Appendix A

Some classical (para-)differential inequalities on $\mathbb{T}^d$ and $\mathbb{T}^d \times \mathbb{R}^d$

We recall and state several classic inequalities of (para-)differential type. First, we have the following tame estimate for commutators (see [111, Lemma 3.4]).

Proposition A.1. *Let $s \geq 1$. There exists $C_s > 0$ such that, for any functions $g, E \in \mathrm{H}^s \cap \mathrm{L}^\infty$, we have*

$$\forall |\alpha| \leq s, \quad \big\| [\partial_x^\alpha, E] g \big\|_{\mathrm{L}^2} \leq C_s \big(\|\nabla_x E\|_{\mathrm{L}^\infty} \|g\|_{\mathrm{H}^{s-1}} + \|E\|_{\mathrm{H}^s} \|g\|_{\mathrm{L}^\infty} \big).$$

The following result is about tame estimates in Sobolev spaces (see e.g. [10, Corollary 2.86]).

Proposition A.2. *Let $s > 0$. There exists $C_s > 0$ such that, for all $w_1, w_2 \in \mathrm{H}^s \cap \mathrm{L}^\infty$, we have*

$$\|w_1 w_2\|_{\mathrm{H}^s_x} \leq C_s \big(\|w_1\|_{\mathrm{L}^\infty} \|w_2\|_{\mathrm{H}^s} + \|w_1\|_{\mathrm{H}^s} \|w_2\|_{\mathrm{L}^\infty} \big).$$

We also recall the following result of Bony about Sobolev continuity of the composition by a smooth function (see e.g. [10, Corollary 2.87] or [53, Proposition 1.4.8] for a more precise version).

Proposition A.3. *Let I be an open interval of $\mathbb{R}$ containing 0. Let $s > 0$ and let σ be the smallest integer such that $\sigma > s$. There exists $C_s > 0$ such that, for any $F \in \mathrm{W}^{\sigma+1,\infty}(I;\mathbb{R})$ with $F(0) = 0$ and for any $w \in \mathrm{H}^s$ taking values in $J \Subset I$, we have*

$$\|F(w)\|_{\mathrm{H}^s} \leq C_s (1 + \|w\|_{\mathrm{L}^\infty})^\sigma \|F'\|_{\mathrm{W}^{\sigma,\infty}(I)} \|w\|_{\mathrm{H}^s}.$$

Remark A.4. Note that if $0 \notin I$ and $w \in \mathrm{H}^s$ takes values in $J \Subset I$, we can extend F outside I by a smooth extension such that $F(0) = 0$, and the previous proposition remains valid. Indeed, by Faà di Bruno's formula, we observe that $\|F(w)\|_{\mathrm{H}^s}$ only involves F through its derivatives evaluated at w.

Lemma A.5. *For all $k \in \mathbb{N}$ and $g : \mathbb{T}^d \to \mathbb{R}^+$ such that $0 < c \leq g \leq C < 1$, we have*

$$\left\| \frac{1}{1-g} \right\|_{\mathrm{H}^k} \leq 1 + \mathrm{C}_k(\|g\|_{\mathrm{L}^\infty}) \|g\|_{\mathrm{H}^k}$$

for some nondecreasing continuous function $\mathrm{C}_k : \mathbb{R}^+ \to \mathbb{R}^+$.

Proof. We rely on Proposition A.3 by writing

$$\left\| \frac{1}{1-g} \right\|_{\mathrm{H}^k} = \|F(g) - F(0)\|_{\mathrm{H}^k} + 1,$$

and this directly concludes the proof. ∎

Let us finally state several product and commutator laws using weighted Sobolev norms.

Lemma A.6. *Let $s \geq 0$. Consider a smooth nonnegative function $\chi = \chi(v)$ such that $|\partial^\gamma \chi| \lesssim_\gamma \chi$ for all $\gamma \in \mathbb{N}^d$ such that $|\gamma| \leq s$.*

- *For any functions $f = f(x,v)$, $g = g(x,v)$ and any $k \geq s/2$, we have*

$$\|\chi f g\|_{\mathrm{H}^k_{x,v}} \lesssim \|f\|_{W^{k,\infty}_{x,v}} \|\chi g\|_{\mathrm{H}^s_{x,v}} + \|g\|_{W^{k,\infty}_{x,v}} \|\chi f\|_{\mathrm{H}^s_{x,v}}. \tag{A.1}$$

- *For any functions $a = a(x)$, $F = F(x,v)$ and any $s_0 > d$, we have*

$$\|\chi a F\|_{\mathrm{H}^s_{x,v}} \lesssim \|a\|_{\mathrm{H}^{s_0}_x} \|\chi F\|_{\mathrm{H}^s_{x,v}} + \|a\|_{\mathrm{H}^s_x} \|\chi F\|_{\mathrm{H}^s_{x,v}}. \tag{A.2}$$

- *For any vector field $E = E(x)$ and any function $f = f(x,v)$, the following holds for all $s_0 > 1+d$ and all $\alpha, \beta \in \mathbb{N}^d$ satisfying $|\alpha| + |\beta| = s \geq 1$:*

$$\left\|\chi\left[\partial_x^\alpha \partial_v^\beta, E(x) \cdot \nabla_v\right] f\right\|_{\mathrm{L}^2_{x,v}} \lesssim \|E\|_{\mathrm{H}^{s_0}_x} \|\chi f\|_{\mathrm{H}^s_{x,v}} + \|E\|_{\mathrm{H}^s_x} \|\chi f\|_{\mathrm{H}^s_{x,v}}. \tag{A.3}$$

- *For any functions $a = a(x)$, $f = f(x,v)$ such that f has a compact support in velocity, the following holds for all $s_0 > 1+d$ and all $\alpha, \beta \in \mathbb{N}^d$ satisfying $|\alpha| + |\beta| = s \geq 1$:*

$$\left\|\chi\left[\partial_x^\alpha \partial_v^\beta, a(x) v \cdot \nabla_v\right] f\right\|_{\mathrm{L}^2_{x,v}} \lesssim (1 + M_f)\|a\|_{\mathrm{H}^{s_0}_x} \|\chi f\|_{\mathrm{H}^s_{x,v}} + \|a\|_{\mathrm{H}^s_x} \|\chi f\|_{\mathrm{H}^s_{x,v}}, \tag{A.4}$$

where

$$\operatorname{supp} f \subset \mathbb{T}^d \times \mathrm{B}(0, M_f), \quad \mathrm{M}_f \in (0, +\infty).$$

Proof. We refer to [90, Lemma 3.1] for the proof of (A.1), (A.2) and (A.3). Let us briefly sketch the proof of (A.4). By expanding the commutator, we have to estimate terms of the form

$$I_{\gamma,\mu} = \left\|\chi \partial_x^\gamma a \partial_v^\mu (v_i) \partial_{v_i} \partial_x^{\alpha-\gamma} \partial_v^{\beta-\mu} f\right\|_{\mathrm{L}^2_{x,v}}$$

for some $1 \leq i \leq d$ and with $(\gamma, \mu) \neq (0,0)$, $\gamma \leq \alpha$, $\mu \leq \beta$ and $|\mu| < 2$. If $\gamma \neq 0$ and $\mu \neq 0$, then $I_{\gamma,\mu} = 0$ or $\partial_v^\mu(v_i) = 1$ and we can conclude as for (A.3). If $\gamma = 0$ and $\mu > 0$, then $\partial_v^\mu(v_i) = 0$ or 1 and we conclude by Sobolev embedding in x, since $I_{\gamma,\mu} \leq \|a\|_{\mathrm{L}^\infty_x} \|\chi f\|_{\mathrm{H}^s_{x,v}}$. Lastly, if $\gamma > 0$ and $\mu = 0$, we rely on the compact support in velocity for f to get

$$I_{\gamma,\mu} \leq M_f \left\|\chi \partial_x^\gamma a \partial_{v_i} \partial_x^{\alpha-\gamma} \partial_v^\beta f\right\|_{\mathrm{L}^2_{x,v}},$$

and we end the proof as for (A.3). ∎