Appendix C

Pseudodifferential calculus with a large parameter on $\mathbb{R}\times\mathbb{T}^d$

In this appendix, we collect several results on pseudodifferential calculus that we shall need in this work. We refer to [2, 119] for a more general and complete approach. Here, our framework is adapted to the physical space $\mathbb{R}\times\mathbb{T}^d$.

For any symbol of the form $a(t,x,\gamma,\tau,k)$ defined on $\mathbb{R}\times\mathbb{T}^d\times(0,+\infty)\times\mathbb{R}\times\mathbb{R}^d\backslash\{0\}$, we consider the quantization

$$\mathrm{Op}^{\gamma}(a)(h)(t,x)=\frac{1}{(2\pi)^{d+1}}\int_{\mathbb{R}\times\mathbb{Z}^d} e^{i(\tau t+k\cdot x)}a(t,x,\gamma,\tau,k)\mathcal{F}_{t,x}h(\tau,k)\,\mathrm{d}\tau\,\mathrm{d}k.$$

Here, we use the discrete measure on $\mathbb{Z}^d$. Above, the variable $\gamma>0$ should be seen as a parameter. The Fourier transform of a function $h(t,x)$ defined on $\mathbb{R}\times\mathbb{T}^d$ is denoted by

$$\mathcal{F}_{t,x}h(\tau,k)=\int_{\mathbb{R}\times\mathbb{Z}^d} e^{-i(\tau t+k\cdot x)}h(t,x)\,\mathrm{d}t\,\mathrm{d}x,$$

while the Fourier transform of a symbol $a(t,x,\gamma,\tau,k)$ defined on $\mathbb{R}\times\mathbb{T}^d\times(0,+\infty)\times\mathbb{R}\times\mathbb{R}^d\backslash\{0\}$ is written as

$$\mathcal{F}_{t,x}a(\theta,\ell,\gamma,\tau,k)=\int_{\mathbb{R}\times\mathbb{Z}^d} e^{-i(\theta t+\ell\cdot x)}a(t,x,\gamma,\tau,k)\,\mathrm{d}t\,\mathrm{d}x.$$

For the sake of readability, we introduce the notation

$$\eta:=(\gamma,\tau,k)\in(0,+\infty)\times\mathbb{R}\times\mathbb{R}^d\backslash\{0\},$$
$$|\eta|:=\left(\gamma^2+\tau^2+|k|^2\right)^{\frac{1}{2}},$$

and we write $\mathrm{L}^{\infty}_{t,x,\eta}$ to denote $\mathrm{L}^{\infty}(\mathbb{R}\times\mathbb{T}^d\times(0,+\infty)\times\mathbb{R}\times\mathbb{R}^d\backslash\{0\})$.

We state L^2 continuity results of Calderón–Vaillancourt type: namely, we shall ask for L^{∞} bounds in all the variables for the symbols of the operators (see [40, 102]). We first introduce the following family of seminorms for our symbols.

Notation C.1. For any $\mathrm{M}\geq 0$ and for any symbol $a(t,x,\eta)$ with $\eta=(\gamma,\tau,k)$, we set

$$\omega[a]:=\sup_{\substack{\alpha\in\mathbb{N}^d\\ \alpha_i\in\{0,1\}}}\|(1+t)\partial_x^{\alpha}a\|_{\mathrm{L}^{\infty}_{t,x,\eta}}+\sup_{\substack{\alpha\in\mathbb{N}^d\\ \alpha_i\in\{0,1\}}}\|(1+t)\partial_t\partial_x^{\alpha}a\|_{\mathrm{L}^{\infty}_{t,x,\eta}}, \tag{C.1}$$

$$\Omega[a] := \sup_{\substack{\alpha\in\mathbb{N}^d \\ \alpha_i\in\{0,1\}}} \left\{ \big\| |\eta| \partial_x^\alpha \nabla_{\tau,k} a \big\|_{L^\infty_{t,x,\eta}} + \big\| |\eta| \partial_x^\alpha \nabla_{\tau,k} \partial_t a \big\|_{L^\infty_{t,x,\eta}} \right\}$$
$$+ \sup_{\substack{\alpha\in\mathbb{N}^d \\ \alpha_i\in\{0,1\}}} \left\{ \big\| |\eta| \partial_x^\alpha \partial_\tau \nabla_{\tau,k} a \big\|_{L^\infty_{t,x,\eta}} + \big\| |\eta| \partial_x^\alpha \partial_\tau \nabla_{\tau,k} \partial_t a \big\|_{L^\infty_{t,x,\eta}} \right\}, \quad \text{(C.2)}$$
$$\Xi[a]_{\mathrm{M}} := \sup_{\substack{1\le|\alpha|\le 1+\mathrm{M} \\ \beta=0,1,2,3,4}} \big\| \partial_x^\alpha \partial_t^\beta a \big\|_{L^\infty_{t,x,\eta}}. \quad \text{(C.3)}$$

The following result states the L^2 continuity of a pseudodifferential operator with symbol having a finite seminorm $\omega[\cdot]$. We refer to [102, Theorem 1] for a proof for symbols with compact support in x, whose adaptation to the physical space $\mathbb{R} \times \mathbb{T}^d$ is fairly straightforward.[1]

Theorem C.2. *There exists $C_d > 0$ such that if a is a symbol satisfying $\omega[a] < +\infty$, then, for every $\gamma > 0$, we have*

$$\forall h \in L^2(\mathbb{R} \times \mathbb{T}^d), \quad \|\mathrm{Op}^\gamma(a)(h)\|_{L^2(\mathbb{R}\times\mathbb{T}^d)} \le C_d\, \omega[a] \|h\|_{L^2(\mathbb{R}\times\mathbb{T}^d)},$$

Next, we have the following symbolic calculus result, with the use of the parameter $\gamma > 0$. Note that taking γ large can be useful in view of an absorption argument. Here, we need to assume that one symbol has a compact support in time, because the time variable $t \in \mathbb{R}$ is unbounded.

Proposition C.3. *There exist $C_d > 0$ and a continuous nonnegative and nondecreasing function Λ such that, for all symbols a, b satisfying*

$$\Omega[a], \Xi[b]_{\mathrm{M}} < +\infty, \quad \mathrm{M} > 1 + 2d,$$
$$\nabla_x b \text{ has compact support in time},$$

and for all $\gamma > 0$ and $h \in L^2(\mathbb{R} \times \mathbb{T}^d)$, we have

$$\big\| \mathrm{Op}^\gamma(a)\mathrm{Op}^\gamma(b)(h) - \mathrm{Op}^\gamma(ab)(h) \big\|_{L^2(\mathbb{R}\times\mathbb{T}^d)}$$
$$\le \frac{C_d}{\gamma} \Lambda(|\mathrm{supp}_t\, \nabla_x b|) \Omega[a] \Xi[b]_{\mathrm{M}} \|h\|_{L^2(\mathbb{R}\times\mathbb{T}^d)}.$$

Proof. A standard formula about the composition of pseudodifferential operators first shows that

$$\mathrm{Op}^\gamma(a)\mathrm{Op}^\gamma(b) = \mathrm{Op}^\gamma(c),$$

[1] More precisely, one can introduce a weight $(1+t)^{-2}$ in [102, proof of Theorem 1, eq. (2.5)] to get some integrability in time. This turns out to be sufficient to consider the seminorm $\omega[\cdot]$ in our statement.

with

$$c(t,x,\gamma,\tau,k) = \frac{1}{(2\pi)^{d+1}}\int_{\mathbb{R}\times\mathbb{T}^d}\int_{\mathbb{R}\times\mathbb{Z}^d} e^{i(\tau'-\tau)(t-t')}e^{i(k'-k)\cdot(x-x')} \times a(t,x,\gamma,\tau',k')b(t',x',\gamma,\tau,k)\,\mathrm{d}t'\mathrm{d}x'\mathrm{d}\tau'\mathrm{d}k'$$

$$= \frac{1}{(2\pi)^{d+1}}\int_{\mathbb{R}\times\mathbb{Z}^d} e^{i(\tau't+k'\cdot x)}a(t,x,\gamma,\tau+\tau',k+k') \times \mathcal{F}_{t,x}b(\tau',k',\gamma,\tau,k)\,\mathrm{d}\tau'\mathrm{d}k'.$$

Therefore, for $\eta=(\gamma,\tau,k)\in(0,+\infty)\times\mathbb{R}\times\mathbb{R}^d\backslash\{0\}$, we have

$$c(t,x,\eta)-a(t,x,\eta)b(t,x,\eta) = \frac{1}{(2\pi)^{d+1}}\int_{\mathbb{R}\times\mathbb{Z}^d} e^{i(\tau't+k'\cdot x)}\mathcal{F}_{t,x}b(\tau',k',\eta) \times\left\{\int_0^1 \nabla_{\tau,k}a(t,x,\gamma,\tau+s\tau',k+sk')\cdot(\tau',k')\,\mathrm{d}s\right\}\mathrm{d}\tau'\mathrm{d}k' =: \frac{1}{\gamma}\mathrm{m}(t,x,\eta),$$

where

$$\mathrm{m}(t,x,\eta) := \frac{1}{(2\pi)^{d+1}}\int_{\mathbb{R}\times\mathbb{Z}^d} e^{i(\tau't+k'\cdot x)}\mathcal{F}_{t,x}b(\tau',k',\eta)\mathcal{J}(a)(t,x,\eta,\tau',k')\,\mathrm{d}\tau'\mathrm{d}k',$$

with

$$\mathcal{J}(a)(t,x,\eta,\tau',k') := \int_0^1 \gamma\nabla_{\tau,k}a(t,x,\gamma,\tau+s\tau',k+sk')\cdot(\tau',k')\,\mathrm{d}s.$$

Since $\mathrm{Op}^\gamma(a)\mathrm{Op}^\gamma(b)-\mathrm{Op}^\gamma(ab)=\frac{1}{\gamma}\mathrm{Op}^\gamma(\mathrm{m})$, and in view of the continuity property stated in Theorem C.2, it remains to estimate the seminorm $\omega[\mathrm{m}]$. For all $\alpha\in\mathbb{N}^d$ such that $\alpha_i\in\{0,1\}$, we now have

$$(1+t)\mathrm{m}(t,x,\eta) = \frac{1}{(2\pi)^{d+1}}\int_{\mathbb{R}\times\mathbb{Z}^d}(1-i\partial_{\tau'})(e^{i\tau't})e^{ik'\cdot x} \times\mathcal{F}_{t,x}b(\tau',k',\eta)\mathcal{J}(a)(t,x,\eta,\tau',k')\,\mathrm{d}\tau'\,\mathrm{d}k',$$

and

$$(1+t)\partial_t\mathrm{m}(t,x,\eta) = \frac{1}{(2\pi)^{d+1}}\int_{\mathbb{R}\times\mathbb{Z}^d} i\tau'(1-i\partial_{\tau'})\big(e^{i\tau't}\big)e^{ik'\cdot x}\mathcal{F}_{t,x}b(\tau',k',\eta) \times\mathcal{J}(a)(t,x,\eta,\tau',k')\,\mathrm{d}\tau'\,\mathrm{d}k'$$
$$+\frac{1}{(2\pi)^{d+1}}\int_{\mathbb{R}\times\mathbb{Z}^d}(1-i\partial_{\tau'})\big(e^{i\tau't}\big)e^{ik'\cdot x}\mathcal{F}_{t,x}b(\tau',k',\eta) \times\partial_t(\mathcal{J}(a))(t,x,\eta,\tau',k')\,\mathrm{d}\tau'\,\mathrm{d}k'.$$

Tedious but standard computations then show that for $\mathrm{M} > d$ we have

$$\begin{aligned}\omega[\mathrm{m}] &\lesssim \Omega[a] \sup_{\substack{\alpha\in\mathbb{N}^d\\ \alpha_i\in\{0,1\}}} \big\|(1+|\tau'|^2)|k'|^{1+|\alpha|}\mathcal{F}_{t,x}[(1+t)b]\big\|_{\mathrm{L}^1(\mathbb{R}_{\tau'}\times\mathbb{Z}^d_{k'};\mathrm{L}^\infty_\eta)} \\ &\lesssim \Omega[a]\Big(\sup_{1\le|\alpha|\le 1+\mathrm{M}} \big\|\mathcal{F}_{t,x}\big[(1+t)\partial_x^\alpha b\big]\big\|_{\mathrm{L}^1(\mathbb{R}\times\mathbb{Z}^d;\mathrm{L}^\infty_\eta)} \\ &\qquad + \sup_{\substack{1\le|\alpha|\le 1+\mathrm{M}\\ \beta=1,2}} \big\|\mathcal{F}_{t,x}\big[(1+t)\partial_x^\alpha\partial_t^\beta b\big]\big\|_{\mathrm{L}^1(\mathbb{R}\times\mathbb{Z}^d;\mathrm{L}^\infty_\eta)}\Big).\end{aligned}$$

As a consequence, we obtain, for $\mathrm{M} > 1 + 2d$,

$$\begin{aligned}\omega[\mathrm{m}] &\lesssim \Omega[a]\Big(\sup_{\substack{1\le|\alpha|\le 1+\mathrm{M}\\ \beta=0,1,2}} \big\|\mathcal{F}_{t,x}\big[(1+t)\partial_x^\alpha\partial_t^\beta b\big]\big\|_{\mathrm{L}^\infty(\mathbb{R}\times\mathbb{Z}^d;\mathrm{L}^\infty_\eta)} \\ &\qquad + \sup_{\substack{1\le|\alpha|\le 1+\mathrm{M}\\ \beta=0,1,2,3,4}} \big\|\mathcal{F}_{t,x}\big[(1+t)\partial_x^\alpha\partial_t^\beta b\big]\big\|_{\mathrm{L}^\infty(\mathbb{R}\times\mathbb{Z}^d;\mathrm{L}^\infty_\eta)}\Big) \\ &\lesssim \Omega[a] \sup_{\substack{1\le|\alpha|\le 1+\mathrm{M}\\ \beta=0,1,2,3,4}} \big\|(1+t)\partial_x^\alpha\partial_t^\beta b\big\|_{\mathrm{L}^1(\mathbb{R}\times\mathbb{T}^d;\mathrm{L}^\infty_\eta)},\end{aligned}$$

and therefore

$$\omega[\mathrm{m}] \lesssim \Lambda(|\mathrm{supp}_t\, \nabla_x b|)\Omega[a]\Xi[b]_{\mathrm{M}}.$$

From Theorem C.2, we have, for all $h \in \mathrm{L}^2(\mathbb{R}\times\mathbb{T}^d)$,

$$\begin{aligned}\|\mathrm{Op}^\gamma(\mathrm{m})h\|_{\mathrm{L}^2(\mathbb{R}\times\mathbb{T}^d)} &\le C_d\,\omega[\mathrm{m}]\|h\|_{\mathrm{L}^2(\mathbb{R}\times\mathbb{T}^d)} \\ &\lesssim C_d\,\Lambda(|\mathrm{supp}_t\, \nabla_x b|)\Omega[a]\Xi[b]_{\mathrm{M}}\|h\|_{\mathrm{L}^2(\mathbb{R}\times\mathbb{T}^d)}.\end{aligned}$$

We obtain the conclusion since $\mathrm{Op}^\gamma(a)\mathrm{Op}^\gamma(b) - \mathrm{Op}^\gamma(ab) = \frac{1}{\gamma}\mathrm{Op}^\gamma(\mathrm{m})$. ■