Appendix B

Local well-posedness for (S_ε): Proof of Proposition 2.18

Recall that we want to find a solution $(f_\varepsilon, \varrho_\varepsilon, u_\varepsilon)$, with $\varepsilon > 0$, defined on some interval $[0, T_\varepsilon^*)$ to the system

$$\begin{cases} \partial_t f_\varepsilon + v \cdot \nabla_x f_\varepsilon - p'(\varrho_\varepsilon)\nabla_x\big[(I - \varepsilon^2\Delta_x)^{-1}\varrho_\varepsilon\big] \cdot \nabla_v f_\varepsilon + \mathrm{div}_v[f_\varepsilon(u_\varepsilon - v)] = 0, \\ \partial_t(\alpha_\varepsilon\varrho_\varepsilon) + \mathrm{div}_x(\alpha_\varepsilon\varrho_\varepsilon u_\varepsilon) = 0, \\ \partial_t u_\varepsilon + (u_\varepsilon \cdot \nabla_x)u_\varepsilon + \dfrac{1}{\varrho_\varepsilon}\nabla_x p(\varrho_\varepsilon) - \dfrac{1}{\varrho_\varepsilon(1-\rho_{f_\varepsilon})}(\Delta_x + \nabla_x \mathrm{div}_x)u_\varepsilon \\ \qquad\qquad\qquad = \dfrac{1}{\varrho_\varepsilon(1-\rho_{f_\varepsilon})}(j_{f_\varepsilon} - \rho_{f_\varepsilon}u_\varepsilon), \\ f_{\varepsilon|t=0} = f^{\mathrm{in}}, \quad \varrho_{\varepsilon|t=0} = \varrho^{\mathrm{in}}, \quad u_{\varepsilon|t=0} = u^{\mathrm{in}}, \end{cases}$$

where

$$\rho_{f_\varepsilon}(t,x) = \int_{\mathbb{R}^d} f_\varepsilon(t,x,v)\,\mathrm{d}v, \quad j_{f_\varepsilon}(t,x) = \int_{\mathbb{R}^d} f_\varepsilon(t,x,v)v\,\mathrm{d}v$$

and $\alpha_\varepsilon(t,x) = 1 - \rho_{f_\varepsilon}(t,x)$. To do so, we rely on a standard iterative scheme. We will derive two types of estimates on the inductive solutions to that scheme:

- first, a *uniform bound in high regularity* which allows us to obtain, through a weak compactness argument, a weak converging (sub)sequence in spaces with high regularity. This will be possible if we consider a small enough time of existence.
- second, *contraction estimates in low regularity* which aim at proving that this sequence of solutions is a Cauchy sequence in spaces with lower regularity, thus strongly converging.

This will be enough in order the pass to the limit in the iterative scheme, obtaining a solution with the aforementioned high order regularity. Note that the first type of estimates is actually required to prove the second one. Uniqueness will classically follow from the same computations leading to the contraction estimates.

We fix $\varepsilon > 0$ and now drop the dependence on ε (though we recall that the time of existence obtained below will still depend on ε): we set

$$f^0 = f^{\mathrm{in}}, \quad \varrho^0 = \varrho^{\mathrm{in}}, \quad u^0 = u^{\mathrm{in}},$$

and, for all $n \in \mathbb{N}$, given a triplet (f^n, u^n, ϱ^n) and a time $T_n > 0$ with

$$(f^n, \varrho^n, u^n) \in \mathcal{C}(0, T_n; \mathcal{H}_r^m) \times \mathcal{C}(0, T_n; \mathrm{H}^m) \times \mathcal{C}(0, T_n; \mathrm{H}^m) \cap \mathrm{L}^2(0, T_n; \mathrm{H}^{m+1}),$$

we consider the system

$$(\widetilde{S}^{n+1}) \qquad \begin{cases} \partial_t f^{n+1} + v\cdot\nabla_x f^{n+1} + \mathrm{div}_v\big[f^{n+1}(E^n(t,x)-v)\big] = 0,\\ \partial_t \mathfrak{m}^{n+1} + \mathrm{div}_x(\mathfrak{m}^{n+1}u^n) = 0,\\ \partial_t u^{n+1} - \dfrac{1}{\mathfrak{m}^n}(\Delta_x + \nabla_x\,\mathrm{div}_x)u^{n+1} \\ \qquad = -(u^n\cdot\nabla_x)u^n - \nabla_x\pi(\varrho^n) + \dfrac{1}{\mathfrak{m}^n}(j_{f^n} - \rho_{f^n}u^n),\\ \varrho^{n+1} = \dfrac{1}{1-\rho_{f^n}}\mathfrak{m}^{n+1},\\ E^n := u^n - p'(\varrho^n)\nabla_x\big[(I-\varepsilon^2\Delta_x)^{-1}\varrho^n\big],\\ f^{n+1}{}_{|t=0} = f^{\mathrm{in}}, \quad \mathfrak{m}^{n+1}{}_{|t=0} = \alpha^{\mathrm{in}}\varrho^{\mathrm{in}}, \quad u^{n+1}{}_{|t=0} = u^{\mathrm{in}}, \end{cases}$$

with

$$\pi'(x) := \frac{p'(x)}{x}, \quad \pi(0)=0,$$

and where

$$\rho_{f^n}(t,x) := \int_{\mathbb{R}^d} f^n(t,x,v)\,\mathrm{d}v, \quad j_{f^n}(t,x) := \int_{\mathbb{R}^d} f^n(t,x,v)v\,\mathrm{d}v.$$

We also set

$$R_0 := 100\big(\|f^{\mathrm{in}}\|_{\mathcal{H}^m_r} + \|\mathfrak{m}^{\mathrm{in}}\|_{\mathrm{H}^m} + \|u^{\mathrm{in}}\|_{\mathrm{H}^m}\big).$$

Step 1: Construction of $(f^{n+1}, \varrho^{n+1}, u^{n+1})$ and uniform estimates. In what follows, we construct the next iteration of the scheme, which is a solution to ($\widetilde{S}^{n+1}$), thus proving that our inductive scheme is well defined. At the same time, we derive uniform high-regularity estimates for the sequence of solutions.

Let $n\in\mathbb{N}$. By induction, we assume that there exists $T>0$ (depending on ε) such that, for all $k=0,\dots,n$,

$$\|f^k\|_{\mathrm{L}^\infty(0,T;\mathcal{H}^m_r)} + \|\mathfrak{m}^k\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^m)} + \|u^k\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^m)\cap\mathrm{L}^2(0,T;\mathrm{H}^{m+1})} < R_0,$$

the functions f_k and ϱ_k are nonnegative, and, for $t\in[0,T]$,

$$\rho_{f_k}(t) \le \frac{\Theta+1}{2}, \quad \frac{\mu}{2} \le \varrho_k(t), \quad \frac{\underline{\theta}}{2} \le \mathfrak{m}_k(t) \le 2\overline{\theta}, \tag{B.1}$$

where Θ, μ, $\underline{\theta}$, $\overline{\theta}$ are the constants given in Proposition 2.18. In particular, we have

$$1-\rho_{f_k}(t) \ge \frac{1-\Theta}{2} > 0.$$

Note also that via Sobolev embedding, Lemma A.5 and Lemma 2.1, we have

$$\|\varrho^n\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^m)} \le \left\|\frac{1}{1-\rho_{f^{n-1}}}\right\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^m)} \|\mathfrak{m}^n\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^m)}$$

$$
\begin{aligned}
&\leq \big(1 + \Lambda(\|\rho_{f^{n-1}}\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^m)})\big)\|\rho_{f^{n-1}}\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^m)}\|\mathfrak{m}^n\|_{\mathrm{L}^\infty(0,t;\mathrm{H}^m)} \\
&\leq \big(1 + \Lambda(\|f^{n-1}\|_{\mathrm{L}^\infty(0,T;\mathcal{H}^m_r)})\big)\|f^{n-1}\|_{\mathrm{L}^\infty(0,T;\mathcal{H}^m_r)}\|\mathfrak{m}^n\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^m)} \\
&\leq \Lambda(R_0).
\end{aligned}
$$

In what follows, we rely on the *a priori* estimates of Chapter 2.

First, we can obtain a unique nonnegative solution $f^{n+1} \in \mathcal{C}(0,T;\mathcal{H}^m_r)$ to the Vlasov equation by the method of characteristics. In view of the regularization introduced by J_ε, Proposition 2.14 yields, for all $t \in [0,T]$,

$$
\begin{aligned}
&\|f^{n+1}(t)\|_{\mathcal{H}^m_r} \\
&\quad\leq \|f^{\mathrm{in}}\|_{\mathcal{H}^m_r} \\
&\qquad \times \exp\Big[C\Big(\big(1 + \|u^n\|_{\mathrm{L}^\infty(0,t;\mathrm{H}^m)}\big)t + \frac{t}{\varepsilon}\Lambda\big(\|\varrho^n\|_{\mathrm{L}^\infty(0,t;\mathrm{H}^{m-2})}\big)\|\varrho^n\|_{\mathrm{L}^\infty(0,t;\mathrm{H}^m)}\Big)\Big] \\
&\quad\leq \frac{R_0}{100}\exp\Big[C\Big((1+R_0)t + \frac{t}{\varepsilon}\Lambda(R_0)\Big)\Big].
\end{aligned}
$$

We then define $T_{[1]} = T_{[1]}(R_0) > 0$, with $T_{[1]}(R_0) < T$ (depending on ε) such that

$$
\frac{R_0}{100}\exp\bigg[C\bigg((1+R_0)T_{[1]} + \frac{\sqrt{T_{[1]}}}{\varepsilon}\Lambda(R_0)\bigg)\bigg] < \frac{R_0}{3}.
$$

Second, since $u^n \in \mathcal{C}(0,T;\mathrm{H}^{m+1})$ is given, we can construct a nonnegative solution $\mathfrak{m}^{n+1}$ to the second equation in ($\widetilde{S}^{n+1}$), relying on standard arguments for continuity equations. We obtain a unique solution $\mathfrak{m}^{n+1} \in \mathcal{C}^1(0,T;\mathrm{H}^m)$ such that, for all $t \in [0,T]$,

$$
\begin{aligned}
&\|\mathfrak{m}^{n+1}(t)\|_{\mathrm{H}^m} \\
&\quad \leq e^{\|\mathrm{div}_x u^n\|_{\mathrm{L}^\infty((0,t)\times\mathbb{T}^d)}t/2}\Big(\|\mathfrak{m}^{\mathrm{in}}\|_{\mathrm{H}^m} + \int_0^t \|u^n(\tau)\|_{\mathrm{H}^{m+1}}\|\mathfrak{m}^{n+1}(\tau)\|_{\mathrm{H}^m}\,\mathrm{d}\tau\Big),
\end{aligned}
$$

thanks to Proposition 2.3. Assuming that $e^{R_0 t/2} \leq 2$, we infer, by Grönwall's lemma, that

$$
\begin{aligned}
\|\mathfrak{m}^{n+1}(t)\|_{\mathrm{H}^m} &\leq 2\|\mathfrak{m}^{\mathrm{in}}\|_{\mathrm{H}^m}\exp\Big(2\int_0^t \|u^n(\tau)\|_{\mathrm{H}^{m+1}}\,\mathrm{d}\tau\Big) \\
&\leq 2\|\mathfrak{m}^{\mathrm{in}}\|_{\mathrm{H}^m}\exp\Big(2\sqrt{t}\|u^n\|_{\mathrm{L}^2(0,t;\mathrm{H}^{m+1})}\Big);
\end{aligned}
$$

therefore, for such times t, we have

$$
\|\mathfrak{m}^{n+1}(t)\|_{\mathrm{H}^m} \leq 2\frac{R_0}{100}e^{2\sqrt{t}R_0}.
$$

We then define $T_{[2]} = T_{[2]}(R_0) > 0$ such that $e^{R_0 T_{[2]}/2} \leq 2$ and

$$
2\frac{R_0}{100}e^{2\sqrt{T_{[2]}}R_0} < \frac{R_0}{3}.
$$

Third, we define u^{n+1} as the unique solution of the parabolic equation

$$\partial_t w - \frac{1}{\mathfrak{m}^n}\big(\Delta_x + \nabla_x \operatorname{div}_x\big) w = -(u^n \cdot \nabla_x) u^n - \nabla_x \pi(\varrho^n) + \frac{1}{\mathfrak{m}^n}(j_{f^n} - \rho_{f^n} u^n),$$

starting from u^{in}, which satisfies

$$\begin{aligned}
&\|u^{n+1}\|_{L^\infty(0,t;H^m)} + \|u^{n+1}\|_{L^2(0,t;H^{m+1})} \\
&\quad \leq \big(1 + \sqrt{t}\Lambda(t, R_0)\big)\Bigg(\|u^{\text{in}}\|_{H^m} + \sqrt{t}\Lambda(\|\varrho^n\|_{L^\infty(0,t;H^m)})\|\varrho^n\|_{L^\infty(0,t;H^m)} \\
&\qquad + \sqrt{t}\|u^n\|^2_{L^\infty(0,t;H^m)} \\
&\qquad + \sqrt{t}\left\|\frac{1}{\mathfrak{m}^n}(j_{f^n} - \rho_{f^n} u^n)\right\|_{L^\infty(0,t;H^{m-1})}\Bigg).
\end{aligned}$$

Here, we have applied the same argument as for the proof of Proposition 2.24, making $\|\varrho^n\|_{L^\infty(0,t;H^m)}$ appear to get a factor $\sqrt{t}$. The second term inside the parentheses is controlled by $\Lambda(R_0)$, while for the third term we can rely on the same of kind of arguments (with Remark A.4) to get

$$\begin{aligned}
&\left\|\frac{1}{\mathfrak{m}^n}(j_{f^n} - \rho_{f^n} u^n)\right\|_{L^\infty(0,t;H^{m-1})} \\
&\leq \left\|\frac{1}{\mathfrak{m}^n}\right\|_{L^\infty(0,t;H^{m-1})}\big(\|j_{f^n}\|_{L^\infty(0,t;H^{m-1})} + \|\rho_{f^n}\|_{L^\infty(0,t;H^{m-1})}\|u^n\|_{L^\infty(0,t;H^{m-1})}\big) \\
&\leq \Lambda(R_0).
\end{aligned}$$

We finally obtain

$$\|u^{n+1}\|_{L^\infty(0,t;H^m)} + \|u^{n+1}\|_{L^2(0,t;H^{m+1})} \leq \big(1 + \sqrt{t}\Lambda(t, R_0)\big)\frac{R_0}{100} + \sqrt{t}\Lambda(R_0).$$

We then define $T_{[3]} = T_{[3]}(R_0) > 0$ with $T_{[3]}(R_0) < T$ such that

$$\Big(1 + \sqrt{T_{[3]}}\Lambda(T_{[3]}, R_0)\Big)\frac{R_0}{100} + \sqrt{T_{[3]}}\Lambda(R_0) < \frac{R_0}{3}.$$

Last, we can rely on Lemmas 2.16 and 2.17 to find a time $T_{[4]} = T_{[4]}(R_0) > 0$ such that the condition (B.1) is satisfied for $(f^{n+1}, \mathfrak{m}^{n+1}, u^{n+1})$ on the time interval $[0, \min(T_{[4]}, T))$.

All in all, we define

$$T(R_0) := \min\big(T_{[1]}(R_0), T_{[2]}(R_0), T_{[3]}(R_0), T_{[4]}(R_0)\big) < T,$$

which may depend on ε but which is independent of n. An induction procedure based on the three previous estimates shows that one can obtain a time $T^{[\varepsilon]} > 0$ and a sequence $(f^n, \mathfrak{m}^n, u^n)_{n\in\mathbb{N}}$ satisfying, for all $n \in \mathbb{N}$,

$$
\begin{aligned}
f^n &\in \mathcal{C}(0, T^{[\varepsilon]}; \mathcal{H}^m_r),\\
\mathfrak{m}^n &\in \mathcal{C}(0, T^{[\varepsilon]}; \mathrm{H}^m),\\
u^n &\in \mathcal{C}(0, T^{[\varepsilon]}; \mathrm{H}^m) \cap \mathrm{L}^2(0, T; \mathrm{H}^{m+1}),
\end{aligned}
$$

with (B.1) and the uniform estimate

$$
\|f^n\|_{\mathrm{L}^\infty(0,T^{[\varepsilon]};\mathcal{H}^m_r)} + \|\mathfrak{m}^n\|_{\mathrm{L}^\infty(0,T^{[\varepsilon]};\mathrm{H}^m)} + \|u^n\|_{\mathrm{L}^\infty(0,T^{[\varepsilon]};\mathrm{H}^m)\cap \mathrm{L}^2(0,T^{[\varepsilon]};\mathrm{H}^{m+1})} < R_0. \tag{B.2}
$$

Step 2: Contraction estimates in $\mathbf{L}^\infty_T\mathcal{H}^0_r \times \mathbf{L}^\infty_T\mathbf{L}^2 \times \mathbf{L}^\infty_T\mathbf{L}^2 \cap \mathbf{L}^2_T\mathbf{H}^1$. If $n \in \mathbb{N}\backslash\{0\}$, we set

$$
g^n := f^{n+1} - f^n, \quad \mathfrak{M}^n = \mathfrak{m}^{n+1} - \mathfrak{m}^n, \quad w^n := u^{n+1} - u^n,
$$

which satisfy the system of equations

$$
\begin{cases}
\partial_t g^n + v\cdot\nabla_x g^n + \mathrm{div}_v[g^n(E^n(t,x)-v)] + (E^n - E^{n-1})\cdot\nabla_v f^n = 0,\\
\partial_t \mathfrak{M}^n + \mathrm{div}_x(\mathfrak{M}^n u^n) = -\mathrm{div}_x(\mathfrak{m}^n w^{n-1}),\\
\partial_t w^n - \dfrac{1}{\mathfrak{m}^{n+1}}\big(\Delta_x + \nabla_x\,\mathrm{div}_x\big)w^n = \mathfrak{S}^n,\\
\varrho^{n+1} := \dfrac{1}{1-\rho_{f^n}}\mathfrak{m}^{n+1},\\
E^n := u^n - p'(\varrho^n)\nabla_x\big[(I-\varepsilon^2\Delta_x)^{-1}\varrho^n\big],\\
g^n{}_{|t=0} = 0, \quad \mathfrak{M}^n{}_{|t=0} = 0, \quad w^n{}_{|t=0} = 0,
\end{cases}
$$

where

$$
\begin{aligned}
\mathfrak{S}^n := \left(\frac{1}{\mathfrak{m}^{n+1}} - \frac{1}{\mathfrak{m}^n}\right)\big(\Delta_x + \nabla_x\,\mathrm{div}_x\big)u^n - (w^{n-1}\cdot\nabla_x)u^n - (u^{n-1}\cdot\nabla_x)w^{n-1}\\
- \nabla_x\big[\pi(\varrho^{n-1}) - \pi(\varrho^n)\big] + \frac{1}{\mathfrak{m}^n}(j_{f^n} - \rho_{f^n}u^n) - \frac{1}{\mathfrak{m}^{n-1}}(j_{f^{n-1}} - \rho_{f^{n-1}}u^{n-1}).
\end{aligned}
$$

Let us derive some L^2 estimates on $(g^n, \mathfrak{M}^n, w^n)$. They will be satisfied on $[0, \widetilde{T}]$ for some $\widetilde{T}^{[\varepsilon]} < T^{[\varepsilon]}$.

First, we perform $\mathrm{L}^2_{x,v}$-weighted estimates in the first equation for g^n, and as before (see (2.3)), we get

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\|g^n(t)\|^2_{\mathcal{H}^0_r}\\
&\quad\lesssim \big(1 + \|E^n(t)\|_{\mathrm{H}^m}\big)\|g^n(t)\|^2_{\mathcal{H}^0_r} + \|E^n(t) - E^{n-1}(t)\|_{\mathrm{L}^2}\|f^n(t)\|_{\mathcal{H}^m_r}\|g^n(t)\|_{\mathcal{H}^0_r}.
\end{aligned}
$$

For $t \in (0,T)$, we can infer that

$$
\begin{aligned}
&\|g^n(t)\|_{\mathcal{H}^0_r}\\
&\quad\lesssim \int_0^t \big(1+\|E^n(s)\|_{\mathrm{H}^m}\big)\|g^n(s)\|_{\mathcal{H}^0_r}\,\mathrm{d}s + \int_0^t \|E^n(s)-E^{n-1}(s)\|_{\mathrm{L}^2}\|f^n(s)\|_{\mathcal{H}^m_r}\,\mathrm{d}s\\
&\quad\le \sqrt{t}\Big(\big(\sqrt{t}+\|E^n\|_{\mathrm{L}^2(0,t;\mathrm{H}^m)}\big)\|g^n\|_{\mathrm{L}^\infty(0,t;\mathcal{H}^0_r)} + R_0\|E^n-E^{n-1}\|_{\mathrm{L}^2(0,t;\mathrm{L}^2)}\Big)\\
&\quad\lesssim \sqrt{t}\Big(\Big(\sqrt{t}+\frac{\sqrt{t}}{\varepsilon}R_0\Big)\|g^n\|_{\mathrm{L}^\infty(0,t;\mathcal{H}^0_r)} + R_0\|E^n-E^{n-1}\|_{\mathrm{L}^2(0,t;\mathrm{L}^2)}\Big).
\end{aligned}
$$

Choosing $\widetilde{T}^{[\varepsilon]} < T^{[\varepsilon]}$ small enough, independent of n, we can absorb the first term inside the parentheses on the left-hand side, so that, for all $t \in (0,\widetilde{T}^{[\varepsilon]})$,

$$
\|g^n(t)\|_{\mathcal{H}^0_r} \lesssim \sqrt{t}R_0\|E^n-E^{n-1}\|_{\mathrm{L}^2(0,t;\mathrm{L}^2)}.
$$

Then observe that by Sobolev embedding and Proposition A.3, we have

$$
\begin{aligned}
&\|E^n-E^{n-1}\|_{\mathrm{L}^2(0,t;\mathrm{L}^2)}\\
&\quad\le \|u^n-u^{n-1}\|_{\mathrm{L}^2(0,t;\mathrm{L}^2)} + \big\|p'(\varrho^n)\nabla_x[\mathrm{J}_\varepsilon\varrho^n] - p'(\varrho^{n-1})\nabla_x[\mathrm{J}_\varepsilon\varrho^{n-1}]\big\|_{\mathrm{L}^2(0,t;\mathrm{L}^2)}\\
&\quad\le \|u^n-u^{n-1}\|_{\mathrm{L}^2(0,t;\mathrm{L}^2)} + \|p'(\varrho^n)\|_{\mathrm{L}^\infty(0,t;\mathrm{L}^\infty)}\big\|\nabla_x[\mathrm{J}_\varepsilon(\varrho^n-\varrho^{n-1})]\big\|_{\mathrm{L}^2(0,t;\mathrm{L}^2)}\\
&\qquad + \|p'(\varrho^n)-p'(\varrho^{n-1})\|_{\mathrm{L}^2(0,t;\mathrm{L}^2)}\big\|\nabla_x[\mathrm{J}_\varepsilon\varrho^{n-1}]\big\|_{\mathrm{L}^\infty(0,t;\mathrm{L}^\infty)}\\
&\quad\lesssim \|u^n-u^{n-1}\|_{\mathrm{L}^2(0,t;\mathrm{L}^2)} + \Lambda(R_0)\frac{1}{\varepsilon}\|\varrho^n-\varrho^{n-1}\|_{\mathrm{L}^2(0,t;\mathrm{L}^2)}\\
&\qquad + \Lambda(t,R_0)\frac{1}{\varepsilon}\|p'(\varrho^n)-p'(\varrho^{n-1})\|_{\mathrm{L}^2(0,t;\mathrm{L}^2)}\\
&\quad\lesssim \|u^n-u^{n-1}\|_{\mathrm{L}^2(0,t;\mathrm{L}^2)} + \Lambda(t,R_0)\frac{1}{\varepsilon}\|\mathfrak{m}^n-\mathfrak{m}^{n-1}\|_{\mathrm{L}^2(0,t;\mathrm{L}^2)}.
\end{aligned}
$$

This yields, for all $t \in (0,\widetilde{T}^{[\varepsilon]})$,

$$
\|g^n\|_{\mathrm{L}^\infty(0,t;\mathcal{H}^0_r)} \lesssim \sqrt{t}\Big(R_0\|w^{n-1}\|_{\mathrm{L}^2(0,t;\mathrm{L}^2)} + \Lambda(t,R_0)\frac{1}{\varepsilon}\|\mathfrak{M}^{n-1}\|_{\mathrm{L}^2(0,t;\mathrm{L}^2)}\Big).
$$

Second, an L^2 energy estimate on the second equation for $\mathfrak{M}^n$ also leads to

$$
\begin{aligned}
&\|\mathfrak{M}^n(t)\|_{\mathrm{L}^\infty(0,t;\mathrm{L}^2)}\\
&\qquad\le e^{\|\mathrm{div}_x u_n\|_{\mathrm{L}^\infty(0,t;\mathrm{L}^\infty)}t}\int_0^t \big\|\mathrm{div}_x\, \mathfrak{m}^n w^{n-1}(\tau)\big\|_{\mathrm{L}^2}\,\mathrm{d}\tau\\
&\qquad\le e^{R_0 t}\Big(\int_0^t \big\|w^{n-1}\cdot\nabla_x\mathfrak{m}^n(\tau)\big\|_{\mathrm{L}^2}\,\mathrm{d}\tau + \int_0^t \big\|\mathfrak{m}^n\,\mathrm{div}_x\, w^{n-1}(\tau)\big\|_{\mathrm{L}^2}\,\mathrm{d}\tau\Big)\\
&\qquad\le \sqrt{t}e^{R_0 t}\Big(\|\nabla_x\mathfrak{m}^n\|_{\mathrm{L}^\infty(0,t;\mathrm{L}^\infty)}\|w^{n-1}\|_{\mathrm{L}^2(0,t;\mathrm{L}^2)}\\
&\qquad\qquad + \|\mathfrak{m}^n\|_{\mathrm{L}^\infty(0,t;\mathrm{L}^\infty)}\|\mathrm{div}_x\, w^{n-1}\|_{\mathrm{L}^2(0,t;\mathrm{L}^2)}\Big)\\
&\qquad\lesssim \sqrt{t}\Lambda(t,R_0)\|w^{n-1}\|_{\mathrm{L}^2(0,t;\mathrm{H}^1)}
\end{aligned}
$$

for all $t \in (0,\widetilde{T}^{[\varepsilon]})$.

Third, performing an $L^\infty_T L^2 \cap L^2_T H^1$ estimate by multiplying by w_n in the parabolic equation satisfied by w^n yields

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w^n\|^2_{L^2} + \int_{\mathbb{T}^d} \frac{1}{\mathfrak{m}^{n+1}}\big(|\nabla_x w^n|^2 + |\operatorname{div}_x w^n|^2\big) \\
&\quad = \sum_{k=1}^{5} \int_{\mathbb{T}^d} \mathfrak{S}^n_k \cdot w^n \\
&\qquad - \sum_{i=1}^{d} \left\{ \int_{\mathbb{T}^d} \nabla_x w^n_i \cdot \nabla_x\left(\frac{1}{\mathfrak{m}^{n+1}}\right) w^n_i - \int_{\mathbb{T}^d} \operatorname{div}_x w^n \partial_i\left(\frac{1}{\mathfrak{m}^{n+1}}\right) w^n_i \right\},
\end{aligned}$$

with

$$\begin{aligned}
\mathfrak{S}^n_1 &:= \left(\frac{1}{\mathfrak{m}^{n+1}} - \frac{1}{\mathfrak{m}^n}\right)(\Delta_x + \nabla_x \operatorname{div}_x) u^n, \\
\mathfrak{S}^n_2 &:= -(w^{n-1}\cdot\nabla_x) u^n, \\
\mathfrak{S}^n_3 &:= -(u^{n-1}\cdot\nabla_x) w^{n-1}, \\
\mathfrak{S}^n_4 &:= -\nabla_x\big[\pi(\varrho^{n-1}) - \pi(\varrho^n)\big], \\
\mathfrak{S}^n_5 &:= \frac{1}{\mathfrak{m}^n}(j_{f^n} - \rho_{f^n} u^n) - \frac{1}{\mathfrak{m}^{n-1}}(j_{f^{n-1}} - \rho_{f^{n-1}} u^{n-1}).
\end{aligned}$$

For all $\eta > 0$, we obtain, by Young's inequality,

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w^n\|^2_{L^2} + \int_{\mathbb{T}^d} \left(\frac{1}{\mathfrak{m}^{n+1}} - \eta\left\|\nabla_x\left(\frac{1}{\mathfrak{m}^{n+1}}\right)\right\|^2_{L^\infty}\right)\big(|\nabla_x w^n|^2 + |\operatorname{div}_x w^n|^2\big) \\
&\quad \leq \sum_{k=1}^{5} \int_{\mathbb{T}^d} \mathfrak{S}^n_k \cdot w^n + \frac{1}{\eta}\int_{\mathbb{T}^d} |w^n|^2.
\end{aligned}$$

Let us focus on the source terms $\mathfrak{S}^n_k$. We have, by Sobolev embedding,

$$\begin{aligned}
\int_{\mathbb{T}^d} \mathfrak{S}^n_1 \cdot w^n &\lesssim \big\|(\Delta_x + \nabla_x \operatorname{div}_x) u^n\big\|^2_{L^\infty} \left\|\frac{1}{\mathfrak{m}^{n+1}} - \frac{1}{\mathfrak{m}^n}\right\|^2_{L^2} + \int_{\mathbb{T}^d} |w^n|^2 \\
&\lesssim R_0^2 \left\|\frac{1}{\mathfrak{m}^{n+1}\mathfrak{m}^n}\right\|^2_{L^\infty} \|\mathfrak{m}^{n+1} - \mathfrak{m}^n\|^2_{L^2} + \int_{\mathbb{T}^d} |w^n|^2 \\
&\lesssim \Lambda(R_0)\|\mathfrak{M}^n\|^2_{L^2} + \|w^n\|^2_{L^2}.
\end{aligned}$$

We next have

$$\begin{aligned}
\int_{\mathbb{T}^d} \mathfrak{S}^n_2 \cdot w^n &\lesssim \int_{\mathbb{T}^d} |\mathfrak{S}^n_2|^2 + \int_{\mathbb{T}^d} |w^n|^2 \leq \|\nabla_x u^{n-1}\|^2_{L^\infty} \|w^{n-1}\|^2_{L^2} + \int_{\mathbb{T}^d} |w^n|^2 \\
&\leq R_0^2 \|w^{n-1}\|^2_{L^2} + \|w^n\|^2_{L^2},
\end{aligned}$$

again by Sobolev embedding, while for all $\delta_1 > 0$, we have, by integration by parts and Sobolev embedding,

$$\begin{aligned}\int_{\mathbb{T}^d} \mathfrak{S}_3^n \cdot w^n &\lesssim \|\nabla_x u^{n-1}\|_{\mathrm{L}^\infty}^2 \|w^{n-1}\|_{\mathrm{L}^2}^2 + \|w^n\|_{\mathrm{L}^2}^2 \\ &\quad + \frac{1}{\delta_1}\|u^{n-1}\|_{\mathrm{L}^\infty}^2 \|w^{n-1}\|_{\mathrm{L}^2}^2 + \delta_1 \int_{\mathbb{T}^d} |\nabla_x w^n|^2 \\ &\lesssim \Lambda_{\delta_1}(R_0)\|w^{n-1}\|_{\mathrm{L}^2} + \|w^n\|_{\mathrm{L}^2}^2 + \delta_1 \int_{\mathbb{T}^d} |\nabla_x w^n|^2.\end{aligned}$$

We also have, for all $\delta_2 > 0$,

$$\begin{aligned}\int_{\mathbb{T}^d} \mathfrak{S}_4^n \cdot w^n &\lesssim \frac{1}{\delta_2}\int_{\mathbb{T}^d} \left|[\pi(\varrho^{n-1}) - \pi(\varrho^n)]\right|^2 + \delta_2 \int_{\mathbb{T}^d} |\mathrm{div}_x\, w^n|^2 \\ &\lesssim \Lambda_{\delta_2}(R_0)\|\varrho^{n-1} - \varrho^n\|_{\mathrm{L}^2}^2 + \delta_2 \int_{\mathbb{T}^d} |\mathrm{div}_x\, w^n|^2.\end{aligned}$$

For $\mathfrak{S}_5^n$, we write

$$\begin{aligned}\mathfrak{S}_5^n &= \left(\frac{1}{\mathfrak{m}^n} - \frac{1}{\mathfrak{m}^{n-1}}\right)(j_{f^n} - \rho_{f^n} u^n) + \frac{1}{\mathfrak{m}^{n-1}}(j_{f^n} - j_{f^{n-1}}) \\ &\quad + \frac{1}{\mathfrak{m}^n}(\rho_{f^{n-1}} - \rho_{f^n})u^{n-1} + \frac{1}{\mathfrak{m}^{n-1}}\rho_{f^n}(u^{n-1} - u^n).\end{aligned}$$

Therefore, by Sobolev embedding and Lemma 2.1,

$$\begin{aligned}\int_{\mathbb{T}^d} \mathfrak{S}_5^n \cdot w^n &\lesssim \|j_{f^n} - \rho_{f^n} u^n\|_{\mathrm{L}^\infty}^2 \left\|\frac{1}{\mathfrak{m}^n} - \frac{1}{\mathfrak{m}^{n-1}}\right\|_{\mathrm{L}^2}^2 + \left\|\frac{1}{\mathfrak{m}^n}\right\|_{\mathrm{L}^\infty}^2 \|j_{f^n} - j_{f^{n-1}}\|_{\mathrm{L}^2}^2 \\ &\quad + \left\|\frac{1}{\mathfrak{m}^n}\right\|_{\mathrm{L}^\infty}^2 \|u^{n-1}\|_{\mathrm{L}^\infty}^2 \|\rho_{f^n} - \rho_{f^{n-1}}\|_{\mathrm{L}^2}^2 \\ &\quad + \left\|\frac{1}{\mathfrak{m}^{n-1}}\right\|_{\mathrm{L}^\infty}^2 \|\rho_{f^n}\|_{\mathrm{L}^\infty}^2 \|u^{n-1} - u^n\|_{\mathrm{L}^2}^2 + \int_{\mathbb{T}^d} |w^n|^2 \\ &\lesssim \Lambda(R_0)\Big(\|\mathfrak{m}^n - \mathfrak{m}^{n-1}\|_{\mathrm{L}^2}^2 + \|f^n - f^{n-1}\|_{\mathcal{H}_r^0}^2 + \|u^{n-1} - u^n\|_{\mathrm{L}^2}^2\Big) \\ &\quad + \|w^n\|_{\mathrm{L}^2}^2.\end{aligned}$$

We finally obtain

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\|w^n(t)\|_{\mathrm{L}^2}^2 + \int_{\mathbb{T}^d} \left(\frac{1}{\mathfrak{m}^{n+1}} - \eta R_0^2 - \delta_1 - \delta_2\right)\left(|\nabla_x w^n|^2 + |\mathrm{div}_x\, w^n|^2\right) \\ &\quad \leq \Lambda_{\delta_{1,2},\eta}(R_0)\Big(\left\|\mathfrak{M}^{n-1}(t)\right\|_{\mathrm{L}^2}^2 + \left\|g^{n-1}(t)\right\|_{\mathcal{H}_r^0}^2 + \left\|w^{n-1}(t)\right\|_{\mathrm{L}^2}^2\Big) \\ &\qquad + \Lambda(R_0)\|\mathfrak{M}^n(t)\|_{\mathrm{L}^2}^2 + \Lambda_\eta \|w^n(t)\|_{\mathrm{L}^2}^2\end{aligned} \tag{B.3}$$

for some constant $\Lambda_\eta > 0$. A good choice of η, δ_1 and δ_2 with respect to $\inf \frac{1}{\mathfrak{m}^{n+1}}$, combined with Grönwall's lemma, shows that

$$\begin{aligned}&\|w^n(t)\|_{L^2}^2\\&\quad\lesssim e^{\Lambda_\eta t}\int_0^t \Big(\|\mathfrak{M}^{n-1}(\tau)\|_{L^2}^2 + \|g^{n-1}(\tau)\|_{\mathcal{H}_r^0}^2 + \|w^{n-1}(\tau)\|_{L^2}^2 + \|\mathfrak{M}^n(\tau)\|_{L^2}^2\Big)\,d\tau.\end{aligned}$$

Reducing $\widetilde{T}^{[\varepsilon]}$ if necessary so that $\widetilde{T}^{[\varepsilon]} \le 1/\Lambda_\eta$, we can integrate in time the previous differential inequality (B.3) to obtain, for all $t \in (0, \widetilde{T}^{[\varepsilon]})$,

$$\begin{aligned}&\|w^n(t)\|_{L^\infty(0,t;L^2)}^2 + \|w^n(t)\|_{L^2(0,t;H^1)}^2\\&\qquad\lesssim \Lambda(R_0)t\Big(\|\mathfrak{M}^{n-1}\|_{L^\infty(0,t;L^2)}^2 + \|g^{n-1}\|_{L^\infty(0,t;\mathcal{H}_r^0)}^2 + \|w^{n-1}\|_{L^\infty(0,t;L^2)}^2\Big)\\&\qquad\quad+\Lambda(R_0)t\|\mathfrak{M}^n\|_{L^\infty(0,t;L^2)}^2.\end{aligned}$$

Combining the three previous points, we reach the following inequality:

$$\begin{aligned}&\|g^n(t)\|_{L^\infty(0,t;\mathcal{H}_r^0)} + \|\mathfrak{M}^n(t)\|_{L^\infty(0,t;L^2)} + \|w^n(t)\|_{L^\infty(0,t;L^2)\cap L^2(0,t;H^1)}\\&\quad\lesssim \sqrt{t}\Big(R_0\sqrt{t}\|w^{n-1}\|_{L^\infty(0,t;L^2)} + \Lambda(t,R_0)\frac{1}{\varepsilon}\|\mathfrak{M}^{n-1}\|_{L^2(0,t;L^2)}\Big)\\&\qquad+\sqrt{t}\Lambda(t,R_0)\|w^{n-1}\|_{L^2(0,t;H^1)} + \Lambda(R_0)t\,\|\mathfrak{M}^n\|_{L^\infty(0,t;L^2)}\\&\qquad+\Lambda(R_0)t\Big(\|\mathfrak{M}^{n-1}\|_{L^\infty(0,t;L^2)} + \|g^{n-1}\|_{L^\infty(0,t;\mathcal{H}_r^0)} + \|w^{n-1}\|_{L^\infty(0,t;L^2)}\Big),\end{aligned}$$

which is valid for all $t \in (0, \widetilde{T}^{[\varepsilon]})$. Reducing again $\widetilde{T}^{[\varepsilon]}$ if necessary, this means that for all $n \in \mathbb{N}\backslash\{0\}$,

$$\begin{aligned}&\|g^n\|_{L^\infty(0,\widetilde{T}^{[\varepsilon]};\mathcal{H}_r^0)} + \|\mathfrak{M}^n\|_{L^\infty(0,\widetilde{T}^{[\varepsilon]};L^2)} + \|w^n\|_{L^\infty(0,\widetilde{T}^{[\varepsilon]};L^2)\cap L^2(0,\widetilde{T}^{[\varepsilon]};H^1)}\\&\qquad\le\frac{1}{2}\Big(\|g^{n-1}\|_{L^\infty(0,\widetilde{T}^{[\varepsilon]};\mathcal{H}_r^0)} + \|\mathfrak{M}^{n-1}\|_{L^\infty(0,\widetilde{T}^{[\varepsilon]};L^2)}\\&\qquad\qquad+\|w^{n-1}\|_{L^\infty(0,\widetilde{T}^{[\varepsilon]};L^2)\cap L^2(0,\widetilde{T}^{[\varepsilon]};H^1)}\Big).\end{aligned}\tag{B.4}$$

Note that $\widetilde{T}^{[\varepsilon]}$ has been chosen independent of $n \in \mathbb{N}\backslash\{0\}$.

Step 3: Obtaining a unique solution to (S_ε). Thanks to the uniform bound (B.2) and the contraction estimate (B.4), we can deduce that the sequence $(f_n, \mathfrak{m}_n, u_n)$ is weakly-$\star$ compact in the space

$$L^\infty(0,\widetilde{T}^{[\varepsilon]};\mathcal{H}_r^m)\times L^\infty(0,\widetilde{T}^{[\varepsilon]};H^m)\times L^\infty(0,\widetilde{T}^{[\varepsilon]};H^m)\cap L^2(0,\widetilde{T}^{[\varepsilon]};H^{m+1}),$$

and is a Cauchy sequence in the space

$$L^\infty(0,\widetilde{T}^{[\varepsilon]};\mathcal{H}_r^0)\times L^\infty(0,\widetilde{T}^{[\varepsilon]};L^2)\times L^\infty(0,\widetilde{T}^{[\varepsilon]};L^2)\cap L^2(0,\widetilde{T}^{[\varepsilon]};H^1).$$

This shows that the whole sequence converges weakly-⋆ in the first space. The limit (f, m, u) belongs in particular to this space with high regularity. Using weak–strong convergence principles then allows us to prove that the limit is a solution to the system (S_ε) on $(0, \widetilde{T}^{[\varepsilon]})$ in the sense of distributions. We do not detail this part of the proof. Using the equation and the time derivative of the solution, one can show that the solution actually belongs to

$$\mathrm{X}^m_{\widetilde{T}^{[\varepsilon]}} := \mathcal{C}(0, \widetilde{T}^{[\varepsilon]}; \mathcal{H}^m_r) \times \mathcal{C}(0, \widetilde{T}^{[\varepsilon]}; \mathrm{H}^m) \times \mathcal{C}(0, \widetilde{T}^{[\varepsilon]}; \mathrm{H}^m) \cap \mathrm{L}^2(0, \widetilde{T}^{[\varepsilon]}; \mathrm{H}^{m+1}).$$

The uniqueness of the Cauchy problem for (S_ε) in the former space is obtained by mimicking the contraction estimates of Step 2. In fact, if $(f_1, \mathfrak{m}_1, u_1)$ and $(f_2, \mathfrak{m}_2, u_2)$ are two solutions in $\mathrm{X}^m_{\widetilde{T}^{[\varepsilon]}}$ starting at the same initial condition, then performing the same computations proves that there exists $\overline{T}^{[\varepsilon]}$ (depending on $\| f_{1,2}, \mathfrak{m}_{1,2}, u_{1,2}\|_{\mathrm{X}^m_{\widetilde{T}^{[\varepsilon]}}}$) with $\overline{T}^{[\varepsilon]} < \widetilde{T}^{[\varepsilon]}$ and such that, for all $t \in [0, \overline{T}^{[\varepsilon]}]$, we have

$$\begin{aligned}&\|(f_1 - f_2)(t)\|^2_{\mathcal{H}^0_r} + \|(\mathfrak{m}_1 - \mathfrak{m}_2)(t)\|^2_{\mathrm{L}^2} + \|(u_1 - u_2)(t)\|^2_{\mathrm{L}^2}\\ &\qquad \leq \frac{1}{2}\big(\|(f_1 - f_2)(t)\|^2_{\mathcal{H}^0_r} + \|(\mathfrak{m}_1 - \mathfrak{m}_2)(t)\|^2_{\mathrm{L}^2} + \|(u_1 - u_2)(t)\|^2_{\mathrm{L}^2}\big),\end{aligned}$$

from which we directly infer that $(f_1, \mathfrak{m}_1, u_1) = (f_2, \mathfrak{m}_2, u_2)$ on $[0, \overline{T}^{[\varepsilon]}]$. Repeating this procedure starting from $\overline{T}^{[\varepsilon]} < T'_\varepsilon$, we obtain $(f_1, \mathfrak{m}_1, u_1) = (f_2, \mathfrak{m}_2, u_2)$ on $[0, 2\overline{T}^{[\varepsilon]}]$. After a finite number of steps, we eventually obtain uniqueness on $[0, \widetilde{T}^{[\varepsilon]}]$.