Chapter 3

Marking and gluing along geodesics

In our previous work [25], we proved Theorem 1.1 in the case of disks (for the GH topology) by writing Q_n and $\mathsf{S}^{[0]}_{(L^1)}$ as gluings of appropriate subspaces along geodesic segments, namely so-called *slices* in the discrete setting and their scaling limits in the continuum. The fact that the number of gluings needed was infinite caused some difficulties (which we mainly overcame by noticing that any geodesic between two typical points may be broken down to a finite number of pieces lying in different such subspaces). In contrast, in this work, we will only need to consider gluings of a *finite number* of subspaces along geodesic segments. As this operation is well behaved in a more general setting, we present it in this chapter. But first, we collect a number of useful lemmas on the GHP topology.

We will use the following notation. If $\mu_{\mathcal{X}}$ is a finite positive measure on a set $\mathcal{X}$, we let $\overline{\mu}_{\mathcal{X}} = \mu_{\mathcal{X}}/\mu_{\mathcal{X}}(\mathcal{X})$ be the normalized probability measure. If $\mu_{\mathcal{X}} = 0$, we use the convention $\overline{\mu}_{\mathcal{X}} = 0$. If $\boldsymbol{\mu} = (\mu^1, \dots, \mu^m)$ is a finite family of nonnegative measures, we let $\overline{\boldsymbol{\mu}} = (\overline{\mu}^1, \dots, \overline{\mu}^m)$.

3.1 Useful facts on the Gromov–Hausdorff–Prokhorov topology and markings

Recall the definitions of $(\mathbb{M}^{(\ell,m)}, \mathrm{d}^{(\ell,m)}_{\mathrm{GHP}})$ and $(\mathbb{M}^{(\ell)}, \mathrm{d}^{(\ell)}_{\mathrm{GH}})$ from Section 1.3. If the space $(\mathcal{X}, d_{\mathcal{X}}, \mathbf{A}, \boldsymbol{\mu}_{\mathcal{X}})$ is an element of $\mathbb{M}^{(\ell,m)}$ and $\mathbf{r} \in (\mathbb{Z}_{\geq 0})^m$ is such that

$$r^j = 0 \quad \text{whenever } \mu^j_{\mathcal{X}} = 0,$$

we may consider the variable

$$(\mathcal{X}, d_{\mathcal{X}}, \mathbf{A}(x^1_1, \dots, x^1_{r^1}) \cdots (x^m_1, \dots, x^m_{r^m}))$$

taking values in $\mathbb{M}^{(\ell+\|\mathbf{r}\|)}$, where, for each $j \in \{1, \dots, m\}$, the points $x^j_1, \dots, x^j_{r_j}$ are i.i.d. sampled random variables with law $\overline{\mu}^j_{\mathcal{X}}$ (if the latter measure is 0, then this still makes sense since $r^j = 0$); we denote by $\mathrm{Mark}_{\mathbf{r}}((\mathcal{X}, d_{\mathcal{X}}, \mathbf{A}, \boldsymbol{\mu}_{\mathcal{X}}), \cdot)$ the law of this random marked metric space. Some care is actually needed here since we are considering isometry classes of metric measure spaces. See [79] for an accurate definition of this notion, which is immediately generalized to our setting where we incorporate the extra marks given by $\mathbf{A}$, and several measures. The following lemma states that one can formulate the GHP convergence entirely in terms of the GH convergence of randomly marked spaces.

Lemma 3.1. *Let* $(\mathcal{X}_n, d_{\mathcal{X}_n}, \mathbf{A}_n, \boldsymbol{\mu}_{\mathcal{X}_n})$*,* $n \geq 1$*, and* $(\mathcal{X}, d_{\mathcal{X}}, \mathbf{A}, \boldsymbol{\mu}_{\mathcal{X}})$ *be elements of* $\mathbb{M}^{(\ell,m)}$*. The following statements are equivalent:*

(i) *The space* $(\mathcal{X}_n, d_{\mathcal{X}_n}, \mathbf{A}_n, \boldsymbol{\mu}_{\mathcal{X}_n})$ *converges to the space* $(\mathcal{X}, d_{\mathcal{X}}, \mathbf{A}, \boldsymbol{\mu}_{\mathcal{X}})$ *in* $(\mathbb{M}^{(\ell,m)}, \mathrm{d}_{\mathrm{GHP}}^{(\ell,m)})$.

(ii) *One has* $\boldsymbol{\mu}_{\mathcal{X}_n}(\mathcal{X}_n) \to \boldsymbol{\mu}_{\mathcal{X}}(\mathcal{X})$ *coordinatewise as* $n \to \infty$ *and, for every* $\mathbf{r} \in (\mathbb{Z}_{\geq 0})^m$ *such that* $r^j = 0$ *whenever* $\mu^j_{\mathcal{X}} = 0$*, it holds that*

$$\mathrm{Mark}_{\mathbf{r}}((\mathcal{X}_n, d_{\mathcal{X}_n}, \mathbf{A}_n, \boldsymbol{\mu}_{\mathcal{X}_n}), \cdot) \underset{n\to\infty}{\longrightarrow} \mathrm{Mark}_{\mathbf{r}}((\mathcal{X}, d_{\mathcal{X}}, \mathbf{A}, \boldsymbol{\mu}_{\mathcal{X}}), \cdot)$$

for weak convergence of probability measures on $(\mathbb{M}^{(\ell+\|\mathbf{r}\|)}, \mathrm{d}_{\mathrm{GH}}^{(\ell+\|\mathbf{r}\|)})$.

Proof. The implication (i) ⇒ (ii) is an easy generalization of known results. See [79, Proposition 10] for the case where the measures are probability measures, and [68, Section 2.2] for a generalized context with finite measures; our extended context of marked measured metric spaces adds no difficulty. To show the converse implication, we argue as follows. By taking the trivial case $\mathbf{r} = \mathbf{0}^m$ of (ii), we obtain that $\{(\mathcal{X}_n, d_{\mathcal{X}_n}, \mathbf{A}_n), n \geq 1\}$ is relatively compact in $(\mathbb{M}^{(\ell)}, \mathrm{d}_{\mathrm{GH}}^{(\ell)})$. Since the sequences $(\mu^j_{\mathcal{X}_n}(\mathcal{X}_n), n \geq 1)$ are bounded, this implies that $\{(\mathcal{X}_n, d_{\mathcal{X}_n}, \mathbf{A}_n, \boldsymbol{\mu}_{\mathcal{X}_n}), n \geq 1\}$ is relatively compact in $(\mathbb{M}^{(\ell,m)}, \mathrm{d}_{\mathrm{GHP}}^{(\ell,m)})$. So let $(\mathcal{X}', d_{\mathcal{X}'}, \mathbf{A}', \boldsymbol{\mu}_{\mathcal{X}'})$ be a limit in $\mathbb{M}^{(\ell,m)}$ along some subsequence of $(\mathcal{X}_n, d_{\mathcal{X}_n}, \mathbf{A}_n, \boldsymbol{\mu}_{\mathcal{X}_n})$. By using the implication (i) ⇒ (ii), we obtain that, for every $\mathbf{r}$ such that $r^j = 0$ whenever $\mu^j_{\mathcal{X}} = 0$,

$$\mathrm{Mark}_{\mathbf{r}}((\mathcal{X}, d_{\mathcal{X}}, \mathbf{A}, \boldsymbol{\mu}_{\mathcal{X}}), \cdot) = \mathrm{Mark}_{\mathbf{r}}((\mathcal{X}', d_{\mathcal{X}'}, \mathbf{A}', \boldsymbol{\mu}_{\mathcal{X}'}), \cdot).$$

Now, let m' be the number of nonzero elements of $\boldsymbol{\mu}_{\mathcal{X}}$, fix $r > 0$ and set $r^j = r\mathbf{1}_{\{\mu^j_{\mathcal{X}} \neq 0\}}$. Let $(\mathcal{X}, d_{\mathcal{X}}, (A^1, \ldots, A^\ell, x^1_1, \ldots, x^1_{r^1}, \ldots, x^m_1, \ldots, x^m_{r^m}))$ be the $(\ell + rm')$-marked metric space with law $\mathrm{Mark}_{\mathbf{r}}((\mathcal{X}, d_{\mathcal{X}}, \mathbf{A}, \boldsymbol{\mu}_{\mathcal{X}}), \cdot)$, and set $\boldsymbol{\theta}_r = (\theta^j_r, 1 \leq j \leq m)$, where $\theta^j_r = r^{-1} \sum_{i=1}^r \delta_{x^j_i}$ if $\mu^j_{\mathcal{X}} \neq 0$ and $\theta^j_r = 0$ if $\mu^j_{\mathcal{X}} = 0$. It is a consequence of the law of large numbers that $(\mathcal{X}, d_{\mathcal{X}}, \mathbf{A}, \boldsymbol{\theta}_r)$ converges almost surely in $\mathbb{M}^{(\ell,m)}$, as $r \to \infty$, to $(\mathcal{X}, d_{\mathcal{X}}, \mathbf{A}, \overline{\boldsymbol{\mu}}_{\mathcal{X}})$; see, for instance, [68, Lemma 5]. Applying this same result to $(\mathcal{X}', d_{\mathcal{X}'}, \mathbf{A}', \boldsymbol{\mu}_{\mathcal{X}'})$ allows us to show that $(\mathcal{X}', d_{\mathcal{X}'}, \mathbf{A}', \overline{\boldsymbol{\mu}}'_{\mathcal{X}'})$ is isometry-equivalent to $(\mathcal{X}, d_{\mathcal{X}}, \mathbf{A}, \overline{\boldsymbol{\mu}}_{\mathcal{X}})$. Since $\boldsymbol{\mu}_{\mathcal{X}}(\mathcal{X}) = \boldsymbol{\mu}_{\mathcal{X}'}(\mathcal{X}')$ is the limit of $\boldsymbol{\mu}_{\mathcal{X}_n}(\mathcal{X}_n)$, we conclude. ■

We also recall that, often, the most useful way to estimate GH distances is via the notion of *distortion of a correspondence*. A *correspondence* between two sets $\mathcal{X}$ and $\mathcal{Y}$ is a subset $\mathcal{R} \subseteq \mathcal{X} \times \mathcal{Y}$ whose coordinate projections are $\mathcal{X}$ and $\mathcal{Y}$. We will often write $x \,\mathcal{R}\, y$ instead of $(x, y) \in \mathcal{R}$. If $\mathcal{X}$ and $\mathcal{Y}$ are endowed with the metrics $d_{\mathcal{X}}$ and $d_{\mathcal{Y}}$, the *distortion* of the correspondence $\mathcal{R}$ is the number

$$\mathrm{dis}(\mathcal{R}) = \sup\{|d_{\mathcal{X}}(x, x') - d_{\mathcal{Y}}(y, y')| : x \,\mathcal{R}\, y,\ x' \,\mathcal{R}\, y'\}.$$

If $\mathbf{A} = (A^1, \dots, A^\ell)$ and $\mathbf{B} = (B^1, \dots, B^\ell)$ are markings of $\mathcal{X}$ and of $\mathcal{Y}$, we say that the correspondence $\mathcal{R}$ between $\mathcal{X}$ and $\mathcal{Y}$ is *compatible* with the markings if for every $1 \le i \le \ell$, $\mathcal{R} \cap (A^i \times B^i)$ is a correspondence between A^i and B^i.

Lemma 3.2 ([79, Section 6.4]). *It holds that*

$$\mathrm{d}_{\mathrm{GH}}^{(\ell)}((\mathcal{X}, \mathbf{A}, d_{\mathcal{X}}), (\mathcal{Y}, \mathbf{B}, d_{\mathcal{Y}})) = \frac{1}{2} \inf_{\mathcal{R}} \mathrm{dis}(\mathcal{R}),$$

where the infimum is taken over correspondences compatible with the markings.

Correspondences are also useful for estimating GHP distances when used together with the notion of couplings, which are measures on the product of the two spaces to be compared. The following is a direct adaptation of [68, Lemma 4], which treats the case of $\mathbb{M}^{(0,1)}$.

Lemma 3.3. *Let $(\mathcal{X}, d_{\mathcal{X}}, \mathbf{A}, \boldsymbol{\mu}_{\mathcal{X}})$ and $(\mathcal{Y}, d_{\mathcal{Y}}, \mathbf{B}, \boldsymbol{\mu}_{\mathcal{Y}})$ be elements of $\mathbb{M}^{(\ell,m)}$ for some ℓ, $m \ge 0$. Let $\varepsilon > 0$, and let $\mathcal{R}$ be a correspondence between $\mathcal{X}$ and $\mathcal{Y}$ compatible with the markings and of distortion bounded above by ε. For $1 \le j \le m$, let ν^j be a finite measure on the product $\mathcal{X} \times \mathcal{Y}$ such that $\nu^j(\mathcal{R}^c) < \varepsilon$ and, letting $p_{\mathcal{X}}$, $p_{\mathcal{Y}}$ be the coordinate projections onto $\mathcal{X}$ and $\mathcal{Y}$,*

$$d_{\mathcal{X}}^{\mathrm{P}}(\mu_{\mathcal{X}}^j, (p_{\mathcal{X}})_* \nu^j) \vee d_{\mathcal{Y}}^{\mathrm{P}}(\mu_{\mathcal{Y}}^j, (p_{\mathcal{Y}})_* \nu^j) < \varepsilon.$$

Then $\mathrm{d}_{\mathrm{GHP}}^{(\ell,m)}((\mathcal{X}, d_{\mathcal{X}}, \mathbf{A}, \boldsymbol{\mu}_{\mathcal{X}}), (\mathcal{Y}, d_{\mathcal{Y}}, \mathbf{B}, \boldsymbol{\mu}_{\mathcal{Y}})) \le 3\varepsilon$.

Finally, we state an elementary lemma whose proof is straightforward and omitted.

Lemma 3.4. *The mappings*

$$(\mathcal{X}, d_{\mathcal{X}}, (A^1, \dots, A^\ell), \boldsymbol{\mu}_{\mathcal{X}}) \mapsto (\mathcal{X}, d_{\mathcal{X}}, (A^1 \cup A^2, A^3, \dots, A^\ell), \boldsymbol{\mu}_{\mathcal{X}}),$$
$$(\mathcal{X}, d_{\mathcal{X}}, (A^1, \dots, A^\ell), \boldsymbol{\mu}_{\mathcal{X}}) \mapsto (\mathcal{X}, d_{\mathcal{X}}, (A^1, \dots, A^{\ell-1}), \boldsymbol{\mu}_{\mathcal{X}})$$

are 1*-Lipschitz from* $(\mathbb{M}^{(\ell,m)}, \mathrm{d}_{\mathrm{GHP}}^{(\ell,m)})$ *to* $(\mathbb{M}^{(\ell-1,m)}, \mathrm{d}_{\mathrm{GHP}}^{(\ell-1,m)})$*; the mappings*

$$(\mathcal{X}, d_{\mathcal{X}}, \mathbf{A}, (\mu_{\mathcal{X}}^1, \dots, \mu_{\mathcal{X}}^m)) \mapsto (\mathcal{X}, d_{\mathcal{X}}, \mathbf{A}, (\mu_{\mathcal{X}}^1 + \mu_{\mathcal{X}}^2, \mu_{\mathcal{X}}^3, \dots, \mu_{\mathcal{X}}^m)),$$
$$(\mathcal{X}, d_{\mathcal{X}}, \mathbf{A}, (\mu_{\mathcal{X}}^1, \dots, \mu_{\mathcal{X}}^m)) \mapsto (\mathcal{X}, d_{\mathcal{X}}, \mathbf{A}, (\mu_{\mathcal{X}}^1, \dots, \mu_{\mathcal{X}}^{m-1}))$$

are 2*-Lipschitz and* 1*-Lipschitz from* $(\mathbb{M}^{(\ell,m)}, \mathrm{d}_{\mathrm{GHP}}^{(\ell,m)})$ *to* $(\mathbb{M}^{(\ell,m-1)}, \mathrm{d}_{\mathrm{GHP}}^{(\ell,m-1)})$*; and, for every permutation σ of $\{1, 2, \dots, \ell\}$ and τ of $\{1, 2, \dots, m\}$,*

$$(\mathcal{X}, d_{\mathcal{X}}, (A^1, \dots, A^\ell), (\mu_{\mathcal{X}}^1, \dots, \mu_{\mathcal{X}}^m))$$
$$\mapsto (\mathcal{X}, d_{\mathcal{X}}, (A^{\sigma(1)}, \dots, A^{\sigma(\ell)}), (\mu_{\mathcal{X}}^{\tau(1)}, \dots, \mu_{\mathcal{X}}^{\tau(m)}))$$

is an isometry from $(\mathbb{M}^{(\ell,m)}, \mathrm{d}_{\mathrm{GHP}}^{(\ell,m)})$ *onto itself.*

3.2 Geodesics in metric spaces

We now discuss the important notion of geodesics in metric spaces, as well as its relations with GHP limits.

In a metric space $(\mathcal{X}, d_{\mathcal{X}})$, compact or not, a *geodesic* is a mapping $\chi\colon [0,\ell] \to \mathcal{X}$ defined on some compact interval[1] $[0,\ell]$ and that is isometric, i.e., satisfies

$$d_{\mathcal{X}}(\chi(s), \chi(t)) = |t-s|, \quad 0 \le s, t \le \ell. \tag{3.1}$$

The points $\chi(0)$, $\chi(\ell)$ are called the *extremities* of χ, and $\ell = d_{\mathcal{X}}(\chi(0), \chi(\ell))$ is the *length* of the geodesic, denoted by $\mathrm{length}_{d_{\mathcal{X}}}(\chi)$, or simply $\mathrm{length}(\chi)$ when there is little risk of ambiguity. The space $(\mathcal{X}, d_{\mathcal{X}})$ is called a *geodesic space* if, for every pair of points $x, y \in \mathcal{X}$, there exists a geodesic with extremities x, y.

The range $\chi([0, \mathrm{length}(\chi)])$ of a geodesic path is called a *geodesic segment*. An *oriented geodesic segment* is a pair $(\chi(0), \chi([0, \mathrm{length}(\chi)]))$ made of a geodesic segment and a distinguished extremity, called its *origin*. Note that an oriented geodesic segment uniquely determines the geodesic χ, since $\chi(t)$ is the unique point at distance t away from the origin. For this reason, we will systematically identify geodesics with oriented geodesic segments and use the same piece of notation for both of them.

In a marked measured metric space $(\mathcal{X}, d_{\mathcal{X}}, \mathbf{A}, \boldsymbol{\mu}_{\mathcal{X}})$, some pairs (A^i, A^j) of marks might be oriented geodesic segments; such pairs are called *geodesic marks*.

Geodesic marks and GHP limits. The following proposition states that geodesic marks nicely pass to the limit in the GHP topology.

Proposition 3.5. *Let $(\mathcal{X}_n, d_{\mathcal{X}_n}, \mathbf{A}_n, \boldsymbol{\mu}_{\mathcal{X}_n})$, $n \ge 1$, be a sequence of marked measured compact metric spaces that converges to some limit $(\mathcal{X}, d_{\mathcal{X}}, \mathbf{A}, \boldsymbol{\mu}_{\mathcal{X}})$ in the GHP topology. Suppose that i, j are fixed and that, for every n, the pair of marks $(A^i_n, A^j_n) = \gamma_n$ is a geodesic mark. Then the pair of marks $(A^i, A^j) = \gamma$ of $\mathbf{A}$ is also a geodesic mark. Moreover, it holds that*

$$\mathrm{length}(\gamma) = \lim_{n\to\infty} \mathrm{length}(\gamma_n).$$

Proof. The wanted property deals only with the marks and not with the measures, so it suffices to establish the proposition in the space $\mathbb{M}^{(\ell)}$ of marked, nonmeasured spaces. Without loss of generality (by Lemma 3.4), we may and will assume that $i = 1$ and $j = 2$.

By Lemma 3.2, we may find a sequence of correspondences $\mathcal{R}_n$ between $\mathcal{X}_n$ and $\mathcal{X}$ that is compatible with the markings $\mathbf{A}_n$ and $\mathbf{A}$, and whose distortion

[1] We allow $\ell = 0$ in this definition.

$\varepsilon_n := \mathrm{dis}(\mathcal{R}_n)$ goes to zero. From now on, we will never need to refer to marks other than the first two.

Let $y, z \in A^1$. Since A_n^1 contains a single point, which we denote by $x_n = \gamma_n(0)$, we have $x_n \mathcal{R}_n y$ and $x_n \mathcal{R}_n z$, so that $d_{\mathcal{X}}(y,z) \leq \varepsilon_n$ for every $n \geq 1$, entailing $y = z$. So A^1 is a singleton, which we denote by $A^1 = \{x\}$.

Next, let $a \in A^2$ and $a_n \in A_n^2$ be such that $a_n \mathcal{R}_n a$. Then

$$|d_{\mathcal{X}}(x,a) - d_{\mathcal{X}_n}(x_n, a_n)| \leq \varepsilon_n,$$

which implies that $d_{\mathcal{X}_n}(x_n, a_n) \to d_{\mathcal{X}}(x,a)$ as $n \to \infty$, and in particular, $d_{\mathcal{X}}(x,a) \leq \liminf_{n\to\infty} \mathrm{length}(\gamma_n)$, and therefore

$$\max_{a \in A^2} d_{\mathcal{X}}(x,a) \leq \liminf_{n\to\infty} \mathrm{length}(\gamma_n).$$

In the other direction, let $t \leq \limsup_{n\to\infty} \mathrm{length}(\gamma_n)$. We claim that there exists at least a point $c_t \in A^2$ such that $d_{\mathcal{X}}(x, c_t) = t$. This will entail that $\mathrm{length}(\gamma_n)$ converges to $\ell = \max_{a\in A^2} d_{\mathcal{X}}(x,a)$. To see the claim, observe that, at least along a suitable extraction, there exists a sequence $t_n \to t$ such that $\mathrm{length}(\gamma_n) \geq t_n$. Along this extraction, let $\gamma_n(t_n)$ be the unique point of γ_n such that $t_n = d_{\mathcal{X}_n}(x_n, \gamma_n(t_n))$, and let g_n be an element of A^2 such that $\gamma_n(t_n) \mathcal{R}_n g_n$. Then $|d_{\mathcal{X}}(x, g_n) - t_n| \leq \varepsilon_n$, so that, possibly by further extracting, (g_n) converges to a limit $c_t \in A^2$. It then holds that $d_{\mathcal{X}}(x, c_t) = t$, as claimed.

Now fix $s, t \in [0, \ell]$ with $s \leq t$, and let $a, b \in A^2$ be such that $d_{\mathcal{X}}(x,a) = s$ and $d_{\mathcal{X}}(x,b) = t$. By the triangle inequality, we have $d_{\mathcal{X}}(a,b) \geq t - s$, and, on the other hand, if $a_n \mathcal{R}_n a$ and $b_n \mathcal{R}_n b$ with $a_n, b_n \in A_n^2$, then

$$\begin{aligned} d_{\mathcal{X}}(a,b) \leq d_{\mathcal{X}_n}(a_n, b_n) + \varepsilon_n &= |d_{\mathcal{X}_n}(x_n, b_n) - d_{\mathcal{X}_n}(x_n, a_n)| + \varepsilon_n \\ &\leq |d_{\mathcal{X}}(x,b) - d_{\mathcal{X}}(x,a)| + 3\varepsilon_n \\ &= t - s + 3\varepsilon_n, \end{aligned}$$

where in the second line, we have used the fact that a_n, b_n lie on a geodesic having x_n as one of its extremities. Letting $n \to \infty$, this shows that $d_{\mathcal{X}}(a,b) = t - s$, and in particular, taking $t = s$ shows that the point c_t of the preceding paragraph is the unique point of A^2 at distance t from x. We conclude that γ is an oriented geodesic segment with length ℓ and origin x. ■

Maps as compact geodesic metric spaces. So far, we have been seeing maps as finite metric spaces. We may also interpret a map $\mathbf{m}$ as a compact geodesic metric space, by viewing each edge as isometric to a real segment of length 1 (this is called the metric graph [32] associated with $\mathbf{m}$). Note that the restriction of the metric to the subset corresponding to the vertex set of $\mathbf{m}$ is the graph metric, so that the two metric spaces corresponding to $\mathbf{m}$ are at $\mathrm{d_{GH}}$-distance less than $1/2$. In the scaling limit, this bears no effects.

With this point of view on maps, note that, in the notation of Sections 2.3 and 2.4,

- (ρ, γ) and $(\overline{\rho}, \xi)$ are geodesic marks of **sl**,
- (ρ, γ), $(\overline{\rho}, \xi)$, $(\overline{\rho}, \overline{\gamma})$, and $(\rho, \overline{\xi})$ are geodesic marks of **qd**.

3.3 Gluing along geodesics

Quotient pseudometrics. Let $(\mathcal{X}, d)$ be a pseudometric space, that is, a set equipped with a symmetric function $d\colon \mathcal{X}^2 \to \mathbb{R}_{\geq 0} \sqcup \{\infty\}$ that vanishes on the diagonal and satisfies the triangle inequality. Then $\{d = 0\}$ is an equivalence relation on $\mathcal{X}$, and the quotient set $\mathcal{X}/\{d = 0\}$ equipped with the function induced by d (still denoted by d for simplicity), is a true metric space, meaning that d is also separated.

Let R be an equivalence relation on $\mathcal{X}$. Let d/R be the largest pseudometric on $\mathcal{X}$ such that $d/R \leq d$ and that satisfies $d/R(x, y) = 0$ as soon as $x \mathrel{R} y$. By [32, Theorem 3.1.27], it is given by the formula

$$d/R(x, y) = \inf\Bigg\{ \sum_{i=1}^{m} d(x_i, y_i) : m \geq 1,\ x_1, \ldots, x_m, y_1, \ldots, y_m \in \mathcal{X},\ x_1 = x, \\ y_m = y,\ y_i \mathrel{R} x_{i+1} \text{ for } i \in \{1, \ldots, m-1\} \Bigg\}. \tag{3.2}$$

In this setting, the set $\{d/R = 0\}$ is another equivalence relation on $\mathcal{X}$ that contains R, possibly strictly. We let $(\mathcal{X}, d)/R = (\mathcal{X}/\{d/R = 0\}, d/R)$ and call it the gluing of $(\mathcal{X}, d)$ along R.

A simple observation is that if R_1, R_2 are two equivalence relations on $\mathcal{X}$, then we have the equality of pseudometrics on $\mathcal{X}$

$$(d/R_1)/R_2 = (d/R_2)/R_1 = d/R, \tag{3.3}$$

where R is the coarsest equivalence relation containing $R_1 \cup R_2$. This expression is indeed a direct consequence of (3.2) and the fact that $x \mathrel{R} y$ if and only if there exist some integer m and points $x_0 = x, x_1, \ldots, x_m = y$ such that $(x_{i-1}, x_i) \in R_1 \cup R_2$ for every $i \in \{1, 2, \ldots, m\}$.

Gluing two spaces along geodesics. Let $(\mathcal{X}, d_{\mathcal{X}})$, $(\mathcal{Y}, d_{\mathcal{Y}})$ be two pseudometric spaces and γ, ξ be two geodesics in $\mathcal{X}$ and $\mathcal{Y}$, respectively, where the definition of a geodesic given by (3.1) is naturally extended to pseudometric spaces. The pseudometric of the disjoint union $\mathcal{X} \sqcup \mathcal{Y}$ is defined by

$$d_{\mathcal{X} \sqcup \mathcal{Y}}(x, y) = \begin{cases} d_{\mathcal{X}}(x, y) & \text{if } x, y \in \mathcal{X}, \\ d_{\mathcal{Y}}(x, y) & \text{if } x, y \in \mathcal{Y}, \\ \infty & \text{otherwise.} \end{cases}$$

We define the metric gluing of $\mathcal{X}$ and $\mathcal{Y}$ along γ and ξ by letting $\ell = \text{length}(\gamma) \wedge \text{length}(\xi)$ and by setting

$$G(\mathcal{X}, \mathcal{Y}; \gamma, \xi) = (\mathcal{X} \sqcup \mathcal{Y}, d_{\mathcal{X} \sqcup \mathcal{Y}})/R, \tag{3.4}$$

where R is the coarsest equivalence relation satisfying $\gamma(t)\ R\ \xi(t)$ for every $t \in [0, \ell]$.

In this particular case, the fact that γ and ξ are geodesics greatly simplifies (3.2). Indeed, $y_i\ R\ x_{i+1}$ and $y_{i+1}\ R\ x_{i+2}$ imply that $d_{\mathcal{X} \sqcup \mathcal{Y}}(y_i, x_{i+2}) = d_{\mathcal{X} \sqcup \mathcal{Y}}(x_{i+1}, y_{i+1})$. In other words, using twice the relation R does not create shortcuts. As a result, the pseudometric of the gluing is the function d_G whose restrictions to $\mathcal{X} \times \mathcal{X}$ and $\mathcal{Y} \times \mathcal{Y}$ are $d_{\mathcal{X}}$ and $d_{\mathcal{Y}}$, respectively, and

$$d_G(x, y) = d_G(y, x) = \inf_{t \in [0, \ell]} \{d_{\mathcal{X}}(x, \gamma(t)) + d_{\mathcal{Y}}(\xi(t), y)\} \quad \text{if } x \in \mathcal{X},\ y \in \mathcal{Y}. \tag{3.5}$$

Remark 3.6. In fact, equation (3.5) holds in the more general setting of gluing along isometric subspaces [31, Chapter I.5], the underlying isometry in our context being $\gamma(t) \mapsto \xi(t)$. We will not need this level of generality here.

If $(\mathcal{X}, d_{\mathcal{X}}, \mathbf{A}, \boldsymbol{\mu}_{\mathcal{X}}) \in \mathbb{M}^{(\ell, m)}$ and $(\mathcal{Y}, d_{\mathcal{Y}}, \mathbf{B}, \boldsymbol{\mu}_{\mathcal{Y}}) \in \mathbb{M}^{(\ell', m')}$ are marked measured metric spaces, we may view $G(\mathcal{X}, \mathcal{Y}; \gamma, \xi)$ as an element of $\mathbb{M}^{(\ell+\ell', m+m')}$ by assigning marks and measures

$$(\mathbf{p}(A^1), \ldots, \mathbf{p}(A^\ell), \mathbf{p}(B^1), \ldots, \mathbf{p}(B^{\ell'}))$$

and

$$(\mathbf{p}_* \mu^1_{\mathcal{X}}, \ldots, \mathbf{p}_* \mu^m_{\mathcal{X}}, \mathbf{p}_* \mu^1_{\mathcal{Y}}, \ldots, \mathbf{p}_* \mu^{m'}_{\mathcal{Y}}),$$

where $\mathbf{p}\colon \mathcal{X} \sqcup \mathcal{Y} \to G(\mathcal{X}, \mathcal{Y}; \gamma, \xi)$ is the canonical projection. With a slight abuse of notation, we will keep denoting these by $\mathbf{AB}$ and $\boldsymbol{\mu}_{\mathcal{X}} \boldsymbol{\mu}_{\mathcal{Y}}$. Observe that γ and ξ may themselves be part of the marking, in which case they induce the same marks $\mathbf{p}(\gamma) = \mathbf{p}(\xi)$ in the glued space. Observe also that geodesic marks in $\mathbf{A}$ or in $\mathbf{B}$ remain geodesic marks in $\mathbf{AB}$, due to the fact that, by definition, $(\mathcal{X}, d_{\mathcal{X}})$ and $(\mathcal{Y}, d_{\mathcal{Y}})$ are isometrically embedded in $G(\mathcal{X}, \mathcal{Y}; \gamma, \xi)$.

Finally, observe that the gluing of the point space as an element of $\mathbb{M}^{(\ell, m)}$ with $(\mathcal{Y}, d_{\mathcal{Y}}, \mathbf{B}, \boldsymbol{\mu}_{\mathcal{Y}})$ along ξ only has the effect of prepending ℓ times $\xi(0)$ to $\mathbf{B}$ and m times the zero measure to $\boldsymbol{\mu}_{\mathcal{Y}}$.

Gluing two geodesics in the same space. A similar gluing procedure[2] can be defined for two geodesics γ, ξ in the same pseudometric space $(\mathcal{X}, d_{\mathcal{X}})$. We again set $\ell = \text{length}(\gamma) \wedge \text{length}(\xi)$ and then define

$$G(\mathcal{X}; \gamma, \xi) = (\mathcal{X}, d_{\mathcal{X}})/R,$$

[2] In fact, since we are allowing points at infinite distance, the gluing $G(\mathcal{X}, \mathcal{Y}; \gamma, \xi)$ could be seen as a particular case of gluing of a single space along two geodesics, but we refrain to do so as we will mostly be interested in gluing true metric spaces.

where R is the coarsest equivalence relation satisfying $\gamma(t)\ R\ \xi(t)$ for every $t \in [0, \ell]$. The quotient pseudometric $d_G(x, y)$ may be condensed into

$$d_{\mathcal{X}}(x, y) \wedge \inf_{t\in[0,\ell]} \{d_{\mathcal{X}}(x, \gamma(t)) + d_{\mathcal{X}}(\xi(t), y)\} \\ \wedge \inf_{t\in[0,\ell]} \{d_{\mathcal{X}}(x, \xi(t)) + d_{\mathcal{X}}(\gamma(t), y)\}. \tag{3.6}$$

Similarly to the above, the space $G(\mathcal{X}; \gamma, \xi)$ naturally inherits the marking $\mathbf{A}$ and measures $\boldsymbol{\mu}_{\mathcal{X}}$ that $\mathcal{X}$ may be endowed with, simply by pushing those forward by the canonical projection $\mathcal{X} \to G(\mathcal{X}; \gamma, \xi)$; by a slight abuse of notation, we keep the piece of notation $\mathbf{A}$, $\boldsymbol{\mu}_{\mathcal{X}}$ for these inherited objects. Note however that it is *not true in general* that geodesic marks in $(\mathcal{X}, d_{\mathcal{X}})$ remain geodesic marks in $G(\mathcal{X}; \gamma, \xi)$. Let us state a useful comparison result between $d_{\mathcal{X}}$ and d_G.

Lemma 3.7. *Let $(\mathcal{X}, d_{\mathcal{X}})$ be a pseudometric space with two distinguished geodesics γ, ξ. Denote by d_G the pseudometric on $G(\mathcal{X}; \gamma, \xi)$ as in (3.6).*

(i) *For every $x, y \in \mathcal{X}$,*

$$d_G(x, y) \le d_{\mathcal{X}}(x, y) \le d_G(x, y) + R(\gamma, \xi),$$

where $R(\gamma, \xi)$ is the Hausdorff distance in the space $(\mathcal{X}, d_{\mathcal{X}})$ between the initial segments of γ, ξ that are glued together, i.e., of length ℓ.

(ii) *For every $\varepsilon > 0$ and $x, y \in \mathcal{X}$, if $d_{\mathcal{X}}(x, y) < \varepsilon$ and $d_{\mathcal{X}}(x, \gamma) \wedge d_{\mathcal{X}}(y, \gamma) > \varepsilon$ (or if $d_{\mathcal{X}}(x, \xi) \wedge \mathrm{d}_{\mathcal{X}}(y, \xi) > \varepsilon$), it holds that*

$$d_G(x, y) = d_{\mathcal{X}}(x, y).$$

Proof. Let us first prove (i). The first inequality is a direct consequence of (3.6). To prove the other bound, simply observe that for every $t \in [0, \ell]$ we have

$$d_{\mathcal{X}}(x, y) \le d_{\mathcal{X}}(x, \gamma(t)) + d_{\mathcal{X}}(\gamma(t), \xi(t)) + d_{\mathcal{X}}(\xi(t), y) \\ \le d_{\mathcal{X}}(x, \gamma(t)) + d_{\mathcal{X}}(\xi(t), y) + R(\gamma, \xi).$$

Taking the infimum over t, and then applying the same reasoning with the roles of γ, ξ interchanged, we obtain the result by (3.6).

The proof of (ii) is even more straightforward. Under our assumptions, it holds that both $d_{\mathcal{X}}(x, \gamma(t)) + d_{\mathcal{X}}(y, \xi(t))$ and $d_{\mathcal{X}}(x, \xi(t)) + d_{\mathcal{X}}(y, \gamma(t))$ are greater than $\varepsilon > d_{\mathcal{X}}(x, y)$ for every choice of t, so that $d_G(x, y)$ must be equal to $d_{\mathcal{X}}(x, y)$. ■

Gluing and GHP limits. Henceforth, we mostly focus on compact geodesic spaces. The gluing of one or two geodesic spaces along geodesics is again a geodesic space by general results presented in [32]. The gluing operation also preserves the compactness of the spaces that are glued together. Furthermore, if a compact metric space is the

Gromov–Hausdorff limit of a sequence of compact geodesic spaces, then it is also a compact geodesic space [32, Theorem 7.5.1].

The next result shows that the gluing operations behave well with respect to the GHP metric. For simplicity, we state it with the first marks of the markings but it obviously holds up to index permutations, using Lemma 3.4, for instance.

Proposition 3.8. *Let* $(\mathcal{X}_n, d_{\mathcal{X}_n}, \mathbf{A}_n, \boldsymbol{\mu}_{\mathcal{X}_n})$ *and* $(\mathcal{Y}_n, d_{\mathcal{Y}_n}, \mathbf{B}_n, \boldsymbol{\mu}_{\mathcal{Y}_n})$ *be geodesic marked measured metric spaces that converge in the marked GHP topology to* $(\mathcal{X}, d_{\mathcal{X}}, \mathbf{A}, \boldsymbol{\mu}_{\mathcal{X}})$, $(\mathcal{Y}, d_{\mathcal{Y}}, \mathbf{B}, \boldsymbol{\mu}_{\mathcal{Y}})$. *Assume that the first pairs of marks* $(A_n^1, A_n^2) = \gamma_n$ *and* $(B_n^1, B_n^2) = \xi_n$ *of* $\mathcal{X}_n$ *and of* $\mathcal{Y}_n$ *are geodesic marks for every* $n \geq 0$. *Then the first two marks of* $\mathbf{A}$ *and of* $\mathbf{B}$ *are geodesic marks* γ, ξ, *and*

$$G(\mathcal{X}_n, \mathcal{Y}_n; \gamma_n, \xi_n) \underset{n\to\infty}{\longrightarrow} G(\mathcal{X}, \mathcal{Y}; \gamma, \xi)$$

in the GHP topology.

Similarly, if we now assume that the first four marks are such that $(A_n^1, A_n^2) = \gamma_n$ *and* $(A_n^3, A_n^4) = \xi_n$ *are geodesic marks, then the same holds for the first marks* γ, ξ *of* $\mathbf{A}$, *and*

$$G(\mathcal{X}_n; \gamma_n, \xi_n) \underset{n\to\infty}{\longrightarrow} G(\mathcal{X}; \gamma, \xi)$$

in the GHP topology.

In order to prove this proposition, we are first going to state and prove a useful lemma that allows one to deal only with the situations where the geodesics along which the spaces of interest are glued have the same lengths. While Lemma 3.4 showed that the operation of merging two marks is continuous on $(\mathbb{M}^{(\ell,m)}, \mathrm{d}_{\mathrm{GHP}}^{(\ell,m)})$, this lemma states that in the case of geodesic marks, the natural splitting operation is continuous. If γ is a geodesic mark and $r \in [0, \mathrm{length}(\gamma)]$, the *splitting* of γ at level r is the two geodesic marks $(\gamma(0), \{\gamma(t) : 0 \leq t \leq r\}), (\gamma(r), \{\gamma(t), r \leq t \leq \mathrm{length}(\gamma)\})$.

Lemma 3.9. *Let* $(\mathcal{X}_n, d_{\mathcal{X}_n}, \mathbf{A}_n, \boldsymbol{\mu}_{\mathcal{X}_n})$, $n \geq 0$, *be a sequence of geodesic marked measured metric spaces converging in the GHP topology toward a geodesic marked measured metric space* $(\mathcal{X}, d_{\mathcal{X}}, \mathbf{A}, \boldsymbol{\mu}_{\mathcal{X}})$, *and assume that, say, the first pairs of marks, are geodesic marks* γ_n *and* γ. *Denote by* $\ell_n = \mathrm{length}(\gamma_n)$ *and* $\ell = \mathrm{length}(\gamma)$ *their lengths. Let* $r_n \in (0, \ell_n)$ *be real numbers such that* $r_n \to r \in (0, \ell)$. *Then the convergence* $\mathcal{X}_n \to \mathcal{X}$ *still holds in the GHP topology after replacing the marks* γ_n *and* γ *in* $\mathbf{A}_n$ *and* $\mathbf{A}$, *with their splittings* γ_n', γ_n'' *and* γ', γ'' *at levels* r_n *and* r, *respectively.*

Proof. Since the desired property does not involve the measures, it suffices to establish it in the space of marked, nonmeasured spaces. Let $\mathcal{R}$ be a correspondence between $\mathcal{X}_n$ and $\mathcal{X}$ compatible with the markings. We fix $\varepsilon > \mathrm{dis}(\mathcal{R})$ and consider the enlarged correspondence

$$\mathcal{R}^\varepsilon = \{(x, y) \in \mathcal{X}_n \times \mathcal{X} : \exists (x', y') \in \mathcal{R},\ d_{\mathcal{X}_n}(x, x') \vee \mathrm{d}_{\mathcal{X}}(y, y') < \varepsilon\}.$$

By the triangle inequality, the distortion of $\mathcal{R}^\varepsilon$ is at most $\mathrm{dis}(\mathcal{R}) + 4\varepsilon$. Moreover, we claim that for $s \in [0, \ell_n]$ and $t \in [0, \ell]$ such that $|t - s| < \varepsilon - \mathrm{dis}(\mathcal{R})$, it holds that $\gamma_n(s)\ \mathcal{R}^\varepsilon\ \gamma(t)$. Indeed, since $\mathcal{R}$ is compatible with the markings, for every s, t as above, there exists u such that $\gamma_n(s)\ \mathcal{R}\ \gamma(u)$, so

$$|s - u| = |d_{\mathcal{X}_n}(\gamma_n(s), \gamma_n(0)) - d_{\mathcal{X}}(\gamma(u), \gamma(0))| \leq \mathrm{dis}\,\mathcal{R},$$

and therefore $|d_{\mathcal{X}}(\gamma(t), \gamma(u))| = |t - u| \leq |t - s| + |s - u| < \varepsilon$, as wanted. From this, we conclude that $\mathcal{R}^\varepsilon$ is compatible with the geodesic marks γ_n', γ' and γ_n'', γ'', as soon as $\varepsilon > |r_n - r| \vee |\ell_n - \ell|$. Choosing a sequence of correspondences $\mathcal{R}_n$ with vanishing distortion and taking $\varepsilon_n = (\mathrm{dis}(\mathcal{R}_n) \vee |r_n - r| \vee |\ell_n - \ell|) + 1/n$ yields the result. ∎

Proof of Proposition 3.8. With the help of Lemma 3.9, we may and will assume that γ_n, ξ_n have same length for every n. The fact that the limiting marks are geodesics marks with same length comes from Proposition 3.5. We will first establish the result for marked, nonmeasured spaces, from which we will deduce the full result thanks to Lemma 3.1.

Let $\mathcal{R}$ and $\mathcal{R}'$ be two correspondences between $\mathcal{X}_n$ and $\mathcal{X}$ and between $\mathcal{Y}_n$ and $\mathcal{Y}$, compatible with the considered markings. Identifying $\mathcal{X}_n$ and $\mathcal{Y}_n$ (resp. $\mathcal{X}$ and $\mathcal{Y}$) with their canonical embeddings into $G(\mathcal{X}_n, \mathcal{Y}_n; \gamma_n, \xi_n)$ (resp. into $G(\mathcal{X}, \mathcal{Y}; \gamma, \xi)$), we consider $\mathcal{R}'' = \mathcal{R} \cup \mathcal{R}'$ as a correspondence between $G(\mathcal{X}_n, \mathcal{Y}_n; \gamma_n, \xi_n)$ and $G(\mathcal{X}, \mathcal{Y}; \gamma, \xi)$, obviously compatible with the other marks. In order to bound its distortion, let us take $x_n\ \mathcal{R}\ x$ and $y_n\ \mathcal{R}'\ y$.

We let $\ell_n = \mathrm{length}(\gamma_n) = \mathrm{length}(\xi_n)$ and $\ell = \mathrm{length}(\gamma) = \mathrm{length}(\xi)$ be the lengths of the considered geodesics and we denote by d_n and d the metrics in the previous gluings. From (3.5) and by compactness, there exists $t \in [0, \ell]$ such that

$$d(x, y) = d_{\mathcal{X}}(x, \gamma(t)) + d_{\mathcal{Y}}(\xi(t), y).$$

Then there exist $t_n, t_n' \in [0, \ell_n]$ such that $\gamma_n(t_n)\ \mathcal{R}\ \gamma(t)$ and $\xi_n(t_n')\ \mathcal{R}'\ \xi(t)$. As a result,

$$\begin{aligned} d_n(x_n, y_n) &\leq d_{\mathcal{X}_n}(x_n, \gamma_n(t_n)) + d_n(\gamma_n(t_n), \xi_n(t_n')) + d_{\mathcal{Y}_n}(\xi_n(t_n'), y_n) \\ &\leq d_{\mathcal{X}}(x, \gamma(t)) + \mathrm{dis}(\mathcal{R}) + d_n(\gamma_n(t_n), \xi_n(t_n')) + d_{\mathcal{Y}}(\xi(t), y) + \mathrm{dis}(\mathcal{R}') \\ &\leq d(x, y) + \mathrm{dis}(\mathcal{R}) + \mathrm{dis}(\mathcal{R}') + d_n(\gamma_n(t_n), \xi_n(t_n')). \end{aligned}$$

Using the facts that γ_n, ξ_n, γ, ξ are geodesics and $\gamma_n(0)\ \mathcal{R}\ \gamma(0)$, $\xi_n(0)\ \mathcal{R}'\ \xi(0)$, we easily obtain

$$\begin{aligned} d_n(\gamma_n(t_n), \xi_n(t_n')) &= |d_{\mathcal{X}_n}(\gamma_n(0), \gamma_n(t_n)) - d_{\mathcal{Y}_n}(\xi_n(0), \xi_n(t_n'))| \\ &\leq |d_{\mathcal{X}}(\gamma(0), \gamma(t)) - d_{\mathcal{Y}}(\xi(0), \xi(t))| + \mathrm{dis}(\mathcal{R}) + \mathrm{dis}(\mathcal{R}') \\ &= \mathrm{dis}(\mathcal{R}) + \mathrm{dis}(\mathcal{R}'). \end{aligned}$$

Using a symmetric argument, we obtain

$$|d_n(x_n, y_n) - d(x, y)| \leq 2(\mathrm{dis}(\mathcal{R}) + \mathrm{dis}(\mathcal{R}')).$$

Adding to this the simpler cases where the pairs of points we compare belong both to $\mathcal{R}$ or both to $\mathcal{R}'$, we obtain

$$\mathrm{dis}(\mathcal{R}'') \leq 2(\mathrm{dis}(\mathcal{R}) + \mathrm{dis}(\mathcal{R}'))$$

and the first statement easily follows for the GH topology (without the measures).

Let us show that the result still holds when considering the measures. We assume for simplicity that the terms of $\boldsymbol{\mu}_{\mathcal{X}}, \boldsymbol{\mu}_{\mathcal{Y}}$ are all nonzero, since the case of a vanishing measure, say $\mu^j_{\mathcal{X}}$, is equivalent to the fact that $\mu^j_{\mathcal{X}_n}(\mathcal{X}_n) \to 0$. Denote by m, m' the numbers of coordinates of $\boldsymbol{\mu}_{\mathcal{X}}, \boldsymbol{\mu}_{\mathcal{Y}}$ and sample $r(m + m')$ independent points $\boldsymbol{x} = (x^j_i, 1 \leq i \leq r, 1 \leq j \leq m)$ and $\boldsymbol{y} = (y^j_i, 1 \leq i \leq r, 1 \leq j \leq m')$, where x^j_i has law $\overline{\mu}^j_{\mathcal{X}}$ and y^j_i has law $\overline{\mu}^j_{\mathcal{Y}}$. We identify these points with their images in the glued space by the canonical projection $\mathcal{X} \sqcup \mathcal{Y} \to G(\mathcal{X}, \mathcal{Y}; \gamma, \xi)$. We assume that $\mu_{\mathcal{X}_n}(\mathcal{X}_n) > 0$ and $\mu_{\mathcal{Y}_n}(\mathcal{Y}_n) > 0$, which hold for n sufficiently large since, as $n \to \infty$, $\mu_{\mathcal{X}_n}(\mathcal{X}_n) \to \mu_{\mathcal{X}}(\mathcal{X}) > 0$ and $\mu_{\mathcal{Y}_n}(\mathcal{Y}_n) \to \mu_{\mathcal{Y}}(\mathcal{Y}) > 0$. We proceed similarly to sample $r(m + m')$ random points $\boldsymbol{x}_n = (x^j_{n,i})$, $\boldsymbol{y}_n = (y^j_{n,i})$ in $G(\mathcal{X}_n, \mathcal{Y}_n; \gamma_n, \xi_n)$ with laws $\overline{\mu}^j_{\mathcal{X}_n}$ and $\overline{\mu}^j_{\mathcal{Y}_n}$ as appropriate. Lemma 3.1 guarantees that the marked spaces $(\mathcal{X}_n, d_{\mathcal{X}_n}, \mathbf{A}_n\boldsymbol{x}_n)$ and $(\mathcal{Y}_n, d_{\mathcal{Y}_n}, \mathbf{B}_n\boldsymbol{y}_n)$ converge to $(\mathcal{X}, d_{\mathcal{X}}, \mathbf{A}\boldsymbol{x})$ and $(\mathcal{Y}, d_{\mathcal{Y}}, \mathbf{B}\boldsymbol{y})$ in distribution in the GH topology. Applying the result of Proposition 3.8 proved above in the case without measures, we obtain the convergence in distribution of the glued space $G(\mathcal{X}_n, \mathcal{Y}_n; \gamma_n, \xi_n)$ with markings $\mathbf{A}_n\mathbf{B}_n\boldsymbol{x}_n\boldsymbol{y}_n$ to $G(\mathcal{X}, \mathcal{Y}; \gamma, \xi)$ with the marking $\mathbf{AB}\boldsymbol{x}\boldsymbol{y}$. Since $\boldsymbol{x}_n$, $\boldsymbol{y}_n$ and $\boldsymbol{x}$, $\boldsymbol{y}$ are also independent samples from the renormalized measures $\overline{\boldsymbol{\mu}}_{\mathcal{X}_n}, \overline{\boldsymbol{\mu}}_{\mathcal{Y}_n}$ and $\overline{\boldsymbol{\mu}}_{\mathcal{X}}, \overline{\boldsymbol{\mu}}_{\mathcal{Y}}$ viewed as measures on the glued spaces, an application of the converse implication of Lemma 3.1 implies the result.

The second part of the statement dealing with metric spaces that are glued along two marked geodesics is shown in a similar fashion. We leave the details to the reader. ■

3.4 Proof of Theorem 1.1

Let $g, k \in \mathbb{Z}_{\geq 0}$ be fixed; as in Section 2.2, we exclude the cases $(g, k) \in \{(0,0), (0,1)\}$ of the sphere, the pointed sphere, or the disk. Recall that the case $(g, k) = (0, 0)$ of the sphere is already known. The case $(g, b, k) = (0, 1, 1)$ of the disk is partially known but has not been treated in the complete setting of Theorem 1.1. In fact, it may actually enter the following framework: in this case, the decomposition of Section 2.2 yields only one well-labeled forest (and thus one unique slice) indexed by a degenerate scheme with one external face having one unique vertex with a self-loop edge. This creates a small ambiguity coming from the choice of the first tree in the forest, which can be overcome by randomization when considering random maps. The case $(g, b, k) = (0, 0, 1)$ of the pointed sphere would yield an even more degenerate decomposition with a one-vertex map as a scheme and a unique composite slice of width 0, object that is not introduced in this work.

Instead of considering these extensions and objects, we rather obtain these cases by using Proposition 1.3 at the end of Section 1.4. More precisely, provided Theorem 1.1 holds for $(g, k) = (0, 2)$, we infer the case $(g, k) = (0, 1)$ as follows. Let $l_n^1 \in \mathbb{Z}_{\geq 0}$ be such that $l_n^1/\sqrt{2n} \to L^1$ as $n \to \infty$ and Q_n be uniform in $\vec{\mathbf{Q}}^{[0]}_{n,(l_n^1)}$. Then choose a vertex uniformly at random among the internal vertices of Q_n and denote by $Q_n^\bullet$ the map obtained from Q_n by declaring the chosen vertex as a second hole. Since $Q_n^\bullet$ is clearly uniform in $\vec{\mathbf{Q}}^{[0]}_{n,(l_n^1,0)}$, the case $(g, k) = (0, 2)$ of Theorem 1.1 implies the convergence of the corresponding metric measure space toward $\mathsf{S}^{[0]}_{(L^1,0)}$, and finally the convergence of the metric measure space corresponding to Q_n toward the space $\mathsf{S}^{[0]}_{(L^1)}$, defined as $\mathsf{S}^{[0]}_{(L^1,0)}$ with its second mark and second boundary measure forgotten.

3.5 Gluing quadrangulations from elementary pieces

We start by interpreting the observations of Section 2.2 in the light of the previous section for a deterministic map. Let $n \in \mathbb{Z}_{\geq 0}$ and $\boldsymbol{l} \in \mathbb{N}^k$ be fixed, let $\mathbf{q} \in \mathbf{Q}^{[g]}_{n,\boldsymbol{l}}$ and $v_* \in V(\mathbf{q})$. The CVS construction being one-to-one, there is a unique labeled map $(\mathbf{m}, \lambda) \in \mathbf{M}^{[g]}_{n,\boldsymbol{l}}$ that corresponds to $(\mathbf{q}, v_*)$. We denote by $\mathbf{s}$ the scheme of $\mathbf{m}$ and by $(\mathrm{EP}^e, e \in \vec{E}(\mathbf{s}))$ the collection of elementary pieces of $(\mathbf{q}, v_*)$. We emphasize that the decomposition strongly depends on the distinguished point v_* and not only on $\mathbf{q}$.

If $e \in \vec{B}(\mathbf{s})$, we let γ^e, ξ^e and β^e be the maximal geodesic, shuttle, and base of the slice EP^e, and we let μ^e, ν^e be the associated area and base measures defined at (2.2). If $e \in \vec{I}(\mathbf{s})$, we let $\gamma^e, \xi^e, \overline{\gamma}^e, \overline{\xi}^e$ be the maximal geodesics and shuttles of EP^e, where the first two correspond to I_e and the latter two to $I_{\bar{e}}$, in the notation of Section 2.2; we also let μ^e be the associated area measure defined at (2.3). Note that $\mathrm{EP}^{\bar{e}}$ yields the same map as EP^e with the same measure; only the marks are ordered differently, namely $\overline{\gamma}^e, \overline{\xi}^e, \gamma^e, \xi^e$.

The construction of $\mathbf{q}$ from $(\mathbf{m}, \lambda)$ consists in connecting every corner of $\mathbf{m}$ to its successor, and the paths following consecutive successors are geodesic paths all aiming toward v_*. On the other hand, the construction of the elementary pieces from $(\mathbf{m}, \lambda)$ consists in performing the interval CVS bijection on every interval I_e, in the notation of Section 2.2. The only difference between these constructions lies on the shuttles of these elementary pieces: if c is a corner in some interval I_e whose successor in the interval bijection belongs to the shuttle ξ^e, then in order to obtain $\mathbf{q}$ we should rather connect c to its successor $s(c)$ in the contour order around f_*. This successor will belong to some interval $I_{e'}$ arriving later in contour order around f_* – note that this interval can be I_e itself. Note also that this successor $s(c)$ belongs to the maximal geodesic $\gamma^{e'}$. Moreover, interpreting $\mathbf{q}$ and its elementary pieces as compact geodesic marked metric spaces, it is straightforward to see that the identifications correspond to metric gluings along geodesics.

Iterative gluing procedure. In order to reconstruct $\mathbf{q}$ from EP^e, $e \in \vec{E}(\mathbf{s})$, rather than connecting the shuttle vertices to their actual successors all at once, we will proceed progressively by first connecting only those whose successors belong to the maximal geodesic of the elementary piece that arrives immediately after in contour order around f_*.

We now formalize this idea. Let κ denote the cardinality of $\vec{E}(\mathbf{s})$. We arrange the half-edges $e_1, \dots, e_\kappa$ incident to the internal face of $\mathbf{s}$ according to the contour order, starting at an arbitrarily chosen half-edge. While following the contour of the internal face f_* of $\mathbf{m}$, we successively visit the elementary pieces EP^{e_i}, $1 \le i \le \kappa$, which are themselves viewed as marked measured geodesic metric spaces. The reconstruction of $\mathbf{q}$ will be done recursively in κ steps, resulting in a sequence of marked $(k+1)$-measured metric spaces $\mathbf{q}_0, \dots, \mathbf{q}_\kappa$. At the i-th step, $\mathbf{q}_{i+1}$ will be obtained from $\mathbf{q}_i$ by gluing EP^{e_i} along (part of) its marked maximal geodesic γ^{e_i}. At the same time, we will do some operations on the markings and measures, namely reorderings, unions of marks and sums of measures, which are all continuous by Lemma 3.4.

We need to keep track of the boundary marks and the geodesics yet to be glued as marks. More precisely, the marking of $\mathbf{q}_i$ is $(\gamma_i^0, \xi_i^0, \gamma_i^1, \xi_i^1, \dots, \gamma_i^{u_i}, \xi_i^{u_i}, \beta_i^1, \dots, \beta_i^k)$, where

- γ_i^j, ξ_i^j, $0 \le j \le u_i$, are geodesic marks,
- $\beta_i^1, \dots, \beta_i^k$ are called the *boundary marks*. By convention, certain of these marks may be empty, in which case they are simply discarded from the marks.

The mark ξ_i^0 is the mark along which the subsequent gluing producing $\mathbf{q}_{i+1}$ will occur, and u_i represents the number of quadrilaterals that have been involved only once in the gluing procedures up to the i-th step. Each of these quadrilateral yields two marks γ_i^j, ξ_i^j for some $j \in \{1, \dots, u_i\}$, corresponding to the unvisited half of the quadrilateral, which will have to be glued at a further step. Finally, each $\mathbf{q}_i$ will come with measures μ_i, $\boldsymbol{\nu}_i$, where μ_i is called the *area measure* and $\boldsymbol{\nu}_i = (\nu_i^1, \dots, \nu_i^k)$ is the k-tuple of *boundary measures*.

We initiate the construction by letting $u_0 = 0$, $\mathbf{q}_0 \in \mathbb{M}^{(2,k+1)}$ be the point space with the two marks γ_0^0, ξ_0^0 being the unique point, and measures $\mu_0 = 0$, $\boldsymbol{\nu}_0 = \mathbf{0}^k$. We also let all the boundary marks be empty.

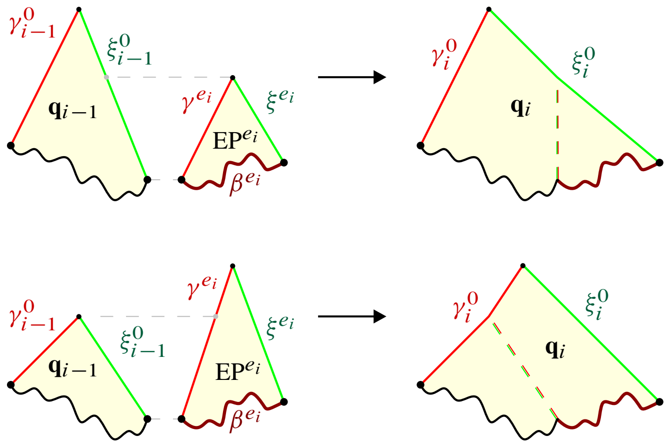

Figure 3.1. The gluing procedure in the case where EP^{e_i} is a slice. In this picture and the following ones, the black wiggly curve depicts all the marks different from $\gamma^0_{i-1}, \xi^0_{i-1}$. The reader should bear in mind that, in general, $\mathbf{q}_{i-1}$ has no reason to present a planar topology as in these pictures. The boundary ξ^0_{i-1} of $\mathbf{q}_{i-1}$ is glued to the maximal geodesic γ^{e_i}, and the base of EP^{e_i} is added to the r-th boundary mark of $\mathbf{q}_{i-1}$ whenever $\overline{e}_i$ is incident to h_r. The first two geodesic marks are updated according to (3.7), which leads to the two alternative situations described in this figure, depending on which of ξ^0_{i-1} and γ^{e_i} is the longest: the unglued part of these geodesics becomes part of ξ^0_i or of γ^0_i.

Next, provided that $\mathbf{q}_{i-1}$ has been constructed for some $i \in \{1, \dots, \kappa\}$, we define $\mathbf{q}_i$ by considering the following cases, depicted in Figures 3.1–3.3.

- If $e_i \in \vec{B}_r(\mathbf{s})$ for some $r \in \{1, \dots, k\}$, meaning in particular that EP^{e_1} is a slice, we set

$$\mathbf{q}_i = G(\mathbf{q}_{i-1}, \mathrm{EP}^{e_i}; \xi^0_{i-1}, \gamma^{e_i}),$$

and mark it as follows. We update the boundary marks by setting

$$\beta^r_i = \beta^r_{i-1} \cup \beta^{e_i}, \quad \beta^{r'}_i = \beta^{r'}_{i-1} \text{ for } r' \in \{1, \dots, k\} \setminus \{r\}.$$

We update the geodesic marks by letting[3]

$$\gamma^0_i = \gamma^0_{i-1} \cup (\gamma^{e_i} \setminus \xi^0_{i-1}), \quad \xi^0_i = \xi^{e_i} \cup (\xi^0_{i-1} \setminus \gamma^{e_i}), \tag{3.7}$$

and, setting $u_i = u_{i-1}$, we let $\gamma^j_i = \gamma^j_{i-1}$ and $\xi^j_i = \xi^j_{i-1}$ for $1 \le j \le u_i$. Finally, we update the measures by

$$\mu_i = \mu_{i-1} + \mu^{e_i}, \quad \nu^r_i = \nu^r_{i-1} + \nu^{e_i}, \quad \nu^{r'}_i = \nu^{r'}_{i-1} \text{ for } r' \in \{1, \dots, k\} \setminus \{r\}.$$

This case is illustrated in Figure 3.1.

[3] In (3.7), we use the convention set in Section 3.3 for marks in a glued space: in particular, after gluing, one of the marks $\gamma^{e_i}, \xi^0_{i-1}$ is contained in the other, depending on which is longest.

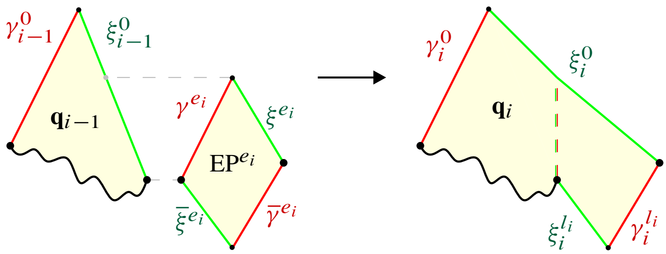

Figure 3.2. The gluing procedure in the case where EP^{e_i} is a quadrilateral that was not involved previously in the construction. The boundary ξ^0_{i-1} of $\mathbf{q}_{i-1}$ is glued to the maximal geodesic γ^{e_i}. The first two geodesic marks are again updated according to (3.7), leading to two possible situations depending on which of ξ^0_{i-1} and γ^{e_i} is the longest. Only one of these situations is represented on this figure. In this case, the two geodesic boundary marks $\overline{\gamma}^{e_i}$ and $\overline{\xi}^{e_i}$ are added to the marking; they will be involved in a later construction step.

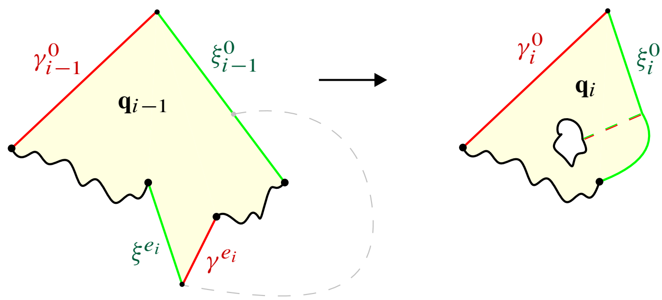

Figure 3.3. The gluing procedure in the case where EP^{e_i} is a quadrilateral, one side of which was already involved in a previous construction step. The boundary ξ^0_{i-1} of $\mathbf{q}_{i-1}$ is glued to the maximal geodesic γ^{e_i}, which had been introduced as a geodesic mark in this previous construction step. The first two geodesic marks are again updated according to (3.7), and the geodesic marks γ^{e_i}, ξ^{e_i} are removed from the remaining marks.

- If $e_i \in \vec{I}(\mathbf{s})$, meaning in particular that EP^{e_1} is a quadrilateral, we keep the boundary marks unchanged by setting $\beta^r_i = \beta^r_{i-1}$ for $1 \le r \le k$, we set $\boldsymbol{\nu}_i = \boldsymbol{\nu}_{i-1}$, and consider the following two possible situations.
  - If $e_i \notin \{\overline{e}_j, 1 \le j < i\}$, that is, if the unoriented edge corresponding to e_i is visited for the *first time*, we let again

$$\mathbf{q}_i = G(\mathbf{q}_{i-1}, \mathrm{EP}^{e_i}; \xi^0_{i-1}, \gamma^{e_i}),$$

and update its geodesic marks as follows. We update the first two geodesic marks by (3.7). We set $u_i = u_{i-1} + 1$ and let $\gamma^j_i = \gamma^j_{i-1}$ and $\xi^j_i = \xi^j_{i-1}$ for $1 \le j \le u_i - 1$. Finally, we set $\gamma^{u_i}_i = \overline{\gamma}^{e_i}$, $\xi^{u_i}_i = \overline{\xi}^{e_i}$, and $\mu_i = \mu_{i-1} + \mu^{e_i}$. This case is illustrated in Figure 3.2.

- If $e_i \in \{\overline{e}_j, 1 \leq j < i\}$, say $e_i = \overline{e}_\ell$, that is, if the unoriented edge corresponding to e_i is visited for the *second time*, then $\gamma^{e_i} = \overline{\gamma}^{e_\ell}$ is a mark of $\mathbf{q}_{i-1}$: it is the mark $\gamma^{e_i} = \gamma_\ell^{u_\ell}$ of $\mathbf{q}_\ell$ and stays a mark of the subsequent spaces $\mathbf{q}_{\ell+1}, \dots, \mathbf{q}_{i-1}$. Similarly, $\xi^{e_i} = \overline{\xi}^{e_\ell}$ is a mark of $\mathbf{q}_{i-1}$. We let
$$\mathbf{q}_i = G(\mathbf{q}_{i-1}; \xi_{i-1}^0, \gamma^{e_i}),$$
we update the first two geodesic marks by (3.7), and, setting $u_i = u_{i-1} - 1$, we let $(\gamma_i^j, \xi_i^j, 1 \leq j \leq u_i)$ be the sequence $(\gamma_{i-1}^j, \xi_{i-1}^j, 1 \leq j \leq u_{i-1})$ from which the terms γ^{e_i} and ξ^{e_i} have been removed. Finally, we set $\mu_i = \mu_{i-1}$. This case is illustrated in Figure 3.3.

It is important to notice that, in $\mathbf{q}_i$, all the marks $\gamma_i^j, \xi_i^j, 0 \leq j \leq u_i$, are geodesic marks. Indeed, each of these paths always take the form of a chain of consecutive successors, which therefore must be a geodesic; more precisely, these are the maximal geodesics and shuttles of the interval CVS bijection on the intervals $I_{e_1} \cup \dots \cup I_{e_i}$ and $I_{\overline{e}_j}$ for each $j \leq i$ such that $e_j \in \vec{I}(\mathbf{s}) \setminus \{\overline{e}_1, \dots, \overline{e}_i\}$.

At the end of this inductive procedure, we have connected all shuttle corners of some interval I_{e_i}, to their actual successors in $\mathbf{m}$ whenever these lie on some $I_{e_{i'}}$ with $1 \leq i < i' \leq \kappa$. It remains to connect the shuttle corners in some I_{e_i} whose actual successor in $\mathbf{m}$ lies in some $I_{e_{i'}}$ with $1 \leq i' \leq i \leq \kappa$. But one can observe that $u_\kappa = 0$, so that $\mathbf{q}_\kappa$ carries exactly two geodesic marks γ_κ^0, ξ_κ^0. The shuttle corners yet to be connected are exactly those of ξ_κ^0, and should be matched to the successive corners of γ_κ^0. Therefore, as marked metric spaces, we have $\mathbf{q} = G(\mathbf{q}_\kappa; \gamma_\kappa^0, \xi_\kappa^0)$, with marks β_κ^r, $1 \leq r \leq k$, which are precisely the connected components of the boundary $\partial\mathbf{q}$, ordered as they should.

It is also possible to view all these gluing operations at once, as shown in Figure 3.4.

Measures. We claim that the previous equality $\mathbf{q} = G(\mathbf{q}_\kappa; \gamma_\kappa^0, \xi_\kappa^0)$ only holds as k-marked, $(k+1)$-measured metric spaces up to a difference in the supports consisting of a bounded number of vertices. When considering rescaled measures through the operator Ω_n, this small difference will be of no importance in our limiting arguments.

First of all, the boundary measures of the external faces match. This is because the boundary of a given external face of $\mathbf{q}$ is made of the bases, say $\beta^{e_{i_1}}, \dots, \beta^{e_{i_j}}$, of several slices that satisfy $\beta^{e_{i_\ell}} \cap \beta^{e_{i_{\ell+1}}} = \{\overline{\rho}^{e_{i_\ell}}\}$ for $1 \leq \ell \leq j$ with $i_{j+1} = i_1$ and where we denoted by $\overline{\rho}^e$ the final point of the base of EP^e. Since the base measure of the slice EP^e is the counting measure on $\beta^e \setminus \{\overline{\rho}^e\}$, the resulting measure in the gluing is the counting measure on the boundary of the considered external face of $\mathbf{q}$, as desired.

For the boundary measures of the external vertices, the measure in $\mathbf{q}$ is the counting measure on the singleton consisting of the external vertex, while the corresponding boundary measure in the gluing is the zero measure.

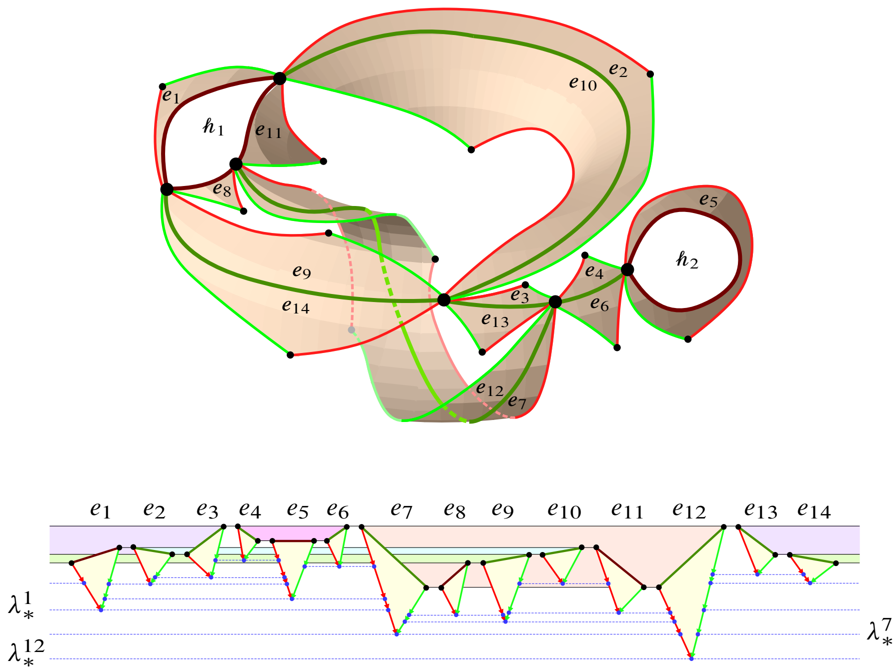

Figure 3.4. Reconstructing **q** by gluing its elementary pieces along geodesics. On this example, we have $\kappa = 14$. Although we used the same scheme as in Figure 2.2 without $\hbar_3$ (for lower complexity), beware that the labels here do not match those from the top of Figure 2.2. On the top, the half-edges of $\vec{E}(\mathbf{s})$ are arranged according to the contour order. With each one of them corresponds a "triangle," which is either a slice (here, with e_1, e_5, e_8, e_{11}) or "half" a quadrilateral, depending on whether the half-edge belongs to $\vec{B}(\mathbf{s})$ or $\vec{I}(\mathbf{s})$. The five matchings of the corresponding "halves" of quadrilaterals, that is, the matchings of half-edges in $\vec{I}(\mathbf{s})$, are represented with light colors. The vertices of these triangles are represented at a height corresponding to their label, where $\lambda_*^i = \min_{I_{e_i}} \lambda - 1$. The geodesic marks of the pieces may be involved in multiple gluings. For instance, the shuttle of the leftmost triangle is involved in three gluings; it is split in three parts, corresponding to the triangles that can be "seen" to its right (those corresponding to e_2, e_5, and e_7).

For the area measure, by convention, we decided that in the elementary pieces, the measures were taken on vertices *outside of the shuttles*. In doing so, after each gluing operation where a piece of a shuttle is glued to a piece of a maximal geodesic, the corresponding vertices are counted only once, as they should, *except possibly for the vertices* ρ, $\overline{\rho}$ *of the quadrilaterals*, since they lie both on a maximal geodesic and on a shuttle, and are therefore not part of the counting measures by convention. Therefore, the final gluing is naturally equipped with an area measure that is the counting measure on all but at most $2(2g + k - 1)$ vertices, which is an upper bound on the number of vertices of the scheme by Lemma 2.1.

Conclusion. We finally observe that the number κ of gluing operations necessary to obtain $\mathbf{q}$ is uniformly bounded in n. Indeed, the number of edges of $\mathbf{s}$ is, by Lemma 2.1, smaller than $3(2g+k-1)$.

As a result, Theorem 1.1 will directly follow from subsequent applications of Proposition 3.8 once we will have shown that, after a proper scaling, the elementary pieces of a uniform quadrangulation jointly converge in distribution in the GHP topology.

3.6 Scaling limit of the collection of elementary pieces

We now fix $\boldsymbol{L} = (L^1, \ldots, L^k) \in [0,\infty)^k$. We let b and p be the numbers of indices i such that $L^i > 0$ and $L^i = 0$ respectively. In order to ease notation, we assume that $L^1, \ldots, L^b > 0$ while $L^{b+1}, \ldots, L^k = 0$.

Limiting measure for size parameters. We denote by $\vec{\mathbf{S}}^\star$ the set of rooted genus g schemes with k holes, $\hbar_1, \ldots, \hbar_b$ being faces, $\hbar_{b+1}, \ldots, \hbar_k$ being vertices of degree 1, and whose internal vertices are all of degree exactly 3. These are called *dominant* schemes. Let $\mathbf{s} \in \vec{\mathbf{S}}^\star$ be fixed and denote its root vertex by v_0. We let $\mathcal{T}_\mathbf{s}$ be the set of tuples

$$((\mathsf{a}^e)_{e\in\vec{E}(\mathbf{s})}, (\mathsf{h}^e)_{e\in\vec{I}(\mathbf{s})}, (\mathsf{l}^e)_{e\in\vec{B}(\mathbf{s})}, (\lambda^v)_{v\in V(\mathbf{s})})$$

in $(\mathbb{R}_{\geq 0})^{\vec{E}(\mathbf{s})} \times (\mathbb{R}_{\geq 0})^{\vec{I}(\mathbf{s})} \times \mathbb{R}^{\vec{B}(\mathbf{s})} \times \mathbb{R}^{V(\mathbf{s})}$ such that

- $\sum_{e\in\vec{E}(\mathbf{s})} \mathsf{a}^e = 1$,
- $\mathsf{h}^{\bar{e}} = \mathsf{h}^e$ for all $e \in \vec{I}(\mathbf{s})$,
- $\sum_{e\in\vec{B}_i(\mathbf{s})} \mathsf{l}^e = L^i$ for $1 \leq i \leq b$,
- $\lambda^{v_0} = 0$.

There is a natural Lebesgue measure $\mathcal{L}_\mathbf{s}$ on $\mathcal{T}_\mathbf{s}$ defined as follows. First, if J is a finite set, and $L > 0$ a positive real number, we let Δ_J^L be the Lebesgue measure on the simplex $\{(\mathsf{x}^j, j \in J) \in (\mathbb{R}_{\geq 0})^J : \sum_{j\in J} \mathsf{x}^j = L\}$. The latter measure can be defined as the image of the measure $\bigotimes_{j\in J'} d\mathsf{x}^j \mathbf{1}_{\{\sum_{j\in J'} \mathsf{x}^j < L\}}$, where J' is obtained from J by removing one arbitrary element j', by the mapping $(\mathsf{x}^j, j \in J') \mapsto (\mathsf{x}^j, j \in J)$, where $\mathsf{x}^{j'} = 1 - \sum_{j\in J'} \mathsf{x}^j$.

Next, let $I(\mathbf{s})$ be an orientation of $\vec{I}(\mathbf{s})$, that is, a set containing exactly one element from $\{e, \bar{e}\}$ for every $e \in \vec{I}(\mathbf{s})$. We let $\mathcal{L}^+_{I(\mathbf{s})}$ be the measure $\bigotimes_{e\in I(\mathbf{s})} d\mathsf{h}^e \mathbf{1}_{\{\mathsf{h}^e \geq 0\}}$. Similarly, we let $\mathcal{L}_{V(\mathbf{s})}$ be the measure $\bigotimes_{v\in V'(\mathbf{s})} d\lambda^v$, where $V'(\mathbf{s}) = V(\mathbf{s}) \setminus \{v_0\}$.

Finally, the measure $\mathcal{L}_\mathbf{s}$ is the image measure of

$$\Delta^1_{\vec{E}(\mathbf{s})} \otimes \mathcal{L}^+_{I(\mathbf{s})} \otimes \bigotimes_{i=1}^b \Delta^{L_i}_{\vec{B}_i(\mathbf{s})} \otimes \mathcal{L}_{V(\mathbf{s})}$$

by the mapping that associates with

$$((\mathsf{a}^e)_{e\in\vec{E}(\mathbf{s})}, (\mathsf{h}^e)_{e\in I(\mathbf{s})}, ((\mathsf{l}^e)_{e\in\vec{B}_i(\mathbf{s})}, 1\le i\le b), (\lambda^v)_{v\in V'(\mathbf{s})})$$

the unique compatible element of $\mathcal{T}_{\mathbf{s}}$, that is, such that $\mathsf{h}^e = \mathsf{h}^{\bar{e}}$ for every $e \in \vec{I}(\mathbf{s})$, and such that $\lambda^{v_0} = 0$.

We let $\mathrm{Param}_{\boldsymbol{L}}$ be the probability measure on $\bigcup_{\mathbf{s}\in\vec{\mathbf{S}}^\star}\{\mathbf{s}\}\times\mathcal{T}_{\mathbf{s}}$ whose density with respect to the measure $\sum_{\mathbf{s}\in\vec{\mathbf{S}}^\star}\delta_{\mathbf{s}}\otimes\mathcal{L}_{\mathbf{s}}$ is

$$\frac{1}{\mathcal{Z}_{\boldsymbol{L}}}\prod_{e\in\vec{I}(\mathbf{s})} q_{\mathsf{h}^e}(\mathsf{a}^e)\prod_{e\in\vec{B}(\mathbf{s})} q_{\mathsf{l}^e}(\mathsf{a}^e)\prod_{e\in I(\mathbf{s})} p_{\mathsf{h}^e}(\delta\lambda^e)\prod_{e\in\vec{B}(\mathbf{s})} p_{3\mathsf{l}^e}(\delta\lambda^e), \tag{3.8}$$

where $p_t(x) = \exp(-x^2/(2t))/\sqrt{2\pi t}$ is the Gaussian density, $q_x(t) = (x/t)p_t(x)\times\mathbf{1}_{\{t>0\}}$ is the (stable 1/2) density for the hitting time of level $-x$ by standard Brownian motion, $\delta\lambda^e = \lambda^{e^+} - \lambda^{e^-}$ for $e \in \vec{E}(\mathbf{s})$, and $\mathcal{Z}_{\boldsymbol{L}}$ is a normalizing constant, equal to the integral of the remaining display. Beware that the third product is over $I(\mathbf{s})$, not $\vec{I}(\mathbf{s})$.

Scaling limits for size parameters. Next, let $(\boldsymbol{l}_n) = (l_n^1, \dots, l_n^k)$ and Q_n be as in the statement of Theorem 1.1. We let v_n^* be uniformly distributed over the set of internal vertices of Q_n, whose cardinality given by (1.3) only depends on the parameters. Consequently, (Q_n, v_n^*) is uniformly distributed over the set of quadrangulations from $\vec{\mathbf{Q}}^{[g]}_{n,\boldsymbol{l}_n0}$, seeing v_* as a $(k+1)$-th hole. The rooted labeled map (M_n, λ_n) corresponding via the CVS correspondence is thus uniformly distributed over $\vec{\mathbf{M}}^{[g]}_{n,\boldsymbol{l}_n}$. We denote by S_n the scheme of the nonrooted map corresponding to M_n, and we root S_n uniformly at random among its half-edges, incident to internal or external faces, but such that the corresponding edge does not belong to the boundary of $\hbar_{b+1}, \dots, \hbar_k$. Note that we could have rooted S_n from the root of M_n by asking that the root of M_n belongs to the forest indexed by the root of S_n but this would have introduced an undesirable bias. Here instead, from the unrooted map corresponding to M_n, the map M_n is rooted at a uniform corner incident to its internal face. Furthermore, the boundaries of the holes $\hbar_{b+1}, \dots, \hbar_k$ are excluded from the possible rootings of S_n since they should be thought of as having null length in the limit.

For $i \in \{b+1, \dots, k\}$, the hole $\hbar_i$ of M_n is called a *vanishing face* if it is a face, that is, if $l_n^i > 0$. The corresponding hole $\hbar_i$ of the scheme S_n is called a *tadpole* if it is made of a single self-loop edge incident to a single vertex of degree 3. We let S_n° be the map S_n in which every tadpole corresponding to a vanishing face $\hbar_i$ has been shrunk into a single vertex, still denoted by $\hbar_i$, in the sense that the corresponding self-loop has been removed. Note that the root of S_n is never removed in this operation, so that S_n° is always rooted.

Forgetting the root of Q_n, we let $(\mathrm{EP}_n^e, e \in \vec{E}(S_n))$ be the collection of elementary pieces of (Q_n, v_n^*). For the half-edges $e \in \vec{B}(S_n)$, we let A_n^e, L_n^e be the area and width of the slice EP_n^e. For $e \in \vec{I}(S_n)$, we let A_n^e, H_n^e be the first half-area and width of the quadrilateral EP_n^e; note that $A_n^{\bar{e}}$ is the second half-area of EP_n^e. Recall that the vertices of S_n are in one-to-one correspondence with the nodes of M_n: for every $v \in V(S_n)$, we denote by Λ_n^v the label of the corresponding node, where we choose for the labeling function λ_n the representative giving label 0 to the root vertex of S_n.

Proposition 3.10. *As $n \to \infty$, with probability tending to one, every vanishing face of M_n induces a tadpole in S_n, and, on this likely event, for every $e \in \bigsqcup_{i=b+1}^{k} \vec{B}_i(S_n)$, it holds that $A_n^e + L_n^e = \Theta((L_n^e)^2)$ in probability.*

Moreover, the following convergence in distribution holds:

$$\Big(S_n^\circ, \Big(\frac{A_n^e}{n}\Big)_{e\in\vec{E}(S_n^\circ)}, \Big(\frac{H_n^e}{\sqrt{2n}}\Big)_{e\in\vec{I}(S_n^\circ)}, \Big(\frac{L_n^e}{\sqrt{2n}}\Big)_{e\in\vec{B}(S_n^\circ)}, \Big(\Big(\frac{9}{8n}\Big)^{1/4}\Lambda_n^v\Big)_{v\in V(S_n^\circ)}\Big)$$
$$\xrightarrow[n\to\infty]{(d)} (S, (A^e)_{e\in\vec{E}(S)}, (H^e)_{e\in\vec{I}(S)}, (L^e)_{e\in\vec{B}(S)}, (\Lambda^v)_{v\in V(S)}), \tag{3.9}$$

where the limiting random variable has the law $\mathrm{Param}_{\boldsymbol{L}}$ *described in the previous paragraph.*

This proposition is a generalization of [22, Proposition 15]. Given its technical nature, we postpone its proof to Appendix B.

Scaling limits of the elementary pieces. As (M_n, λ_n) is uniformly distributed over the set $\vec{\mathbf{M}}_{n,\boldsymbol{l}_n}^{[g]}$, conditionally given (3.9), the random variables EP_n^e, $e \in \vec{E}(S_n)$, are only dependent through the relations linking $\mathrm{EP}_n^{\bar{e}}$ with EP_n^e for $e \in \vec{I}(S_n)$. Moreover,

- if $e \in \vec{B}(S_n)$, then EP_n^e is uniformly distributed among slices with area A_n^e, width L_n^e and tilt $\Lambda_n^{e^+} - \Lambda_n^{e^-}$,
- if $e \in \vec{I}(S_n)$, then EP_n^e is uniformly distributed among quadrilaterals with half-areas A_n^e and $A_n^{\bar{e}}$, width H_n^e and tilt $\Lambda_n^{e^+} - \Lambda_n^{e^-}$.

Applying the Skorokhod representation theorem, we may and will assume that the convergence of (3.9) holds almost surely. Since, by Lemma 2.1, there are finitely many possible schemes, this furthermore implies that $S_n^\circ = S$ for n sufficiently large. Together with Theorems 2.6 and 2.8, the above observations entail that the collection of rescaled random metric spaces $(\Omega_n(\mathrm{EP}_n^e), e \in \vec{E}(S_n^\circ))$, converge in distribution in the GHP topology toward a family $(\mathsf{EP}^e, e \in \vec{E}(S))$ of continuum elementary pieces with the following law conditionally given the right-hand side of (3.9):

- if $e \in \vec{B}(S)$, then EP^e is a continuum slice with area A^e, width H^e and tilt $\Lambda^{e^+} - \Lambda^{e^-}$,
- if $e \in \vec{I}(S)$, then EP^e is a continuum quadrilateral with half-areas A^e and $A^{\bar{e}}$, width H^e and tilt $\Lambda^{e^+} - \Lambda^{e^-}$.

We write $\gamma(\mathsf{EP}^e)$, $\xi(\mathsf{EP}^e)$, $\mu(\mathsf{EP}^e)$, and either $\beta(\mathsf{EP}^e)$, $\nu(\mathsf{EP}^e)$ or $\bar{\gamma}(\mathsf{EP}^e)$, $\bar{\xi}(\mathsf{EP}^e)$ the marks and measures of EP^e, with an obvious choice of notation. As continuum elementary pieces have only been defined as limits in the GHP topology of discrete elementary pieces so far, these marks and measures are the limits of the corresponding marks and measures of the discrete pieces.

We treat the elementary pieces corresponding to the vanishing faces thanks to Corollary 2.7. We define, for every hole $\hbar_i$ of S_n with $b+1 \le i \le k$, the elementary piece $\mathrm{EP}_n^{\hbar_i}$ as EP_n^e if $\vec{B}_i(S_n) = \{e\}$ or the vertex map otherwise, marked five times at its unique vertex and endowed twice with the zero measure (thinking of it as an "empty slice"). By Proposition 3.10 and Corollary 2.7, with probability tending to 1, each one of the rescaled random metric spaces $\Omega_n(\mathrm{EP}_n^{\hbar_i})$, $b+1 \le i \le k$, converges in distribution in the GHP topology toward the point space (note that the tilt of $\mathrm{EP}_n^{\hbar_i}$ is always equal to 0).

3.7 Gluing pieces together

We can now complete the proof of Theorem 1.1 by applying Proposition 3.8 at every step of the inductive construction of Section 3.5.

We work on the event of asymptotic full probability of Proposition 3.10 and assume that n is large enough so that $S_n^\circ = S$. We denote by $\kappa = |\vec{E}(S)| + p$ and let $e_1, \dots, e_\kappa$ be the sequence made of the half-edges of $\vec{E}(S)$, as well as the external vertices of S, listed in contour order. Since S is dominant, these external vertices are $\hbar_{b+1}, \dots, \hbar_k$. Applying the construction of Section 3.5 to the random quadrangulation Q_n, up to adding the gluing of the point space for each external vertex (not changing the markings and measures), we obtain a sequence $Q_{n,1}, \dots, Q_{n,\kappa}$ of marked measured metric spaces.

The limiting space $\mathsf{S}_{\boldsymbol{L}}^{[g]}$ is obtained from $(S, (\mathsf{EP}^e, e \in \vec{E}(S)))$ by recursively defining a sequence of marked measured metric spaces $\mathsf{S}_0, \dots, \mathsf{S}_\kappa$, in the following way. For $0 \le i \le \kappa$, the marked measured metric space S_i will carry geodesic marks γ_i^j, ξ_i^j, $0 \le j \le u_i$, boundary marks $\beta_i^1, \dots, \beta_i^k$, an area measure μ_i, and boundary measures $\nu_i^1, \dots, \nu_i^k$.

We initiate the construction by letting $u_0 = 0$, $\mathsf{S}_0 \in \mathbb{M}^{(2,k+1)}$ be the point space with the two marks γ_0^0, ξ_0^0 being the unique point, and measures $\mu_0 = 0$, $\boldsymbol{\nu}_0 = \mathbf{0}^k$. We also let all the boundary marks be empty.

Next, given S_{i-1} for some $i \in \{1, \dots, \kappa\}$, consider the following cases:

- If $e_i \in \vec{B}_r(S)$ for some $r \in \{1, \dots, k\}$, set

$$\begin{aligned} &\mathsf{S}_i = G(\mathsf{S}_{i-1}, \mathsf{EP}^{e_i}; \xi_{i-1}^0, \gamma(\mathsf{EP}^{e_i})), \\ &\beta_i^r = \beta_{i-1}^r \cup \beta(\mathsf{EP}^{e_i}), \qquad \beta_i^{r'} = \beta_{i-1}^{r'} \text{ for } r' \in \{1, \dots, k\} \setminus \{r\}, \\ &\gamma_i^0 = \gamma_{i-1}^0 \cup (\gamma(\mathsf{EP}^{e_i}) \setminus \xi_{i-1}^0), \qquad \xi_i^0 = \xi(\mathsf{EP}^{e_i}) \cup (\xi_{i-1}^0 \setminus \gamma(\mathsf{EP}^{e_i})), \end{aligned} \tag{3.10}$$

and, setting $u_i = u_{i-1}$, let $\gamma_i^j = \gamma_{i-1}^j$ and $\xi_i^j = \xi_{i-1}^j$ for $1 \le j \le u_i$. Finally, we let $\mu_i = \mu_{i-1} + \mu(\mathsf{EP}^{e_i})$, $\nu_i^r = \nu_{i-1}^r + \nu(\mathsf{EP}^{e_i})$ and $\nu_i^{r'} = \nu_{i-1}^{r'}$ for $r' \neq r$.

- If $e_i \in \vec{I}(S)$, set $\beta_i^r = \beta_{i-1}^r$, $\nu_i^r = \nu_{i-1}^r$ for $1 \le r \le k$, and consider the following two possible situations:
  - If $e_i \notin \{\overline{e}_j, 1 \le j < i\}$, let

$$\mathsf{S}_i = G(\mathsf{S}_{i-1}, \mathsf{EP}^{e_i}; \xi_{i-1}^0, \gamma(\mathsf{EP}^{e_i})),$$

    update the first two geodesic marks by (3.10), and, setting $u_i = u_{i-1} + 1$, let $\gamma_i^j = \gamma_{i-1}^j$ and $\xi_i^j = \xi_{i-1}^j$ for $1 \le j \le u_i - 1$, and $\gamma_i^{u_i} = \overline{\gamma}(\mathsf{EP}^{e_i})$, $\xi_i^{u_i} = \overline{\xi}(\mathsf{EP}^{e_i})$. Finally, set $\mu_i = \mu_{i-1} + \mu(\mathsf{EP}^{e_i})$.
  - If $e_i \in \{\overline{e}_j, 1 \le j < i\}$, let

$$\mathsf{S}_i = G(\mathsf{S}_{i-1}; \xi_{i-1}^0, \gamma(\mathsf{EP}^{e_i})),$$

    update the first two geodesic marks by (3.10), and, setting $u_i = u_{i-1} - 1$, let $(\gamma_i^j, \xi_i^j, 1 \le j \le u_i)$ be the sequence $(\gamma_{i-1}^j, \xi_{i-1}^j, 1 \le j \le u_{i-1})$ from which the terms $\gamma(\mathsf{EP}^{e_i})$ and $\xi(\mathsf{EP}^{e_i})$ have been removed. Finally, set $\mu_i = \mu_{i-1}$.
- If e_i is an external vertex of S, set $\mathsf{S}_i = \mathsf{S}_{i-1}$.

Finally, we let $\mathsf{S}_{\boldsymbol{L}}^{[g]} = G(\mathsf{S}_\kappa; \xi_\kappa^0, \gamma_\kappa^0)$, seen as an element of $\mathbb{M}^{(k,k+1)}$, equipped with the marking $(\beta_\kappa^1, \dots, \beta_\kappa^k)$ and measures $\mu_\kappa, \boldsymbol{\nu}_\kappa$. An application of Proposition 3.8 and of Lemma 3.4 at every step of the construction shows that, for every $i \in \{1, \dots, \kappa\}$, the rescaled marked measured metric space $\Omega_n(Q_{n,i})$ converges to S_i in the marked GHP topology, and finally $\Omega_n(Q_n)$ converges to $\mathsf{S}_{\boldsymbol{L}}^{[g]}$ by a final application of Proposition 3.8 and the observation regarding the measure supports in the final gluing mentioned at the end of Section 3.5. This completes the proof of Theorem 1.1.

3.8 Topology and Hausdorff dimension

In this section, we derive from Proposition 1.2 in the case $(g, k) \in \{(0, 0), (0, 1)\}$ an alternate proof of Proposition 1.2 in the other cases. In the spherical case, Proposition 1.2 was obtained by Le Gall and Paulin [73] thanks to a theorem of Moore by seeing the Brownian sphere as a rather wild quotient of the sphere by some equivalence relation. The same result was later obtained in [78] through the theory of regularity of sequences developed by Begle and studied by Whyburn. The latter approach was generalized in [20–22] in order to obtain the general cases.

Our approach of decomposition into elementary pieces gives a rather direct and transparent proof of Proposition 1.2 in the case $(g, k) \notin \{(0, 0), (0, 1)\}$ provided the following lemma, which will be obtained in Chapters 4 and 5, and which amounts to Proposition 1.2 for the noncompact analogs of the cases $(g, k) \in \{(0, 0), (0, 1)\}$.

Lemma 3.11. *Almost surely, a slice or a quadrilateral is homeomorphic to a disk and is locally of Hausdorff dimension* 4*. Its boundary as a topological manifold consists in the union of its marks, the intersection of any two marks being empty or a singleton. Furthermore, in the case of a slice, the base is locally of Hausdorff dimension* 2*.*

Now, a quadrangulation from $\mathbf{Q}_{n,\boldsymbol{l}_n}^{[g]}$ is "not far" from being homeomorphic to $\boldsymbol{\Sigma}_{b_n}^{[g]}$, where b_n is the number of external faces, in the sense that we can "fill in" the internal faces with small topological disks and "fill in" the external faces by thin topological annuli without altering the metric and in such a way that the resulting object is homeomorphic to $\boldsymbol{\Sigma}_{b_n}^{[g]}$ and at bounded GH distance from the quadrangulation (see [22, Section 4.3.3] for more details about this procedure). The decomposition of the quadrangulation into elementary pieces gives a decomposition of this surface into pieces that are no other than the elementary pieces of the quadrangulation with faces filled in and a thin rectangle added on the bases of the slices; see Figure 3.5.

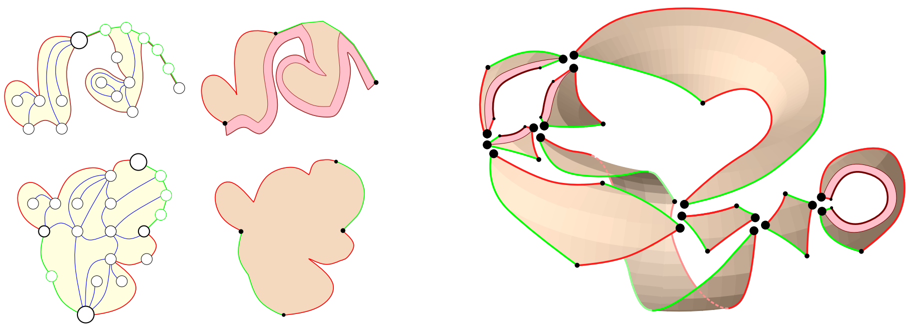

Figure 3.5. *Left.* Topological disk corresponding to an elementary piece of a quadrangulation. *Right.* Decomposition of the surface associated with a quadrangulation into surfaces homeomorphic to disks. Here also, we used the same scheme as in Figure 2.2 without $\hbar_3$.

With the notation of the previous section, Q_n yields a surface homeomorphic to $\boldsymbol{\Sigma}_{b_n}^{[g]}$ and its elementary pieces EP_n^e, $e \in \vec{E}(S_n)$, yield surfaces homeomorphic to disks.

- If $e \in \vec{I}(S_n)$, then the boundary of the surface associated with EP_n^e consists in the two maximal geodesics and the two shuttles of EP_n^e.
- If $e \in \vec{B}(S_n)$, then the boundary of the surface associated with EP_n^e consists in the maximal geodesic, the shuttle, as well as three sides of the added thin rectangle.

Then gluing back these disks along their boundaries in the same way as they were cut gives back the surface $\boldsymbol{\Sigma}_{b_n}^{[g]}$. Forgetting the slice corresponding to a tadpole topologically amounts to fill in the corresponding vanishing face. Assuming that n is sufficiently large, all the vanishing faces correspond to tadpoles in the scheme. So the gluing of the elementary pieces without the slices corresponding to tadpoles yields $\boldsymbol{\Sigma}_{b}^{[g]}$.

In the limit, we glue topological disk exactly in the same way (using markings that are topologically equivalent), so we obtain the same surface. The result about the topology follows.

The statement about the Hausdorff dimension is even more straightforward as we glue along geodesics a finite number of objects that are locally of dimension 4 and the boundary is the union of the bases of the slices, which are all locally of Hausdorff dimension 2.