Applying the commutator estimate from Proposition A.1, we get

$$\begin{aligned}\|\mathcal{C}_2\|_{\mathrm{L}^2} &\lesssim \left\|\nabla_x \frac{\varrho}{1-\rho_f}\right\|_{\mathrm{L}^\infty} \|j_f - \rho_f u\|_{\mathrm{H}^m} + \left\|\frac{\varrho}{1-\rho_f}\right\|_{\mathrm{H}^m} \|\mathrm{div}_x(j_f - \rho_f u)\|_{\mathrm{L}^\infty} \\ &= \mathcal{C}_{2,1} + \mathcal{C}_{2,2}.\end{aligned}$$

For $\mathcal{C}_{2,1}$, we infer from Sobolev embedding (since $m - 2 > 1 + d/2$) that

$$\begin{aligned}\left\|\nabla_x \frac{\varrho}{1-\rho_f}\right\|_{\mathrm{L}^\infty} &\leq \left\|\frac{1}{1-\rho_f}\right\|_{\mathrm{L}^\infty} \|\nabla_x \varrho\|_{\mathrm{L}^\infty} + \left\|\frac{\varrho}{(1-\rho_f)^2}\right\|_{\mathrm{L}^\infty} \|\nabla_x \rho_f\|_{\mathrm{L}^\infty} \\ &\lesssim \left\|\frac{1}{1-\rho_f}\right\|_{\mathrm{L}^\infty} \|\varrho\|_{\mathrm{H}^{m-2}} + \left\|\frac{1}{(1-\rho_f)}\right\|_{\mathrm{L}^\infty}^2 \|\rho\|_{\mathrm{H}^{m-2}} \|\nabla_x \rho_f\|_{\mathrm{L}^\infty} \\ &\leq \Lambda(T, R),\end{aligned}$$

thanks to Lemma 2.20 and Corollary 4.2. Therefore, the tame estimate coming from Proposition A.2 entails

$$\begin{aligned}\|\mathcal{C}_{2,1}\|_{\mathrm{L}^2(0,T)} \lesssim \Lambda(T, R)\big(&\|j_f\|_{\mathrm{L}^2(0,T;\mathrm{H}^m)} + \|f\|_{\mathrm{L}^\infty(0,T;\mathcal{H}_r^{m-1})} \|u\|_{\mathrm{L}^2(0,T;\mathrm{H}^m)} \\ &+ \|u\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^m)} \|\rho_f\|_{\mathrm{L}^2(0,T;\mathrm{H}^m)}\big),\end{aligned}$$

since $m - 1 > d/2$. Invoking Lemma 2.1 and Corollary 4.2, we get

$$\|\mathcal{C}_{2,1}\|_{\mathrm{L}^2(0,T)} \leq \Lambda(T, R).$$

For $\mathcal{C}_{2,2}$, we observe that since $m - 1 > 1 + d/2$,

$$\|\mathrm{div}_x(j_f - \rho_f u)\|_{\mathrm{L}^\infty} \lesssim \|j_f\|_{\mathrm{H}^{m-1}} + \|\rho_f\|_{\mathrm{H}^{m-1}} \|u\|_{\mathrm{H}^{m-1}};$$

therefore we obtain

$$\|\mathcal{C}_{2,2}\|_{\mathrm{L}^2(0,T)} \leq \Lambda(T, R),$$

by using what we have done for $\mathfrak{S}_2$ above.

Estimate of $\mathscr{C}_3$. We have

$$\mathscr{C}_3 = \frac{\varrho}{1-\rho_f}\big([\partial_x^\alpha, \operatorname{div}_x u\cdot](\rho_f) + [\partial_x^\alpha, u\cdot](\nabla_x \rho_f)\big) = \frac{\varrho}{1-\rho_f}(\mathscr{C}_{3,1} + \mathscr{C}_{3,2}).$$

Applying Proposition A.1, we get, for $m+1 > 2 + d/2$,

$$\|\mathscr{C}_{3,1}\|_{\mathrm{L}^2} \lesssim \|\nabla_x \operatorname{div}_x u\|_{\mathrm{L}^\infty} \|\rho_f\|_{\mathrm{H}^{m-1}} + \|\operatorname{div}_x u\|_{\mathrm{H}^m} \|\rho_f\|_{\mathrm{L}^\infty} \lesssim \|u\|_{\mathrm{H}^{m+1}} \|\rho_f\|_{\mathrm{H}^{m-1}}$$

and

$$\begin{aligned}\|\mathscr{C}_{3,2}\|_{\mathrm{L}^2} &\lesssim \|\nabla_x u\|_{\mathrm{L}^\infty} \|\nabla_x \rho_f\|_{\mathrm{H}^{m-1}} + \|u\|_{\mathrm{H}^m} \|\nabla_x \rho_f\|_{\mathrm{L}^\infty} \\ &\lesssim \|u\|_{\mathrm{H}^m} \|\rho_f\|_{\mathrm{H}^m} + \|u\|_{\mathrm{H}^m} \|\rho_f\|_{\mathrm{H}^{m-1}}.\end{aligned}$$

Taking the L^2-norm in time and using Lemma 2.1, we have, for $\mathscr{C}_{3,1}$,

$$\|\mathscr{C}_{3,1}\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)} \lesssim \|u\|_{\mathrm{L}^2(0,T;\mathrm{H}^{m+1})} \|\rho_f\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^{m-1})} \leq \Lambda(T, R),$$

while for $\mathscr{C}_{3,2}$ we have

$$\begin{aligned}\|\mathscr{C}_{3,2}\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)} &\lesssim \|u\|_{\mathrm{L}^\infty(0,T,\mathrm{H}^m)} \|\rho_f\|_{\mathrm{L}^2(0,T;\mathrm{H}^m)} \\ &\quad + \|u\|_{\mathrm{L}^2(0,T;\mathrm{H}^{m+1})} \|\rho_f\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^{m-1})} \\ &\leq \Lambda(T, R),\end{aligned}$$

thanks to Corollary 4.2. ■

Let us now transform the equation for the derivatives of ϱ obtained in Lemma 5.2. To ease readability, let us temporarily set

$$\mathrm{K}^{\text{free}}_{1,G}[F](t,x) := \int_0^t \int_{\mathbb{R}^d} v[\nabla_x F](s, x-(t-s)v) \cdot G(t,s,x,v)\, \mathrm{d}v\, \mathrm{d}s. \tag{5.2}$$

Lemma 5.3. *For* $h = \partial_x^\alpha \varrho$ *with* $|\alpha| \leq m$*, one has*

$$\partial_t h + u \cdot \nabla_x h + \frac{\varrho}{1-\rho_f} \operatorname{div}_x \big[\mathrm{K}^{\text{free}}_{1,G}(\mathrm{J}_\varepsilon h) - \mathrm{K}^{\text{free}}_G(\mathrm{J}_\varepsilon h) u\big] = \mathscr{R},$$

with

$$G(t,x,v) := p'(\varrho(t,x)) \nabla_v f(t,x,v),$$

and where the remainder $\mathscr{R}$ *satisfies*

$$\|\mathscr{R}\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{T}^d))} \leq \Lambda(T, R).$$

Proof. By Lemma 5.2, we have

$$\partial_t h + u \cdot \nabla_x h + \frac{\varrho}{1-\rho_f} \operatorname{div}_x [j_{\partial_x^\alpha f} - \rho_{\partial_x^\alpha f} u] = R^\alpha,$$

with

$$\|R^\alpha\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)} \leq \Lambda(T, R).$$

Thanks to Proposition 4.1, we can write

$$\rho_{\partial_x^\alpha f} = \mathrm{K}_G^{\mathrm{free}}(\mathrm{J}_\varepsilon \partial_x^\alpha \varrho) + \mathrm{R}^\alpha[\rho_f], \quad j_{\partial_x^\alpha f} = \mathrm{K}_{1,G}^{\mathrm{free}}(\mathrm{J}_\varepsilon \partial_x^\alpha \varrho) + \mathrm{R}^\alpha[j_f],$$

with $G(t,x,v) := p'(\varrho(t,x))\nabla_v f(t,x,v)$ and where

$$\left\|\mathrm{R}^\alpha[\rho_f]\right\|_{\mathrm{L}^2(0,T,\mathrm{H}^1)} \leq \Lambda(T,R), \quad \left\|\mathrm{R}^\alpha[j_f]\right\|_{\mathrm{L}^2(0,T,\mathrm{H}^1)} \leq \Lambda(T,R).$$

We obtain

$$\begin{aligned}\partial_t h + u\cdot\nabla_x h &+ \frac{\varrho}{1-\rho_f}\operatorname{div}_x\left[\mathrm{K}_{1,G}^{\mathrm{free}}(\mathrm{J}_\varepsilon h) - \mathrm{K}_G^{\mathrm{free}}(\mathrm{J}_\varepsilon h)u\right]\\ &= R^\alpha - \operatorname{div}_x\left[\mathrm{R}^\alpha[j_f] - \mathrm{R}^\alpha[\rho_f]u\right].\end{aligned}$$

Thanks to the aforementioned estimates in $\mathrm{L}^2_T\mathrm{H}^1_x$, we obtain the desired estimate on the remainder. ■

Remark 5.4. In what follows, we shall rely on the following estimate: for all $\ell > 0$ and $\sigma > 0$ such that $\ell < m - d/2 - 2$, we have

$$\sup_{0\leq t\leq T} \|G(t)\|_{\mathcal{H}^\ell_\sigma} \leq \Lambda(T,R), \tag{5.3}$$

where we recall that

$$G(t,x,v) = p'(\varrho(t,x))\nabla_v f(t,x,v).$$

Indeed, we have

$$\begin{aligned}&\|p'(\varrho(t))\nabla_v f(t)\|^2_{\mathcal{H}^\ell_\sigma}\\ &\lesssim \sum_{|\mu|+|\nu|\leq\ell}\sum_{\gamma=0}^{\mu+\nu}\|\partial_x^\gamma(p'(\varrho(t)))\|^2_{\mathrm{L}^\infty}\int_{\mathbb{T}^d\times\mathbb{R}^d}\langle v\rangle^{2\sigma}\left|\partial_{x,v}^{\mu+\nu-\gamma}\nabla_v f(t,x,v)\right|^2\mathrm{d}x\,\mathrm{d}v\\ &\lesssim \|p'(\varrho(t))\|^2_{\mathrm{H}^k}\|f(t)\|^2_{\mathcal{H}^{m-1}_\sigma}\end{aligned}$$

if $m-1\geq\ell$ and $k > \frac{d}{2}+\ell$. Invoking Lemma 2.20 by choosing also $k \leq m-2$, we obtain the estimate (5.3).

Our goal is now to understand the term

$$\operatorname{div}_x\left[\mathrm{K}_{1,G}^{\mathrm{free}}(\mathrm{J}_\varepsilon h) - \mathrm{K}_G^{\mathrm{free}}(\mathrm{J}_\varepsilon h)u\right].$$

We will show that up to good remainders, it is related to the transport part $\partial_t h + u\cdot\nabla_x h$ appearing in the equation of Lemma 5.3. This is the object of the following Lemmas 5.5 and 5.6, which are crucial commutation results.

Lemma 5.5. *For all* $|\alpha| \le m$ *and* $h = \partial_x^\alpha \varrho$, *we have*

$$\operatorname{div}_x\big(\mathrm{K}_G^{\mathrm{free}}[\mathrm{J}_\varepsilon h]u\big) = \mathrm{K}_G^{\mathrm{free}}[\mathrm{J}_\varepsilon(u\cdot\nabla_x h)] + \mathscr{R},$$

with

$$\|\mathscr{R}\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{T}^d))} \le \Lambda(T,R).$$

Proof. First, let us prove that for any smooth function $\mathfrak{h}(t,x)$ we have

$$\operatorname{div}_x\big(\mathrm{K}_G^{\mathrm{free}}[\mathfrak{h}]u\big) = \mathrm{K}_G^{\mathrm{free}}[(u\cdot\nabla_x\mathfrak{h})] + \mathscr{R}, \tag{5.4}$$

with

$$\|\mathscr{R}\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{T}^d))} \le \Lambda(T,R,\|\mathfrak{h}\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{T}^d))}). \tag{5.5}$$

We have

$$\begin{aligned}\operatorname{div}_x\big(u\mathrm{K}_G^{\mathrm{free}}[\mathfrak{h}]\big) &= u\cdot\nabla_x\mathrm{K}_G^{\mathrm{free}}[\mathfrak{h}] + (\operatorname{div}_x u)\mathrm{K}_G^{\mathrm{free}}[\mathfrak{h}]\\ &= \mathrm{K}_G^{\mathrm{free}}[(u\cdot\nabla_x\mathfrak{h})] + (\operatorname{div}_x u)\mathrm{K}_G^{\mathrm{free}}[\mathfrak{h}] + \big[u\cdot\nabla_x,\mathrm{K}_G^{\mathrm{free}}\big][\mathfrak{h}].\end{aligned}$$

Using the notation $\partial_i = \partial_{x_i}$, we have

$$\begin{aligned}[u\cdot\nabla_x,\mathrm{K}_G^{\mathrm{free}}][\mathfrak{h}](t,x) = &\sum_{i=1}^d\int_{\mathbb{R}^d} u_i(t,x)\partial_i\big(\mathrm{K}_G^{\mathrm{free}}[\mathfrak{h}]\big)(t,x)\\ &-\sum_{i=1}^d\int_{\mathbb{R}^d}\mathrm{K}_G^{\mathrm{free}}[(u_i\partial_i\mathfrak{h})](t,x),\end{aligned}$$

and then

$$\begin{aligned}&\big[u\cdot\nabla_x,\mathrm{K}_G^{\mathrm{free}}\big][\mathfrak{h}](t,x)\\ &=\sum_{i=1}^d\int_{\mathbb{R}^d}u_i(t,x)\int_0^t\int_{\mathbb{R}^d}\nabla_x(\partial_i\mathfrak{h})(s,x-(t-s)v)\cdot G(t,x,v)\,\mathrm{d}v\,\mathrm{d}s\\ &\quad+\sum_{i=1}^d\int_{\mathbb{R}^d}u_i(t,x)\int_0^t\int_{\mathbb{R}^d}\nabla_x\mathfrak{h}(s,x-(t-s)v)\cdot\partial_i G(t,x,v)\,\mathrm{d}v\,\mathrm{d}s\\ &\quad-\sum_{i=1}^d\int_{\mathbb{R}^d}\int_0^t\int_{\mathbb{R}^d}\partial_i\mathfrak{h}(s,x-(t-s)v)\nabla_x u_i(s,x-(t-s)v)\cdot G(t,x,v)\,\mathrm{d}v\,\mathrm{d}s\\ &\quad-\sum_{i=1}^d\int_{\mathbb{R}^d}\int_0^t\int_{\mathbb{R}^d}u_i(s,x-(t-s)v)\nabla_x(\partial_i\mathfrak{h})(s,x-(t-s)v)\cdot G(t,x,v)\,\mathrm{d}v\,\mathrm{d}s,\end{aligned}$$

so

$$\begin{aligned}
&\big[u\cdot\nabla_x, \mathrm{K}^{\mathrm{free}}_G\big][\mathfrak{h}](t,x)\\
&= \mathrm{K}^{\mathrm{free}}_{(u\cdot\nabla_x)G}[\mathfrak{h}](t,x) - \mathrm{K}^{\mathrm{free}}_{(G\cdot\nabla_x)\widetilde{u}}[\mathfrak{h}](t,x)\\
&\quad+\sum_{i=1}^d \int_{\mathbb{R}^d} \nabla_x(\partial_i\mathfrak{h})(s,x-(t-s)v)\cdot\big((u_i(t,x)-\widetilde{u}_i(s,t,x,v))G(t,x,v)\big)\,\mathrm{d}v\,\mathrm{d}s,
\end{aligned}$$

where we have set $\widetilde{u}(s,t,x,v) = u(s,x-(t-s)v)$. We thus get

$$\begin{aligned}
\mathrm{div}_x\big(u\mathrm{K}^{\mathrm{free}}_G[\mathfrak{h}]\big) &= \mathrm{K}^{\mathrm{free}}_G[(u\cdot\nabla_x\mathfrak{h})] + (\mathrm{div}_x\, u)\mathrm{K}^{\mathrm{free}}_G[\mathfrak{h}] + \mathrm{K}^{\mathrm{free}}_{(u\cdot\nabla_x)G}[\mathfrak{h}] - \mathrm{K}^{\mathrm{free}}_{(G\cdot\nabla_x)\widetilde{u}}[\mathfrak{h}]\\
&\quad+\sum_{i=1}^d \mathrm{K}^{\mathrm{free}}_{(u_i-\widetilde{u}_i)G}[\partial_i\mathfrak{h}],
\end{aligned}$$

which gives a decomposition as (5.4). In what follows, we shall constantly use the estimate (5.3) of Remark 5.4, that is, for all $0 < p < m - d/2 - 2$ and $\sigma > 0$,

$$\sup_{0\le t\le T}\|G(t)\|_{\mathcal{H}^p_\sigma} \le \Lambda(T,R).$$

Let us estimate the different terms of the previous decomposition in order to prove (5.5). For the first one, we rely on the smoothing estimate of Proposition 3.1 to directly get, for $\ell > 1+d$ and $\sigma > d/2$,

$$\begin{aligned}
&\big\|(\mathrm{div}_x\, u)\mathrm{K}^{\mathrm{free}}_G[\mathfrak{h}]\big\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)}\\
&\qquad\le \Lambda(T,R)\sup_{0\le t\le T}\|G(t)\|_{\mathcal{H}^\ell_\sigma}\|\mathfrak{h}\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)} \le \Lambda(T,R)\|\mathfrak{h}\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)},
\end{aligned}$$

since $m > 3 + 3d/2$. For the second and third ones, Proposition 3.1 yields

$$\begin{aligned}
&\big\|\mathrm{K}^{\mathrm{free}}_{(u\cdot\nabla_x)G}[\mathfrak{h}]\big\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)} + \big\|\mathrm{K}^{\mathrm{free}}_{(G\cdot\nabla_x)\widetilde{u}}[\mathfrak{h}]\big\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)}\\
&\quad\lesssim \Big(\sup_{0\le s,t\le T}\|(u\cdot\nabla_x)G(t,s)\|_{\mathcal{H}^p_\sigma}. + \sup_{0\le s,t\le T}\|(G\cdot\nabla_x)\widetilde{u}(t,s)\|_{\mathcal{H}^p_\sigma}\Big)\|\mathfrak{h}\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)}
\end{aligned}$$

if $\sigma > d/2$ and $p > 1+d$. With the same arguments as in Remark 5.4 to estimate the terms inside the parentheses, we get (since $m > 4 + 3d/2$)

$$\big\|\mathrm{K}^{\mathrm{free}}_{(u\cdot\nabla_x)G}[\mathfrak{h}]\big\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)} + \big\|\mathrm{K}^{\mathrm{free}}_{(G\cdot\nabla_x)\widetilde{u}}[\mathfrak{h}]\big\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)} \lesssim \Lambda(T,R)\|\mathfrak{h}\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)}.$$

For the last term, we observe that the kernel $(u_i(t) - \widetilde{u}_i(t,s,x,v))G(t,x,v)$ vanishes at $s = t$; therefore, Remark 3.6 implies

$$\begin{aligned}
&\big\|\mathrm{K}^{\mathrm{free}}_{(u_i-\widetilde{u}_i)G}[\partial_i\mathfrak{h}]\big\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)}\\
&\quad\lesssim \Lambda(T)\sup_{0\le s,t\le T}\|\partial_s(u_i-\widetilde{u}_i)G(t,s)\|_{\mathcal{H}^p_\sigma}\|\mathfrak{h}\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)}\\
&\quad= \Lambda(T)\sup_{0\le s,t\le T}\|\partial_s\widetilde{u}_i(t,s)\|_{\mathrm{H}^k}\sup_{0\le t\le T}\|G(t)\|_{\mathcal{H}^p_\sigma}\|\mathfrak{h}\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)}
\end{aligned}$$

for $p > 7 + d/2$, $k > p + d/2$ and $\sigma > d/2$. Using the equation for u, we obtain (since $m > 9 + d$)

$$\big\| \mathrm{K}^{\text{free}}_{(u_i - \tilde{u}_i)G}[\partial_i \mathfrak{h}] \big\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)} \leq \Lambda(T, R) \|\mathfrak{h}\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)}.$$

All in all, this yields the claimed estimate (5.5).

Now, applying (5.4) and (5.5) with $\mathfrak{h} = \mathrm{J}_\varepsilon h$, we get

$$\mathrm{div}_x \big(u \mathrm{K}^{\text{free}}_G [\mathrm{J}_\varepsilon h] \big) = \mathrm{K}^{\text{free}}_G [(u \cdot \nabla_x \mathrm{J}_\varepsilon h)] + \mathcal{R},$$

with

$$\|\mathcal{R}\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{T}^d))} \leq \Lambda(T, R, \|\mathrm{J}_\varepsilon h\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{T}^d))}) \leq \Lambda(T, R, \|h\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{T}^d))}).$$

Finally, observe that

$$\mathrm{K}^{\text{free}}_G [(u \cdot \nabla_x \mathrm{J}_\varepsilon h)] = \mathrm{K}^{\text{free}}_G [\mathrm{J}_\varepsilon (u \cdot \nabla_x h)] + \mathrm{K}^{\text{free}}_G \big[[u \cdot \nabla_x, \mathrm{J}_\varepsilon] h \big].$$

Relying once again on Proposition 3.1, we thus have

$$\big\| \mathrm{K}^{\text{free}}_G \big[[u \cdot \nabla_x, \mathrm{J}_\varepsilon] h \big] \big\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{T}^d))} \leq \Lambda(T, R) \big\| [u \cdot \nabla_x, \mathrm{J}_\varepsilon] h \big\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{T}^d))}.$$

Invoking (a variant of the proof of) [21, Theorem C.14] about the commutator between a differential operator of order 1 and a regularizing operator, we get

$$\big\| [u \cdot \nabla_x, \mathrm{J}_\varepsilon] h \big\|_{\mathrm{L}^2(\mathbb{T}^d)} \lesssim \|u\|_{\mathrm{W}^{1,\infty}(\mathbb{T}^d)} \|h\|_{\mathrm{L}^2(\mathbb{T}^d)},$$

where this estimate is independent of ε. We obtain

$$\big\| \mathrm{K}^{\text{free}}_G \big[[u \cdot \nabla_x, \mathrm{J}_\varepsilon] h \big] \big\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{T}^d))} \leq \Lambda(T, R),$$

which concludes the proof. ■

Lemma 5.6. *For all* $|\alpha| \leq m$, $h = \partial_x^\alpha \varrho$ *and* $G(s,t,x,v) = p'(\varrho(t,x)) \nabla_v f(t,x,v)$, *we have*

$$\mathrm{div}_x \, \mathrm{K}^{\text{free}}_{1,G} [\mathrm{J}_\varepsilon h] = -\mathrm{K}^{\text{free}}_G [\mathrm{J}_\varepsilon \partial_s h] + \mathcal{R},$$

with

$$\|\mathcal{R}\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{T}^d))} \leq \Lambda(T, R, \|h(0)\|_{\mathrm{H}^1(\mathbb{T}^d)}).$$

Proof. Recall the definition (5.2) of $\mathrm{K}^{\text{free}}_{1,G}$. We first write

$$\begin{aligned} &\mathrm{div}_x \, \mathrm{K}^{\text{free}}_{1,G} [\mathrm{J}_\varepsilon h] \\ &\quad = p'(\varrho(t,x)) \int_0^t \int_{\mathbb{R}^d} \nabla_x \, \mathrm{div}_x [v \mathrm{J}_\varepsilon h](s, x - (t-s)v) \cdot \nabla_v f(t,x,v) \, \mathrm{d}v \, \mathrm{d}s + \mathcal{R}, \end{aligned}$$

with

$$\mathcal{R}(t,x) := \int_{\mathbb{R}^d} \nabla_x[\mathrm{J}_\varepsilon h](s, x-(t-s)v)\cdot[(v\cdot\nabla_x)(p'(\varrho(t,x))\nabla_v f(t,x,v))]\,\mathrm{d}v\,\mathrm{d}s$$
$$= \mathrm{K}^{\mathrm{free}}_{(v\cdot\nabla_x)(p'(\varrho)\nabla_v f)}[\mathrm{J}_\varepsilon h].$$

Thanks to Proposition 3.1, we thus have, for $\ell > 1+d$ and $\sigma > d/2$,

$$\|\mathcal{R}\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{T}^d))} \lesssim \|\mathrm{J}_\varepsilon h\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{T}^d))} \sup_{0\le s,t\le T} \|(v\cdot\nabla_x p'(\varrho))\nabla_v f(t,s)\|_{\mathcal{H}^\ell_\sigma}$$
$$\le \Lambda(T,R).$$

Now observe the identity

$$\partial_s\big[\mathrm{J}_\varepsilon h(s,x-(t-s)v)\big] = (\partial_s \mathrm{J}_\varepsilon h)(s,x-(t-s)v) + \mathrm{div}_x(v\mathrm{J}_\varepsilon h)(s,x-(t-s)v).$$

Since

$$\int_{\mathbb{R}^d} \nabla_x \mathrm{J}_\varepsilon h(t,x)\cdot\nabla_v f(t,x,v)\,\mathrm{d}v = 0,$$

we have

$$\int_0^t \int_{\mathbb{R}^d} \partial_s\big[\nabla_x \mathrm{J}_\varepsilon h(s,x-(t-s)v)\big]\cdot\nabla_v f(t,x,v)\,\mathrm{d}v\mathrm{d}s$$
$$= -\int_{\mathbb{R}^d} \nabla_x \mathrm{J}_\varepsilon h(0,x-tv)\cdot\nabla_v f(t,x,v)\,\mathrm{d}v.$$

Using the generalized Minkowski inequality and the Sobolev embedding, the last term is estimated in $\mathrm{L}^2(0,T,\mathrm{L}^2)$ by

$$\|\nabla_x \mathrm{J}_\varepsilon h(0)\|_{\mathrm{L}^2(\mathbb{T}^d)} \left\| \int_{\mathbb{R}^d} \sup_{x\in\mathbb{T}^d} |\nabla_v f(\cdot,x,v)|\,\mathrm{d}v \right\|_{\mathrm{L}^2(0,T)} \le \Lambda(T,R)\|h(0)\|_{\mathrm{H}^1(\mathbb{T}^d)}.$$

We deduce that we can write

$$\mathrm{div}_x\, \mathrm{K}^{\mathrm{free}}_{1,G}[\mathrm{J}_\varepsilon h] = -\mathrm{K}^{\mathrm{free}}_G[\partial_s \mathrm{J}_\varepsilon h] + \mathcal{R},$$

with

$$\|\mathcal{R}\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{T}^d))} \le \Lambda(T,R,\|h(0)\|_{\mathrm{H}^1(\mathbb{T}^d)}).$$ ∎

Combining the results of Lemmas 5.3, 5.5 and 5.6 leads to the proof of Proposition 5.1.

5.2 Propagation of the Penrose condition for short times

We now show how to propagate the Penrose stability condition (P) for short times. This will allow us to study the operator

$$\mathrm{Id} - \frac{\varrho}{1-\rho_f}\mathrm{K}_G^{\mathrm{free}} \circ \mathrm{J}_\varepsilon$$

in the following sections; the goal will be to prove its ellipticity.

First, we shall need several estimates on the time derivatives of the solutions. We have the following basic lemma.

Lemma 5.7. *Let $s \geq 0$ and $\sigma \geq 0$. For all $T \in (0, T_\varepsilon)$, the following holds.*

(1) *If $s > d$ and $s+1 > d/2$, we have*

$$\begin{aligned}
&\|\partial_t f\|_{\mathrm{L}^\infty(0,T;\mathcal{H}^s_\sigma)} \\
&\quad \lesssim \|f\|_{\mathrm{L}^\infty(0,T;\mathcal{H}^{s+1}_{\sigma+1})} \\
&\qquad + \|f\|_{\mathrm{L}^\infty(0,T;\mathcal{H}^{s+1}_\sigma)}\big(\|u\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^s)} + \Lambda(\|\varrho\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^{s+1})})\big).
\end{aligned}$$

(2) *If $s > d/2$ and $\sigma > 1 + d/2$, we have*

$$\|\partial_t \rho_f\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty)} \lesssim \|\partial_t \rho_f\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^s)} \lesssim \|f\|_{\mathrm{L}^\infty(0,T;\mathcal{H}^{1+s}_\sigma)}.$$

(3) *If $s > d/2$ and $\sigma > 1 + d/2$, we have*

$$\begin{aligned}
\|\partial_t \varrho\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty)} &\lesssim \|u\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^s)}\|\varrho\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^{1+s})} \\
&\quad + \left\|\frac{1}{1-\rho_f}\right\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty)} \|\varrho\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^s)} \\
&\qquad \times \big(\|f\|_{\mathrm{L}^\infty(0,T;\mathcal{H}^{1+s}_\sigma)} + \|u\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^{1+s})} \\
&\qquad\quad + \|f\|_{\mathrm{L}^\infty(0,T;\mathcal{H}^{1+s}_\sigma)}\|u\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^{1+s})}\big).
\end{aligned}$$

Proof. (1) Using the Vlasov equation satisfied by f, we get

$$\begin{aligned}
\|\partial_t f\|_{\mathcal{H}^s_\sigma} &\lesssim \|f\|_{\mathcal{H}^s_\sigma} + \|v\cdot\nabla_x f\|_{\mathcal{H}^s_\sigma} + \|v\cdot\nabla_v f\|_{\mathcal{H}^s_\sigma} + \|E^{u,\varrho}_{\mathrm{reg},\varepsilon}\cdot\nabla_v f\|_{\mathcal{H}^s_\sigma} \\
&\lesssim \|f\|_{\mathcal{H}^{s+1}_{\sigma+1}} + \|E^{u,\varrho}_{\mathrm{reg},\varepsilon}\cdot\nabla_v f\|_{\mathcal{H}^s_\sigma}.
\end{aligned}$$

Next, combining the estimates (A.2) of Lemma A.6 and (2.2) of Lemma 2.7, we have, for $s > d$ such that $s > 3 + d/2$,

$$\left\|E^{u,\varrho}_{\mathrm{reg},\varepsilon}\cdot\nabla_v f\right\|_{\mathcal{H}^s_\sigma} \lesssim \left\|E^{u,\varrho}_{\mathrm{reg},\varepsilon}\right\|_{\mathrm{H}^s}\|f\|_{\mathcal{H}^{s+1}_\sigma} \lesssim \|f\|_{\mathcal{H}^{s+1}_\sigma}\big(\|u(t)\|_{\mathrm{H}^s} + \Lambda(\|\varrho(t)\|_{\mathrm{H}^{s+1}})\big),$$

hence providing the first estimate.

(2) To estimate $\partial_t \rho_f$, we use the fact that $\partial_t \rho_f = -\operatorname{div}_x j_f$ so that we have, by Sobolev embedding,

$$\|\partial_t \rho_f\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty)} \lesssim \|\partial_t \rho_f\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^s)} \lesssim \|j_f\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^{1+s})} \lesssim \|f\|_{\mathrm{L}^\infty(0,T;\mathcal{H}^{1+s}_\sigma)},$$

thanks to Lemma 2.1. This gives the second estimate.

(3) To estimate $\partial_t \varrho$, we use the equation

$$\partial_t \varrho = -u \cdot \nabla_x \varrho - \frac{1}{1-\rho_f} \operatorname{div}_x (j_f - \rho_f u + u)\varrho,$$

from which we infer that

$$\begin{aligned}
&\|\partial_t \varrho\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty)} \\
&\quad \lesssim \|u\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^s)} \|\nabla_x \varrho\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^s)} \\
&\qquad + \left\| \frac{1}{1-\rho_f} \right\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty)} \|\varrho\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^s)} \|j_f - \rho_f u + u\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^{1+s})}.
\end{aligned}$$

We conclude by using Lemma 2.1. ■

Starting from initial data satisfying the Penrose condition (P), we now show that it is propagated for short times.

Lemma 5.8. *There exists $\widetilde{T}_0(R) > 0$ independent of ε such that the following holds: if the initial data $(f^{\mathrm{in}}, \varrho^{\mathrm{in}})$ satisfies the c-Penrose stability condition $(\mathrm{P})_c$ for some $c > 0$ then the function $(f(t), \varrho(t))$ satisfies the $\frac{c}{2}$-Penrose stability condition $(\mathrm{P})_{c/2}$ for all $t \in [0, \min(\widetilde{T}_0(R), T^\varepsilon)]$.*

Proof. Let $T < T^\varepsilon$ and recall the definition (1.4) of the Penrose function $\mathcal{P}$. We start by writing, for all $t \in [0, T]$,

$$\begin{aligned}
&1 - \mathcal{P}_{f(t),\varrho(t)}(x, \gamma, \tau, k) \\
&\quad = 1 - \frac{p'(\varrho(t,x))\rho(t,x)}{1-\rho_f(t,x)} \int_0^{+\infty} e^{-(\gamma+i\tau)s} \frac{ik}{1+|k|^2} \cdot (\mathcal{F}_v \nabla_v f)(t,x,ks)\,\mathrm{d}s \\
&\quad = 1 - \mathcal{P}_{f^{\mathrm{in}},\varrho^{\mathrm{in}}}(x,\gamma,\tau,k) + \mathbf{A}_0(t,x,\gamma,\tau,k) + \mathbf{B}_0(t,x,\gamma,\tau,k),
\end{aligned}$$

where

$$\begin{aligned}
&\mathbf{A}_0(t,x,\gamma,\tau,k) \\
&:= -\frac{p'(\varrho^{\mathrm{in}}(x))\varrho^{\mathrm{in}}(x)}{1-\rho_{f^{\mathrm{in}}}(x)} \\
&\qquad \times \left(\int_0^{+\infty} e^{-(\gamma+i\tau)s} \frac{ik}{1+|k|^2} \cdot \left[(\mathcal{F}_v \nabla_v f)(t,x,ks) - (\mathcal{F}_v \nabla_v f^{\mathrm{in}})(x,ks) \right] \mathrm{d}s \right)
\end{aligned}$$

and

$$\mathbf{B}_0(t,x,\gamma,\tau,k) := \left(\frac{p'(\varrho^{\mathrm{in}}(x))\varrho^{\mathrm{in}}(x)}{1-\rho_{f^{\mathrm{in}}}(x)} - \frac{p'(\varrho(t,x))\varrho(t,x)}{1-\rho_f(t,x)}\right) \times \int_0^{+\infty} e^{-(\gamma+i\tau)s} \frac{ik}{1+|k|^2}\cdot(\mathcal{F}_v\nabla_v f)(t,x,ks)\,\mathrm{d}s.$$

Estimate of $\mathbf{A}_0$. Using Taylor's formula, we have

$$\begin{aligned}
|\mathbf{A}_0(t,x,\gamma,\tau,k)| &\leq \left\|\frac{p'(\varrho^{\mathrm{in}})\varrho^{\mathrm{in}}}{1-\rho_{f^{\mathrm{in}}}}\right\|_{\mathrm{L}^\infty} \int_0^{+\infty} |k| \int_0^T |(\mathcal{F}_v\nabla_v\partial_t f)(\theta,x,ks)|\,\mathrm{d}\theta\,\mathrm{d}s \\
&\leq \left\|\frac{p'(\varrho^{\mathrm{in}})\varrho^{\mathrm{in}}}{1-\rho_{f^{\mathrm{in}}}}\right\|_{\mathrm{L}^\infty} \int_0^T \int_0^{+\infty} \left|(\mathcal{F}_v\nabla_v\partial_t f)\left(\theta,x,\frac{k}{|k|}s\right)\right|\mathrm{d}s\,\mathrm{d}\theta \\
&\lesssim \left\|\frac{p'(\varrho^{\mathrm{in}})\varrho^{\mathrm{in}}}{1-\rho_{f^{\mathrm{in}}}}\right\|_{\mathrm{L}^\infty} \int_0^T \sup_{s\in\mathbb{R}+}(1+s^2)\left|(\mathcal{F}_v\nabla_v\partial_t f)\left(\theta,x,\frac{k}{|k|}s\right)\right|\mathrm{d}\theta.
\end{aligned}$$

Therefore we get, for $\sigma > d/2$,

$$\begin{aligned}
&|\mathbf{A}_0(t,x,\gamma,\tau,k)| \\
&\quad\lesssim \left\|\frac{p'(\varrho^{\mathrm{in}})\varrho^{\mathrm{in}}}{1-\rho_{f^{\mathrm{in}}}}\right\|_{\mathrm{L}^\infty} \int_0^T \sup_{s\in\mathbb{R}+}\sum_{|\beta|\leq 2}\left|(\mathcal{F}_v\partial_v^\beta\nabla_v\partial_t f)\left(\theta,x,\frac{k}{|k|}s\right)\right|\mathrm{d}\theta \\
&\quad\lesssim \left\|\frac{p'(\varrho^{\mathrm{in}})\varrho^{\mathrm{in}}}{1-\rho_{f^{\mathrm{in}}}}\right\|_{\mathrm{L}^\infty} \int_0^T \left(\sum_{|\beta|\leq 2}\int_{\mathbb{R}^d}(1+|v|^2)^\sigma\left|\partial_v^\beta\nabla_v\partial_t f(\theta,x,v)\right|^2\mathrm{d}v\right)^{\frac{1}{2}}\mathrm{d}\theta \\
&\quad\lesssim \left\|\frac{p'(\varrho^{\mathrm{in}})\varrho^{\mathrm{in}}}{1-\rho_{f^{\mathrm{in}}}}\right\|_{\mathrm{L}^\infty} T\|\partial_t f\|_{\mathrm{L}^\infty(0,T;\mathcal{H}^{3+s}_\sigma)}
\end{aligned}$$

for all $s > d/2$, thanks to the Sobolev embedding. Lemma 5.7 (with $3+s>d$ and $3+s+1>d/2$) yields

$$\|\partial_t f\|_{\mathrm{L}^\infty(0,T;\mathcal{H}^{3+s}_\sigma)} \leq \Lambda(T,R),$$

thanks to Lemma 2.20, choosing $s+6\leq m$. Therefore, there exists a universal constant $C>0$ such that

$$|\mathbf{A}_0(t,x,\gamma,\tau,k)| \leq C\left\|\frac{p'(\varrho^{\mathrm{in}})\varrho^{\mathrm{in}}}{1-\rho_{f^{\mathrm{in}}}}\right\|_{\mathrm{L}^\infty} T\Lambda(T,R).$$

Estimate of $\mathbf{B}_0$. Likewise, we have, for all $s > d/2$ such that $3 + s < m - 1$,

$$\begin{aligned}|\mathbf{B}_0(t,x,\gamma,\tau,k)| &\lesssim \|f\|_{\mathrm{L}^\infty(0,T;\mathcal{H}^{3+s}_\sigma)} \left| \frac{p'(\varrho(t,x))\rho(t,x)}{1-\varrho_f(t,x)} - \frac{p'(\varrho^{\mathrm{in}}(x))\varrho^{\mathrm{in}}(x)}{1-\rho_{f^{\mathrm{in}}}(x)} \right| \\ &\leq R\int_0^T \left| \partial_t \left\{ \frac{p'(\varrho(\theta,x))\varrho(\theta,x)}{1-\rho_f(\theta,x)} \right\} \right| \mathrm{d}\theta \\ &\leq RT \left\| \partial_t \left\{ \frac{p'(\varrho)\varrho}{1-\rho_f} \right\} \right\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty)}.\end{aligned}$$

Now observe that

$$\partial_t \left\{ \frac{p'(\varrho)\varrho}{1-\rho_f} \right\} = \frac{1}{1-\rho_f}\big(\varrho\partial_t\varrho\, p''(\varrho) + p'(\varrho)\partial_t\varrho\big) + p'(\varrho)\varrho\frac{\partial_t\rho_f}{(1-\rho_f)^2}.$$

Therefore, by Sobolev embedding, we get for all $s > d/2$,

$$\begin{aligned}&\left\| \partial_t \left\{ \frac{p'(\varrho)\varrho}{1-\rho_f} \right\} \right\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty)} \\ &\quad\leq \left\| \frac{1}{1-\rho_f} \right\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty)} \|\partial_t\varrho\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty)} \\ &\qquad\times \big(\|\varrho\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^s)} \|p''(\varrho)\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^s)} + \|p'(\varrho)\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^s)} \big) \\ &\quad + \left\| \frac{1}{1-\rho_f} \right\|^2_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty)} \|p'(\varrho)\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^s)} \|\varrho\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^s)} \|\partial_t\rho_f\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty)}.\end{aligned}$$

Using Lemma 5.7 with Lemma 2.20, we obtain as before

$$|\mathbf{B}_0(t,x,\gamma,\tau,k)| \leq CT\Lambda(T,R).$$

All in all, we have, for all $t \in [0,T]$,

$$|\mathbf{A}_0(t,x,\gamma,\tau,k)| + |\mathbf{B}_0(t,x,\gamma,\tau,k)| \leq C\left(\left\| \frac{p'(\varrho^{\mathrm{in}})\varrho^{\mathrm{in}}}{1-\rho_{f^{\mathrm{in}}}} \right\|_{\mathrm{L}^\infty} + 1 \right) T\Lambda(T,R).$$

Consequently, there exists $\eta_0 > 0$ independent of ε (and depending on c) such that, for $\widetilde{T}_0 = \widetilde{T}_0(R)$ satisfying

$$C\left(\left\| \frac{p'(\varrho^{\mathrm{in}})\varrho^{\mathrm{in}}}{1-\rho_{f^{\mathrm{in}}}} \right\|_{\mathrm{L}^\infty} + 1 \right) \widetilde{T}_0\Lambda(\widetilde{T}_0,R) \leq \eta_0,$$

the $\frac{c}{2}$-Penrose stability condition $(\mathrm{P})_{c/2}$ holds true for the function $(f(t),\varrho(t))$ whenever $t \in [0,(\widetilde{T}_0(R),T^\varepsilon)]$, provided that $(f^{\mathrm{in}},\varrho^{\mathrm{in}})$ satisfies the c-Penrose stability condition $(\mathrm{P})_c$. ■

5.3 Extension of the solution

In this section, our goal is to construct a suitable extension in time of the solution $(f_\varepsilon, \varrho_\varepsilon, u_\varepsilon)$ on the whole line $\mathbb{R}$. This technical step is required in view of the subsequent pseudodifferential analysis (in time-space) of Sections 5.4 and 5.5 – the symbols being dependent on our solutions. A main issue is to obtain an extension still satisfying the Penrose stability condition (for all times): we refer to the later Proposition 5.17.

Define $T_\varepsilon^\star$ (depending on R) as

$$T_\varepsilon^\star := \min\big(T_\varepsilon(R), \overline{T}(R), \widetilde{T}_0(R)\big).$$

In particular, the Penrose stability condition holds on $[0, T_\varepsilon^\star]$ thanks to Lemma 5.8.

Consider two nonnegative nonincreasing cutoffs $\chi, \underline{\chi} \in \mathcal{C}^\infty(\mathbb{R})$ such that

$$\forall t \in \mathbb{R}, \quad \chi(t) = \begin{cases} 1, & t \leq 0, \\ 0, & t \geq 1, \end{cases} \quad \text{and} \quad \underline{\chi}(t) = \begin{cases} 1, & t \leq 0, \\ 1/2, & t \geq 1. \end{cases}$$

We set, for $\delta > 0$ to be fixed later, $\chi_\delta(t) := \chi(t/\delta)$.

Given a solution (f, ϱ, u) to the system (S_ε), we consider its extension $(\widetilde{f}, \widetilde{\varrho}, \widetilde{u})$ as follows. Given $(N_u, N_f, N_\varrho) \in \mathbb{N}^3$ to be determined later on (by the number of derivatives we will use), we define:

Extension in time for u. We set

$$\widetilde{u}(t) = \begin{cases} u(t), & t \in [0, T_\varepsilon^\star], \\ \chi(t - T_\varepsilon^\star) \sum_{k=0}^{N_u} \partial_t^k u(T_\varepsilon^\star) \frac{(t-T_\varepsilon^\star)^k}{k!}, & t \geq T_\varepsilon^\star, \\ \chi(-t) \sum_{k=0}^{N_u} \partial_t^k u(0) \frac{t^k}{k!}, & t \leq 0. \end{cases}$$

In particular, the extension is 0 after $t = T_\varepsilon^\star + 1$ and before $t = -1$.

Extension in time for f. We set

$$\widetilde{f}(t) = \begin{cases} f(t), & t \in [0, T_\varepsilon^\star], \\ \chi_\delta(t - T_\varepsilon^\star) f(T_\varepsilon^\star) + \chi_\delta(t - T_\varepsilon^\star) \sum_{k=1}^{N_f} \partial_t^k f(T_\varepsilon^\star) \frac{(t-T_\varepsilon^\star)^k}{k!}, & t \geq T_\varepsilon^\star, \\ \chi_\delta(-t) f^{\mathrm{in}} + \chi_\delta(-t) \sum_{k=1}^{N_f} \partial_t^k f(0) \frac{t^k}{k!}, & t \leq 0. \end{cases}$$

In particular, the extension is 0 after $t = T_\varepsilon^\star + \delta$ and before $t = -\delta$.

Extension in time for ϱ. We set

$$\widetilde{\varrho}(t) = \begin{cases} \varrho(t), & t \in [0, T_\varepsilon^\star], \\ \underline{\chi}(t - T_\varepsilon^\star)\varrho(T_\varepsilon^\star) + \chi_\delta(t - T_\varepsilon^\star) \sum_{k=1}^{N_\varrho} \partial_t^k \varrho(T_\varepsilon^\star) \frac{(t-T_\varepsilon^\star)^k}{k!}, & t \geq T_\varepsilon^\star, \\ \underline{\chi}(-t)\varrho^{\mathrm{in}} + \chi_\delta(-t) \sum_{k=1}^{N_\varrho} \partial_t^k \varrho(0) \frac{t^k}{k!}, & t \leq 0. \end{cases}$$

In particular, the extension is constant in time equal to $\varrho(T_\varepsilon^\star)/2$ after $t = T_\varepsilon^\star + 1$ and equal to $\varrho^{\mathrm{in}}/2$ before $t = -1$.

The bounds from above and below ($\mathrm{B}_\Theta^{\mu,\theta}(T)$) on ϱ and ρ_f are still valid for $\widetilde{\varrho}$ and $\rho_{\widetilde{f}}$, provided that we choose the parameter δ small enough.

Remark 5.9. Note that if $T_\varepsilon^\star \leq 1$, then the Penrose function $\mathcal{P}_{\widetilde{f}(t),\widetilde{\varrho}(t)}$ has a compact support in time contained in $[0, 2]$.

Hereafter, we drop out the tilde notation and we shall always consider this extension of our solutions. Let us conclude this chapter by explaining how we will deal with such an extension:

- Replacing the former solution (defined on $[0, T_\varepsilon^\star]$) by its extension (f, ϱ, u) on $\mathbb{R}$, we observe that (f, ϱ, u) satisfies (S_ε) with the addition of a new source term S^{new} on the right-hand side which has a support contained in $\mathbb{R} \backslash [0, T_\varepsilon^\star]$.
- The results of Chapter 4 and Section 5.1 remain true on $[0, T_\varepsilon^\star)$.

We also refer to Proposition 5.17 below, where we will prove that if δ is chosen appropriately, the extension (f, ϱ, u) satisfies a Penrose stability condition for all times (the proof requiring some technical estimates from the upcoming Section 5.4).

5.4 Bounds on the symbols

The aim of this section is twofold:

- to obtain some bounds in terms of the initial data for some symbol seminorms of the Penrose function introduced in (1.4) (depending on the extension (f, ϱ));
- to propagate the Penrose stability condition (P) on the whole line in time for the extension (f, ϱ, u).

These two ingredients are required to obtain crucial elliptic estimates in Section 5.5.

Before stating the next lemma, consider the symbol seminorms (C.1), (C.2) and (C.3) introduced in Appendix C: for any $\mathrm{M} \geq 0$ and for any symbol $a(t, x, \eta)$ with

$\eta = (\gamma, \tau, k) \in (0, +\infty) \times \mathbb{R} \times \mathbb{R}^d \backslash \{0\}$, we set

$$\begin{aligned}
\omega[a] &:= \sup_{\substack{\alpha \in \mathbb{N}^d \\ \alpha_i \in \{0,1\}}} \|(1+t)\partial_x^\alpha a\|_{\mathrm{L}^\infty_{t,x,\eta}} + \sup_{\substack{\alpha \in \mathbb{N}^d \\ \alpha_i \in \{0,1\}}} \|(1+t)\partial_t \partial_x^\alpha a\|_{\mathrm{L}^\infty_{t,x,\eta}}, \\
\Omega[a] &:= \sup_{\substack{\alpha \in \mathbb{N}^d \\ \alpha_i \in \{0,1\}}} \left\{ \left\| |\eta| \partial_x^\alpha \nabla_{\tau,k} a \right\|_{\mathrm{L}^\infty_{t,x,\eta}} + \left\| |\eta| \partial_x^\alpha \nabla_{\tau,k} \partial_t a \right\|_{\mathrm{L}^\infty_{t,x,\eta}} \right\} \\
&\quad + \sup_{\substack{\alpha \in \mathbb{N}^d \\ \alpha_i \in \{0,1\}}} \left\{ \left\| |\eta| \partial_x^\alpha \partial_\tau \nabla_{\tau,k} a \right\|_{\mathrm{L}^\infty_{t,x,\eta}} + \left\| |\eta| \partial_x^\alpha \partial_\tau \nabla_{\tau,k} \partial_t a \right\|_{\mathrm{L}^\infty_{t,x,\eta}} \right\}, \\
\Xi[a]_{\mathrm{M}} &:= \sup_{\substack{1 \le |\alpha| \le 1+\mathrm{M} \\ \beta = 0,1,2,3,4}} \left\| \partial_x^\alpha \partial_t^\beta a \right\|_{\mathrm{L}^\infty_{t,x,\eta}}.
\end{aligned}$$

Lemma 5.10. *For* $(t, x, \gamma, \tau, k) \in \mathbb{R} \times \mathbb{T}^d \times (0, +\infty) \times \mathbb{R} \times \mathbb{R}^d \backslash \{0\}$*, set*

$$a_f(t, x, \gamma, \tau, k) := \int_0^{+\infty} e^{-(\gamma + i\tau)s} ik \cdot (\mathcal{F}_v \nabla_v f)(t, x, ks)\, \mathrm{d}s. \tag{5.6}$$

The symbol a_f *is (positively) homogeneous of degree zero in* (γ, τ, k) *in the sense that*

$$\forall (t, x),\ \forall \eta = (\gamma, \tau, k),\ \forall \lambda > 0, \quad a_f(t, x, \lambda \eta) = a_f(t, x, \eta).$$

Furthermore, for any $A > 0$ *and* $r > d/2 + 4$*, we have*

$$\omega[a_f] \lesssim \sup_{i=0,1} \|(1+t)\partial_t^i f\|_{\mathrm{L}^\infty(\mathbb{R}; \mathcal{H}^\ell_r)}, \quad 3 + \frac{3d}{2} < \ell, \tag{5.7}$$

$$\Xi[a_f]_M \lesssim \sup_{i=0,1,2,3,4} \|\partial_t^i f\|_{\mathrm{L}^\infty(\mathbb{R}; \mathcal{H}^\ell_r)}, \quad 4 + M + \frac{d}{2} < \ell, \tag{5.8}$$

$$\Omega[a_f] \lesssim \sup_{i=0,1} \|\partial_t^i f\|_{\mathrm{L}^\infty(\mathbb{R}; \mathcal{H}^\ell_r)}, \quad 7 + \frac{3d}{2} < \ell. \tag{5.9}$$

Proof. The homogeneity is obtained by performing the change of variable $s = \frac{s'}{\lambda}$ (for $\lambda > 0$) in the integrals in s defining $a_f(t, x, \eta)$. In what follows, we will rely on the estimate

$$\begin{aligned}
&\left| \partial_t^\delta \partial_x^\alpha \partial_\xi^\beta \mathcal{F}_v \nabla_v f(t, x, \xi) \right| \\
&\quad \lesssim \frac{1}{1 + |\xi|^q} \left(\int_{\mathbb{R}^d} (1 + |v|^2)^{\sigma + |\beta|} \left| \nabla_v \partial_t^\delta \partial_x^\alpha (I - \Delta_v)^{\frac{q}{2}} f(t, x, v) \right|^2 \mathrm{d}v \right)^{\frac{1}{2}},
\end{aligned} \tag{5.10}$$

which is valid for all $\sigma > d/2$, $q > 0$ and any $(\delta, \alpha, \beta) \in \mathbb{N} \times \mathbb{N}^d \times \mathbb{N}^d$.

Consequently, for any $\alpha \in \mathbb{N}^d$ with $\alpha_i = 0, 1$, we apply (5.10) with $\delta = 0, 1$, $q = 2$ and $\beta = 0$, and obtain, for $\chi(v) = (1 + |v|^2)^{\frac{\sigma}{2}}$,

$$
\begin{aligned}
&\left|\partial_t^\delta \partial_x^\alpha a_f(t, x, \eta)\right| \\
&\quad \lesssim \left(\int_{\mathbb{R}^d} (1 + |v|^2)^\sigma \left|\nabla_v \partial_t^\delta \partial_x^\alpha (I - \Delta_v) f(t, x, v)\right|^2 \mathrm{d}v\right)^{\frac{1}{2}} \int_0^{+\infty} \frac{|k|}{1 + s^2|k|^2} \,\mathrm{d}s \\
&\quad \lesssim \left\|\chi \partial_t^\delta \partial_x^\alpha f(t, x)\right\|_{\mathrm{H}^3_v(\mathbb{R}^d)} \int_0^{+\infty} \frac{1}{1 + s^2} \,\mathrm{d}s \\
&\quad \lesssim \left\|\partial_t^\delta f(t)\right\|_{\mathcal{H}_\sigma^{3+|\alpha|+\frac{d}{2}+\kappa}}
\end{aligned}
$$

for all $\kappa > 0$, thanks to the Sobolev embedding. We then deduce that

$$
\omega[a_f] \lesssim \left\|(1 + t) f\right\|_{\mathrm{L}^\infty\left(\mathbb{R};\mathcal{H}_\sigma^{3+d+\frac{d}{2}+\kappa}\right)} + \left\|(1 + t)\partial_t f\right\|_{\mathrm{L}^\infty\left(\mathbb{R};\mathcal{H}_\sigma^{3+d+\frac{d}{2}+\kappa}\right)},
$$

hence the claimed inequality (5.7). The inequality (5.8) can be obtained in the same way.

Let us now turn to the proof of (5.9). First, observe that by the homogeneity of a_f in $\eta = (\gamma, \tau, k)$, it is enough to estimate the quantities $\|\partial_x^\alpha \nabla_{\tau,k} \partial_t^\delta a_f\|_{\mathrm{L}^\infty_{t,x}\mathrm{L}^\infty_\eta(S^+)}$ and $\|\partial_x^\alpha \partial_\tau \nabla_{\tau,k} \partial_t^\delta a\|_{\mathrm{L}^\infty_{t,x}\mathrm{L}^\infty_\eta(S^+)}$ with $\delta = 0, 1$, $\alpha \in \mathbb{N}^d$ with $\alpha_i = 0, 1$, and where

$$
S^+ := \left\{\widetilde{\eta} = (\widetilde{\gamma}, \widetilde{\tau}, \widetilde{k}) \in (0, +\infty) \times \mathbb{R} \times \mathbb{R}^d \backslash \{0\} \mid \widetilde{\gamma}^2 + \widetilde{\tau}^2 + |\widetilde{k}|^2 = 1\right\}.
$$

We thus need to estimate the following symbols:

$$
\begin{aligned}
I_1^{\alpha,\delta}(t, x, \widetilde{\eta}) &= \int_0^{+\infty} e^{-(\widetilde{\gamma}+i\widetilde{\tau})s} n \cdot \left(\partial_x^\alpha \partial_t^\delta \mathcal{F}_v \nabla_v f\right)(t, x, \widetilde{k}s) \,\mathrm{d}s, \quad n \in \mathbb{R}^d, |n| = 1, \\
I_2^{\alpha,\delta}(t, x, \widetilde{\eta}) &= \int_0^{+\infty} e^{-(\widetilde{\gamma}+i\widetilde{\tau})s} s\widetilde{k} \cdot \left(\partial_\xi^\beta \partial_x^\alpha \partial_t^\delta \mathcal{F}_v \nabla_v f\right)(t, x, \widetilde{k}s) \,\mathrm{d}s, \quad |\beta| \in \{0, 1\},
\end{aligned}
$$

and

$$
\begin{aligned}
J_{1,q_1}^{\alpha,\delta}(t, x, \widetilde{\eta}) &= \int_0^{+\infty} e^{-(\widetilde{\gamma}+i\widetilde{\tau})s} s^{q_1} n \cdot \left(\partial_x^\alpha \partial_t^\delta \mathcal{F}_v \nabla_v f\right)(t, x, \widetilde{k}s) \,\mathrm{d}s, \\
&\qquad n \in \mathbb{R}^d, |n| = 1, q_1 \in \{1, 2\}, \\
J_{2,q_2}^{\alpha,\delta}(t, x, \widetilde{\eta}) &= \int_0^{+\infty} e^{-(\widetilde{\gamma}+i\widetilde{\tau})s} s^{q_2} \widetilde{k} \cdot \left(\partial_\xi \partial_x^\alpha \partial_t^\delta \mathcal{F}_v \nabla_v f\right)(t, x, \widetilde{k}s) \,\mathrm{d}s, \\
&\qquad q_2 \in \{2, 3\},
\end{aligned}
$$

for $\widetilde{\eta} \in S^+$.

We focus on the terms $J_{1,q_1}^{\alpha,\delta}$ and $J_{2,q_2}^{\alpha,\delta}$ by following [90, Lemma 5.5], the treatment of the symbols $I_1^{\alpha,\delta}$ and $I_2^{\alpha,\delta}$ being similar and involving fewer derivatives.

If $|\widetilde{k}| \geq 1/2$, invoking (5.10) with $q = q_1 + 3$ and $\beta = 0$ yields as before

$$\begin{aligned} \left|J^{\alpha,\delta}_{1,q_1}(t,x,\widetilde{\eta})\right| &\lesssim \left(\int_{\mathbb{R}^d}(1+|v|^2)^{\sigma}\left|\nabla_v \partial_t^{\delta}\partial_x^{\alpha}(I-\Delta_v)^{\frac{q_1+3}{2}} f(t,x,v)\right|^2 \mathrm{d}v\right)^{\frac{1}{2}} \\ &\quad\times \int_0^{+\infty}\frac{s^{q_1}}{1+s^{q_1+3}|\widetilde{k}|^{q_1+3}}\,\mathrm{d}s \\ &\lesssim \|\partial_t^{\delta} f(t)\|_{\mathcal{H}_{\sigma}^{4+q_1+|\alpha|+\frac{d}{2}}} + \int_0^{+\infty}\frac{s^{q_1}}{1+s^{q_1+3}}\,\mathrm{d}s, \end{aligned}$$

since $|\widetilde{k}|$ is bounded from below. We thus obtain a uniform estimate in this case. Otherwise, if $|\widetilde{k}| \leq 1/2$, then $\widetilde{\gamma}^2 + \widetilde{\tau}^2 \geq 3/4$ and we can therefore rely on the exponential to integrate by parts in s in the integral defining $J^{\alpha,\delta}_{1,q_1}(t,x,\widetilde{\eta})$. If $q_1 = 1$, we first get

$$\begin{aligned} J^{\alpha,\delta}_{1,1}(t,x,\widetilde{\eta}) &= \frac{1}{\widetilde{\gamma}+i\widetilde{\tau}}\int_0^{+\infty} e^{-(\widetilde{\gamma}+i\widetilde{\tau})s} n\cdot\left(\partial_x^{\alpha}\partial_t^{\delta}\mathcal{F}_v\nabla_v f\right)(t,x,\widetilde{k}s)\,\mathrm{d}s \\ &\quad+\frac{1}{\widetilde{\gamma}+i\widetilde{\tau}}\int_0^{+\infty} e^{-(\widetilde{\gamma}+i\widetilde{\tau})s} sn\cdot\left(\mathrm{D}_{\xi}\partial_x^{\alpha}\partial_t^{\delta}\mathcal{F}_v\nabla_v f\right)(t,x,\widetilde{k}s)\widetilde{k}\,\mathrm{d}s, \end{aligned}$$

and integrating by parts once again yields the estimate (since $\widetilde{\gamma}^2 + \widetilde{\tau}^2 \geq 3/4$)

$$\begin{aligned} \left|J^{\alpha,\delta}_{1,1}(t,x,\widetilde{\eta})\right| &\lesssim \left|\partial_x^{\alpha}\partial_t^{\delta}\mathcal{F}_v\nabla_v f(t,x,0)\right| + \int_0^{+\infty}|\widetilde{k}|\left|\mathrm{D}_{\xi}\partial_x^{\alpha}\partial_t^{\delta}\mathcal{F}_v\nabla_v f(t,x,\widetilde{k}s)\right|\mathrm{d}s \\ &\quad+\int_0^{+\infty} s|\widetilde{k}|^2\left|\mathrm{D}^2_{\xi}\partial_x^{\alpha}\partial_t^{\delta}\mathcal{F}_v\nabla_v f(t,x,\widetilde{k}s)\right|\mathrm{d}s. \end{aligned}$$

Using (5.10) with $q = 2$ and $|\beta| = 0, 1$, or $q = 3$ and $|\beta| = 2$, now provides, for all $\kappa > 0$,

$$\begin{aligned} \left|J^{\alpha,\delta}_{1,1}(t,x,\widetilde{\eta})\right| &\lesssim \left\|\partial_t^{\delta} f(t)\right\|_{\mathcal{H}_{\sigma}^{4+|\alpha|+\frac{d}{2}+\kappa}} \\ &\quad\times\left(1+\int_0^{+\infty}\frac{|\widetilde{k}|}{1+|\widetilde{k}|^2 s^2}\,\mathrm{d}s+\int_0^{+\infty}\frac{|\widetilde{k}|^2 s}{1+|\widetilde{k}|^3 s^3}\,\mathrm{d}s\right) \\ &\lesssim \left\|\partial_t^{\delta} f(t)\right\|_{\mathcal{H}_{\sigma+2}^{4+|\alpha|+\frac{d}{2}+\kappa}}\left(1+\int_0^{+\infty}\frac{1}{1+s^2}\,\mathrm{d}s+\int_0^{+\infty}\frac{s}{1+s^3}\,\mathrm{d}s\right), \end{aligned}$$

whence the uniform estimate for this symbol. If $q_1 = 2$, we rely on the same strategy with additional integration by parts and (5.10) with $q = 2$ and $|\beta| = 0, 1$, or $q = 3$

and $|\beta| = 2$, or $q = 4$ and $|\beta| = 3$ to get

$$
\begin{aligned}
&\left|J_{1,2}^{\alpha,\delta}(t,x,\widetilde{\eta})\right| \\
&\quad \lesssim \left|J_{1,1}^{\alpha,\delta}(t,x,\widetilde{\eta})\right| + \int_0^{+\infty} |\widetilde{k}|\left|\mathrm{D}_\xi \partial_x^\alpha \partial_t^\delta \mathcal{F}_v \nabla_v f(t,x,\widetilde{k}s)\right| \mathrm{d}s \\
&\qquad + \int_0^{+\infty} s\left|\widetilde{k}\right|^2 \left|\mathrm{D}_\xi^2 \partial_x^\alpha \partial_t^\delta \mathcal{F}_v \nabla_v f(t,x,\widetilde{k}s)\right| \mathrm{d}s \\
&\qquad + \int_0^{+\infty} s^2\left|\widetilde{k}\right|^3 \left|\mathrm{D}_\xi^3 \partial_x^\alpha \partial_t^\delta \mathcal{F}_v \nabla_v f(t,x,\widetilde{k}s)\right| \mathrm{d}s \\
&\quad \lesssim \left\|\partial_t^\delta f(t)\right\|_{\mathcal{H}_{\sigma+3}^{5+|\alpha|+\frac{d}{2}+}} \\
&\qquad \times \left(1 + \int_0^{+\infty} \frac{|\widetilde{k}|}{1+|\widetilde{k}|^2 s^2} \mathrm{d}s + \int_0^{+\infty} \frac{|\widetilde{k}|^2 s}{1+|\widetilde{k}|^3 s^3} \mathrm{d}s + \int_0^{+\infty} \frac{|\widetilde{k}|^3 s^2}{1+|\widetilde{k}|^4 s^4} \mathrm{d}s\right) \\
&\quad \lesssim \left\|\partial_t^\delta f(t)\right\|_{\mathcal{H}_{\sigma+3}^{5+|\alpha|+\frac{d}{2}+}} \\
&\qquad \times \left(1 + \int_0^{+\infty} \frac{1}{1+s^2} \mathrm{d}s + \int_0^{+\infty} \frac{s}{1+s^3} \mathrm{d}s + \int_0^{+\infty} \frac{s^2}{1+s^4} \mathrm{d}s\right).
\end{aligned}
$$

Gathering the two cases, we deduce the estimate

$$
\left\| J_{1,q_1}^{\alpha,\delta} \right\|_{\mathrm{L}^\infty_{t,x}\mathrm{L}^\infty_\eta(S+)} \lesssim \left\| \partial_t^\delta f \right\|_{\mathrm{L}^\infty\left(\mathbb{R};\mathcal{H}_{\sigma+3}^{6+|\alpha|+\frac{d}{2}+\kappa}\right)}
$$

for all $\kappa > 0$. Likewise, we can apply these arguments to $J_{2,q_2}^{\alpha,\delta}$, by invoking again (5.10) with $q = q_2 + 3$ and $\beta = 1$ if $|\widetilde{k}| \leq 1/2$, or with at most $q = 5$ and $\beta = 4$ if $|\widetilde{k}| \geq 1/2$. All in all, we obtain

$$
\left\| J_{2,q_1}^{\alpha,\delta} \right\|_{\mathrm{L}^\infty_{t,x}\mathrm{L}^\infty_\eta(S+)} \lesssim \left\| \partial_t^\delta f \right\|_{\mathrm{L}^\infty\left(\mathbb{R};\mathcal{H}_{\sigma+4}^{7+|\alpha|+\frac{d}{2}+\kappa}\right)}.
$$

We have finally proved the estimate (5.9). ∎

In view of Lemma 5.10, it will be useful to have the following estimates on f.

Lemma 5.11. *For $k > d$ and $\sigma > 0$, we have*

$$
\sup_{i=0,1} \left\|(1+t)\partial_t^i f\right\|_{\mathrm{L}^\infty(\mathbb{R};\mathcal{H}_\sigma^k)} \leq \Lambda(1+M_{\text{in}}), \quad \sigma+4<r,\ k+3<m, \tag{5.11}
$$

$$
\sup_{i=0,1,2,3,4} \left\|\partial_t^i f\right\|_{\mathrm{L}^\infty(\mathbb{R};\mathcal{H}_\sigma^k)} \leq \Lambda(1+M_{\text{in}}), \quad \sigma+4<r,\ k+6<m. \tag{5.12}
$$

Proof. Thanks to our choice of extension for (f, ϱ, u), and picking $N_f = 4$, $N_\varrho = 3$ and $N_u = 3$, it is sufficient to study the estimates on $[0, T]$ with $T \in [0, T_\varepsilon^\star]$. Here, we shall constantly apply Remarks 2.21 and 2.25. Taking all the exponents k large enough in the following, we can always assume that the Sobolev spaces that we consider are algebras.

We proceed inductively, relying on the equations satisfied by f, ϱ and u. For $i = 0$, we directly apply Lemma 2.22 with $k \leq m - 1$ to get

$$\|f\|_{\mathrm{L}^\infty(0,T;\mathcal{H}^k_\sigma)} \leq \|f^{\mathrm{in}}\|_{\mathcal{H}^{m-1}_\sigma} + T^{\frac{1}{4}}\Lambda(T, R) \leq M_{\mathrm{in}} + 1.$$

For $i = 1$, we apply Lemma 5.7 to get, for $k > d$ and $k + 1 > d/2$,

$$\|\partial_t f\|_{\mathrm{L}^\infty(0,T;\mathcal{H}^k_\sigma)} \lesssim \|f\|_{\mathrm{L}^\infty(0,T;\mathcal{H}^{k+1}_{\sigma+1})} + \|f\|_{\mathrm{L}^\infty(0,T;\mathcal{H}^{k+1}_\sigma)}\big(\|u\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^k)} + \Lambda(\|\varrho\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^{k+1})})\big).$$

Therefore, using Remarks 2.21 and 2.25 and the previous estimate on f, we deduce that

$$\|\partial_t f\|_{\mathrm{L}^\infty(0,T;\mathcal{H}^k_\sigma)} \leq \Lambda(1 + M_{\mathrm{in}})$$

if $k \leq m - 3$. For $i = 2$, we take one derivative in time in the Vlasov equation and get

$$\begin{aligned}\|\partial_t^2 f\|_{\mathrm{L}^\infty(0,T;\mathcal{H}^k_\sigma)} &\lesssim \|\partial_t f\|_{\mathrm{L}^\infty(0,T;\mathcal{H}^{k+1}_{\sigma+1})} \\ &\quad + \|E^{\varrho,u}_{\mathrm{reg},\varepsilon}\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^k)}\|\partial_t f\|_{\mathrm{L}^\infty(0,T;\mathcal{H}^{k+1}_\sigma)} \\ &\quad + \|\partial_t E^{\varrho,u}_{\mathrm{reg},\varepsilon}\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^k)}\|f\|_{\mathrm{L}^\infty(0,T;\mathcal{H}^{k+1}_\sigma)} \\ &\leq \Lambda(1 + M_{\mathrm{in}}) + \Lambda(1 + M_{\mathrm{in}})\|\partial_t E^{\varrho,u}_{\mathrm{reg},\varepsilon}\|_{\mathrm{L}^\infty(0,T;\mathcal{H}^k)}\end{aligned}$$

if we take $k \leq m - 4$. Since

$$\partial_t E^{\varrho,u}_{\mathrm{reg},\varepsilon} = \partial_t u - \partial_t \varrho p''(\varrho)\mathrm{J}_\varepsilon \nabla_x \varrho - p'(\varrho)\mathrm{J}_\varepsilon \nabla_x \partial_t \varrho,$$

it is enough to estimate $\|\partial_t u\|_{\mathrm{H}^k}$ and $\|\partial_t \varrho\|_{\mathrm{H}^{k+1}}$. Thanks to the equation for u and ϱ, we easily get the fact that this involves ρ_f, j_f, ϱ and u in $\mathrm{L}^\infty(0, T; \mathrm{H}^{k+2})$. Using Lemma 2.22 and Remarks 2.21 and 2.25 with $k \leq m - 4$ we obtain

$$\|\partial_t^2 f\|_{\mathrm{L}^\infty(0,T;\mathcal{H}^k_\sigma)} \lesssim \Lambda(1 + M_{\mathrm{in}}).$$

Now, for $i = 3$, we observe that $\|\partial_t^3 f\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^k)}$ is estimated by, at most, the quantities

$$\|\partial_t^2 f\|_{\mathrm{L}^\infty(0,T;\mathcal{H}^{k+1}_{\sigma+1})} \qquad \text{and} \qquad \|\partial_t^2 E^{\varrho,u}_{\mathrm{reg},\varepsilon}\|_{\mathrm{L}^\infty(0,T;\mathcal{H}^k_\sigma)},$$

thus requiring to estimate at most $\|\partial_t^2 u\|_{\mathrm{H}^k}$ and $\|\partial_t^2 \varrho\|_{\mathrm{H}^{k+1}}$. Using again the equation for u and ϱ, this now involves u in $\mathrm{L}^\infty(0,T;\mathrm{H}^{k+4})$ and ρ_f, j_f, ϱ in $\mathrm{L}^\infty(0,T;\mathrm{H}^{k+3})$. Using Lemma 2.22 and Remarks 2.21 and 2.25 with $k \leq m-5$ we obtain

$$\|\partial_t^3 f\|_{\mathrm{L}^\infty(0,T;\mathcal{H}^k_\sigma)} \leq \Lambda(1+M_{\mathrm{in}}).$$

The same can be done for $i=4$, implying, for $k \leq m-6$,

$$\|\partial_t^4 f\|_{\mathrm{L}^\infty(0,T;\mathcal{H}^k_\sigma)} \leq \Lambda(1+M_{\mathrm{in}}).$$

This allows us to conclude the proof. ■

We are now in a position to prove the following result concerning the Penrose symbol $\mathcal{P}_{f,\varrho}$, whose expression we recall here:

$$\mathcal{P}_{f,\varrho}(t,x,\gamma,\tau,\eta) = \frac{p'(\varrho(t,x))\varrho(t,x)}{1-\rho_f(t,x)} \int_0^{+\infty} e^{-(\gamma+i\tau)s} \frac{ik}{1+|k|^2} \cdot (\mathcal{F}_v \nabla_v f)(t,x,\eta s)\,\mathrm{d}s.$$

The symbol seminorms (C.1), (C.2) and (C.3) of $\mathcal{P}_{f,\varrho}$ are first estimated as follows.

Lemma 5.12. *For any $k > d/2$ and $\mathrm{M} \geq 0$, we have*

$$\omega[\mathcal{P}_{f,\varrho}] \leq \Lambda\Big(\sup_{i=0,1}\|\partial_t^i\varrho\|_{\mathrm{L}^\infty(\mathbb{R};\mathrm{H}^{d+k})}, \sup_{i=0,1}\|\partial_t^i\rho_f\|_{\mathrm{L}^\infty(\mathbb{R};\mathrm{H}^{d+k})}\Big)\omega[a_f], \tag{5.13}$$

$$\Xi[P_{f,\varrho}]_{\mathrm{M}} \leq \Lambda\Big(\sup_{i=0,1,2,3,4}\|\partial_t^i\varrho\|_{\mathrm{L}^\infty(\mathbb{R};\mathrm{H}^{1+M+k})}, \sup_{i=0,1,2,3,4}\|\partial_t^i\rho_f\|_{\mathrm{L}^\infty(\mathbb{R};\mathrm{H}^{1+\mathrm{M}+k})}\Big)\Xi[a_f]_{\mathrm{M}}, \tag{5.14}$$

$$\Omega[\mathcal{P}_{f,\varrho}] \leq \Lambda\Big(\sup_{i=0,1}\|\partial_t^i\varrho\|_{\mathrm{L}^\infty(\mathbb{R};\mathrm{H}^{d+k})}, \sup_{i=0,1}\|\partial_t^i\rho_f\|_{\mathrm{L}^\infty(\mathbb{R};\mathrm{H}^{d+k})}\Big)\Omega[a_f]. \tag{5.15}$$

Proof. Since the symbol $\mathcal{P}_{f,\varrho}$ depends only on (γ,τ,k) through a_f, we only prove the estimate (5.13), the treatment of (5.14) and (5.15) being similar.

First, we write the symbol $\mathcal{P}_{f,\varrho}$ as

$$\mathcal{P}_{f,\varrho}(t,x,\gamma,\tau,k) = \mathfrak{m}(\varrho(t,x),\rho_f(t,x))\frac{1}{1+|k|^2}a_f(t,x,\gamma,\tau,k),$$

where a_f has been defined in (5.6) and

$$\mathfrak{m}(\varrho(t,x),\rho_f(t,x)) := \frac{p'(\varrho(t,x))\varrho(t,x)}{1-\rho_f(t,x)}.$$

We have

$$
\begin{aligned}
&\omega[\mathcal{P}_{f,\varrho}]\\
&\lesssim \sup_{\substack{\alpha\in\mathbb{N}^d\\ \alpha_i\in\{0,1\}}} \big\|(1+t)\partial_x^\alpha(\mathfrak{m}(\varrho,\rho_f)a_f)\big\|_{\mathrm{L}^\infty(\mathbb{R}\times\mathbb{T}^d;\mathrm{L}^\infty_\eta)}\\
&\quad+\sup_{\substack{\alpha\in\mathbb{N}^d\\ \alpha_i\in\{0,1\}}} \big\|(1+t)\partial_x^\alpha(\mathfrak{m}(\varrho,\rho_f)\partial_t a_f)\big\|_{\mathrm{L}^\infty(\mathbb{R}\times\mathbb{T}^d;\mathrm{L}^\infty_\eta)}\\
&\quad+\sup_{\substack{\alpha\in\mathbb{N}^d\\ \alpha_i\in\{0,1\}}} \big\|(1+t)\partial_x^\alpha((\partial_1\mathfrak{m}(\varrho,\rho_f)\partial_t\varrho+\partial_2\mathfrak{m}(\varrho,\rho_f)\partial_t\rho_f)a_f)\big\|_{\mathrm{L}^\infty(\mathbb{R}\times\mathbb{T}^d;\mathrm{L}^\infty_\eta)}.
\end{aligned}
$$

Using the Leibniz rule, we get, for all $\alpha\in\mathbb{N}^d$ such that $\alpha_i\in\{0,1\}$,

$$
\begin{aligned}
&(1+t)\big|\partial_x^\alpha(\mathfrak{m}(\varrho,\rho_f)a_f)\big|\\
&\qquad\lesssim\sum_{\beta\le\alpha}\big|\partial_x^\beta(\mathfrak{m}(\varrho,\rho_f))\big|\big|\partial_x^{\alpha-\beta}a_f\big|\le\omega[a_f]\sum_{\beta\le\alpha}\big|\partial_x^\beta(\mathfrak{m}(\varrho,\rho_f))\big|,
\end{aligned}
$$

and one can observe that for all $k>\frac{d}{2}$,

$$
\sum_{\beta\le\alpha}\big|\partial_x^\beta(\mathfrak{m}(\varrho,\rho_f))\big|\lesssim\|\mathfrak{m}(\varrho,\rho_f)\|_{\mathrm{H}^{d+k}}\le\Lambda(\|\varrho\|_{\mathrm{L}^\infty},\|\rho_f\|_{\mathrm{L}^\infty})\|\varrho\|_{\mathrm{H}^{d+k}}\|\rho_f\|_{\mathrm{H}^{d+k}},
$$

where we have applied Proposition A.3 and the fact that H^{k+d} is an algebra. Doing the same with the term involving $\partial_t a_f$, we obtain

$$
\begin{aligned}
&\sup_{\substack{\alpha\in\mathbb{N}^d\\ \alpha_i\in\{0,1\}}} \big\|(1+t)\partial_x^\alpha(\mathfrak{m}(\varrho,\rho_f)a_f)\big\|_{\mathrm{L}^\infty(\mathbb{R}\times\mathbb{T}^d;\mathrm{L}^\infty_\eta)}\\
&\quad+\sup_{\substack{\alpha\in\mathbb{N}^d\\ \alpha_i\in\{0,1\}}} \big\|(1+t)\partial_x^\alpha(\mathfrak{m}(\varrho,\rho_f)\partial_t a_f)\big\|_{\mathrm{L}^\infty(\mathbb{R}\times\mathbb{T}^d;\mathrm{L}^\infty_\eta)}\\
&\qquad\le\Lambda\big(\|\varrho\|_{\mathrm{L}^\infty(\mathbb{R};\mathrm{L}^\infty)},\|\rho_f\|_{\mathrm{L}^\infty(\mathbb{R};\mathrm{L}^\infty)}\big)\|\varrho\|_{\mathrm{L}^\infty(\mathbb{R};\mathrm{H}^{d+k})}\|\rho_f\|_{\mathrm{L}^\infty(\mathbb{R};\mathrm{H}^{d+k})}\omega[a_f].
\end{aligned}
$$

Likewise, similar computations – which we omit – show that for all $k>\frac{d}{2}$,

$$
\begin{aligned}
&\sup_{\substack{\alpha\in\mathbb{N}^d\\ \alpha_i\in\{0,1\}}} \big\|\partial_x^\alpha\big([\partial_1 G(\varrho,\rho_f)\partial_t\varrho+\partial_2 G(\varrho,\rho_f)\partial_t\rho_f]a_f\big)\big\|_{\mathrm{L}^\infty(\mathbb{R}\times\mathbb{T}^d;\mathrm{L}^\infty_\eta)}\\
&\le\Lambda\big(\|\varrho\|_{\mathrm{L}^\infty(\mathbb{R};\mathrm{L}^\infty)},\|\rho_f\|_{\mathrm{L}^\infty(\mathbb{R};\mathrm{L}^\infty)}\big)\\
&\quad\times\Lambda\big(\|\varrho\|_{\mathrm{L}^\infty(\mathbb{R};\mathrm{H}^{d+k})},\|\partial_t\varrho\|_{\mathrm{L}^\infty(\mathbb{R};\mathrm{H}^{d+k})},\|\rho_f\|_{\mathrm{L}^\infty(\mathbb{R};\mathrm{H}^{d+k})},\|\partial_t\rho_f\|_{\mathrm{L}^\infty(\mathbb{R};\mathrm{H}^{d+k})}\big)\\
&\quad\times\omega[a_f].
\end{aligned}
$$

This concludes the proof of the estimate (5.13) of the lemma, thanks to Sobolev embedding. ∎

We finally obtain the following result, yielding some control on the seminorms of the Penrose function in terms of the initial data only.

Corollary 5.13. *The following hold:*

$$\omega[\mathcal{P}_{f,\varrho}] \lesssim \Lambda(1+M_{\mathrm{in}}), \tag{5.16}$$

$$\Xi[\mathcal{P}_{f,\varrho}]_{\mathrm{M}} \lesssim \Lambda(1+M_{\mathrm{in}}), \quad 2\mathrm{M} < m - 11 - d/2, \tag{5.17}$$

$$\Omega[\mathcal{P}_{f,\varrho}] \lesssim \Lambda(1+M_{\mathrm{in}}). \tag{5.18}$$

Proof. We combine the estimates (5.7), (5.8) and (5.9) of Lemma 5.10 with (5.13), (5.14) and (5.15) of Lemma 5.12. We first get, for $3 + 3d/2 \leq \ell < m - 3$,

$$\begin{aligned}\omega[\mathcal{P}_{f,\varrho}] &\leq \Lambda\Big(\sup_{i=0,1} \|\partial_t^i \varrho\|_{\mathrm{L}^\infty(\mathbb{R};\mathrm{H}^{d+\ell})}, \sup_{i=0,1} \|\partial_t^i \rho_f\|_{\mathrm{L}^\infty(\mathbb{R};\mathrm{H}^{d+\ell})}\Big) \\ &\quad \times \sup_{i=0,1} \|(1+t)\partial_t^i f\|_{\mathrm{L}^\infty(\mathbb{R};\mathcal{H}^\ell_r)} \\ &\leq \Lambda\Big(\sup_{i=0,1} \|\partial_t^i \varrho\|_{\mathrm{L}^\infty(\mathbb{R};\mathrm{H}^{d+\ell})}, \sup_{i=0,1} \|\partial_t^i f\|_{\mathrm{L}^\infty(\mathbb{R};\mathcal{H}^{d+\ell}_\sigma)}\Big)\Lambda(1+M_{\mathrm{in}})\end{aligned}$$

for $\sigma > d/2$. Hence, by using the equation for ϱ with Remark 2.21 and Lemma 2.22, and by taking $\ell < m - 3 - d$, we can rely on (5.11) from Lemma 5.11 to obtain (5.16). Next, we have, for $4 + \mathrm{M} + d/2 < \ell < m - 6$,

$$\begin{aligned}&\Xi[\mathcal{P}_{f,\varrho}]_{\mathrm{M}} \\ &\leq \Lambda\Big(\sup_{i=0,1,2,3,4} \|\partial_t^i \varrho\|_{\mathrm{L}^\infty(\mathbb{R};\mathrm{H}^{1+\mathrm{M}+\ell})}, \sup_{i=0,1,2,3,4} \|\partial_t^i \rho_f\|_{\mathrm{L}^\infty(\mathbb{R};\mathrm{H}^{1+\mathrm{M}+\ell})}\Big) \\ &\quad \times \sup_{i=0,1,2,3,4} \|\partial_t^i f\|_{\mathrm{L}^\infty(\mathbb{R};\mathcal{H}^\ell_r)} \\ &\leq \Lambda\Big(\sup_{i=0,1,2,3,4} \|\partial_t^i \varrho\|_{\mathrm{L}^\infty(\mathbb{R};\mathrm{H}^{1+\mathrm{M}+\ell})}, \sup_{i=0,1,2,3,4} \|\partial_t^i f\|_{\mathrm{L}^\infty(\mathbb{R};\mathcal{H}^{1+\mathrm{M}+\ell}_\sigma)}\Big)\Lambda(1+M_{\mathrm{in}})\end{aligned}$$

for $\sigma > d/2$. Using (5.12) from Lemma 5.11, the equation for ϱ, Remark 2.21 and Lemma 2.22 with $\ell + 1 + \mathrm{M} < m - 6$, we deduce (5.17). Finally, we also have, for $7 + 3d/2 < \ell < m - 3 - d$,

$$\Omega[\mathcal{P}_{f,\varrho}] \leq \Lambda\Big(\sup_{i=0,1} \|\partial_t^i \varrho\|_{\mathrm{L}^\infty(\mathbb{R};\mathrm{H}^{d+\ell})}, \sup_{i=0,1} \|\partial_t^i f\|_{\mathrm{L}^\infty(\mathbb{R};\mathcal{H}^{d+\ell}_\sigma)}\Big)\Lambda(1+M_{\mathrm{in}})$$

for $\sigma > d/2$ and as before, we obtain (5.18). ∎

5.5 Elliptic estimates through pseudodifferential analysis

Let $T \in (0, T_\varepsilon^\star)$. In view of the equation (5.1) on $h = \partial_x^\alpha \varrho$ obtained in Proposition 5.1, we initiate the study of the equation

$$\left(\mathrm{Id} - \frac{\varrho}{1-\rho_f}\mathrm{K}_G^{\mathrm{free}} \circ \mathrm{J}_\varepsilon\right)\big[\widetilde{H}\big] = \widetilde{\mathcal{R}}, \quad 0 \leq t \leq T, \tag{5.19}$$

where $\widetilde{\mathcal{R}}$ is a given source term defined on $(0, T)$. Given a solution $\widetilde{H}$ to this equation, we want to derive an $\mathrm{L}^2(0, T; \mathrm{L}^2)$ estimate of $\widetilde{H}$ in terms of $\widetilde{R}$. This will be possible thanks to the Penrose stability condition satisfied by $(f(t), \varrho(t))$ (see the forthcoming Proposition 5.17).

Note that the operator involved in the equation (5.19) depends on ϱ and f, which are defined for all times.

Following [90], we would like to link the operator $\mathrm{Id} - \frac{\varrho}{1-\rho_f}\mathrm{K}_G^{\mathrm{free}} \circ \mathrm{J}_\varepsilon$ which appears in (5.19) to a pseudodifferential operator of order 0 in time-space. We will use the following notation for the Fourier transform in time-space, and we also refer to Appendix C:

$$\forall (\tau, k) \in \mathbb{R} \times \mathbb{Z}^d, \quad \mathcal{F}_{t,x} g(\tau, k) = \int_{\mathbb{R}\times\mathbb{T}^d} e^{-i(\tau t + k\cdot x)} g(t, x)\, \mathrm{d}t\, \mathrm{d}x.$$

For symbols of the form $a(t, x, \gamma, \tau, k)$ on $[0, T] \times \mathbb{T}^d \times \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^d \backslash \{0\}$ and in the Schwartz class, we rely on the quantification

$$\mathrm{Op}^\gamma(a)(h)(t, x) = \frac{1}{(2\pi)^{d+1}} \int_{\mathbb{R}\times\mathbb{Z}^d} e^{i(\tau t + k\cdot x)} a(t, x, \gamma, \tau, k) \mathcal{F}_{t,x} h(\tau, k)\, \mathrm{d}\tau\, \mathrm{d}k.$$

The measure on $\mathbb{Z}^d$ is the discrete measure. Note that the symbols we shall handle are defined on $\mathbb{R} \times \mathbb{T}^d$ in the physical space, thanks to the extension procedure from Section 5.3. Note also that we shall handle symbols defined in the whole space $\mathbb{R}^d$ for the k variable, even if we only use them for $k \in \mathbb{Z}^d$ in the formula.

For $\gamma > 0$ (which will be chosen large enough in the end, but always independent of ε), we set

$$\widetilde{H}(t, x) := e^{\gamma t} H(t, x), \quad \widetilde{\mathcal{R}}(t, x) := e^{\gamma t} \mathcal{R}(t, x).$$

We can rewrite (5.19) as

$$H(t, x) - \frac{\varrho}{1-\rho_f} e^{-\gamma t} \mathrm{K}_G^{\mathrm{free}}[e^{\gamma \cdot} \mathrm{J}_\varepsilon H](t, x) = \mathcal{R}(t, x), \quad 0 \leq t \leq T. \tag{5.20}$$

First, let us extend the solution H by zero for times $t < 0$ and let us choose any smooth and compactly supported extension in time of H after time T. We still call this solution H. The following lemma now provides the link between the integro-differential equation (5.20) and a pseudodifferential equation.

Lemma 5.14. *We have*

$$e^{-\gamma t}\mathrm{K}_G^{\mathrm{free}}[e^{\gamma\cdot}H](t,x) = p'(\varrho(t,x))\mathrm{Op}^{\gamma}(a_f)(H)(t,x) \quad \text{on } \mathbb{R}\times\mathbb{T}^d,$$

where a_f is defined in (5.6). In particular, the equation (5.20) on H reads

$$H - \frac{p'(\varrho)\varrho}{1-\rho_f}\mathrm{Op}^{\gamma}(a_f)(\mathrm{J}_\varepsilon H) = S \quad \text{on } \mathbb{R}\times\mathbb{T}^d, \tag{5.21}$$

where S is a source term defined on $\mathbb{R}\times\mathbb{T}^d$ such that $S_{|(-\infty,0)} = 0$ and $S_{|[0,T]} = \mathcal{R}$.

Proof. We write

$$H(t,x) = \frac{1}{(2\pi)^{d+1}}\int_{\mathbb{R}\times\mathbb{Z}^d} e^{i(\tau t+k\cdot x)}\mathcal{F}_{t,x}H(\tau,k)\,\mathrm{d}\tau\,\mathrm{d}k;$$

therefore,

$$\begin{aligned} &e^{-\gamma t}\mathrm{K}_G^{\mathrm{free}}(e^{\gamma\cdot}H)(t,x)\\ &\quad= \frac{p'(\varrho(t,x))}{(2\pi)^{d+1}}\int_{-\infty}^{t}\int_{\mathbb{R}^d}\int_{\mathbb{R}\times\mathbb{Z}^d} e^{-\gamma(t-s)}e^{i(\tau s+k\cdot(x-(t-s)v))}ik\\ &\qquad\qquad\cdot\nabla_v f(t,x,v)\mathcal{F}_{t,x}H(\tau,k)\,\mathrm{d}v\,\mathrm{d}s\,\mathrm{d}\tau\,\mathrm{d}k, \end{aligned}$$

because H is 0 on negative times. We apply the Fubini theorem (which holds since $\gamma>0$) and get

$$\begin{aligned} &e^{-\gamma t}\mathrm{K}_G^{\mathrm{free}}(e^{\gamma\cdot}H)(t,x)\\ &= \frac{p'(\varrho(t,x))}{(2\pi)^{d+1}}\int_{\mathbb{R}\times\mathbb{Z}^d} e^{i(\tau t+k\cdot x)}\\ &\quad\times\left(\int_{-\infty}^{t} e^{-(\gamma+i\tau)(t-s)}ik\cdot\int_{\mathbb{R}^d} e^{-ik\cdot v(t-s)}\nabla_v f(t,x,v)\,\mathrm{d}v\,\mathrm{d}s\right)\mathcal{F}_{t,x}H(\tau,k)\,\mathrm{d}\tau\,\mathrm{d}k. \end{aligned}$$

Therefore, setting $s' = t-s$,

$$\begin{aligned} &e^{-\gamma t}\mathrm{K}_G^{\mathrm{free}}(e^{\gamma\cdot}H)(t,x)\\ &= \frac{p'(\varrho(t,x))}{(2\pi)^{d+1}}\int_{\mathbb{R}\times\mathbb{Z}^d} e^{i(\tau t+k\cdot x)}\left(\int_0^{+\infty} e^{-(\gamma+i\tau)s}ik\cdot\int_{\mathbb{R}^d} e^{-ik\cdot vs}\nabla_v f(t,x,v)\,\mathrm{d}v\,\mathrm{d}s\right)\\ &\qquad\qquad\times\mathcal{F}_{t,x}H(\tau,k)\,\mathrm{d}\tau\,\mathrm{d}k\\ &= \frac{p'(\varrho(t,x))}{(2\pi)^{d+1}}\int_{\mathbb{R}\times\mathbb{Z}^d} e^{i(\tau t+k\cdot x)}\left(\int_0^{+\infty} e^{-(\gamma+i\tau)s}ik\cdot(\mathcal{F}_v\nabla_v f)(t,x,ks)\,\mathrm{d}s\right)\\ &\qquad\qquad\times\mathcal{F}_{t,x}H(\tau,k)\,\mathrm{d}\tau\,\mathrm{d}k\\ &= p'(\varrho(t,x,))\mathrm{Op}(a_f)(H)(t,x). \end{aligned}$$

This provides the desired equality. That the source term S in the final pseudodifferential equation (5.21) satisfies $S_{|(-\infty,0)} = 0$ follows from the equation and the fact that H is zero for negative times. ■

Having in mind a semiclassical approach (see also Apppendix C), we introduce the following quantization.

Notation 5.15. For any symbol $b(t,x,\gamma,\tau,k)$ on $R \times \mathbb{T}^d \times \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^d \backslash \{0\}$, we set, for $\varepsilon \in (0,1)$,

$$b^\varepsilon(t,x,\gamma,\tau,k) := b(t,x,\varepsilon\gamma,\varepsilon\tau,\varepsilon k),$$
$$\mathrm{Op}^{\gamma,\varepsilon}(b) := \mathrm{Op}^\gamma(b^\varepsilon).$$

Lemma 5.16. *We have*

$$p'(\varrho)\mathrm{Op}^\gamma(a_f)(\mathrm{J}_\varepsilon H) = \mathrm{Op}^{\gamma,\varepsilon}(\mathfrak{a}_{f,\varrho})(H), \tag{5.22}$$

where

$$\mathfrak{a}_{f,\varrho}(t,x,\eta) := \frac{1}{1+|k|^2} p'(\varrho(t,x)) a_f(t,x,\eta).$$

Proof. We have the exact composition formula

$$p'(\varrho)\mathrm{Op}^\gamma(a_f)(\mathrm{J}_\varepsilon H) = \mathrm{Op}^\gamma(\widetilde{a}_{\varepsilon,f,\varrho})(H),$$
$$\widetilde{a}_{\varepsilon,f,\varrho}(t,x,\eta) := p'(\varrho(t,x)) a_f(t,x,\eta) \frac{1}{1+|\varepsilon k|^2},$$

since we are composing on the right by a Fourier multiplier. Since $a_{f,\varrho}$ is homogeneous of degree 0 in the variable η (see Lemma 5.10), we have

$$\widetilde{a}_{\varepsilon,f,\varrho}(t,x,\eta) = p'(\varrho(t,x)) a_f(t,x,\varepsilon\eta) \frac{1}{1+|\varepsilon k|^2} = \mathfrak{a}_{f,\varrho}(t,x,\varepsilon\eta),$$

and the conclusion follows. ■

In the following proposition, we show that we can choose an extension of (f,ϱ,u) as in Section 5.3 and such that it satisfies a Penrose condition for all times. By the definition of $T_\varepsilon^\star$, we know that

$$\forall t \in [0, T_\varepsilon^\star], \qquad \inf_{(x,\gamma,\tau,k)} |1 - \mathcal{P}_{f(t),\varrho(t)}(x,\gamma,\tau,k)| \geq c_0/2,$$

where we recall the expression of the Penrose symbol:

$$\mathcal{P}_{f(t),\varrho(t)}(x,\gamma,\tau,k) = \frac{p'(\varrho(t,x))\varrho(t,x)}{1-\rho_f(t,x)} \int_0^{+\infty} e^{-(\gamma+i\tau)s} \frac{ik}{1+|k|^2} \cdot (\mathcal{F}_v \nabla_v f)(t,x,ks)\,\mathrm{d}s.$$

We want this condition to be true for the extension of (f,ϱ). We have the following result, which requires the technical assumption (1.2) on the pressure.

Proposition 5.17. *There exists $\delta^\star = \delta(c_0, M_{\mathrm{in}}) > 0$ small enough such that, considering the extension of f and ϱ with respect to $T_\varepsilon^\star$ and $\delta^\star$, we have*

$$\forall t \in \mathbb{R}, \qquad \inf_{(x,\gamma,\tau,k)} |1 - \mathcal{P}_{f(t),\varrho(t)}(x,\gamma,\tau,k)| \geq c_0/4.$$

Proof. We only treat the case $t \in (T_\varepsilon^\star, +\infty)$, as the case of negative times is identical. Let us set

$$\chi_\delta^\star(t) := \chi_\delta(t - T_\varepsilon^\star), \qquad \underline{\chi}^\star(t) := \underline{\chi}(t - T_\varepsilon^\star),$$

where we employ the notation of Section 5.3. By the definition of the extension for f, we have

$$\begin{aligned} &\mathcal{P}_{f(t),\varrho(t)} \\ &\quad = \frac{\chi_\delta^\star(t) p'(\varrho(t,x))\varrho(t,x)}{1-\rho_f(t,x)} \\ &\qquad \times \int_0^{+\infty} e^{-(\gamma+i\tau)s} \frac{ik}{1+|k|^2} \cdot (\mathcal{F}_v \nabla_v f)(T_\varepsilon^\star, x, ks)\,\mathrm{d}s \\ &\qquad + \frac{\chi_\delta^\star(t) p'(\varrho(t,x))\varrho(t,x)}{1-\rho_f(t,x)} \\ &\qquad\quad \times \sum_{k=1}^{N_f} \frac{(t-T_\varepsilon^\star)^k}{k!} \int_0^{+\infty} e^{-(\gamma+i\tau)s} \frac{ik}{1+|k|^2} \cdot \big(\mathcal{F}_v \nabla_v \partial_t^k f\big)(T_\varepsilon^\star, x, ks)\,\mathrm{d}s. \end{aligned} \tag{5.23}$$

We next proceed to the following decompositions: we have

$$\begin{aligned} \frac{1}{1-\rho_f(t)} &= \frac{1}{1 - \chi_\delta^\star(t)\rho_f(T_\varepsilon^\star) - \chi^\star(t)\sum_{k=1}^{N_f} \partial_t^k \rho_f(T_\varepsilon^\star)\frac{(t-T_\varepsilon^\star)^k}{k!}} \\ &= \frac{1}{1-\chi_\delta^\star(t)\rho_f(T_\varepsilon^\star)} + \frac{1}{1-\chi_\delta^\star(t)\rho_f(T_\varepsilon^\star)} \frac{Q^\star(t)}{1-Q^\star(t)}, \end{aligned}$$

where

$$Q^\star(t) := \frac{\chi_\delta^\star(t)\sum_{k=1}^{N_f} \partial_t^k \rho_f(T_\varepsilon^\star)\frac{(t-T_\varepsilon^\star)^k}{k!}}{1-\chi_\delta^\star(t)\rho_f(T_\varepsilon^\star)},$$

and by writing

$$\varrho(t) = \underline{\chi}^\star(t)\varrho(T_\varepsilon^\star) + R^\star(t), \qquad R^\star(t) := \chi_\delta(t-T_\varepsilon^\star)\sum_{k=1}^{N_\varrho} \varrho^{(k)}(T_\varepsilon^\star)\frac{(t-T_\varepsilon^\star)^k}{k!},$$

we also have

$$p'(\varrho(t))\varrho(t) = p'\big(\underline{\chi}^\star(t)\varrho(T_\varepsilon^\star)\big)\underline{\chi}^\star(t)\varrho(T_\varepsilon^\star) + S^\star(t),$$

where

$$S^\star(t) := \big[p'\big(\underline{\chi}^\star(t)\varrho(T_\varepsilon^\star) + R^\star(t)\big) - p'\big(\underline{\chi}^\star(t)\varrho(T_\varepsilon^\star)\big)\big]\underline{\chi}^\star(t)\varrho(T_\varepsilon^\star) \\ + p'\big(\underline{\chi}^\star(t)\varrho(T_\varepsilon^\star) + R^\star(t)\big)R^\star(t).$$

Note that

$$S^\star(t) = \underset{t\to T_\varepsilon^\star}{\mathcal{O}}(t - T_\varepsilon^\star).$$

Now, the equality (5.23) turns into

$$1 - \mathcal{P}_{f(t),\varrho(t)} = 1 - \frac{\chi_\delta^\star(t)p'\big(\underline{\chi}^\star(t)\varrho(T_\varepsilon^\star)\big)\underline{\chi}^\star(t)\varrho(T_\varepsilon^\star)}{1 - \chi_\delta^\star(t)\rho_f(T_\varepsilon^\star)} \\ \times \frac{1}{1+|k|^2}\int_0^{+\infty} e^{-(\gamma+i\tau)s} ik\cdot(\mathcal{F}_v\nabla_v f)(T_\varepsilon^\star, x, ks)\,\mathrm{d}s \\ + \mathfrak{E}^\star(t),$$

with a remainder $\mathfrak{E}^\star(t) := -\chi_\delta^\star(t)(\mathrm{I}+\mathrm{II}+\mathrm{III}+\mathrm{IV})$, where

$$\mathrm{I} = \frac{p'\big(\underline{\chi}^\star(t)\varrho(T_\varepsilon^\star)\big)\underline{\chi}^\star(t)\varrho(T_\varepsilon^\star)}{1 - \chi_\delta^\star(t)\rho_f(T_\varepsilon^\star)}\frac{Q^\star(t)}{1-Q^\star(t)} \\ \times \int_0^{+\infty} e^{-(\gamma+i\tau)s}\frac{ik}{1+|k|^2}\cdot(\mathcal{F}_v\nabla_v f)(T_\varepsilon^\star, x, ks)\,\mathrm{d}s,$$

$$\mathrm{II} = \frac{S^\star(t)}{1 - \chi_\delta^\star(t)\rho_f(T_\varepsilon^\star)}\int_0^{+\infty} e^{-(\gamma+i\tau)s}\frac{ik}{1+|k|^2}\cdot(\mathcal{F}_v\nabla_v f)(T_\varepsilon^\star, x, ks)\,\mathrm{d}s,$$

$$\mathrm{III} = \frac{S^\star(t)}{1-\chi_\delta^\star(t)\rho_f(T_\varepsilon^\star)}\frac{Q^\star(t)}{1-Q^\star(t)}\int_0^{+\infty} e^{-(\gamma+i\tau)s}\frac{ik}{1+|k|^2}\cdot(\mathcal{F}_v\nabla_v f)(T_\varepsilon^\star, x, ks)\,\mathrm{d}s,$$

$$\mathrm{IV} = \frac{p'(\varrho(t,x))\varrho(t,x)}{1-\rho_f(t,x)} \\ \times \sum_{k=1}^{N_f}\frac{(t-T_\varepsilon^\star)^k}{k!}\int_0^{+\infty} e^{-(\gamma+i\tau)s}\frac{ik}{1+|k|^2}\cdot\big(\mathcal{F}_v\nabla_v\partial_t^k f\big)(T_\varepsilon^\star, x, ks)\,\mathrm{d}s.$$

We claim that a homogeneity argument shows that

$$\inf_{(x,\gamma,\tau,k)}\Bigg|1 - \frac{\chi_\delta^\star(t)p'\big(\underline{\chi}^\star(t)\varrho(T_\varepsilon^\star,x)\big)\underline{\chi}^\star(t)\varrho(T_\varepsilon^\star,x)}{1-\chi_\delta^\star(t)\rho_f(T_\varepsilon^\star,x)} \\ \times \frac{1}{1+|k|^2}\int_0^{+\infty} e^{-(\gamma+i\tau)s} ik\cdot(\mathcal{F}_v\nabla_v f)(T_\varepsilon^\star, x, ks)\,\mathrm{d}s\Bigg|$$

$$\geq \inf_{(x,\gamma,\tau,k)} \left| 1 - \frac{p'(\varrho(T_\varepsilon^\star, x))\varrho(T_\varepsilon^\star, x)}{1-\rho_f(T_\varepsilon^\star, x)} \times \frac{1}{1+|k|^2} \int_0^{+\infty} e^{-(\gamma+i\tau)s} ik \cdot (\mathcal{F}_v \nabla_v f)(T_\varepsilon^\star, x, ks)\,\mathrm{d}s \right|$$
$$\geq c_0/2.$$

Indeed, we know from Lemma 5.10 that for all $x \in \mathbb{T}^d$, the function

$$(\gamma, \tau, k) \mapsto \int_0^{+\infty} e^{-(\gamma+i\tau)s} ik \cdot (\mathcal{F}_v \nabla_v f)(T_\varepsilon^\star, x, ks)\,\mathrm{d}s$$

is homogeneous of degree 0, and since by the assumption (1.2) and by construction we have

$$0 \leq \frac{\chi_\delta^\star(t) p'\big(\underline{\chi}^\star(t)\varrho(T_\varepsilon^\star, x)\big)\underline{\chi}^\star(t)\varrho(T_\varepsilon^\star, x)}{1-\chi_\delta^\star(t)\rho_f(T_\varepsilon^\star, x)} \leq \frac{p'\left(\varrho(T_\varepsilon^\star, x)\right)\varrho(T_\varepsilon^\star, x)}{1-\rho_f(T_\varepsilon^\star, x)},$$

we can rely on Remark 1.10 (see (1.5)) to obtain the previous claim. By writing

$$\inf_{(x,\gamma,\tau,k)} |1 - \mathcal{P}_{f(t),\varrho(t)}(x,\gamma,\tau,k)| \geq \frac{c_0}{2} - \sup_{(x,\gamma,\tau,k)} |\mathfrak{E}^\star(t,x,\gamma,\tau,k)|$$

thanks to the triangular inequality, it remains to prove that a suitable choice of δ can lead to

$$\forall t \in [T_\varepsilon^\star, +\infty), \qquad \sup_{(x,\gamma,\tau,k)} |\mathfrak{E}^\star(t,x,\gamma,\tau,k)| \leq \frac{c_0}{4}.$$

First note that the remainder $\mathfrak{E}^\star$ has a factor $\chi_\delta^\star(t)$ multiplying all the terms in its expression, and is therefore compactly supported in time, with support in the interval $(T_\varepsilon^\star, T_\varepsilon^\star + \delta)$. Relying on the bounds for f and ϱ depending only on M_{in} (see Remark 2.21 and Lemma 2.22), we can proceed as in the proof of the estimates (5.16), (5.17) and (5.18) and show that

$$|\mathfrak{E}^\star(t)| \leq \Lambda(M_{\mathrm{in}})\chi_\delta^\star(t)|t - T_\varepsilon^\star| \leq \Lambda(M_{\mathrm{in}})\delta.$$

This procedure is allowed since the extension and the remainder $\mathfrak{E}^\star$ only involve a finite number of derivatives in time of ϱ and f at $t = T_\varepsilon^\star$. Choosing δ small enough is now sufficient to conclude. ∎

We are now in a position to provide the key $\mathrm{L}^2(\mathbb{R}, \mathrm{L}^2_x)$ estimate for a solution to the equation (5.21). We will actually consider a slightly different pseudodifferential equation, which is

$$\mathcal{H} - \frac{p'(\varrho)\varrho}{1-\rho_f}\mathrm{Op}^\gamma(a_f)(\mathrm{J}_\varepsilon \mathcal{H}) = \mathcal{S} \quad \text{on } \mathbb{R} \times \mathbb{T}^d, \tag{5.24}$$

where $\mathcal{S} = S$ on $[0, T]$ and is zero outside $[0, T]$. The main part of our analysis will provide an estimate in $\mathrm{L}^2(\mathbb{R}; \mathrm{L}^2_x)$ for the solution $\mathcal{H}$ of the equation (5.24), and we will show subsequently that it will provide an $\mathrm{L}^2(0, T, \mathrm{L}^2_x)$ estimate on H, the solution to the original equation (5.20). This will be based on a causality principle for the pseudodifferential equation (5.20) (see Lemma 5.19 below).

Proposition 5.18. *Assume that $\mathcal{H}$ is a solution of the equation* (5.24) *on $\mathbb{R} \times \mathbb{T}^d$. There exists $\Lambda(M_{\mathrm{in}}) > 0$ such that, for any $\gamma \geq \Lambda(M_{\mathrm{in}})$, we have*

$$\|\mathcal{H}\|_{\mathrm{L}^2(\mathbb{R}\times\mathbb{T}^d)} \leq \Lambda(M_{\mathrm{in}})\|\mathcal{S}\|_{\mathrm{L}^2(\mathbb{R}\times\mathbb{T}^d)}.$$

In particular, we have

$$\|\mathcal{H}\|_{\mathrm{L}^2((0,T)\times\mathbb{T}^d)} \leq \Lambda(M_{\mathrm{in}})\|\mathcal{S}\|_{\mathrm{L}^2((0,T)\times\mathbb{T}^d)}.$$

Proof. Thanks to (5.22) and Lemma 5.14, the equation (5.24) can be rewritten as

$$\left(\mathrm{Id} - \frac{\varrho}{1-\rho_f}\mathrm{Op}^{\gamma,\varepsilon}(\mathfrak{a}_{f,\varrho})\right)(\mathcal{H}) = \mathcal{S},$$

where $\mathfrak{a}_{f,\varrho}$ has been defined in (5.22). Now observe that, recalling the definition (1.4) of the Penrose symbol $\mathcal{P}_{f,\varrho}$, we have

$$\frac{\varrho}{1-\rho_f}\mathfrak{a}_{f,\varrho} = \mathcal{P}_{f,\varrho};$$

therefore, H satisfies

$$\mathrm{Op}^{\gamma,\varepsilon}(1-\mathcal{P}_{f,\varrho})(\mathcal{H}) = \mathcal{S}. \tag{5.25}$$

Relying on Proposition 5.17 on the Penrose condition satisfied by the (extension) of $(f(t), \varrho(t))$ on $\mathbb{R}$, we can consider

$$c_{f,\varrho} := \frac{1}{1-\mathcal{P}_{f,\varrho}}.$$

Note that the symbol $c_{f,\varrho} - 1$ vanishes outside a compact set in time and hence, in view of the estimates (5.16), (5.17) and (5.18) of Corollary 5.13 on the symbol $\mathcal{P}_{f,\varrho}$ and Faà di Bruno's formula, we get

$$\omega\big[c^{\varepsilon}_{f,\varrho} - 1\big] + \Omega\big[c^{\varepsilon}_{f,\varrho} - 1\big] \leq \Lambda(1 + M_{\mathrm{in}}). \tag{5.26}$$

Applying $\mathrm{Op}^{\gamma,\varepsilon}(c_{f,\varrho})$ to the equation (5.25) yields

$$\begin{aligned}
\mathcal{H} &= \mathcal{S} + \mathrm{Op}^{\gamma,\varepsilon}(c_{f,\varrho}-1)(\mathcal{S}) \\
&\quad + \big[\mathrm{Op}^{\gamma,\varepsilon}\big((c_{f,\varrho}-1)(1-\mathcal{P}_{f,\varrho})\big) - \mathrm{Op}^{\gamma,\varepsilon}(c_{f,\varrho}-1)\mathrm{Op}^{\gamma,\varepsilon}(1-\mathcal{P}_{f,\varrho})\big](\mathcal{H}) \\
&= \mathcal{S} + \mathrm{Op}^{\gamma,\varepsilon}(c_{f,\varrho}-1)(\mathcal{S}) \\
&\quad - \big[\mathrm{Op}^{\gamma,\varepsilon}\big((c_{f,\varrho}-1)\mathcal{P}_{f,\varrho}\big) - \mathrm{Op}^{\gamma,\varepsilon}(c_{f,\varrho}-1)\mathrm{Op}^{\gamma,\varepsilon}(\mathcal{P}_{f,\varrho})\big](\mathcal{H}).
\end{aligned}$$

Using the L^2 continuity property from Theorem C.2 and the commutation estimates from Proposition C.3 (recall that $\mathcal{P}_{f,\rho}$ has compact support in time), we get, for all $\gamma > 0$ and $M > 1 + 2d$,

$$\|\mathcal{H}\|_{L^2(\mathbb{R}\times\mathbb{T}^d)} \\ \leq \big(1 + C\omega[c_{f,\varrho} - 1]\big)\|\mathcal{S}\|_{L^2(\mathbb{R}\times\mathbb{T}^d)} + \frac{C}{\gamma}\Omega[c^{\varepsilon}_{f,\varrho} - 1]\,\Xi\big[\mathcal{P}^{\varepsilon}_{f,\varrho}\big]_M \|\mathcal{H}\|_{L^2(\mathbb{R}\times\mathbb{T}^d)}$$

for some constant $C > 0$ depending only on the dimension. By homogeneity of the previous seminorms with respect to the semiclassical quantification in ε (in particular, the fact that $\Omega[c^{\varepsilon}_{f,\varrho} - 1] \leq \Omega[c_{f,\varrho} - 1]$ for $\varepsilon \leq 1$), we infer that

$$\|\mathcal{H}\|_{L^2(\mathbb{R}\times\mathbb{T}^d)} \\ \leq \big(1 + C\omega[c_{f,\varrho} - 1]\big)\|\mathcal{S}\|_{L^2(\mathbb{R}\times\mathbb{T}^d)} + \frac{C}{\gamma}\Omega[c_{f,\varrho} - 1]\,\Xi\big[\mathcal{P}_{f,\varrho}\big]_M \|\mathcal{H}\|_{L^2(\mathbb{R}\times\mathbb{T}^d)}.$$

Thanks to (5.26) and the estimate (5.17) of Corollary 5.13, we get

$$\|\mathcal{H}\|_{L^2(\mathbb{R}\times\mathbb{T}^d)} \leq C\Lambda(1 + M_{\text{in}})\|\mathcal{S}\|_{L^2(\mathbb{R}\times\mathbb{T}^d)} + \frac{C}{\gamma}\Lambda(1 + M_{\text{in}})\|\mathcal{H}\|_{L^2(\mathbb{R}\times\mathbb{T}^d)}.$$

Taking γ large enough with respect to M_{in} allows us to apply an absorption argument: we get the existence of some $\Lambda(M_{\text{in}}) > 0$ such that, for any $\gamma \geq \Lambda(M_{\text{in}})$, we have

$$\|\mathcal{H}\|_{L^2(\mathbb{R}\times\mathbb{T}^d)} \lesssim \Lambda(M_{\text{in}})\|\mathcal{S}\|_{L^2(\mathbb{R}\times\mathbb{T}^d)},$$

which was the desired conclusion. The last inequality stated in the proposition follows from the fact that $\mathcal{S}$ is zero outside $[0, T]$. ∎

Let us briefly explain how to obtain a solution $\mathcal{H}$ of the equation (5.24) with source term $\mathcal{S}$. The main idea is to use the estimate derived in Proposition 5.18. Indeed, setting

$$A_\gamma := \mathrm{Op}^\gamma(1 - \mathcal{P}_{f,\varrho}), \quad B_\gamma := \mathrm{Op}^\gamma\left(\frac{1}{1 - \mathcal{P}_{f,\varrho}}\right),$$

one can repeat the proof of Proposition 5.18 to show that

$$B_\gamma A_\gamma = \mathrm{Id} + \frac{1}{\gamma}C_\gamma,$$

where C_γ is a bounded operator on $L^2(\mathbb{R}\times\mathbb{T}^d)$ whose norm is uniformly bounded independently of γ. Hence, for γ large enough, we obtain that $B_\gamma A_\gamma$ is invertible on $L^2(\mathbb{R}\times\mathbb{T}^d)$, and so A_γ has a left inverse. By the same argument for $A_\gamma B_\gamma$, we obtain that A_γ is invertible on $L^2(\mathbb{R}\times\mathbb{T}^d)$ for γ large enough. This leads to the existence of a solution to the equation (5.24).

Let us now show how to relate the solutions of the equations (5.21) and (5.24) on $[0, T]$. This comes from the following causality principle.

Lemma 5.19. *Let $T > 0$. Consider the solution H to the equation (5.21) and a solution $\mathscr{H}$ to the equation (5.24) on $\mathbb{R} \times \mathbb{T}^d$. Then*

$$H|_{[0,T]} = \mathscr{H}|_{[0,T]}.$$

Proof. First introduce, for $\gamma > 0$,

$$\widetilde{H}(t,x) := e^{\gamma t} H(t,x), \quad \widetilde{S}(t,x) := e^{\gamma t} S(t,x),$$

and

$$\widetilde{\mathscr{H}}(t,x) := e^{\gamma t} H(t,x), \quad \widetilde{\mathcal{S}}(t,x) := e^{\gamma t} \mathcal{S}(t,x).$$

In view of the equations (5.21) and (5.24), we have by linearity

$$e^{-\gamma t}(\widetilde{H} - \widetilde{\mathscr{H}}) - \frac{p'(\varrho)\varrho}{1-\rho_f} \mathrm{Op}^\gamma(a_f)\big(\mathrm{J}_\varepsilon e^{-\gamma \cdot}(\widetilde{H} - \widetilde{\mathscr{H}})\big) = e^{-\gamma t}(\widetilde{S} - \widetilde{\mathcal{S}}) \quad \text{on } \mathbb{R} \times \mathbb{T}^d.$$

Note that the source term of this equation is zero on $(-\infty, T]$ by the definition of S and $\mathcal{S}$, which satisfy

$$S|_{(-\infty,T]} = \mathcal{S}|_{(-\infty,T]}.$$

By Proposition 5.18, there exists $\Lambda(M_{\text{in}})$ such that, for all $\gamma \geq \Lambda(M_{\text{in}})$,

$$\begin{aligned}\int_0^T e^{-2\gamma t} \|\widetilde{H}(t) - \widetilde{\mathscr{H}}(t)\|^2_{\mathrm{L}^2(\mathbb{T}^d)} \,\mathrm{d}t &\leq \|e^{-\gamma \cdot}(\widetilde{H} - \widetilde{\mathscr{H}})\|^2_{\mathrm{L}^2(\mathbb{R}\times\mathbb{T}^d)} \\ &\leq \Lambda(M_{\text{in}})^2 \int_T^{+\infty} e^{-2\gamma t} \|\widetilde{S}(t) - \widetilde{\mathcal{S}}(t)\|^2_{\mathrm{L}^2(\mathbb{T}^d)} \,\mathrm{d}t\end{aligned}$$

for some $\gamma_0 > 0$ and $C_0 \geq 0$. We thus infer that

$$\int_0^T \|\widetilde{H}(t) - \widetilde{\mathscr{H}}(t)\|^2_{\mathrm{L}^2(\mathbb{T}^d)} \,\mathrm{d}t \leq \Lambda(M_{\text{in}})^2 \int_T^{+\infty} e^{2\gamma(T-t)} \|\widetilde{S}(t) - \widetilde{\mathcal{S}}(t)\|^2_{\mathrm{L}^2(\mathbb{T}^d)} \,\mathrm{d}t.$$

By letting $\gamma \to +\infty$, we deduce that $\widetilde{H}|_{[0,T]} = \widetilde{\mathscr{H}}|_{[0,T]}$, hence the result. ■

As a consequence, we can finally obtain an estimate for the equation (5.19) on $(0, T) \times \mathbb{T}^d$.

Corollary 5.20. *Consider $\widetilde{H}$ the solution to (5.19) on $(0,T) \times \mathbb{T}^d$. We have*

$$\|\widetilde{H}\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{T}^d))} \leq \Lambda(T, M_{\text{in}}) \|\widetilde{\mathcal{R}}\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{T}^d))}.$$

Proof. First recall that defining

$$\widetilde{H}(t,x) := e^{\gamma t} H(t,x) \quad \text{and} \quad \widetilde{\mathcal{R}}(t,x) := e^{\gamma t}\mathcal{R}(t,x)$$

and extending H by zero for $t < 0$, it is solution of (5.21) on $\mathbb{R} \times \mathbb{T}^d$ with a source term S satisfying $S_{|(-\infty,0)} = 0$ and $S_{|[0,T]} = \mathcal{R}$. By Lemma 5.19 and Proposition 5.18, we thus get, for all $\gamma \geq \Lambda(M_{\mathrm{in}})$,

$$\int_0^T e^{-2\gamma t}\|\widetilde{H}(t)\|^2_{\mathrm{L}^2(\mathbb{T}^d)}\,\mathrm{d}t \leq \Lambda(M_{\mathrm{in}})^2 \int_0^T e^{-2\gamma t}\|\widetilde{\mathcal{R}}(t)\|^2_{\mathrm{L}^2(\mathbb{T}^d)}\,\mathrm{d}t.$$

Therefore, taking $\gamma = \Lambda(M_{\mathrm{in}})$ provides

$$\|\widetilde{H}\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{T}^d))} \leq e^{\Lambda(M_{\mathrm{in}})T}\Lambda(M_{\mathrm{in}})\|\widetilde{\mathcal{R}}\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{T}^d))},$$

which concludes the proof. ∎

5.6 Final hyperbolic estimates

To conclude this chapter, it remains to perform an energy estimate on the hyperbolic part $(\partial_t + u\cdot\nabla_x)(h)$.

Let us observe that by Remark 2.25, we have

$$\|\mathrm{div}_x\, u\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty)} \lesssim \big(1 + T^{1/2}\Lambda(T,R)\big)M_{\mathrm{in}} + T^{1/2}\Lambda(T,R),$$

by Sobolev embedding (since $m > 1 + d/2$); therefore, for all $t \in (0, T_\varepsilon^\star)$,

$$\|\mathrm{div}_x\, u\|_{\mathrm{L}^\infty(0,t;\mathrm{L}^\infty)} \leq 1 + 2M_{\mathrm{in}}. \tag{5.27}$$

We can therefore state the following lemma.

Lemma 5.21. *Let* $T \in (0, \min(T_\varepsilon(R), \overline{T}(R), \widetilde{T}_0(R)))$. *Assume that* h *is a solution of the equation*

$$(\partial_t + u\cdot\nabla_x)(h) = \widetilde{H}, \quad t \in [0,T].$$

The following estimate holds:

$$\|h\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{T}^d))} \leq e^{(1+M_{\mathrm{in}})T}T^{\frac{1}{2}}\big(\|h(0)\|_{\mathrm{L}^2} + T^{\frac{1}{2}}\|\widetilde{H}\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{T}^d))}\big).$$

Proof. The proof is standard. Due to the hyperbolic nature of the equation, an energy estimate provides pointwise-in-time L^2 bounds (in the same spirit as the proof of Proposition 2.3). For all $t \in [0,T]$, we have

$$\|h(t)\|_{\mathrm{L}^2} \leq e^{\|\mathrm{div}_x\, u\|_{\mathrm{L}^\infty(0,t;\mathrm{L}^\infty)}t}\|h(0)\|_{\mathrm{L}^2} + \int_0^t e^{\|\mathrm{div}_x\, u\|_{\mathrm{L}^\infty(0,t;\mathrm{L}^\infty)}(t-\tau)}\|\widetilde{H}(\tau)\|_{\mathrm{L}^2}\,\mathrm{d}\tau.$$

By (5.27), we get by the Cauchy–Schwarz inequality in time that for all $t \in [0,T]$,

$$\|h(t)\|_{\mathrm{L}^2} \leq e^{(1+M_{\mathrm{in}})T}\left(\|h(0)\|_{\mathrm{L}^2} + T^{\frac{1}{2}}\int_0^T \|\widetilde{H}(\tau)\|_{\mathrm{L}^2}^2 \,\mathrm{d}\tau\right),$$

hence the conclusion, taking the L^2 norm in time on $(0,T)$ in the above inequality. ■

Gathering the results presented in Lemma 5.2, Proposition 5.1, Corollary 5.20 and Lemma 5.21, we directly infer the following statement.

Corollary 5.22. *For all $|\alpha| \leq m$ and all $T \in (0, \min(T_\varepsilon(R), \overline{T}(R), \widetilde{T}_0(R)))$, we have the estimate*

$$\|\partial_x^\alpha \varrho\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{T}^d))} \leq T^{\frac{1}{2}}\Lambda(M_{\mathrm{in}}, T, R).$$