## Chapter 7

# Generalization to several models of thick sprays

In this chapter, we show how the strategy developed in this work can be applied to treat several variants of the system (TS), which were presented in Section 1.3.

## 7.1 Generalization to the non-barotropic case

In this section, we show how to handle the case of the full Navier–Stokes system for non-barotropic fluids, where we consider the additional internal energy $e \in \mathbb{R}^+$ for the fluid and where the pressure depends on $\varrho$ and $e$. As explained in the introduction, the system which is at stake is the following:

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + \mathrm{div}_v\big[f(u-v) - f\nabla_x p(\varrho, e)\big] = 0, \\ \partial_t(\alpha\varrho) + \mathrm{div}_x(\alpha\varrho u) = 0, \\ \partial_t(\alpha\varrho u) + \mathrm{div}_x(\alpha\varrho u \otimes u) + \alpha\nabla_x p(\varrho, e) - \Delta_x u - \nabla_x \mathrm{div}_x u = j_f - \rho_f u, \\ \partial_t(\alpha\varrho e) + \mathrm{div}_x(\alpha\varrho e u) + p(\varrho, e)(\partial_t\alpha + \mathrm{div}_x(\alpha u)) = \displaystyle\int_{\mathbb{R}^d} |u-v|^2 f \, \mathrm{d}v, \\ \alpha = 1 - \rho_f. \end{cases}$$

Here, we will assume[1] that the pressure law is given as

$$p(\varrho, e) = \pi(\varrho e)$$

for some given function $\pi : \mathbb{R}^+ \to \mathbb{R}$ such that $\pi \in \mathcal{C}(\mathbb{R}^+) \cap \mathcal{C}^\infty(\mathbb{R}^+ \backslash \{0\})$. For instance, the relation $p(\varrho, e) = b\varrho e$ (with $b > 0$) is a perfect gas pressure law. Similarly to the technical hypothesis (1.2), we shall assume that

$$y \mapsto \pi'(y)(y + \pi(y)) \text{ is nondecreasing on } \mathbb{R}.$$

Setting

$$\vartheta := \varrho e,$$

the system in $(f, \varrho, u, \vartheta)$ can be rewritten as

$$(\mathrm{TS}_e) \quad \begin{cases} \partial_t f + v \cdot \nabla_x f + \mathrm{div}_v\big[f(u - v - \nabla_x \pi(\vartheta))\big] = 0, \\ \partial_t(\alpha\varrho) + \mathrm{div}_x(\alpha\varrho u) = 0, \\ \partial_t(\alpha\varrho u) + \mathrm{div}_x(\alpha\varrho u \otimes u) + \alpha\nabla_x \pi(\vartheta) - \Delta_x u - \nabla_x \mathrm{div}_x u = j_f - \rho_f u, \\ \partial_t(\alpha\vartheta) + \mathrm{div}_x(\alpha\vartheta u) + \pi(\vartheta)(\partial_t\alpha + \mathrm{div}_x(\alpha u)) = \displaystyle\int_{\mathbb{R}^d} |v-u|^2 f \, \mathrm{d}v, \\ \alpha = 1 - \rho_f. \end{cases}$$

[1] It is likely that more general pressures $p(\varrho, e)$ could be treated by our method.

As we shall see below, it is significant to define the following Penrose symbol of a function $(f(x,v),\vartheta(x))$ as

$$\mathbf{P}^{\text{energy}}_{f,\vartheta}(x,\gamma,\tau,\eta) := \frac{\pi'(\vartheta(x))[\vartheta(x)+\pi(\vartheta(x))]}{1-\rho_f(x)} \int_0^{+\infty} e^{-(\gamma+i\tau)s} \frac{ik}{1+|k|^2} \cdot (\mathcal{F}_v \nabla_v f)(x,s\eta)\,\mathrm{d}s. \tag{7.1}$$

We can now introduce the following Penrose condition, adapted to ($\mathrm{TS}_e$): there exists $c>0$ such that

$$(\mathrm{P}^{\text{energy}}) \qquad \forall x \in \mathbb{T}^d, \qquad \inf_{(\gamma,\tau,\eta)\in(0,+\infty)\times\mathbb{R}\times\mathbb{R}^d\setminus\{0\}} \left|1-\mathbf{P}^{\text{energy}}_{f,\vartheta}(x,\gamma,\tau,\eta)\right| > c.$$

Our main result for the system ($\mathrm{TS}_e$) reads as follows.

**Theorem 7.1.** *There exist $m_0>0$ and $r_0>0$, depending only on the dimension, such that the following holds for all $m \geq m_0$ and $r \geq r_0$. Let*

$$f^{\text{in}} \in \mathcal{H}^m_r, \quad \varrho^{\text{in}} \in \mathrm{H}^{m-1}, \quad u^{\text{in}} \in \mathrm{H}^m, \quad \vartheta^{\text{in}} \in \mathrm{H}^{m+1},$$

*such that $(f^{\text{in}},\vartheta^{\text{in}})$ satisfies the $c$-Penrose stability condition $(\mathrm{P}^{\text{energy}})_c$ (with $c>0$) and*

$$0 \leq f^{\text{in}}, \quad \rho_{f^{\text{in}}} < \Upsilon < 1, \quad 0 < \mu \leq \varrho^{\text{in}}, \vartheta^{\text{in}},$$
$$0 < \underline{\nu} \leq (1-\rho_{f^{\text{in}}})\varrho^{\text{in}}, \quad (1-\rho_{f^{\text{in}}})\vartheta^{\text{in}} \leq \overline{\nu}$$

*for some fixed constants $\Upsilon, \mu, \underline{\nu}, \overline{\nu}$. Then there exist $T>0$ and a solution $(f,\varrho,u,\vartheta)$ to ($\mathrm{TS}_e$) with initial condition $(f^{\text{in}},\varrho^{\text{in}},u^{\text{in}},\vartheta^{\text{in}})$ such that*

$$f \in \mathcal{C}([0,T];\mathcal{H}^{m-1}_r), \quad \varrho \in \mathcal{C}([0,T];\mathrm{H}^{m-1}),$$
$$u \in \mathcal{C}([0,T];\mathrm{H}^m) \cap \mathrm{L}^2(0,T;\mathrm{H}^{m+1}), \quad \vartheta \in \mathrm{L}^2(0,T;\mathrm{H}^m),$$

*and with $(f(t),\vartheta(t))$ satisfying the $\frac{c}{2}$-Penrose stability condition $(\mathrm{P}^{\text{energy}})_{c/2}$ for all $t \in [0,T]$. In addition, this solution is unique in this class.*

Let us explain the main strategy for the proof of Theorem 7.1. First, we observe that the equation for $\vartheta$ can be rewritten as

$$\begin{aligned}\partial_t\vartheta + u\cdot\nabla_x\vartheta + \frac{\vartheta+\pi(\vartheta)}{1-\rho_f}\,\mathrm{div}_x(j_f-\rho_f u) \\ = -\frac{\vartheta+\pi(\vartheta)}{1-\rho_f}\,\mathrm{div}_x\, u + \frac{1}{1-\rho_f}\int_{\mathbb{R}^d}|v-u|^2 f\,\mathrm{d}v.\end{aligned} \tag{7.2}$$

Apart from the last term, this equation has exactly the same structure as that for $\varrho$ in Lemma 2.2. Hence, the following estimate holds:

$$\|\vartheta(t)\|_{\mathrm{H}^m} \leq \|\varrho^{\text{in}}\|_{\mathrm{H}^m}\,\Phi\big(T,\dots,\|u\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^{m+1})},\|f\|_{\mathrm{L}^\infty(0,T;\mathcal{H}^{m+1}_r)}\big),$$

and it features the same loss of derivative as for $\varrho$ in (TS). Note also that the equation for $\varrho$ can be directly solved once $f$ and $u$ are given. As in Section 2.2, we consider the regularization $-\pi'(\vartheta)\mathrm{J}_\varepsilon\nabla_x\vartheta$ of the term $-\nabla_x\pi(\vartheta)$ in the kinetic equation of ($\mathrm{TS}_e$) and introduce the quantity

$$\mathbf{N}_{m,r}(f_\varepsilon,\varrho_\varepsilon,u_\varepsilon,\vartheta_\varepsilon,T):=\|f_\varepsilon\|_{\mathrm{L}^\infty(0,T;\mathcal{H}^{m-1}_r)}+\|\varrho_\varepsilon\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^{m-1})}$$
$$+\|u_\varepsilon\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^m)\cap\mathrm{L}^2(0,T;\mathrm{H}^{m+1})}+\|\vartheta_\varepsilon\|_{\mathrm{L}^2(0,T;\mathrm{H}^m)}.$$

Following the bootstrap procedure we have set for the case of (TS), we mainly want to control the quantity $\|\vartheta_\varepsilon\|_{\mathrm{L}^2(0,T;\mathrm{H}^m)}$.

Using (7.2) (and dropping the dependence on $\varepsilon$) it is possible to obtain the following equation for $h=\partial_x^\alpha\vartheta$ with $|\alpha|\leq m$ (see Proposition 5.1):

$$\left(\mathrm{Id}-\frac{\vartheta+\pi(\vartheta)}{1-\rho_f}\mathrm{K}_G^{\mathrm{free}}\circ\mathrm{J}_\varepsilon\right)\big[\partial_t h+u\cdot\nabla_x h\big]=\mathcal{R},\quad t\in(0,T),$$

with $G(t,x,v)=\pi'(\vartheta(t,x))\nabla_v f(t,x,v)$ and

$$\|\mathcal{R}\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{T}^d))}\leq\Lambda(T,R,\|h(0)\|_{\mathrm{H}^1(\mathbb{T}^d)}).$$

Following the arguments of Chapter 5, we are thus led to the study of the pseudodifferential equation

$$H-\frac{\pi'(\vartheta)(\vartheta+\pi(\vartheta))}{1-\rho_f}\mathrm{Op}^\gamma(a_f)(\mathrm{J}_\varepsilon H)=\mathcal{R}\quad\text{on }(0,T)\times\mathbb{T}^d,$$

where $a_f$ is defined in (5.6), that is,

$$\mathrm{Op}^{\gamma,\varepsilon}\big(1-\mathbf{P}_{f,\vartheta}^{\mathrm{energy}}\big)(H)=\mathcal{R}.$$

In particular, this explains the introduction of the Penrose symbol (7.1) above, which allows us to invert the previous equation for $H$, up to a small remainder.

Some additional arguments also need to be given to treat the last term in (7.2). Since

$$\int_{\mathbb{R}^d}|v-u|^2 f\,\mathrm{d}v=\int_{\mathbb{R}^d}|v|^2 f\,\mathrm{d}v+|u|^2\rho_f-2u\cdot j_f,$$

we have to include the treatment of the second order moment in velocity $m_2 f(t,x):=\int_{\mathbb{R}^d}|v|^2 f(t,x,v)\,\mathrm{d}v$ in the analysis of Chapter 4. In addition to Proposition 4.1, we have the following result.

**Proposition 7.2.** *For all $|I|\leq m$, we have, for any $t\in(0,T)$,*

$$\partial_x^I m_2 f(t,x)$$
$$=p'(\varrho(t,x))\int_0^t\int_{\mathbb{R}^d}|v|^2\nabla_x[\mathrm{J}_\varepsilon\partial_x^I\varrho](s,x-(t-s)v)\cdot\nabla_v f(t,x,v)\,\mathrm{d}v\,\mathrm{d}s$$
$$+R^I[m_2 f](t,x),$$

*where the remainder* $R^I[m_2 f]$ *satisfies*

$$\left\| R^I[m_2 f] \right\|_{\mathrm{L}^2(0,T;\mathrm{H}^1_x)} \leq \Lambda(T,R).$$

*In particular, we have*

$$\left\| \partial_x^I \int_{\mathbb{R}^d} |v-u|^2 f \,\mathrm{d}v \right\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)} \leq \Lambda(T,R).$$

The fairly straightforward adaptation of the analysis of Chapter 4 is left to the reader.

## 7.2 Generalization to the inelastic Boltzmann case

In this section, we consider the case where one takes into account inelastic collisions between particles. This corresponds to the following Vlasov–Boltzmann equation in the coupling (the other equations for $(\varrho, u)$ being unchanged):

$$\text{(TS-Coll)} \quad \begin{cases} \partial_t f + v \cdot \nabla_x f + \mathrm{div}_v\big[f(u-v) - f\nabla_x p(\varrho)\big] = \mathcal{Q}_\lambda(f,f), \\ \partial_t(\alpha\varrho) + \mathrm{div}_x(\alpha\varrho u) = 0, \\ \partial_t(\alpha\varrho u) + \mathrm{div}_x(\alpha\varrho u \otimes u) + \alpha\nabla_x p - \Delta_x u - \nabla_x \mathrm{div}_x u = j_f - \rho_f u, \end{cases}$$

where $\mathcal{Q}_\lambda(f,f)$ stands for a quadratic collision operator of Boltzmann type, in an inelastic hard-spheres regime. Here, the fixed parameter $\lambda \in (0,1)$ corresponds to the so-called *restitution coefficient*: if $'v$ and $'v_\star$ denote the velocities of two particles before collision, their respective velocities $v$ and $v_\star$ after collision are given by

$$\begin{cases} v = {}'v - \dfrac{1+\lambda}{2}({}'u \cdot n)n, \\ v_\star = {}'v_\star + \dfrac{1+\lambda}{2}({}'u \cdot n)n, \end{cases}$$

where $'u := {}'v - {}'v_\star$ is the relative pre-collision velocity and $n \in \mathbb{S}^{d-1}$ is a unit vector that points from the particle center with velocity $v$ to the particle center with velocity $v_\star$ at the impact. Note that $\lambda = 1$ corresponds to the standard elastic case.

In this representation, given two distribution functions $f = f(v)$ and $g = g(v)$, we can consider the following expression for the Boltzmann collision operator, as a difference of a gain and a loss term,

$$\mathcal{Q}_\lambda(f,g) = \mathcal{Q}_\lambda^+(f,g) - \mathcal{Q}_\lambda^-(f,g),$$

where, setting $u = v - v_\star$ and $\widehat{u} = u/|u|$, we define

$$\begin{aligned} \mathcal{Q}_\lambda^+(f,g)(v) &= \frac{1}{\lambda^2}\int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} |u\cdot n|\, b(\widehat{u}\cdot n) f({}'v) g({}'v_\star)\,\mathrm{d}v_\star\,\mathrm{d}n, \\ \mathcal{Q}_\lambda^-(f,g)(v) &= f(v)\int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} |u\cdot n|\, b(\widehat{u}\cdot n) g(v_\star)\,\mathrm{d}v_\star\,\mathrm{d}n. \end{aligned} \tag{7.3}$$

Here, the function $b \in \mathrm{L}^1([-1,1])$ is a given angular collision kernel. For the sake of simplicity, we consider $b \equiv 1$. Note that, in the (truly) inelastic case $\lambda \in (0,1)$, we have

$$|v|^2 + |v_\star|^2 = |'v|^2 + |'v_\star|^2 - \frac{1-\lambda^2}{2}('u \cdot n)^2,$$

thus inducing a loss of kinetic energy at each collision, while mass and momentum are conserved. We refer to [135] (see also the introduction of [3]) for more details on this model, which comes from the theory of *granular media* and describes a cloud of macroscopic particles whose size is larger than that usually described by the standard Boltzmann equation with elastic collisions (for so-called molecular gases). To include dissipative effects, inelastic collisions are thus considered. Note that the presence of such a collision operator (with a large parameter $\varepsilon^{-1}$ in front of it) formally leads to a biphasic fluid model when starting from (TS-Coll) (see [61]). We also refer to [13, 58, 124] for its applications in the study of sprays.

Our main result (which also includes the elastic case $\lambda = 1$) reads as follows.

**Theorem 7.3.** *Let $\lambda \in (0,1]$. There exists $m_0 > 0$, depending only on the dimension, such that the following holds for all $m \geq m_0$. Let*

$$e^{|v|^2} f^{\mathrm{in}} \in \mathcal{H}^m_0, \quad \varrho^{\mathrm{in}} \in \mathrm{H}^{m+1}, \quad u^{\mathrm{in}} \in \mathrm{H}^m,$$

*such that $(f^{\mathrm{in}}, \varrho^{\mathrm{in}})$ satisfies the $c$-Penrose stability condition $(\mathrm{P})_c$ (with $c > 0$) and*

$$0 \leq f^{\mathrm{in}}, \quad \rho_{f^{\mathrm{in}}} < \Theta < 1, \quad 0 < \mu \leq \varrho^{\mathrm{in}}, \quad 0 < \underline{\theta} \leq (1 - \rho_{f^{\mathrm{in}}})\varrho^{\mathrm{in}} \leq \overline{\theta}$$

*for some fixed constants $\Theta$, $\mu$, $\underline{\theta}$, $\overline{\theta}$. Then there exist $T > 0$ and a solution $(f, \varrho, u)$ to (TS-Coll) with initial condition $(f^{\mathrm{in}}, \varrho^{\mathrm{in}}, u^{\mathrm{in}})$ such that*

$$\begin{aligned} e^{|v|^2} f \in \mathcal{C}\big([0,T]; \mathcal{H}^{m-1}_0\big), \quad \varrho \in \mathrm{L}^2(0,T;\mathrm{H}^m), \\ u \in \mathcal{C}\big([0,T]; \mathrm{H}^m\big) \cap \mathrm{L}^2(0,T;\mathrm{H}^{m+1}), \end{aligned}$$

*with $(f(t), \varrho(t))$ satisfying the $\frac{c}{2}$-Penrose stability condition $(\mathrm{P})_{c/2}$ for all $t \in [0,T]$. In addition, this solution is unique in this class.*

As seen below, the friction term in the kinetic equation comes in handy in order to treat some of the new terms due to the collision operator. This was remarked already in [115].

Let us present the main changes that must be considered in our strategy of proof and that are due to the collision operator. It mainly concerns:

- the energy estimates for $f$ from Chapter 2;
- the integro-differential system for the derivatives of $f$ from Section 4.1.

The rest of Chapter 2 and of Chapter 4 then remains unchanged, as do Chapters 5 and 6.

### 7.2.1 New energy estimates for the kinetic part

Let us focus on the energy estimates for the new kinetic equation

$$\partial_t f + v \cdot \nabla_x f + \mathrm{div}_v[f(u-v) - f\nabla_x p(\varrho) f] = \mathcal{Q}_\lambda(f, f). \tag{7.4}$$

Following [115], we first define $g(t,x,v) = e^{|v|^2} f(t,x,v)$ with $f$ solving (7.4). Setting

$$E^{u,\varrho} = u - \nabla_x p(\varrho),$$

it implies that $g$ satisfies the following modified Vlasov–Boltzmann equation:

$$\partial_t g + v \cdot \nabla_x g + \mathrm{div}_v[(E^{u,\varrho} - v)g] - 2v \cdot (E^{u,\varrho} - v)g = \Gamma_\lambda[g,g], \tag{7.5}$$

where, for all functions $h_1(v), h_2(v)$, the operator $\Gamma_\lambda[g,g] = \Gamma_\lambda^+[g,g] - \Gamma_\lambda^-[g,g]$ is defined via

$$\Gamma_\lambda^+[h_1,h_2](v) := \frac{1}{\lambda^2}\int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} e^{-|v_\star|^2 - \frac{1-\lambda^2}{2}(('v-'v_\star)\cdot n)^2} |u \cdot n|\, h_1('v) h_2('v_\star)\, \mathrm{d}v_\star\, \mathrm{d}n,$$

$$\Gamma_\lambda^-[h_1,h_2](v) := h_2(v)\int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} e^{-|v_\star|^2} |u \cdot n|\, h_1(v_\star)\, \mathrm{d}v_\star\, \mathrm{d}n.$$

Note that the additional term $2v \cdot (E^{u,\varrho} - v)g$ comes from the friction term in (7.4). Using

$$'u = 'v_\star - 'v = v_\star - v - (1+\lambda)('u \cdot n)n = v_\star - v + (1+\lambda)\lambda(u\cdot n)n,$$

we note that for all $\lambda \in (0,1)$ there exists a constant $c(\lambda) > 0$ (and $c(1) = 0$) such that

$$\Gamma_\lambda^+[h_1,h_2] := \frac{1}{\lambda^2}\int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} e^{-|v_\star|^2 - c(\lambda)((v-v_\star)\cdot n)^2} |(v-v_\star)\cdot n|\, h_1('v) h_2('v_\star)\, \mathrm{d}v_\star\, \mathrm{d}n.$$

The exponential inside the integral is roughly behaving as $e^{-|v_\star|^2 - c(\lambda)|v-v_\star|^2}$. We have the following bilinear estimates on the previous collision operators, where some loss of weights in velocity classically shows up.

**Lemma 7.4.** *There exists $s = s(d) > 0$ large enough such that, for all $\sigma \geq 0$, we have, for any smooth nonnegative function $g = g(x,v)$,*

$$\sum_{|\alpha|+|\beta|\leq s} \int_{\mathbb{T}^d}\int_{\mathbb{R}^d} \langle v\rangle^{2\sigma} \partial_x^\alpha \partial_v^\beta [\Gamma_\lambda(g,g)] \partial_x^\alpha \partial_v^\beta g\, \mathrm{d}x\, \mathrm{d}v \lesssim \|g\|_{\mathcal{H}_\sigma^s}^2 \|g\|_{\mathcal{H}_{\sigma+1}^s}.$$

*Proof.* We refer to [115, Lemma 2.3] combined with [3, Proof of Theorem A.1]. ■

The key estimate allowing us to recover the previous loss of weight then comes from the following lemma bearing on the extra term $2v \cdot (E^{u,\varrho} - v)g$.

**Lemma 7.5.** *Let $s \geq 0$ and $\sigma > 0$. For any smooth nonnegative function $g = g(x, v)$ and $\delta \in (0, 1)$, we have*

$$\sum_{|\alpha|+|\beta|\leq s} \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} \langle v\rangle^{2\sigma} \partial_x^\alpha \partial_v^\beta \big[2v \cdot (E^{u,\varrho} - v)g\big] \partial_x^\alpha \partial_v^\beta g \,\mathrm{d}x\,\mathrm{d}v$$
$$\lesssim -(1-\delta)\, \|g\|^2_{\mathcal{H}^s_{\sigma+1}} + \left(1 + \frac{1}{\delta}\|E^{u,\varrho}\|^2_{\mathrm{H}^s}\right)\|g\|^2_{\mathcal{H}^s_\sigma}.$$

*Proof.* We refer to [115, Lemma 2.7]. ∎

Let us show how one can now obtain an *a priori* energy estimate for a solution $g$ to (7.5), that reads

$$\mathcal{T}^{u,\varrho}(g) - 2v \cdot (E^{u,\varrho} - v)g = \Gamma_\lambda(g, g),$$

where

$$\mathcal{T}^{u,\varrho} = \partial_t + v \cdot \nabla_x - v \cdot \nabla_v + E^{u,\varrho}(t, x) \cdot \nabla_v - d\,\mathrm{Id}.$$

The result is the following.

**Lemma 7.6.** *For all $r \geq 0$, $m > 3 + d/2$, $c > 0$ and $T > 0$, and for all smooth functions $(f, \varrho, u)$ satisfying*

$$\partial_t g + v \cdot \nabla_x g - v \cdot \nabla_v g + E^{u,\varrho}(t, x) \cdot \nabla_v - 2v \cdot (E^{u,\varrho} - v)g - dg$$
$$= \Gamma_\lambda(g, g) \quad \text{on } [0, T]$$

*and $\varrho \geq c$ on $[0, T]$, the following holds for all $t \in [0, T]$:*

$$\mathcal{E}_{m,\sigma}[g(t)] \leq \|g(0)\|^2_{\mathcal{H}^m_\sigma} \exp\Big[C\Big(\big(1 + \|u\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^m)} + \|\mathcal{E}_{m,\sigma}[g]\|_{\mathrm{L}^\infty(0,T)}\big)T$$
$$+ \sqrt{T}\Lambda\big(\|\varrho\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^{m-2})}\big)\|\varrho\|_{\mathrm{L}^2(0,T;\mathrm{H}^{m+1})}\Big)\Big] \tag{7.6}$$

*for some universal constant $C > 0$, where*

$$\mathcal{E}_{m,\sigma}[g(t)] := \|g(t)\|^2_{\mathcal{H}^m_\sigma} + \frac{1}{4}\int_0^t \|g(\tau)\|^2_{\mathcal{H}^m_{\sigma+1}}\,\mathrm{d}\tau.$$

*Proof.* By Lemma 2.9, we first have

$$\mathcal{T}^{u,\varrho}(\partial_x^\alpha \partial_v^\beta g) = -\sum_{\substack{i=1\\ \beta_i\neq 0}}^{d} \big(\partial_x^{\widehat{\alpha}^i}\partial_v^{\overline{\beta}^i} g - \partial_x^\alpha\partial_v^\beta g\big) - \big[\partial_x^\alpha \partial_v^\beta, E^{u,\varrho}(t, x) \cdot \nabla_v\big]g$$
$$+ \partial_x^\alpha \partial_v^\beta[2v \cdot (E^{u,\varrho} - v)g] + \partial_x^\alpha \partial_v^\beta[\Gamma_\lambda(f, f)].$$

The analog of inequality (2.3) from the proof of Proposition 2.10 is, thanks to Lemmas 7.4 and 7.5, for all $\delta \in (0,1)$,

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\|g(t)\|^2_{\mathcal{H}^m_\sigma} + (1-\delta)\|g(t)\|^2_{\mathcal{H}^m_{\sigma+1}} \\
&\quad \lesssim \left(1 + \|E^{u,\varrho}(t)\|_{\mathrm{H}^m} + \frac{1}{\delta}\|E^{u,\varrho}(t)\|^2_{\mathrm{H}^m}\right)\|g(t)\|^2_{\mathcal{H}^m_\sigma} + \|g(t)\|^2_{\mathcal{H}^m_\sigma}\|g(t)\|_{\mathcal{H}^m_{\sigma+1}} \\
&\quad \lesssim \left(1 + \|E^{u,\varrho}(t)\|_{\mathrm{H}^m} + \frac{1}{\delta}\|E^{u,\varrho}(t)\|^2_{\mathrm{H}^m}\right)\|g(t)\|^2_{\mathcal{H}^m_\sigma} \\
&\qquad + \frac{2}{1-\delta}\|g(t)\|^4_{\mathcal{H}^m_\sigma} + \frac{1-\delta}{2}\|g(t)\|^2_{\mathcal{H}^m_{\sigma+1}}.
\end{aligned}$$

Therefore, after absorbing the last term with $\delta = 1/2$, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_{m,\sigma}[g(t)] \lesssim \Big(1 + \|E^{u,\varrho}(t)\|^2_{\mathrm{H}^m} + \mathcal{E}_{m,\sigma}[g(t)]\Big)\mathcal{E}_{m,\sigma}[g(t)].$$

Concluding as in the proof of Proposition 2.10, we finally obtain the result. ■

The bootstrap argument from Section 2.2 is then carried out with the quantity

$$\begin{aligned}
&\mathcal{N}_{m,r}(g,\varrho,u,T) \\
&\qquad := \|\mathcal{E}_{m-1,\sigma}[g]\|_{\mathrm{L}^\infty(0,T)} + \|\varrho\|_{\mathrm{L}^2(0,T;\mathrm{H}^m)} + \|u\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^m)\cap \mathrm{L}^2(0,T;\mathrm{H}^{m+1})},
\end{aligned}$$

instead of $\mathcal{N}_{m,r}(f,\varrho,u,T)$, by considering the same regularization $\mathrm{J}_\varepsilon \nabla_x \varrho$ in the equation (7.5). Taking into account (7.6) and the quantity $\mathcal{N}_{m,r}(g,\varrho,u,T)$, the content of Chapter 2 can be modified accordingly. Concerning the local-in-time existence for the regularized system from Appendix B, we just modify the kinetic part of the scheme of approximation:

$$\begin{cases}
\partial_t g^{n+1} + v\cdot\nabla_x g^{n+1} + \mathrm{div}_v[(E^n(t,x)-v)g^{n+1}] - 2v\cdot(E^{u,\varrho}_{\mathrm{reg},\varepsilon} - v)g^{n+1} \\
\qquad\qquad = \Gamma^+_\lambda[g^n,g^n] - \Gamma^-_\lambda[g^n,g^{n+1}], \\
f^{n+1}{}_{|t=0} = f^{\mathrm{in}},
\end{cases}$$

where $\varepsilon > 0$ is given. We refer to [115] for more details.

### 7.2.2 Equation for the augmented variable $\mathcal{F}$

Let us highlight the main changes that occur at the end of Section 4.1, more precisely in Notation 4.5. Firstly, recall that the goal is to consider a new unknown $\mathcal{F} = \big(\partial_x^K\partial_v^K f\big)_{|K|+|L|\in\{m-1,m\}}$. Here, one has to consider a new coupling matrix

$$\mathbb{M} = \big(\mathbb{M}_{(I,J),(K,L)}\big),$$

which takes into account the collision operator $\mathcal{Q}_\lambda$ and is defined by

$$\mathbb{M}_{(I,J),(K,L)} := \mathcal{M}_{(I,J),(K,L)} + \mathcal{M}^{\mathcal{Q}_\lambda}_{(I,J),(K,L)},$$

with $|I| + |J|, |K| + |L| \in \{m-1, m\}$, where

- $\mathcal{M}_{(I,J),(K,L)} \in \mathbb{R}$ stands for the former terms of the coupling matrix already appearing in Notation 4.5;
- $\mathcal{M}^{\mathcal{Q}_\lambda}_{(I,J),(K,L)}$ is a new operator term coming from the collision operator, defined by the relation

$$\begin{aligned}\partial_x^I \partial_v^J \mathcal{Q}_\lambda(f,f) &= \sum_{\substack{0\le\alpha\le I\\ 0\le\beta\le J}} \binom{I}{\alpha}\binom{J}{\beta} \mathcal{Q}_\lambda\big(\partial_x^\alpha \partial_v^\beta f, \partial_x^{I-\alpha}\partial_v^{J-\beta} f\big)\\ &= \sum_{|K|+|L|\in\{m-1,m\}} \mathcal{M}^{\mathcal{Q}}_{(I,J),(K,L)}\big[\partial_x^K \partial_v^L f\big],\end{aligned}$$

that is,

$$\mathcal{M}^{\mathcal{Q}_\lambda}_{(I,J),(K,L)}(\cdot) := \sum_{\substack{0\le\alpha\le I\\ 0\le\beta\le J\\ |\alpha|+|\beta|\le 1}} \binom{I}{\alpha}\binom{J}{\beta} \mathbf{1}_{\substack{K=I-\alpha\\ L=J-\beta}} \mathcal{Q}_\lambda\big(\partial_x^\alpha \partial_v^\beta f, \cdot\big).$$

Hence, $\mathcal{M}^{\mathcal{Q}_\lambda}$ is a matrix with operator coefficients, acting on

$$\mathcal{F} = \big(\partial_x^K \partial_v^K f\big)_{|K|+|L|\in\{m-1,m\}}.$$

Using the same notation as in Section 4.1, we obtain the following equation for $\mathcal{F} = \big(\partial_x^K \partial_v^K f\big)_{|K|+|L|\in\{m-1,m\}}$ (see (4.3)):

$$\mathcal{T}^{u,\varrho}_{\mathrm{reg},\varepsilon}\mathcal{F} + \mathbb{M}\mathcal{F} + \mathcal{L} = -\mathcal{R}_0 - \mathcal{R}_1.$$

After the composition by $(t,x,v) \mapsto (t, \mathrm{X}^{t;0}(x,v), \mathrm{V}^{t;0}(x,v))$, where $\mathrm{Z} = (\mathrm{X},\mathrm{V})$ is the solution to (4.4), the following equation holds:

$$\partial_t \widetilde{\mathcal{F}} + \widetilde{\mathbb{M}}\widetilde{\mathcal{F}} + \widetilde{\mathcal{L}} = d\widetilde{\mathcal{F}} - \widetilde{\mathcal{R}}_0 - \widetilde{\mathcal{R}}_1,$$

with $\widetilde{g}(t,x,v) = g(t, \mathrm{X}^{t;0}(x,v), \mathrm{V}^{t;0}(x,v))$. We can still consider the resolvent associated to the previous operator $\mathbb{M} - d\,\mathrm{Id}$, that is, the solution $s \mapsto \mathfrak{N}^{s,t}(x,v)$ of

$$\begin{cases}\partial_s \mathfrak{N}^{s;t} + \big[\mathbb{M}\circ \mathrm{Z}^{s;0} - d\,\mathrm{Id}\big]\mathfrak{N}^{s;t} = 0,\\ \mathfrak{N}^{t;t} = \mathrm{Id},\end{cases}$$

whose existence and uniqueness is still provided by the Cauchy–Lipschitz theorem. Hence, this shows that the contribution of the collision operator $\mathcal{Q}_\lambda$ can be handled by the modified operator $\mathbb{M}$. The strategy of proof followed in the rest of Chapter 4 applies *mutatis mutandis*.

## 7.3 Generalization to the density-dependent drag case

In this section, we show how one can deal with the case of density-dependent drag in the force acting on the particles, that is, with the notation of the introduction, when the force in the kinetic equation is

$$\Gamma(t,x,v) = \varrho(t,x)(u(t,x)-v) - \nabla_x[p(\varrho)](t,x).$$

This additional factor also appears in the feedback of particles on the fluid, that is, in the source term in the Navier–Stokes equations. We are thus led to consider the following system:

$$\begin{cases} \partial_t f + v\cdot\nabla_x f - \nabla_x p\cdot\nabla_v f + \mathrm{div}_v[f\varrho(u-v) - f\nabla_x p(\varrho)] = 0, \\ \partial_t(\alpha\varrho) + \mathrm{div}_x(\alpha\varrho u) = 0, \\ \partial_t(\alpha\varrho u) + \mathrm{div}_x(\alpha\varrho u\otimes u) + \alpha\nabla_x p - \Delta_x u - \nabla_x \mathrm{div}_x u = \varrho(j_f - \rho_f u), \\ \alpha = 1-\rho_f. \end{cases} \tag{7.7}$$

Our main result on local well-posedness is the following.

**Theorem 7.7.** *Consider the same assumptions of Theorem 1.6. Assume also that $f^{\mathrm{in}}$ is compactly supported in velocity. Then the conclusion of Theorem 1.6 holds for the density-dependent drag case of system (7.7).*

To fix notation, let us assume that

$$\mathrm{supp}\, f^{\mathrm{in}} \subset \mathbb{T}^d \times \mathrm{B}(0, M^{\mathrm{in}}), \quad M^{\mathrm{in}} > 0. \tag{7.8}$$

In what follows, we shall only present the main modifications that have to be added to the strategy used in this work.

### 7.3.1 Modification of the energy estimates and of the bootstrap argument

Our first goal is to adapt the energy estimates of Chapter 2 and the bootstrap procedure to the density-dependent drag case. The main change comes from the estimate for $f$ from Proposition 2.10.

Indeed, let us set

$$\mathcal{T}^{u,\varrho}_{\mathrm{drag}} = \partial_t + v\cdot\nabla_x - \varrho v\cdot\nabla_v + E^{u,\varrho}(t,x)\cdot\nabla_v - d\varrho\mathrm{Id},$$

where $E^{u,\varrho}_{\mathrm{drag}} := \varrho u - \nabla_x p(\varrho)$. Observe that because of the term $\varrho v\cdot\nabla_v$ (coming from friction), a growth in velocity can occur in the analysis, that is, if $f$ is controlled in $\mathcal{H}^m_r$, then this term would *a priori* require a control in $\mathcal{H}^m_{r+1}$. This explains the additional assumption of compact support in velocity in the next proposition. We mention though that the use of exponential-weighted norms in velocity could relax this assumption (see the work [9] on the Vlasov–Maxwell equations, and also [47] in the context of fluid-kinetic equations).

**Proposition 7.8.** *For all* $r \geq 0$, $m > 3 + d/2$, $c > 0$, $T > 0$ *and all smooth functions* $(f, \varrho, u)$ *with* $f$ *having a compact support in velocity:*

$$\forall t \in [0,T], \quad \operatorname{supp} f(t) \subset \mathbb{T}^d \times \mathrm{B}(0, M(t))$$

*for some* $M \in \mathrm{L}^\infty(0,T)$*, satisfying*

$$\mathcal{T}^{u,\varrho}_{\text{drag}}(f) = 0 \quad \text{on } [0,T]$$

*and* $\varrho \geq c$ *on* $[0,T]$*, the following holds. For all* $t \in [0,T]$*, we have*

$$\begin{aligned}\|f(t)\|^2_{\mathcal{H}^m_r} \leq \|f(0)\|^2_{\mathcal{H}^m_r} \exp\Big[C\big(1 + \|M\|_{\mathrm{L}^\infty(0,T)}\big) \\ \times \Big(T + \sqrt{T}\|u\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^m)}\|\varrho\|_{\mathrm{L}^2(0,T;\mathrm{H}^m)} \\ + \sqrt{T}\Lambda\big(\|\varrho\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^{m-2})}\big)\|\varrho\|_{\mathrm{L}^2(0,T;\mathrm{H}^{m+1})}\Big)\Big].\end{aligned}$$

*Proof.* Let us suppose that there exists $M \in \mathrm{L}^\infty(0,T)$ such that

$$\forall t \in [0,T], \quad \operatorname{supp} f(t) \subset \mathbb{T}^d \times \mathrm{B}(0, M(t)).$$

Since $\mathcal{T}^{u,\varrho}_{\text{drag}}(f) = 0$, we have, by Lemma 2.9,

$$\begin{aligned}\mathcal{T}^{u,\varrho}_{\text{drag}}\big(\partial^\alpha_x \partial^\beta_v f\big) = -\sum_{\substack{i=1\\ \beta_i \neq 0}}^{d} \partial^{\widehat{\alpha}^i}_x \partial^{\widehat{\beta}^i}_v f + \big[\partial^\alpha_x \partial^\beta_v, \varrho(t,x) v \cdot \nabla_v\big] \\ - \big[\partial^\alpha_x \partial^\beta_v, E^{u,\varrho}_{\text{drag}}(t,x)\cdot\nabla_v\big] f + d\big[\partial^\alpha_x \partial^\beta_v, \varrho \mathrm{Id}\big] f\end{aligned}$$

for all $\alpha, \beta \in \mathbb{N}^d$. We take the scalar product of this equality with $(1+|v|^2)^r \partial^\alpha_x \partial^\beta_v f$, sum for all $|\alpha| + |\beta| \leq m$ and then integrate on $\mathbb{T}^d \times \mathbb{R}^d$. For the left-hand side, we have, as in Proposition 2.10,

$$\begin{aligned}&\sum_{|\alpha|+|\beta|\leq m} \int_{\mathbb{T}^d\times\mathbb{R}^d} (1+|v|^2)^r \mathcal{T}^{u,\varrho}_{\text{drag}}\big(\partial^\alpha_x \partial^\beta_v f\big) \partial^\alpha_x \partial^\beta_v f \\ &\quad = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|f(t)\|^2_{\mathcal{H}^m_r} - \sum_{|\alpha|+|\beta|\leq m} \int_{\mathbb{T}^d\times\mathbb{R}^d} \nabla_v (1+|v|^2)^r \cdot \big(E^{u,\varrho}_{\text{drag}} - \varrho v\big) \frac{|\partial^\alpha_x \partial^\beta_v f|^2}{2},\end{aligned}$$

the last term satisfying

$$\begin{aligned}\sum_{|\alpha|+|\beta|\leq m} \int_{\mathbb{T}^d\times\mathbb{R}^d} \nabla_v (1+|v|^2)^r \cdot \big(E^{u,\varrho}_{\text{drag}} - \varrho v\big) \frac{|\partial^\alpha_x \partial^\beta_v f|^2}{2} \\ \leq \big(\|\varrho(t)\|_{\mathrm{L}^\infty} + \|E^{u,\varrho}_{\text{drag}}(t)\|_{\mathrm{L}^\infty}\big)\|f(t)\|^2_{\mathcal{H}^m_r}.\end{aligned}$$

We now look at the four terms on the right-hand side. For the first, third and fourth ones, we proceed as in the proof of Proposition 2.10 (with a variant of (A.3)) and get

$$\sum_{|\alpha|+|\beta|\leq m} \int_{\mathbb{T}^d\times\mathbb{R}^d} (1+|v|^2)^r \times \left( -\sum_{\substack{i=1\\ \beta_i\neq 0}}^{d} \partial_x^{\widehat{\alpha}^i} \partial_v^{\widehat{\beta}^i} f + \left[\partial_x^\alpha \partial_v^\beta, E^{u,\varrho}_{\mathrm{drag}}(t,x)\cdot\nabla_v + d\varrho \mathrm{Id}\right] f \right) \partial_x^\alpha \partial_v^\beta f \lesssim \left(1+\|E^{u,\varrho}_{\mathrm{drag}}(t)\|_{\mathrm{H}^m} + \|\varrho(t)\|_{\mathrm{H}^m}\right)\|f(t)\|^2_{\mathcal{H}^m_r}.$$

The treatment of the third term requires the use of the compact support in velocity of $f$. Invoking the inequality (A.4), we have

$$\sum_{|\alpha|+|\beta|\leq m} \int_{\mathbb{T}^d\times\mathbb{R}^d} (1+|v|^2)^r \left[\partial_x^\alpha \partial_v^\beta, \varrho(t,x) v\cdot\nabla_v\right] \partial_x^\alpha \partial_v^\beta f \lesssim (1+M(t))\|\varrho(t)\|_{\mathrm{H}^m}\|f(t)\|^2_{\mathcal{H}^m_r}.$$

All in all, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|f(t)\|^2_{\mathcal{H}^m_r} \lesssim (1+M(t))\left(1+\|\varrho(t)\|_{\mathrm{H}^m} + \|E^{u,\varrho}(t)\|_{\mathrm{H}^m}\right)\|f(t)\|^2_{\mathcal{H}^m_r} \tag{7.9}$$

if $m > d/2$. As in the proof of Proposition 2.10, and by Sobolev embedding, we have

$$\|E^{u,\varrho}(t)\|_{\mathrm{H}^m} \lesssim \|\varrho(t)\|_{\mathrm{H}^m}\|u(t)\|_{\mathrm{H}^m} + \Lambda\left(\|\varrho(t)\|_{\mathrm{H}^{m-2}}\right)\|\varrho(t)\|_{\mathrm{H}^{m+1}}.$$

By integrating in time the inequality (7.9), we get

$$\|f(t)\|^2_{\mathcal{H}^m_r} \leq \|f(0)\|^2_{\mathcal{H}^m_r} + C\int_0^t (1+M(s)) \times \left(1+\|\varrho(s)\|_{\mathrm{H}^m}\|u(s)\|_{\mathrm{H}^m} + \Lambda\left(\|\varrho\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^{m-2})}\right)\|\varrho(s)\|_{\mathrm{H}^{m+1}}\right)\|f(s)\|^2_{\mathcal{H}^m_r}\,\mathrm{d}s$$

for all $t\in[0,T)$ and for some constant $C>0$ independent of $\varepsilon$. Using the Cauchy–Schwarz inequality and Grönwall's inequality, this implies, for all $t\in[0,T)$,

$$\|f(t)\|^2_{\mathcal{H}^m_r} \leq \|f(0)\|^2_{\mathcal{H}^m_r}\exp\Big[C\left(1+\|M\|_{\mathrm{L}^\infty(0,T)}\right)\Big(T+\sqrt{T}\|u\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^m)}\|\varrho\|_{\mathrm{L}^2(0,T;\mathrm{H}^m)} + \sqrt{T}\Lambda\left(\|\varrho\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^{m-2})}\right)\|\varrho\|_{\mathrm{L}^2(0,T;\mathrm{H}^{m+1})}\Big)\Big],$$

and this concludes the proof. ∎

The proof of the other estimates from Chapter 2 is mainly unchanged and details are left to the reader. We need to adapt the bootstrap procedure, by taking into account the need of a compact support in velocity. We thus define the following modified condition.

**Definition 7.9.** Let $T > 0$. For any nonnegative functions $f(t,x,v)$ and $\varrho(t,x)$ on $[0,T]$, we define the following property:

$$(\mathrm{B}^{\mu,\theta}_{\Theta,M^{\mathrm{in}}}(T)) \quad \forall t \in [0,T], \quad \begin{cases} \rho_f(t) \leq \dfrac{\Theta+1}{2}, \ \dfrac{\mu}{2} \leq \varrho(t), \ \dfrac{\theta}{2} \leq (1-\rho_f(t))\varrho(t) \leq 2\overline{\theta}, \\ \operatorname{supp} f(t) \subset \mathbb{T}^d \times \mathrm{B}(0, 1+M^{\mathrm{in}}), \end{cases}$$

where $\Theta$, $\mu$, $\underline{\theta}$, $\overline{\theta}$ are given in the statement of Theorem 1.6 and $M^{\mathrm{in}}$ has been introduced in (7.8).

If $T^*_\varepsilon > 0$ is the maximal time of existence for the system ($\mathrm{S}_\varepsilon$) (with density-dependent drag term), we introduce the following time for all $\varepsilon > 0$:

$$T_\varepsilon = T_\varepsilon(R) := \sup\left\{T \in [0, T^*_\varepsilon),\ \mathcal{N}_{m,r}(f_\varepsilon, \varrho_\varepsilon, u_\varepsilon, T) \leq R \text{ and } (\mathrm{B}^{\mu,\theta}_{\Theta,M^{\mathrm{in}}}(T)) \text{ holds}\right\},$$

where $R > 0$ has to be chosen large enough and independent of $\varepsilon$. Here, the quantity $\mathcal{N}_{m,r}(f_\varepsilon, \varrho_\varepsilon, u_\varepsilon, T)$ is exactly the same as in Notation 2.19.

### 7.3.2 Modification in the straightening change of variable

As a matter of fact, the main difference in the analysis appears in the part related to characteristics, namely Section 2.3. Our purpose here is to explain how to modify the arguments of Section 2.3 about the straightening change of variable in velocity, that is, Lemma 2.26. It turns out that, in the density-dependent drag case, obtaining a suitable diffeomorphism $\psi^{s;t}_x$ is not as straightforward as in Lemma 2.26. Indeed, again because of the term $-\varrho v \cdot \nabla_v f$, there could be a growth in velocity in the dynamics that prevents our proof of Lemma 2.26, which is based on a perturbative approach, from directly holding. This is where the assumption of compact support in velocity appears to be crucial; we do not know whether it is possible to replace it here by an exponential moment assumption.

With the notation of Section 2.3, we will actually directly straighten the total kinetic operator

$$\mathcal{T}_{\mathrm{drag,F}} = \partial_t + v \cdot \nabla_x - \varrho(t,x) v \cdot \nabla_v + \mathrm{F}(t,x) \cdot \nabla_v - d\,\mathrm{Id}$$

into the free-transport operator

$$\mathcal{T}^{\mathrm{free}} = \partial_t + v \cdot \nabla_x.$$

For $(x, v) \in \mathbb{T}^d \times \mathrm{B}(0, 1 + \mathrm{M}^{\mathrm{in}})$ and $t \in [0, T]$, we consider the solution

$$s \mapsto \big(\mathrm{X}^{s;t}_{\mathrm{drag}}, \mathrm{V}^{s;t}_{\mathrm{drag}}\big)(x, v)$$

to the following system of differential equations:

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}s}\mathrm{X}^{s;t}_{\mathrm{drag}} = \mathrm{V}^{s;t}_{\mathrm{drag}}, \quad \mathrm{X}^{t;t}_{\mathrm{drag}}(x, v) = x, \\ \dfrac{\mathrm{d}}{\mathrm{d}s}\mathrm{V}^{s;t}_{\mathrm{drag}} = -\varrho\big(s, \mathrm{X}^{s;t}_{\mathrm{drag}}\big)\mathrm{V}^{s;t}_{\mathrm{drag}} + \mathrm{F}\big(s, \mathrm{X}^{s;t}_{\mathrm{drag}}\big), \quad \mathrm{V}^{t;t}_{\mathrm{drag}}(x, v) = v. \end{cases}$$

**Lemma 7.10.** *Let $T > 0$ and $k \geq 1$. Let $\mathrm{F} \in \mathrm{L}^2(0, T; \mathrm{W}^{k,\infty}(\mathbb{T}^d))$ be a vector field such that*

$$\|\mathrm{F}\|_{\mathrm{L}^2(0,T;\mathrm{W}^{k,\infty}(\mathbb{T}^d))} \leq \Lambda(T, \mathrm{R})$$

*for some $\mathrm{R} > 0$. Then there exists $\overline{T}(\mathrm{R}) > 0$ such that, for all $x \in \mathbb{T}^d$ and all $s, t \in [0, \min(\overline{T}(\mathrm{R}), T)]$, there exists a diffeomorphism*

$$\psi_{s,t}(x, \cdot) : \mathrm{B}(0, 1 + \mathrm{M}^{\mathrm{in}}) \to \mathrm{B}(0, 1 + \mathrm{M}^{\mathrm{in}})$$

*satisfying, for all $v \in \mathrm{B}(0, 1 + \mathrm{M}^{\mathrm{in}})$,*

$$\mathrm{X}^{s;t}_{\mathrm{drag}}(x, \psi_{s,t}(x, v)) = x - (t - s)v,$$

*and which furthermore satisfies the estimates*

$$\frac{1}{C} \leq \det(\mathrm{D}_v \psi_{s,t}(x, v)) \leq C,$$

*and*

$$\sup_{s,t\in[0,T]} \left\| \partial^{\alpha}_{x,v} \left(\psi_{s,t}(x, v) - v\right) \right\|_{\mathrm{L}^\infty(\mathbb{T}^d\times\mathbb{R}^d)} \leq \varphi(T)\Lambda(T, \mathrm{R}), \quad |\alpha| \leq k,$$
$$\sup_{s,t\in[0,T]} \left\|\partial^{\beta}_{x,v}\partial_s\psi_{s,t}(x, v)\right\|_{\mathrm{L}^\infty(\mathbb{T}^d\times\mathbb{R}^d)} \leq \varphi(T)\Lambda(T, \mathrm{R}), \quad |\beta| \leq k - 1,$$

*for some $C > 0$ and some nondecreasing continuous function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ vanishing at 0.*

*Proof.* Let us sketch the proof. Dropping the $(x, v)$ dependence on the trajectories, we have

$$\mathrm{V}^{s;t}_{\mathrm{drag}} = \exp\left(\int_s^t \varrho\big(\tau, \mathrm{X}^{\tau;t}_{\mathrm{drag}}\big)\,\mathrm{d}\tau\right)v - \int_s^t \exp\left(\int_s^\tau \varrho\big(\tau', \mathrm{X}^{\tau';t}_{\mathrm{drag}}\big)\,\mathrm{d}\tau'\right)\mathrm{F}\big(\tau, \mathrm{X}^{\tau;t}_{\mathrm{drag}}\big)\,\mathrm{d}\tau,$$

from which we deduce that

$$\begin{aligned} \mathrm{X}^{s,t}_{\mathrm{drag}} = x - &\left(\int_s^t \exp\left(\int_{s'}^t \varrho\big(\tau, \mathrm{X}^{\tau;t}_{\mathrm{drag}}\big)\,\mathrm{d}\tau\right)\mathrm{d}s'\right)v \\ &+ \int_s^t \int_{s'}^t \exp\left(\int_{s'}^\tau \varrho\big(\tau', \mathrm{X}^{\tau';t}_{\mathrm{drag}}\big)\,\mathrm{d}\tau'\right)\mathrm{F}\big(\tau, \mathrm{X}^{\tau;t}_{\mathrm{drag}}\big)\,\mathrm{d}\tau\,\mathrm{d}s' \end{aligned}$$

$$= x - (t-s)\Big[v + \Big(\frac{1}{t-s}\int_s^t \exp\Big(\int_{s'}^t \varrho\big(\tau, \mathrm{X}_{\mathrm{drag}}^{\tau;t}\big)\,\mathrm{d}\tau\Big)\,\mathrm{d}s' - 1\Big)v$$
$$- \frac{1}{t-s}\int_s^t \int_{s'}^t \exp\Big(\int_{s'}^{\tau} \varrho\big(\tau', \mathrm{X}_{\mathrm{drag}}^{\tau';t}\big)\,\mathrm{d}\tau'\Big)\mathrm{F}\big(\tau, \mathrm{X}_{\mathrm{drag}}^{\tau;t}\big)\,\mathrm{d}\tau\,\mathrm{d}s'\Big].$$

Taking one derivative in velocity, we observe that because of the term

$$\Big(\frac{1}{t-s}\int_s^t \exp\Big(\int_{s'}^t \varrho\big(\tau, \mathrm{X}_{\mathrm{drag}}^{\tau;t}\big)\,\mathrm{d}\tau\Big)\,\mathrm{d}s' - 1\Big)v,$$

we obtain a term where the derivative is not falling on the $v$ factor. Hence, the latter is *a priori* unbounded, inducing a potential linear growth in velocity of the derivative. But since we restrict ourselves to $v \in \mathrm{B}(0, 1+\mathrm{M}^{\mathrm{in}})$, we can however deduce a rough bound on $\nabla_v \mathrm{X}_{\mathrm{drag}}^{s;t}$ by Grönwall's inequality, which takes the form

$$\begin{aligned}\big|\nabla_v \mathrm{X}_{\mathrm{drag}}^{s;t}\big| &\lesssim \exp(T\Lambda(T,R)(|v|+1))T\Lambda(T,R)\\ &\lesssim \exp(T\Lambda(T,R)(2+M^{\mathrm{in}}))T\Lambda(T,R).\end{aligned}$$

We can now check that for $T$ small enough, for all $0 \leq s, t \leq T$, the map

$$v \mapsto v + \Big(\frac{1}{t-s}\int_s^t \exp\Big(\int_{s'}^t \varrho\big(\tau, \mathrm{X}_{\mathrm{drag}}^{\tau;t}\big)\,\mathrm{d}\tau\Big)\,\mathrm{d}s' - 1\Big)v$$
$$- \frac{1}{t-s}\int_s^t \int_{s'}^t \exp\Big(\int_{s'}^{\tau} \varrho\big(\tau', \mathrm{X}_{\mathrm{drag}}^{\tau';t}\big)\,\mathrm{d}\tau'\Big)E_{\mathrm{drag}}^{u,\varrho}\big(\tau, \mathrm{X}_{\mathrm{drag}}^{\tau;t}\big)\,\mathrm{d}\tau\,\mathrm{d}s'$$

is a $\mathcal{C}^1$ diffeomorphism from $\mathrm{B}(0, 1+\mathrm{M}^{\mathrm{in}})$ onto its image, since it is a small Lipschitz perturbation of the identity map. Details are left to the reader. ∎

As a result, for small times, we can directly come down to the free-transport case.

### 7.3.3 Modifications in remainder terms and conclusion of the bootstrap

To conclude, let us point out the last main modifications that have to be made to conclude the bootstrap argument.

In Chapter 4, one shall be careful when handling the different remainder terms $\mathcal{R}_I^{\mathrm{Diff}}$, $\mathcal{R}_{I,1}^{\mathrm{Duha}}$, $\mathcal{R}_{I,2}^{\mathrm{Duha}}$, $\mathcal{R}_I$ in $\mathrm{L}^2(0,T;\mathrm{H}^1)$, because they now involve some terms with at most $m+1$ derivatives on $\varrho u$ stemming from the force field $E_{\mathrm{drag}}^{u,\varrho}$. It was somehow harmless in the linear-drag case because the corresponding terms involved only $u$, which has an additional regularity provided by the Navier–Stokes equation, namely a control in $\mathrm{L}^\infty(0,T;\mathrm{H}^m) \cap \mathrm{L}^2(0,T;\mathrm{H}^{m+1})$ (see the terms $\mathbf{S}_2$, $\mathbf{S}_5$, $\mathbf{S}_{12}$, $\mathbf{S}_{16}$). Since we only have a control of $\varrho$ in $\mathrm{L}^2(0,T;\mathrm{H}^m)$, this requires an additional argument.

The main idea is to rely on the same type of decomposition as in Lemma 4.13 combined with the use of the smoothing estimates from Chapter 3. The expression $\partial_x^K(\varrho u)$ $(K \in \mathbb{N}^d)$ will involve terms or sum of terms

- of the form
$$\varrho \partial_x^K u \quad \text{or} \quad \partial_x^{\overline{K}^i-\beta} u \nabla_x(\partial_x^\beta \varrho) \cdot e_i,$$
with $i = 1, \dots, d$ and $|\beta| = 0, \dots, \lfloor \frac{|K|-1}{2} \rfloor$. They are treated using $\mathrm{L}^\infty$ bounds on $\varrho$ (with Sobolev embeddings) and $\mathrm{L}^2$ bounds on $u$;
- of the form
$$\nabla_x(\partial_x^\beta \varrho) \cdot e_i \partial_x^{\overline{K}^i-\beta} u,$$
with $i = 1, \dots, d$ and $|\beta| = \lfloor \frac{|K|-1}{2} \rfloor + 1, \dots, |K| - 1$. These terms are addressed thanks to Proposition 3.4. We refer to Chapter 4 for the same kind of procedure for the estimates on remainders via the smoothing estimates of Chapter 3.

Let us briefly conclude by showing how the bootstrap procedure ends when one considers the condition ($\mathrm{B}^{\mu,\theta}_{\Theta,M^{\mathrm{in}}}(T)$). Namely, we have to show how to control the compact support in velocity for $f$ for short times. We rely on the Lagrangian structure of the equation, which implies a finite propagation in time of the support of $f$ in velocity, namely

$$\forall t > 0, \quad \operatorname{supp} f(t) \subset \mathbb{T}^d \times \mathrm{V}^{t;0}_{\mathrm{drag}}(\operatorname{supp} f^{\mathrm{in}}).$$

Since

$$\begin{aligned}\mathrm{V}^{t;0}_{\mathrm{drag}} &= \exp\left(-\int_0^t \varrho\big(\tau, \mathrm{X}^{\tau;0}_{\mathrm{drag}}\big)\,\mathrm{d}\tau\right) v \\ &\quad + \int_0^t \exp\left(-\int_\tau^t \varrho\big(\tau', \mathrm{X}^{\tau';0}_{\mathrm{drag}}\big)\,\mathrm{d}\tau'\right) E^{u,\varrho}_{\mathrm{drag}}\big(\tau, \mathrm{X}^{\tau;0}_{\mathrm{drag}}\big)\,\mathrm{d}\tau,\end{aligned}$$

the same kind of estimates on the trajectories as those performed in the proof of Lemma 7.10 give, for all $(x, v) \in \mathbb{T}^d \times \mathrm{B}_v(0, M^{\mathrm{in}})$,

$$\left|\mathrm{V}^{t;0}_{\mathrm{drag}}\right| \leq |v| + T\Lambda(T, R) \leq M^{\mathrm{in}} + T\Lambda(T, R).$$

As a consequence, choosing $T$ small enough (with respect to $R$) is sufficient to control the size of the support of $f(t)$ for short times.