# Chapter 6

# End of the proof

## 6.1 Conclusion of the bootstrap

Let us conclude the bootstrap argument. We choose

$$T \in \big(0, \min\big(T_\varepsilon(R), \overline{T}(R), \widetilde{T}_0(R)\big)\big).$$

We can consider the following explicit quantity, which appears in all the estimates from Chapter 5:

$$M_{\text{in}} := \|f^{\text{in}}\|_{\mathcal{H}^m_r} + \|\varrho^{\text{in}}\|_{\mathrm{H}^{m+1}} + \|u^{\text{in}}\|_{\mathrm{H}^m}.$$

We now apply Corollary 5.22 to get

$$\|\varrho\|_{\mathrm{L}^2(0,T;\mathrm{H}^m)} \leq T^{1/2}\Lambda(M_{\text{in}}, T, R).$$

We also invoke the energy estimate of Lemma 2.22 on $f$ that yields

$$\|f\|_{\mathrm{L}^\infty(0,T;\mathcal{H}^{m-1}_r)} \leq M_{\text{in}} + T^{\frac{1}{4}}\Lambda(T, R),$$

and the energy estimate of Proposition 2.24 on $u$ giving

$$\|u\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^m)\cap\mathrm{L}^2(0,T;\mathrm{H}^{m+1})} \leq M_{\text{in}} + T^{1/2}\Lambda(M_{\text{in}}, T, R) + \Lambda(T, R)\|\varrho\|_{\mathrm{L}^2(0,T;\mathrm{H}^m)},$$

and therefore, using the previous estimate on $\varrho$, we have

$$\|u\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^m)\cap\mathrm{L}^2(0,T;\mathrm{H}^{m+1})} \leq M_{\text{in}} + T^{1/2}\Lambda(M_{\text{in}}, T, R).$$

Combining all these estimates together, we finally obtain the key estimate

$$\mathcal{N}_{m,r}(f, \varrho, u, T) \leq C\big(M_{\text{in}} + T^{1/4}\Lambda(T, R) + T^{1/2}\Lambda(M_{\text{in}}, T, R)\big)$$

for some universal constant $C > 0$. Note that the previous right-hand side is independent of $\varepsilon$. Next, we choose $R$ large enough so that

$$CM_{\text{in}} < \frac{1}{2}R.$$

Now, with $R$ being fixed, we rely on the continuity at 0 of the function

$$s \mapsto s^{1/4}\Lambda(s, R) + s^{1/2}\Lambda(M_{\text{in}}, s, R)$$

to find some time $T^\sharp > 0$ independent of $\varepsilon$ with

$$T^\sharp \in \big(0, \min\big(\overline{T}(R), \widetilde{T}_0(R)\big)\big)$$

and such that, for every $T \in [0, T^\sharp]$,

$$C\big(T^{1/4}\Lambda(M_{\text{in}}, T, R) + T^{1/2}\Lambda(M_{\text{in}}, T, R)\big) < \frac{1}{2}R.$$

We deduce that for all $T \in [0, \min(T^\sharp, T_\varepsilon(R))]$, we have $\mathcal{N}_{m,r}(f, \varrho, u, T) < R$. In addition, thanks to Lemmas 2.16 and 2.17, we easily get the fact that the condition ($\mathrm{B}_{\Theta}^{\mu,\theta}(T)$) is satisfied, up to reducing $T^\sharp$ so that $\Theta + T^\sharp R < 1$ and

$$T^\sharp \in \left(0, \frac{1}{R}\min\left(\frac{1-\Theta}{2}, \ln(2), \ln\left(\frac{2\theta}{\mu}\right)\right)\right).$$

Since we were assuming that $\mathcal{N}_{m,r}(f, \varrho, u, T_\varepsilon(R)) = R$, this shows that we must have $T_\varepsilon > T^\sharp$.

In conclusion, we have found $R > 0$ and $T > 0$ such that, for all $\varepsilon > 0$,

$$\mathcal{N}_{m,r}(f, \varrho, u, T) \leq R.$$

## 6.2 Existence of a solution

Let us first focus on the existence part of Theorem 1.6. We will rely on a standard compactness argument, that we briefly detail. In view of Section 6.1, there exist $T > 0$, $R > 0$ and

$$(f_\varepsilon, \varrho_\varepsilon, u_\varepsilon) \in \mathcal{C}([0,T]; \mathcal{H}_r^m) \times \mathcal{C}([0,T]; \mathrm{H}^m) \times \mathcal{C}([0,T]; \mathrm{H}^m) \cap \mathrm{L}^2(0,T; \mathrm{H}^{m+1})$$

a solution to ($\mathrm{S}_\varepsilon$) with initial data $(f^{\text{in}}, \varrho^{\text{in}}, u^{\text{in}})$ such that

$$\sup_{\varepsilon \in (0,1]} \mathcal{N}_{m,r}(f_\varepsilon, \varrho_\varepsilon, u_\varepsilon, T) \leq R.$$

Hence, we deduce that $(f_\varepsilon)_\varepsilon$ is bounded in $\mathrm{L}^\infty(0,T; \mathcal{H}_r^{m-1})$, $(\varrho_\varepsilon)_\varepsilon$ is bounded in $\mathrm{L}^2(0,T; \mathrm{H}^m)$ and $(u_\varepsilon)_\varepsilon$ is bounded in $\mathrm{L}^\infty(0,T; \mathrm{H}^m)$ and in $\mathrm{L}^2(0,T; \mathrm{H}^{m+1})$. From Lemma 2.20, we also know that $(\varrho_\varepsilon)_\varepsilon$ is bounded in $\mathrm{L}^\infty(0,T; \mathrm{H}^{m-2})$. We deduce that $(f_\varepsilon, \varrho_\varepsilon, u_\varepsilon)$ has a weak-$\star$ limit $(f, \varrho, u)$ (up to some extraction) in the previous spaces.

Furthermore, using the equations for $f_\varepsilon$, $\varrho_\varepsilon$ and $u_\varepsilon$, we know that

$$\begin{aligned}
&(\partial_t f_\varepsilon)_{\varepsilon \leq 1} \text{ is bounded in } \mathrm{L}^\infty(0,T; \mathcal{H}_{r-1}^{m-3}),\\
&(\partial_t \varrho_\varepsilon)_{\varepsilon \leq 1} \text{ is bounded in } \mathrm{L}^\infty(0,T; \mathrm{H}^{m-3}),\\
&(\partial_t u_\varepsilon)_{\varepsilon \leq 1} \text{ is bounded in } \mathrm{L}^\infty(0,T; \mathrm{H}^{m-3}).
\end{aligned}$$

Invoking the Aubin–Lions–Simon lemma (see e.g. [32, Theorem II.5.16]), we therefore deduce that, up to some extraction, we have

$$f_\varepsilon \xrightarrow[\varepsilon\to 0]{} f \quad \text{in } \mathcal{C}([0,T];\mathcal{H}^{m-2}_{r-1}),$$
$$\varrho_\varepsilon \xrightarrow[\varepsilon\to 0]{} \varrho \quad \text{in } \mathcal{C}([0,T];\mathrm{H}^{m-3}),$$
$$u_\varepsilon \xrightarrow[\varepsilon\to 0]{} u \quad \text{in } \mathcal{C}([0,T];\mathrm{H}^{m-1}).$$

These strong convergences allow us to pass to the limit in ($\mathrm{S}_\varepsilon$) and to obtain that $(f,\varrho,u)$ is a solution to the thick spray equations (TS) on $[0,T]$, and that $(f,\varrho)$ satisfies a Penrose stability condition (P) on the same interval of time.

It remains to prove that $f \in \mathcal{C}([0,T];\mathcal{H}^{m-1}_r)$ and $u \in \mathcal{C}([0,T];\mathrm{H}^m)$, as we only have $f \in \mathrm{L}^\infty(0,T;\mathcal{H}^{m-1}_r)$ and $u \in \mathrm{L}^\infty(0,T;\mathrm{H}^m)$ for the moment. Since $f \in \mathrm{L}^\infty(0,T;\mathcal{H}^{m-1}_r) \cap \mathcal{C}_w([0,T];\mathcal{H}^{m-2}_{r-1})$ and $u \in \mathrm{L}^\infty(0,T;\mathrm{H}^m) \cap \mathcal{C}_w([0,T];\mathrm{H}^{m-1})$, we know (see e.g. [32, Lemma II.5.9]) that

$$f \in \mathcal{C}_w([0,T];\mathcal{H}^{m-1}_r) \quad \text{and} \quad u \in \mathcal{C}_w([0,T];\mathrm{H}^m).$$

It is now sufficient to prove that $t \mapsto \|f(t)\|_{\mathcal{H}^{m-1}_r}$ and $t \mapsto \|u(t)\|_{\mathrm{H}^m}$ are continuous functions on $[0,T]$ to conclude. Coming back to the energy estimates of Section 2.2, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|f(t)\|^2_{\mathcal{H}^{m-1}_r} < +\infty, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2_{\mathrm{H}^m} < +\infty.$$

As $t \mapsto \|f(t)\|^2_{\mathcal{H}^{m-1}_r}$ and $t \mapsto \|u(t)\|^2_{\mathrm{H}^m}$ are integrable (because $f \in \mathrm{L}^\infty(0,T;\mathcal{H}^{m-1}_r)$ and $u \in \mathrm{L}^\infty(0,T;\mathrm{H}^m)$), we obtain that $t \mapsto \|f(t)\|^2_{\mathcal{H}^{m-1}_r}$ and $t \mapsto \|u(t)\|^2_{\mathrm{H}^m}$ belong to $\mathrm{W}^{1,1}(0,T)$, hence the continuity in time of these quantities. This finally yields the desired continuity for $f$ and $u$ and concludes the proof.

## 6.3 Uniqueness of the solution

Let us turn to the uniqueness part of Theorem 1.6. Let $(f_1,\varrho_1,u_1)$ and $(f_2,\varrho_2,u_2)$ be two solutions of (TS) belonging to

$$\mathrm{L}^\infty(0,\boldsymbol{T};\mathcal{H}^{m-1}_r) \times \mathrm{L}^2(0,\boldsymbol{T};\mathrm{H}^m) \times \mathrm{L}^\infty(0,\boldsymbol{T};\mathrm{H}^m) \cap \mathrm{L}^2(0,\boldsymbol{T};\mathrm{H}^{m+1})$$

for some $\boldsymbol{T} > 0$, with the same initial condition $(f^{\mathrm{in}},\varrho^{\mathrm{in}},u^{\mathrm{in}})$ and such that $t \mapsto (f_1(t),\varrho_1(t))$ satisfies the Penrose stability condition (P) on $[0,\boldsymbol{T}]$. Let us set

$$f := f_1 - f_2, \quad \alpha_i := 1 - \rho_{f_i}, \quad \alpha := \alpha_1 - \alpha_2, \quad \varrho := \varrho_1 - \varrho_2, \quad u := u_1 - u_2,$$

and

$$R := \max_{i=1,2}\left(\|f_i\|_{\mathrm{L}^\infty(0,\boldsymbol{T};\mathcal{H}^{m-1}_r)} + \|\varrho_i\|_{\mathrm{L}^2(0,\boldsymbol{T};\mathrm{H}^m)} + \|u_i\|_{\mathrm{L}^\infty(0,\boldsymbol{T};\mathrm{H}^m)\cap\mathrm{L}^2(0,\boldsymbol{T};\mathrm{H}^{m+1})}\right).$$

Note that $R$ depends on $\boldsymbol{T}$.

**Step 1.** Let us show that $\varrho_1 = \varrho_2$, at least on a small interval of time. The key is to obtain $\mathrm{L}^2(0, T; \mathrm{L}^2)$ estimates for $\varrho$ for some $T = T(R) \leq \boldsymbol{T}$. To this end, we write the equation satisfied by $\varrho$ as

$$\begin{aligned}
&\alpha_1(\partial_t \varrho + u_1 \cdot \nabla_x \varrho) + \varrho_1(\partial_t \alpha + u_1 \cdot \nabla_x \alpha) \\
&\quad = -\big(\partial_t \varrho_2 \alpha + \partial_t \alpha_2 \varrho + [\varrho u_1 + \varrho_2 u] \cdot \nabla_x \alpha_2 + [\alpha u_1 + \alpha_2 u] \cdot \nabla_x \varrho_2 \\
&\qquad\qquad + \alpha \rho_1 \operatorname{div}_x u_1 + \alpha_2 \varrho \operatorname{div}_x u_1 + \alpha_2 \varrho_2 \operatorname{div}_x u\big),
\end{aligned}$$

and since $\partial_t \alpha_i = \operatorname{div}_x j_{f_i}$, we get

$$\partial_t \varrho + u_1 \cdot \nabla_x \varrho + \frac{\varrho_1}{1 - \rho_{f_1}} \operatorname{div}_x (j_f - \rho_f u_1) = S_{1,2}(f, \varrho, u), \tag{6.1}$$

where

$$\begin{aligned}
S_{1,2}(f, \varrho, u) := -\frac{1}{1 - \rho_{f_1}} \big(&-\partial_t \varrho_2 \rho_f + \partial_t \alpha_2 \varrho + [\varrho u_1 + \varrho_2 u] \cdot \nabla_x \alpha_2 \\
&+ [-\rho_f u_1 + \alpha_2 u] \cdot \nabla_x \varrho_2 - \rho_f \rho_1 \operatorname{div}_x u_1 \\
&+ \alpha_2 \varrho \operatorname{div}_x u_1 + \alpha_2 \varrho_2 \operatorname{div}_x u - \varrho_2 \rho_f \operatorname{div}_x u_1\big).
\end{aligned} \tag{6.2}$$

The right-hand side $S_{1,2}(f, \varrho, u)$ can be seen as a source term whose $\mathrm{L}^2(0, T; \mathrm{L}^2)$ norm will be estimated by that of $\varrho$, without any loss of derivative in $(f, \varrho, u)$. Note that we have rewritten the equation as above in order to perform the right pseudodifferential factorization. This is of course reminiscent of what we have done in Chapter 5. Next, the equations for the differences $f$ and $u$ read

$$\begin{cases}
\partial_t f + v \cdot \nabla_x f + \operatorname{div}_v\big[(u_2 - v) f - \nabla_x p(\varrho_2) f\big] \\
\qquad + \big(u - \nabla_x p(\varrho_1) + \nabla_x p(\varrho_2)\big) \cdot \nabla_v f_1 = 0, \\
\partial_t u + (u_2 \cdot \nabla_x) u + (u \cdot \nabla_x) u_1 + \dfrac{1}{\varrho_1} \nabla_x p(\varrho_1) - \dfrac{1}{\varrho_2} \nabla_x p(\varrho_2) \\
\qquad - \dfrac{1}{\alpha_1 \varrho_1} (\Delta_x + \nabla_x \operatorname{div}_x) u - \left(\dfrac{1}{\alpha_1 \varrho_1} - \dfrac{1}{\alpha_2 \varrho_2}\right) (\Delta_x + \nabla_x \operatorname{div}_x) u_2 \\
\qquad\qquad = j_f - \rho_f u_1 - \rho_{f_2} u,
\end{cases} \tag{6.3}$$

and, in particular,

$$\begin{aligned}
&\partial_t f + v \cdot \nabla_x f + \operatorname{div}_v\big[(u_2 - v) f - \nabla_x p(\varrho_2) f\big] \\
&\quad = p'(\varrho_1) \nabla_x \varrho \cdot \nabla_v f_1 + (p'(\varrho_1) - p'(\varrho_2)) \nabla_x \varrho_2 \cdot \nabla_v f_1 - u \cdot \nabla_x f_1.
\end{aligned}$$

Again, the last two terms on the right-hand side should be seen as some source terms, without any loss of derivative in $(f, \varrho, u)$. One can show that

$$\begin{aligned}
&\big\| (p'(\varrho_1) - p'(\varrho_2)) \nabla_x \varrho_2 \cdot \nabla_v f_1 \big\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)} + \| u \cdot \nabla_x f_1 \|_{\mathrm{L}^2(0,T;\mathrm{L}^2)} \\
&\quad \leq \Lambda(T, R) \| \varrho \|_{\mathrm{L}^2(0,T;\mathrm{L}^2)}.
\end{aligned}$$

The proof of the estimate for the second term will be similar to that for the term $S[\varrho]$ we will treat below.

Arguing as in Section 2.3 and Chapters 3 and 4, but for a force field

$$\mathfrak{F}(t,x) := u_2(t,x) - \nabla_x p(\varrho_2(t,x))$$

instead of $E^{\varrho,u}_{\mathrm{reg},\varepsilon}$, one can also prove that for $T(R)$ small enough, we have, for all $t \in [0, T(R)]$,

$$\begin{aligned}
&\rho_f(t,x) \\
&\quad = p'(\varrho_1(t,x)) \int_0^t \int_{\mathbb{R}^d} \nabla_x \varrho(s, x-(t-s)v) \cdot \nabla_v f_1(t,x,v)\, \mathrm{d}s\, \mathrm{d}v + \mathcal{R}[\rho_f](t,x), \\
&j_f(t,x) \\
&\quad = p'(\varrho_1(t,x)) \int_0^t \int_{\mathbb{R}^d} v \nabla_x \varrho(s, x-(t-s)v) \cdot \nabla_v f_1(t,x,v)\, \mathrm{d}s\, \mathrm{d}v + \mathcal{R}[j_f](t,x),
\end{aligned}$$

where

$$\left\|\mathcal{R}[\rho_f]\right\|_{\mathrm{L}^2(0,T;\mathrm{H}^1)} + \left\|\mathcal{R}[j_f]\right\|_{\mathrm{L}^2(0,T;\mathrm{H}^1)} \leq \Lambda(T,R) \|\varrho\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)}.$$

Here, we have also used the fact that $f_{|t=0} = 0$. Injecting these expressions in equation (6.1) on $\varrho$, we get, as in Chapter 5,

$$\partial_t \varrho + u_1 \cdot \nabla_x \varrho + \frac{\varrho_1}{1-\rho_{f_1}} \operatorname{div}_x \left[\mathrm{K}^{\mathrm{free}}_{1,G_1}(\varrho) - \mathrm{K}^{\mathrm{free}}_{G_1}(\varrho) u_1\right] = S[\varrho],$$

with

$$G_1(t,x,v) := p'(\varrho_1(t,x)) \nabla_v f_1(t,x,v)$$

and

$$S[\varrho] = -\frac{\varrho_1}{1-\rho_{f_1}} \operatorname{div}_x \left(\mathcal{R}[j_f] - \mathcal{R}[\rho_f] u_1\right) + S_{1,2}(f, \varrho, u)$$

(the operator $\mathrm{K}^{\mathrm{free}}_{1,G_1}$ being defined in (5.2)), and then

$$\left(\mathrm{Id} - \frac{\varrho_1}{1-\rho_{f_1}} \mathrm{K}^{\mathrm{free}}_{G_1}\right)\left[\partial_t \varrho + u_1 \cdot \nabla_x \varrho\right] = S[\varrho], \quad t \in (0,T).$$

It is straightforward to prove that the source $S[\varrho]$ satisfies

$$\|S[\varrho]\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)} \leq \Lambda(T,R) \|\varrho\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)}.$$

The main issue comes from obtaining the same estimate for the term $S_{1,2}(u,f)$ defined in (6.2). We observe that it is made of a sum of three types of terms, of the following form:

- some terms where $\varrho$ appears as a factor, so that we directly obtain the estimate;

- some terms where $\rho_f$ appears as a factor: we can rely on the previous decomposition of $\rho_f$ as $\rho_f = \mathrm{K}_{G_1}^{\mathrm{free}}[\varrho] + \mathcal{R}[\rho_f]$ and an $\mathrm{L}^2(0,T;\mathrm{L}^2)$ estimate is then provided by the smoothing estimate of Proposition 3.4;
- some terms where $u$ or $\operatorname{div}_x u$ appears as a factor: to estimate them, we perform the same kind of energy estimate as in Appendix B, following the steps leading to the equality (B.3), which gives, after integration in time (since $u_{|t=0} = 0$),

$$\begin{aligned}
\|u(t)\|_{\mathrm{L}^2}^2 &+ \int_0^t \|\operatorname{div}_x u(s)\|_{\mathrm{L}^2}^2 \,\mathrm{d}s \\
&\lesssim \int_0^t \|u(s)\|_{\mathrm{L}^2}^2 \,\mathrm{d}s \\
&\quad + \int_0^t \Big( \|\rho_f(s)\|_{\mathrm{L}^2}^2 + \|j_f(s)\|_{\mathrm{L}^2}^2 + \|\alpha_1 \varrho_1(s) - \alpha_2 \varrho_2(s)\|_{\mathrm{L}^2}^2 \Big) \,\mathrm{d}s \\
&\lesssim \int_0^t \|u(s)\|_{\mathrm{L}^2}^2 \,\mathrm{d}s + \Lambda(T,R) \|\varrho\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)}.
\end{aligned}$$

  Here, we have also used the smoothing estimate from Proposition 3.4 combined with the previous decomposition of $\rho_f$ and $j_f$. We obtain the desired control on $u$ after an application of Grönwall's lemma.

  As in Section 5.5, we then study the equation

$$\left( \mathrm{Id} - \frac{\varrho_1}{1-\rho_{f_1}} \mathrm{K}_{G_1}^{\mathrm{free}} \right) [\widetilde{H}] = \widetilde{\mathcal{R}}, \quad 0 \leq t \leq T,$$

where $\widetilde{\mathcal{R}}$ is a given source term and we want to derive an $\mathrm{L}^2(0,T;\mathrm{L}^2)$ estimate on the solution $\widetilde{H}$. After applying the same extension procedure for the coefficients (depending on $(f_1, \varrho_1, u_1)$) in the equation as in Section 5.3, and by setting $\widetilde{H}(t,x) := e^{\gamma t} H(t,x)$ and $\widetilde{\mathcal{R}}(t,x) = e^{\gamma t} \mathcal{R}(t,x)$ for $\gamma > 0$ (with the same continuation by zero outside $[0,T]$ as in Section 5.5), we are led to the study of the pseudodifferential equation

$$\mathrm{Op}^{\gamma}(1 - \mathcal{P}_{f_1,\varrho_1})(H) = \mathcal{R},$$

where

$$\mathcal{P}_{f_1(t),\varrho_1(t)}(x,\gamma,\tau,k) := \frac{p'(\varrho_1(x))\varrho_1(x)}{1-\rho_{f_1}(x)} \int_0^{+\infty} e^{-(\gamma+i\tau)s} ik \cdot (\mathcal{F}_v \nabla_v f_1)(t,x,ks) \,\mathrm{d}s.$$

Observe that there is no factor $(1+|k|^2)^{-1}$ in the definition of this symbol, because there is no regularization operator in the equation. Note also that the estimates (5.13), (5.14) and (5.15) hold true for $\mathcal{P}_{f_1,\varrho_1}$, in view of the regularity of $(f_1, \varrho_1)$. Likewise, as in Corollary 5.20, we have

$$\|\widetilde{H}\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{T}^d))} \leq \Lambda(T,R) \|\widetilde{\mathcal{R}}\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{T}^d))},$$

provided that one can apply $\mathrm{Op}^\gamma\left(\frac{1}{1-\mathcal{P}_{f_1,\varrho_1}}\right)$ to the previous pseudodifferential equation and take $\gamma$ large enough in order to invert it up to a small remainder. To do so, we prove that if the condition (P)

$$\text{(P)}\quad \forall t\in\mathbb{R},\ \forall x\in\mathbb{T}^d,\qquad \inf_{(\gamma,\tau,k)\in(0,+\infty)\times\mathbb{R}\times\mathbb{R}^d\backslash\{0\}} |1-\mathcal{P}_{f_1(t),\varrho_1(t)}(x,\gamma,\tau,k)|>c$$

holds for some $c>0$, then the condition

$$\text{(P')}\quad \forall t\in\mathbb{R},\ \forall x\in\mathbb{T}^d,\qquad \inf_{(\gamma,\tau,k)\in(0,+\infty)\times\mathbb{R}\times\mathbb{R}^d\backslash\{0\}} |1-\mathscr{P}_{f_1(t),\varrho_1(t)}(x,\gamma,\tau,k)|>c$$

holds as well. Here, we implicitly consider the extension in time of $f_1$ and $\varrho_1$, as was done in Section 5.3. To this end, we rely on a homogeneity argument, as in [90]: we define the function

$$\widetilde{\mathcal{P}}_{f_1,\varrho_1}(x,\tilde{\gamma},\tilde{\tau},\tilde{k},\sigma):=\mathcal{P}_{f_1,\varrho_1}(x,\sigma\tilde{\gamma},\sigma\tilde{\tau},\sigma\tilde{k}),\quad x\in\mathbb{T}^d,\ (\tilde{\gamma},\tilde{\tau},\tilde{k})\in S^+,\ \sigma>0,$$

where

$$S^+:=\left\{(\tilde{\gamma},\tilde{\tau},\tilde{k})\in(0,+\infty)\times\mathbb{R}\times\mathbb{R}^d\backslash\{0\}\mid \tilde{\gamma}^2+\tilde{\tau}^2+\tilde{k}^2=1\right\}.$$

Since $f_1$ is regular enough, one can prove that $\widetilde{\mathcal{P}}_{f_1,\varrho_1}$ can be extended as a continuous function on $\mathbb{T}^d\times S^+\times[0,+\infty)$ and we obtain

$$\forall t\in\mathbb{R},\ \forall x\in\mathbb{T}^d,\qquad \inf_{(\tilde{\gamma},\tilde{\tau},\tilde{k},\sigma)\in S^+\times[0,+\infty)}\left|1-\widetilde{\mathcal{P}}_{f_1(t),\varrho_1(t)}(x,\tilde{\gamma},\tilde{\tau},\tilde{k},\sigma)\right|>c.$$

In view of the homogeneity of degree 0 of the symbol $a_{f,\varrho}$ with respect to the variable $(\gamma,\tau,\eta)$ (see Lemma 5.16), we also have

$$\widetilde{\mathcal{P}}_{f_1,\varrho_1}(x,\tilde{\gamma},\tilde{\tau},\tilde{k},\sigma) = \frac{1}{1+\sigma^2|\tilde{k}|^2}\frac{p'(\varrho_1(x))\varrho_1(x)}{1-\rho_{f_1}(x)}\int_0^{+\infty} e^{-(\tilde{\gamma}+i\tilde{\tau})s}\, i\tilde{k}\cdot(\mathcal{F}_v\nabla_v f_1)(t,x,\tilde{k}s)\,\mathrm{d}s,$$

hence

$$\widetilde{\mathcal{P}}_{f_1(t),\varrho_1(t)}(x,\tilde{\gamma},\tilde{\tau},\tilde{k},0)=\mathscr{P}_{f_1(t),\varrho_1(t)}(x,\tilde{\gamma},\tilde{\tau},\tilde{k}),$$

from which we infer that (P') holds on $S^+$. Again, the homogeneity of degree 0 of $\mathcal{P}_{f_1(t),\varrho_1(t)}(x,\gamma,\tau,k)$ with respect to the variable $(\gamma,\tau,k)$ implies that (P') holds.

Next, we come up with the transport equation for $\varrho$

$$\partial_t\varrho+u_1\cdot\nabla_x\varrho=\widetilde{H},$$

where $\varrho_{|t=0}=0$ and the source term $\widetilde{H}$ has been shown to satisfy

$$\|\widetilde{H}\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{T}^d))}\le\Lambda(T,R)\|S[\varrho]\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{T}^d))}\le\Lambda(T,R)\|\varrho\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{T}^d))}.$$

Performing an $L^2$ energy estimate gives, for all $t \in [0, T]$ (since $\varrho_{|t=0} = 0$),

$$\|\varrho(t)\|_{L^2} \leq \Lambda(T, R) \int_0^t \|\widetilde{H}(\tau)\|_{L^2} \, d\tau \leq T^{\frac{1}{2}} \Lambda(T, R) \|\varrho\|_{L^2(0,T;L^2(\mathbb{T}^d))},$$

by the Cauchy–Schwarz inequality. We end up with

$$\|\varrho\|_{L^2(0,T;L^2(\mathbb{T}^d))} \leq T\Lambda(T, R)\|\varrho\|_{L^2(0,T;L^2(\mathbb{T}^d))}.$$

We deduce the fact that there exists a small enough $T = T(R) > 0$ which depends only on $R$ such that

$$\forall t \in [0, T(R)], \quad \varrho(t) = 0.$$

**Step 2.** Let us now prove that $f_1 = f_2$ and $u_1 = u_2$ on $[0, T(R)]$. This is in fact a direct consequence of $\varrho = \varrho_1 - \varrho_2 = 0$ on $[0, T(R)]$. Indeed, the previous step has shown that, for all $t \in [0, T(R)]$,

$$\|u(t)\|_{L^2}^2 \lesssim \int_0^t \|u(s)\|_{L^2}^2 \, ds + \Lambda(t, R) \, \|\varrho\|_{L^2(0,t;L^2)} \lesssim \int_0^t \|u(s)\|_{L^2}^2 \, ds,$$

and therefore we directly have $u = 0$ on $[0, T(R)]$. The equations for $f$ in (6.3) now turn into

$$\begin{cases} \partial_t f + v \cdot \nabla_x f - v \cdot \nabla_v f + \big(u_2 - \nabla_x p(\varrho_2)\big) \cdot \nabla_v f = df, \\ f_{|t=0} = 0. \end{cases}$$

The method of characteristics thus shows that $f = 0$ on $[0, T(R)]$.

In conclusion, we have obtained $(f, \varrho, u) = (0, 0, 0)$ on $[0, T(R)]$. We finally observe that we can repeat this procedure starting from $t = T(R)$ instead of $t = 0$. As a matter of fact, $f_1(T(R), \cdot)$ still satisfies the Penrose stability condition. Since the time $T(R)$ only depends on $R$, we obtain $\varrho = 0$ on $[0, 2T(R)]$ and then $(f, u) = (0, 0)$ on $[0, 2T(R)]$. After a finite number of steps, this yields $(f, \varrho, u) = (0, 0, 0)$ on $[0, \boldsymbol{T}]$. This concludes the proof of the uniqueness part of the statement.