Chapter 5

Convergence of quadrilaterals with geodesic sides

The general method to prove Theorem 2.8 is the same as for slices. We start by seeing a discrete quadrilateral as part of a discrete map that is known to converge to a Brownian surface, which in this case is the Brownian plane rather than the Brownian half-plane. However, the lack of an analog of Corollary 4.2, namely that quadrilaterals are only *locally* isometrically embedded in the Brownian plane, makes matters considerably more delicate. For this reason, we adapt the strategy we used in [25, Section 4] when treating the case of noncomposite slices. Beware that, in this chapter, part of the notation we will be using is slightly conflicting with that of Chapter 4: in particular, the random times T_x will be re-defined.

5.1 Quadrilaterals coded by two functions

In contrast with slices, which were coded by a pair of functions defined on a common interval I, a quadrilateral will be coded by a pair of functions defined on a common union of two intervals $I_- \cup I_+$, each interval accounting for one "half" of the quadrilateral. This leads to similar but slightly more intricate definitions. We start with the most convenient setting, asking that $\sup I_- = \inf I_+ = 0$.

Recall the notation of Section 4.1.1. We adapt Section 4.1.2 to quadrilaterals as follows. We now say that a pair $(f, g) \in \mathcal{C}^2$ of functions with common closed interval of definition I is a *quadrilateral trajectory* if they satisfy (4.6) and the following:

- the interval I contains 0 in its interior and is either bounded or equal to the whole real line $\mathbb{R}$, and, letting $I_+ = I \cap \mathbb{R}_{\geq 0}$ and $I_- = I \cap \mathbb{R}_{\leq 0}$,
- we have $\inf_{I_-} f = \inf_{I_+} f$, and
- if $I = \mathbb{R}$, then $\inf_{t\geq 0} f(t) = \inf_{t\leq 0} f(t) = \inf_{t\geq 0} g(t) = \inf_{t\leq 0} g(t) = -\infty$.

We may observe that $(f|_{I_+}, g|_{I_+})$ is a slice trajectory, a fact that will not be used here. For a quadrilateral trajectory (f, g), we set

$$\widehat{d}_g(s,t) = \begin{cases} d_g(s,t) & \text{for } s,t \in I_+ \text{ or } s,t \in I_-, \\ \infty & \text{for } st < 0 \end{cases}$$

and

$$\widehat{D}_{f,g} = \widehat{d}_g / \{d_f = 0\}. \tag{5.1}$$

Note that $\widehat{d}_g$ is the disjoint union pseudometric of the two $\mathbb{R}$-tree pseudometrics $d_{g|_{I_+}}$ and $d_{g|_{I_-}}$. Let$(\widehat{M}_{f,g}, \widehat{D}_{f,g})$ be the quotient space $I/\{\widehat{D}_{f,g} = 0\}$ equipped with the metric induced by $\widehat{D}_{f,g}$, still denoted by the same symbol. We call the metric space

$$\mathsf{Qd}_{f,g} = (\widehat{M}_{f,g}, \widehat{D}_{f,g})$$

the *quadrilateral* coded by (f, g).

We extend the above constructions to unions of two closed intervals $I = I_- \cup I_+$, where $I_- \subseteq \mathbb{R}_{\leq 0}$ and $I_+ \subseteq \mathbb{R}_{\geq 0}$, as follows. First, a pair of functions (f, g) defined on I is a *quadrilateral trajectory* if the pair (f', g') defined by

$$f'(t) = \begin{cases} f(t + \inf I_+) & \text{for } t \in I_+ - \inf I_+, \\ f(t + \sup I_-) & \text{for } t \in I_- - \sup I_-, \end{cases}$$

and similarly for g', is a quadrilateral trajectory as defined above. Note that the continuity hypothesis on f' implies in particular that $f(\sup I_-) = f(\inf I_+)$, and similarly for g. We then define the quadrilateral coded by (f, g) using the exact same definitions as above. Note that the mapping $t \mapsto (t - \inf I_+)\mathbf{1}_{t\in I_+} + (t - \sup I_-)\mathbf{1}_{t\in I_-}$ induces an isometry from $(\widehat{M}_{f,g}, \widehat{D}_{f,g})$ onto $(\widehat{M}_{f',g'}, \widehat{D}_{f',g'})$.

From now on, we work in this extended framework and consider a fixed quadrilateral trajectory (f, g).

Geodesic sides and area measure. For every $t \in I \setminus \{0\}$, we let $I_t = I_-$ if $t < 0$ or $I_t = I_+$ if $t > 0$, and set

$$\Gamma_t(r) = \inf\{s \geq t : g(s) = g(t) - r\} \quad \text{for } r \in \mathbb{R}_{\geq 0} \text{ such that } \inf_{\substack{s\geq t\\ s\in I_t}} g(s) \leq g(t) - r;$$

$$\Xi_t(r) = \sup\{s \leq t : g(s) = g(t) - r\} \quad \text{for } r \in \mathbb{R}_{\geq 0} \text{ such that } \inf_{\substack{s\leq t\\ s\in I_t}} g(s) \leq g(t) - r.$$

If $0 \in I$, we also define Γ_0 and Ξ_0 by the same definition, using $I_0 = I_+$ in the definition of Γ_0, while using $I_0 = I_-$ in the definition of Ξ_0. Observe that, in contrast with the definition for slices, the infimum of g is now taken on a subset of I_t. In particular, this implies that the ranges of Γ_t, Ξ_t are included in I_t. From the same discussion as the one around (4.7), we see that Γ_t, Ξ_t are geodesics for the pseudometrics $\widehat{d}_g$ and $\widehat{D}_{f,g}$. In the case where $\sup I_+ = \infty$, then, for every $t \in I_+$, the range of the path Γ_t is a geodesic ray, and, in the case where $\inf I_- = -\infty$, then the same goes for Ξ_t for every $t \in I_-$. This allows us to define geodesic paths in $\mathsf{Qd}_{f,g}$ by the formulas

$$\gamma_t(r) = \widehat{\mathbf{p}}_{f,g}(\Gamma_t(r)), \quad 0 \leq r \leq g(t) - \underline{g}(t, \sup I_t),\ r \in \mathbb{R},$$

$$\xi_t(r) = \widehat{\mathbf{p}}_{f,g}(\Xi_t(r)), \quad 0 \leq r \leq g(t) - \underline{g}(\inf I_t, t),\ r \in \mathbb{R},$$

where $\widehat{\mathbf{p}}_{f,g}\colon I \to \widehat{M}_{f,g}$ is the canonical projection and, as above, if $0 \in I$, $I_0 = I_+$ in the definition of γ_0 and $I_0 = I_-$ in that of ξ_0. Note that the geodesics γ_t, ξ_t share a common initial part.

The quadrilateral $\mathsf{Qd}_{f,g}$ comes with four or two geodesic sides, defined as follows. If I is bounded, the particular geodesics $\gamma = \gamma_{\inf I_+}$ and $\overline{\gamma} = \gamma_{\inf I_-}$ are called the *maximal geodesics* of $\mathsf{Qd}_{f,g}$, while $\xi = \xi_{\sup I_+}$ and $\overline{\xi} = \xi_{\sup I_-}$ are called the *shuttles* of $\mathsf{Qd}_{f,g}$. In this case, γ, ξ (resp. $\overline{\gamma}, \overline{\xi}$) have a common endpoint $x_* = \widehat{\mathbf{p}}_{f,g}(s_*)$ (resp. $\overline{x}_* = \widehat{\mathbf{p}}_{f,g}(\overline{s}_*)$), where $s_* \in I_+$ is such that $g(s) = \inf_{I_+} g$ (resp. $\overline{s}_* \in I_-$ is such that $g(\overline{s}_*) = \inf_{I_-} g$). The points $x_*, \overline{x}_*$ are called the *apexes* of $\mathsf{Qd}_{f,g}$. If I is unbounded, then $\mathsf{Qd}_{f,g}$ has one *maximal geodesic* $\gamma = \gamma_{\inf I_+}$ and one *shuttle* $\overline{\xi} = \xi_{\sup I_-}$; we set $\xi_\infty = \gamma_{-\infty} = \varnothing$.

Finally, the *area measure* is defined as $\mu = (\widehat{\mathbf{p}}_{f,g})_* \mathrm{Leb}_I$.

Gluing quadrilaterals. For $x \in \mathbb{R}$, we let

$$T_x = \inf\{t \in I_+ : f(t) = -x\} \in \mathbb{R}_{\geq 0} \cup \{+\infty\},$$
$$\overline{T}_x = \sup\{t \in I_- : f(t) = -x\} \in \mathbb{R}_{\leq 0} \cup \{-\infty\},$$

as well as $T_\infty = -\overline{T}_\infty = \infty$. Note that, here again, there is a slight difference with the definition of Chapter 4 since, now, $\mathbb{R}_{\leq 0}$ and $\mathbb{R}_{\geq 0}$ play different roles. Recall that $\inf_{I_-} f = \inf_{I_+} f = \inf_I f$, and let $H, H' \in \mathbb{R}_{\geq 0} \cup \{\infty\}$ be such that

$$0 \leq H \leq H' \leq -\inf_I f.$$

We may define the restrictions $f^{(H,H')}$, $g^{(H,H')}$ of f and g to the union of intervals $I^{(H,H')} = [\overline{T}_{H'}, \overline{T}_H] \cup [T_H, T_{H'}]$, which is a subset of I. The pair $(f^{(H,H')}, g^{(H,H')})$ is another quadrilateral trajectory. The associated quadrilateral is defined as

$$\mathsf{Qd}^{(H,H')} = (\widehat{M}^{(H,H')}, \widehat{D}^{(H,H')}) = \mathsf{Qd}_{f^{(H,H')}, g^{(H,H')}}.$$

We let $\widehat{\mathbf{p}}^{(H,H')} \colon I^{(H,H')} \to \widehat{M}^{(H,H')}$ be the canonical projection, $\mu^{(H,H')}$ be the area measure of $\mathsf{Qd}^{(H,H')}$, and, whenever they exist, $\gamma^{(H,H')}$, $\overline{\gamma}^{(H,H')}$ be the maximal geodesics, $\xi^{(H,H')}$, $\overline{\xi}^{(H,H')}$ be the shuttles.

We refer to Figure 5.1 for an illustration of the following proposition in the upcoming context of random quadrilaterals in the Brownian plane.

Proposition 5.1. *Let* $0 \leq H < H' < H'' \leq -\inf_I f$ *be in the extended positive real line. Then*

$$\mathsf{Qd}^{(H,H'')} = G(G(\mathsf{Qd}^{(H,H')}, \mathsf{Qd}^{(H',H'')}; \xi^{(H,H')}, \gamma^{(H',H'')}); \overline{\gamma}^{(H,H')}, \overline{\xi}^{(H',H'')}), \quad (5.2)$$

and it holds that

$$\gamma^{(H,H'')} = \gamma^{(H,H')} \cup (\gamma^{(H',H'')} \setminus \xi^{(H,H')}),$$
$$\xi^{(H,H'')} = \xi^{(H',H'')} \cup (\xi^{(H,H')} \setminus \gamma^{(H',H'')}),$$
$$\overline{\gamma}^{(H,H'')} = \overline{\gamma}^{(H',H'')} \cup (\overline{\gamma}^{(H,H')} \setminus \overline{\xi}^{(H',H'')}),$$
$$\overline{\xi}^{(H,H'')} = \overline{\xi}^{(H,H')} \cup (\overline{\xi}^{(H',H'')} \setminus \overline{\gamma}^{(H,H')}).$$

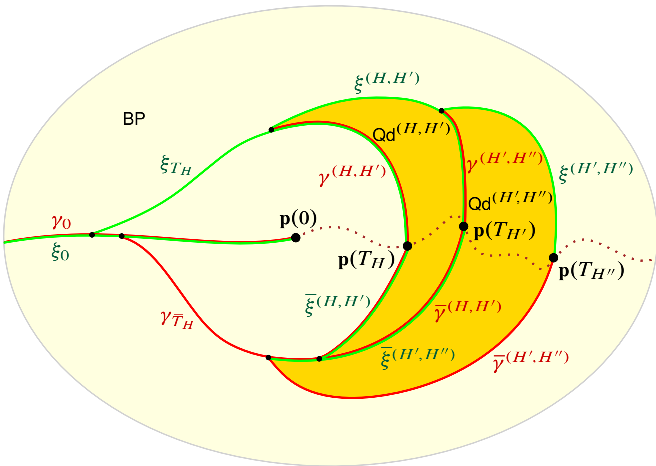

Figure 5.1. Seeing free quadrilaterals in the Brownian plane. The union of the dark yellow regions forms $\mathsf{Qd}^{(H,H'')}$. The dotted brown line is $\{\mathbf{p}(T_h) : h \geq 0\}$. Note that BP is obtained by gluing $\mathsf{Qd}^{(0,\infty)}$ along the geodesics $\gamma^{(0,\infty)}$ and $\xi^{(0,\infty)}$, resulting in the geodesic $\gamma_0 = \xi_0$.

Observe that, after the first gluing operation is performed, the marks $\overline{\gamma}$ and $\overline{\xi}$ remain geodesic, as observed in Section 3.3. Note also that the order of the gluings in (5.2) is not important, due to (3.3).

Proof. The proof is similar to that of Proposition 4.1, and we only sketch the argument. Again, we view $I^{(H,H'')}$ as a disjoint union $I^{(H,H')} \sqcup I^{(H',H'')}$ (denoting elements of these sets with superscripts 0, 1, respectively), where the extremities $T^0_{H'}$, $T^1_{H'}$ and $\overline{T}^0_{H'}$, $\overline{T}^1_{H'}$ are identified. We then observe that the pseudometric $\widehat{d}_g$ can be viewed as a quotient d/R_1, where d is the disjoint union metric on $I^{(H,H')} \sqcup I^{(H',H'')}$ whose restriction to each interval composing this set equals the restriction of d_g to that interval, and R_1 is the coarsest equivalence relation containing

$$\{(\Xi_{T_{H'}}(r)^0, \Gamma_{T_{H'}}(r)^1),\ 0 \leq r \leq g(T_{H'}) - \underline{g}(T_H, T_{H'}) \vee \underline{g}(T_{H'}, T_{H''})\},$$
$$\{(\Gamma_{\overline{T}_{H'}}(r)^0, \Xi_{\overline{T}_{H'}}(r)^1),\ 0 \leq r \leq g(\overline{T}_{H'}) - \underline{g}(\overline{T}_H, \overline{T}_{H'}) \vee \underline{g}(\overline{T}_{H'}, \overline{T}_{H''})\}.$$

Moreover, the equivalence relation $\{d_f = 0\}$ factorizes in the sense that if $d_f(s,t) = 0$ with $s, t \in I^{(H,H'')}$, then it must hold that s, t belong either both to $I^{(H,H')}$ or both to $I^{(H',H'')}$. Therefore, setting R_2 as the equivalence relation on $I^{(H,H')} \sqcup I^{(H',H'')}$ defined by $s^i\ R_2\ t^j$ if and only if $d_f(s,t) = 0$ and $i = j \in \{0,1\}$, we obtain

$$\widehat{D}^{(H,H'')} = (d/R_1)/R_2 = (d/R_2)/R_1.$$

We see that d/R_2 is the pseudometric of the disjoint union of $(I^{(H,H')}, \widehat{D}^{(H,H')})$ and $(I^{(H',H'')}, \widehat{D}^{(H',H'')})$, while R_1 can be seen as the coarsest equivalence relation obtained by first gluing $\xi^{(H,H')}$ with $\gamma^{(H',H'')}$, and then $\overline{\gamma}^{(H,H')}$ with $\overline{\xi}^{(H',H'')}$. ∎

Due to the fact that the second gluing operation in Proposition 5.1 involves two geodesics belonging to the same space, there is no direct analog of Corollary 4.2. However, we have the following alternative, which is an immediate consequence of Lemma 3.7 and a crude estimate of the length of the path $\overline{\xi}^{(H',H'')}$.

Proposition 5.2. *Under the same assumptions as in Proposition 5.1, it holds that for every $s, t \in I^{(H,H')}$,*

$$\widehat{D}^{(H,H'')}(s,t) \le \widehat{D}^{(H,H')}(s,t) \le \widehat{D}^{(H,H'')}(s,t) + \omega(g; I^{(H',H'')}).$$

Finally, we observe that the metric space $(M_{f,g}, D_{f,g})$ obtained by metric gluing of the pseudometric d_g along the relation $\{d_f = 0\}$, rather than using $\widehat{d}_g$ as in the definition of $\mathsf{Qd}_{f,g}$, is related to the latter by a final gluing operation. The proof is analog to that of Proposition 5.1, noting that d_g is the gluing of $\widehat{d}_g$ along the coarsest equivalence relation containing $\{(\Gamma_0(r), \Xi_0(r)), r \ge 0\}$.

Lemma 5.3. *One has $(M_{f,g}, D_{f,g}) = G(\mathsf{Qd}_{f,g}; \gamma, \overline{\xi})$.*

5.2 Random continuum quadrilaterals

Let us now describe the limiting continuum quadrilaterals that appear in Theorem 2.8, by suitably randomizing the quadrilateral trajectory (f, g). We let (X, W) be the canonical process defined on quadrilateral trajectories. We introduce, for any process Y defined on an interval containing 0, the piece of notation $\underline{\underline{Y}}_t = \underline{Y}(0 \wedge t, 0 \vee t)$.

Let us fix $A, \overline{A}, H \in (0, \infty)$ and $\Delta \in \mathbb{R}$. We let $\mathbf{Quad}_{A,\overline{A},H,\Delta}$ be the probability distribution under which

- $(X_t, 0 \le t \le A)$ and $(X_{-t}, 0 \le t \le \overline{A})$ are independent first-passage bridges of standard Brownian motion from 0 to $-H$, with durations A and $\overline{A}$,
- conditionally given X, the process W has same law as $(Z_t + \zeta_{-\underline{\underline{X}}_t}, -\overline{A} \le t \le A)$, where Z is the random snake driven by $X - \underline{\underline{X}}$, and ζ is a standard Brownian bridge of duration H and terminal value Δ, independent of X and Z.

In this way, the probability distribution $\mathbf{Quad}_{A,\overline{A},H,\Delta}$ is carried by quadrilateral trajectories on the interval $[-\overline{A}, A]$. We remark that, in fact, we can view W more directly as the random snake driven by X, conditioned on the event $\{W_A = \Delta\}$, a fact that we leave to the interested reader.

Definition 5.4. The *quadrilateral with half-areas A, $\overline{A}$, width H and tilt* Δ, generically denote by $\mathsf{Qd}_{A,\overline{A},H,\Delta}$, is the 6-marked 1-measured metric space $\mathsf{Qd}_{X,W}$ under the law $\mathbf{Quad}_{A,\overline{A},H,\Delta}$, endowed with its area measure μ, as well as the marking $\partial\mathsf{Qd}_{A,\overline{A},H,\Delta} = (\gamma, \xi, \overline{\gamma}, \overline{\xi})$, where γ, $\overline{\gamma}$ are geodesic marks as usual, while ξ, $\overline{\xi}$ are seen as (nonoriented) geodesic segments, that is, given without their origins.

As for slices, the piece of notation $\partial\mathsf{Qd}_{A,\overline{A},H,\Delta}$ comes from Lemma 3.11, which we will prove at the end of the upcoming section. The boundary of the topological disk $\mathsf{Qd}_{A,\overline{A},H,\Delta}$ is the union of γ, ξ, $\overline{\gamma}$, $\overline{\xi}$, which intersect only at the points $\gamma(0) = \overline{\xi}(0) = \widehat{\mathbf{p}}_{f,g}(0)$, x_*, $\overline{\gamma}(0) = \xi(0) = \widehat{\mathbf{p}}_{f,g}(A) = \widehat{\mathbf{p}}_{f,g}(-\overline{A})$, and $\overline{x}_*$.

5.3 The Brownian plane, and its embedded quadrilaterals

Similarly to the fact that (free) slices can be found in the Brownian half-plane, one can obtain quadrilaterals from the Brownian plane, as we now explain. We let **Plane** be the probability distribution on $\mathcal{C}^2$ under which

- the process X is a two-sided standard Brownian motion,[1] and
- the process W is the random snake driven by X.

The measure **Plane** is carried by quadrilateral trajectories defined over $\mathbb{R}$.

Definition 5.5. The *Brownian plane*, generically denoted by BP, is the metric space $(M_{X,W}, D_{X,W})$ defined by (4.5), considered under **Plane**. Letting $\mathbf{p}\colon \mathbb{R} \to \mathsf{BP}$ be the canonical projection, it is endowed with the area measure $\mu = \mathbf{p}_*\mathrm{Leb}_{\mathbb{R}}$.

In this definition, beware that the metric is indeed defined by (4.5) rather than (5.1), which would produce the metric space $\mathsf{Qd}_{X,W} = \mathsf{Qd}^{(0,\infty)} = (\widehat{M}_{X,W}, \widehat{D}_{X,W})$. Observe that, by Lemma 5.3,

$$\mathsf{BP} = G(\mathsf{Qd}^{(0,\infty)}; \gamma^{(0,\infty)}, \overline{\xi}^{(0,\infty)}); \tag{5.3}$$

see Figure 5.1 below for an illustration. Alternatively, the space $\mathsf{Qd}^{(0,\infty)}$ can be seen as cutting the Brownian plane along the geodesic ray $\gamma_0 = \xi_0$; we do not go into further details as we will not explicitly need this property.

Note also that, despite the similarity between this definition and that of the Brownian half-plane, there is no marking now because, as its name suggests, the Brownian plane is homeomorphic to $\mathbb{R}^2$ and so has an empty boundary as a topological surface.

One should finally mind that this definition is different from the original one given in [40], which will be recalled in Appendix A; in a nutshell, one goes from a definition to the other by changing X into the process obtained by taking its Pitman transform both on $\mathbb{R}_{\geq 0}$ and on $\mathbb{R}_{\leq 0}$.

Free quadrilaterals. Similarly to the discussion of Section 4.3, the Brownian plane satisfies a Markov property which can be interpreted as a "flow" of continuum quadrilaterals. Fixing $0 \leq H \leq H' \leq \infty$, and denoting by

$$\vartheta_H\colon t \in [\overline{T}_{H'} - \overline{T}_H, T_{H'} - T_H] \mapsto (t + \overline{T}_H)\mathbf{1}_{t<0} + (t + T_H)\mathbf{1}_{t\geq 0},$$

[1] This means that $(X_t, t \geq 0)$ and $(X_{-t}, t \geq 0)$ are independent standard Brownian motions.

we see that the process $(X^{(H,H')} \circ \vartheta_H + H, W^{(H,H')} \circ \vartheta_H - W_{T_H})$ is independent of $(X^{(0,H)}, W^{(0,H)})$, $(X^{(H',+\infty)}, W^{(H',+\infty)})$, and has same distribution as the pair $(X^{(H'-H)}, W^{(H'-H)})$. This can be proved by excursion theory of (X, W) separately in positive and negative times; we omit the details, which are similar to those presented in Section 4.3.

Under **Plane**, the process $(X^{(0,H)})$ is a two-sided Brownian motion killed at its first hitting times T_H, $\overline{T}_H$ of $-H$ in positive and negative times, respectively, while the process $(W^{(0,H)}_{T_x}, 0 \le x \le H)$ is a standard Brownian motion killed at time H. This implies that the law of $(X^{(0,H)}, W^{(0,H)})$ under **Plane** equals

$$\mathbf{FQuad}_H = \int_{(0,\infty)^2} q_H(A) q_H(\overline{A})\, \mathrm{d}A\, \mathrm{d}\overline{A} \int_{\mathbb{R}} p_H(\Delta)\, \mathrm{d}\Delta\, \mathbf{Quad}_{A,\overline{A},H,\Delta},$$

where the densities p_t, q_x are defined after (4.9). A random metric space with same law as $\mathsf{Qd}^{(0,H)}$ under $\mathbf{FQuad}_H$ will be referred to as a *free (continuum) quadrilateral* of width H. From these considerations and Proposition 5.1, we obtain the following result.

Proposition 5.6. *Let* $0 \le H < H' < H'' \le \infty$*. Then, under* **Plane***, it holds that*

$$\mathsf{Qd}^{(H,H'')} = G(G(\mathsf{Qd}^{(H,H')}, \mathsf{Qd}^{(H',H'')}; \xi^{(H,H')}, \gamma^{(H',H'')}); \overline{\gamma}^{(H,H')}, \overline{\xi}^{(H',H'')}),$$

where the glued spaces $\mathsf{Qd}^{(H,H')}$ *and* $\mathsf{Qd}^{(H',H'')}$ *are independent. Moreover, the space* $\mathsf{Qd}^{(H,H')}$ *is a free continuum quadrilateral of width* $H' - H$*.*

We refer to Figure 5.1 for an illustration, which suggests, as is proved in the following proposition, that quadrilaterals are topological disks bounded by their geodesic sides. In contrast with our treatment of slices, a difficulty arises from the fact that the quadrilaterals $\mathsf{Qd}^{(H,H')}$ are not isometrically embedded in BP, and, in general, not even locally isometrically embedded (think of a point of BP lying on the geodesic γ_0).

Proposition 5.7. *For every* $H \in (0, \infty)$*, almost surely under* $\mathbf{FQuad}_H$*, the quadrilateral* $\mathsf{Qd}^{(0,H)}$ *is a topological disk with boundary given by the geodesics* $\gamma^{(0,H)}$*,* $\xi^{(0,H)}$*,* $\overline{\gamma}^{(0,H)}$ *and* $\overline{\xi}^{(0,H)}$*, which pairwise meet only at the points* $\gamma(0) = \overline{\xi}(0)$*,* $\xi(0) = \overline{\gamma}(0)$*, and the apexes* $x_*^{(0,H)}$ *and* $\overline{x}_*^{(0,H)}$*.*

In order to prove this proposition and for later use, it will be important to characterize the set $\{D_{X,W} = 0\}$.

Lemma 5.8. *The following holds almost surely under* **Plane***. For every* $s, t \in \mathbb{R}$ *such that* $s \neq t$*, it holds that*

$$D_{X,W}(s,t) = 0$$

if and only if either $d_X(s,t) = 0$ *or* $d_W(s,t) = 0$*, these two cases being mutually exclusive.*

Proof. According to [40, Proposition 11], it holds that $D_{X,W}(s,t) = 0$ implies that $d_X(s,t) = 0$ or $d_W(s,t) = 0$. The fact that these two properties are mutually exclusive is a consequence of the fact from [65, Lemma 2.2] that almost surely, if s is a point such that $X_u \geq X_s$ for every $u \in [s, s+\varepsilon]$ for some $\varepsilon > 0$, then it must hold that $\inf_{u \in [s,s+\delta]} W_u < W_s$ for every $\delta \in (0, \varepsilon)$. In fact, [65, Lemma 2.2] is proved when the process X is distributed as a standard Brownian excursion, and W as a random snake Z driven by this excursion. However, being a local property of the processes at hand, it extends easily to our setting by an absolute continuity argument. Details are left to the reader. ■

To the terminology of Section 4.1.1, we add

$$[\![a,b[\![_f = [\![a,b]\!]_f \setminus \{b\} \quad \text{for } a,b \in \mathcal{T}_f.$$

The important consequence of this lemma for us is the following. Almost surely, if $a,b \in \mathcal{T}_X$ and $c,d \in \mathcal{T}_W$, then the paths $\pi_X([\![a,b]\!]_X)$ and $\pi_W([\![c,d]\!]_W)$ are simple paths. Furthermore, $\pi_X([\![a,b[\![_X)$ may intersect $\pi_W([\![c,d[\![_W)$ only if $\pi_X(a) = \pi_W(c)$, in which case these paths intersect at this point only. In particular, if we denote the geodesic ray $\mathbf{p}_X(\{s \geq t : X_s = \underline{X}(t,s)\})$ of $\mathcal{T}_X$ by $[\![\mathbf{p}_X(t), \infty[\![_X$, then $\pi_X([\![\mathbf{p}_X(t), \infty[\![_X)$ is a simple path in BP. For instance, in Figure 5.1, we represented the simple path $\pi_X([\![\mathbf{p}_X(0), \infty[\![_X)$ with a dotted brown line.

Proof of Proposition 5.7. Let us depart slightly from the setting of the statement and fix for now two numbers $0 \leq H < H' < \infty$.

Claim. *We assume that the geodesics $\gamma^{(H,H')}$ and $\overline{\xi}^{(H,H')}$ do not intersect γ_0 in BP. Then the following holds:*

(i) *The geodesics $\gamma^{(H,H')}$, $\xi^{(H,H')}$, $\overline{\gamma}^{(H,H')}$, $\overline{\xi}^{(H,H')}$ intersect only at the points $x_*^{(H,H')}$, $\overline{\gamma}^{(H,H')}(0) = \xi^{(H,H')}(0)$, $\overline{x}_*^{(H,H')}$ and $\gamma^{(H,H')}(0) = \overline{\xi}^{(H,H')}(0)$ in this cyclic order, and their union forms a Jordan curve C.*

(ii) *The set $\mathbf{p}(I^{(H,H')}) \subseteq \mathsf{BP}$ is the closure of the bounded connected component of $\mathsf{BP} \setminus C$.*

Indeed, note that $\mathbf{p}_X(T_H)$ and $\mathbf{p}_X(T_{H'})$ are two distinct points of $[\![\mathbf{p}_X(0), \infty[\![_X$, so that their images by π_X are distinct in BP. Then the paths $\gamma^{(H,H')}$ and $\xi^{(H,H')}$ are the images by π_W of the two geodesic paths $[\![\mathbf{p}_W(T_H), a_*(W^{(H,H')})]\!]_W$ and $[\![\mathbf{p}_W(T_{H'}), a_*(W^{(H,H')})]\!]_W$ in $\mathcal{T}_W$, which by definition meet only at $a_*(W^{(H,H')})$, and their union is the geodesic $[\![\mathbf{p}_W(T_H), \mathbf{p}_W(T_{H'})]\!]_W$ in $\mathcal{T}_W$, which is thus projected via π_W to a simple path in BP. Therefore, $\gamma^{(H,H')}$ and $\xi^{(H,H')}$ meet only at $x_*^{(H,H')} = \pi_W(a_*(W^{(H,H')}))$. The same reasoning shows that $\overline{\gamma}^{(H,H')}$ and $\overline{\xi}^{(H,H')}$ intersect only at $\overline{x}_*^{(H,H')}$, and gives that the points $x_*^{(H,H')}$ and $\overline{x}_*^{(H,H')}$ are distinct points (because they are distinct points in $\mathcal{T}_W$ lying inside two geodesics).

Next, if the path $\gamma^{(H,H')}$ does not intersect $\gamma_0 = \pi_W([\![\mathbf{p}_W(0), \infty[\![_W)$, then necessarily the path $[\![\mathbf{p}_W(T_H), a_*(W^{(H,H')})]\!]_W$ must be disjoint from $[\![\mathbf{p}_W(0), \infty[\![_W$, which means that

$$\underline{W}(T_H, T_{H'}) > \underline{W}(0, T_H) \quad \text{and} \quad \underline{W}(\overline{T}_{H'}, \overline{T}_H) > \underline{W}(\overline{T}_H, 0).$$

This implies that $[\![\mathbf{p}_W(T_{H'}), a_*(W^{(H,H')})[\![_W$ is also disjoint from $[\![\mathbf{p}_W(0), \infty[\![_W$, and by projecting by π_W, that $\xi^{(H,H')}$ is disjoint from γ_0. A similar argument applies to $\overline{\gamma}^{(H,H')}$ and $\overline{\xi}^{(H,H')}$. Therefore, under the conditions of the claim, the paths $[\![\mathbf{p}_W(T_H), a_*(W^{(H,H')})]\!]_W$ and $[\![\mathbf{p}_W(\overline{T}_H), \overline{a}_*(W^{(H,H')})]\!]_W$ are disjoint paths in $\mathcal{T}_W$, and their projections $\gamma^{(H,H')}$ and $\overline{\xi}^{(H,H')}$ via π_W intersect, if at all, only at their extremities. It is indeed the case that $\mathbf{p}(T_H) = \mathbf{p}(\overline{T}_H)$, while, as we already saw, $x_*^{(H,H')} \neq \overline{x}_*^{(H,H')}$. This proves (i).

The argument for (ii) is similar to that in the proof of Proposition 4.6, where the role of the base is now played by the infinite path $\pi_X([\![\mathbf{p}_X(0), \infty[\![_X) = \{\mathbf{p}(T_h) : h \geq 0\}$. For any $t \in \mathbb{R}$, we let

$$\Sigma_t(r) = \inf\{s \geq t : X_s = X_t - r\} \quad \text{for } 0 \leq r \leq X_t - \underline{\underline{X}}_t,$$

where we recall that $\underline{\underline{X}}_t = \underline{X}(0 \wedge t, 0 \vee t)$. The range of $\mathbf{p}_X \circ \Sigma_t$ is the geodesic path $[\![\mathbf{p}_X(t), \mathbf{p}_X(T_{-\underline{\underline{X}}_t})]\!]_X$ in $\mathcal{T}_X$ and its image by π_X defines a path $\sigma_t = \mathbf{p} \circ \Sigma_t$ from $\mathbf{p}(t)$ to $\mathbf{p}(T_{-\underline{\underline{X}}_t})$. Moreover, by Lemma 5.8, the paths σ_t, $t \in \mathbb{R}$, do not intersect any of the geodesics γ_s, $s \in \mathbb{R}$, except possibly at their starting points. There are now the following possibilities:

- If $t \in I^{(H,H')}$, then σ_t ends on the path $(\mathbf{p}(T_h), H \leq h \leq H')$. This means that if $\mathbf{p}(t)$ does not belong to the four geodesics $\gamma^{(H,H')}$, $\xi^{(H,H')}$, $\overline{\gamma}^{(H,H')}$, $\overline{\xi}^{(H,H')}$, then we may connect it to, say, the point $\mathbf{p}(T_{(H+H')/2})$ of the bounded set $\mathsf{Qd}^{(H,H')}$, without crossing the four mentioned geodesics.
- If $t \notin I^{(H,H')}$, we distinguish two cases:
  - If $t \notin [\overline{T}_{H'}, T_{H'}]$, then σ_t ends on the unbounded path $\{\mathbf{p}(T_h) : h > H\}$.
  - If $t \in (\overline{T}_H, T_H)$, then σ_t ends on $\{\mathbf{p}(T_h) : 0 \leq h < H\}$.

  If $\mathbf{p}(t)$ does not belong to the four geodesics of interest, then it may be joined without crossing the four geodesics either to the unbounded path $\{\mathbf{p}(T_h) : h > H\}$ or to the unbounded path $\{\mathbf{p}(T_h) : 0 \leq h < H\} \cup \gamma_0$, by the assumption that γ_0 does not intersect the four geodesics.

This completes the proof of the claim.

Now fix $H > 0$ and consider another positive number H_0 to be thought of as large. Since we know that $\mathsf{Qd}^{(H_0, H_0+H)}$ under **Plane** has same distribution as $\mathsf{Qd}^{(0,H)}$, we may work with the former space rather than with the latter. For every $\varepsilon > 0$, it holds that there exists some H_0 large enough such that with probability at least

$1-\varepsilon$, the geodesics $\gamma^{(H_0,H_0+H)}$ and $\overline{\xi}^{(H_0,H_0+H)}$ do not intersect γ_0. Indeed, this happens whenever $\underline{W}(T_{H_0}, T_{H_0+H}) > \underline{W}(0, T_{H_0})$ or, equivalently,

$$W_{T_{H_0}} - \underline{W}(0, T_{H_0}) > W_{T_{H_0}} - \underline{W}(T_{H_0}, T_{H_0+H}), \tag{5.4}$$

and similarly in negative times. The two sides of (5.4) are independent by the Markov property stated above; the right-hand side has a distribution that depends only on H, while the left-hand side, which has same distribution as $-\underline{W}(0, T_{H_0})$ by a simple time-reversal argument, converges to ∞ in probability as $H_0 \to \infty$.

By the claim, we obtain that on an event happening with probability at least $1-\varepsilon$, the set $\mathbf{p}(I^{(H_0,H_0+H)})$ is the closure of the connected component of the complement in BP of the paths

$$\gamma^{(H_0,H_0+H)}, \quad \xi^{(H_0,H_0+H)}, \quad \overline{\gamma}^{(H_0,H_0+H)}, \quad \overline{\xi}^{(H_0,H_0+H)},$$

which all together form a Jordan curve. On this event, the identity on $I^{(H_0,H_0+H)}$ induces, by precomposition with the projection mappings $\mathbf{p}$ and $\mathbf{p}^{(H_0,H_0+H)}$, a bijective mapping ϕ from the compact space $\mathsf{Qd}^{(H_0,H_0+H)}$ to $\mathbf{p}(I^{(H_0,H_0+H)})$, which is 1-Lipschitz since $D_{X,W} \leq \widehat{D}^{(H_0,H_0+H)}$ by Lemma 3.7, (5.3) and Proposition 5.6. This shows that ϕ is a homeomorphism, and therefore, with probability at least $1-\varepsilon$, the space $\mathsf{Qd}^{(H_0,H_0+H)}$ has the properties claimed in the statement. Using the fact that $\mathsf{Qd}^{(H_0,H_0+H)}$ has same distribution as $\mathsf{Qd}^{(0,H)}$ and that ε was arbitrary, we conclude. ∎

The continuum quadrilaterals of the preceding section can be linked to the free quadrilaterals embedded in the Brownian plane by an absolute continuity argument, whose proof is similar to that of Lemma 4.7 and is omitted.

Lemma 5.9. *Fix $0 < K < H$, as well as $A > 0$, $\overline{A} > 0$, and $\Delta \in \mathbb{R}$. Then, for every nonnegative function G that is measurable with respect to the σ-algebra generated by $(X^{(0,K)}, W^{(0,K)})$, one has*

$$\mathbf{Quad}_{A,\overline{A},H,\Delta}[G] = \mathbf{Plane}[\psi_{A,\overline{A},H,\Delta}(T_K, -\overline{T}_K, K, W_{T_K}) \cdot G],$$

where

$$\psi_{A,\overline{A},H,\Delta}(A', \overline{A}', H', \Delta') = \frac{q_{H-H'}(A-A')}{q_H(A)} \frac{q_{H-H'}(\overline{A}-\overline{A}')}{q_H(\overline{A})} \frac{p_{H-H'}(\Delta-\Delta')}{p_H(\Delta)}.$$

This allows us to obtain, as stated in Lemma 3.11, the topology of quadrilaterals.

Proof of Lemma 3.11 for quadrilaterals. The proof is similar to that for slices. We use the fact that the Brownian plane is topologically a plane [40, Proposition 13], as well as [40, Proposition 4] to obtain that it is locally of Hausdorff dimension 4 from the analog result about the Brownian sphere [64]. We deduce from there the desired properties for a free quadrilateral. To extend this result to quadrilaterals $\mathsf{Qd}_{A,\bar{A},H,\Delta}$, which we view as $\mathsf{Qd}^{(0,H)}$ under the law $\mathbf{Quad}_{A,\bar{A},H,\Delta}$, we use the fact from Proposition 5.1 that it can be seen as the gluing of $\mathsf{Qd}^{(0,H/2)}$ and $\mathsf{Qd}^{(H/2,H)}$ along the boundaries $\xi^{(0,H/2)}$ and $\gamma^{(H/2,H)}$ on the one hand, and $\overline{\gamma}^{(0,H/2)}$ and $\overline{\xi}^{(H/2,H)}$ on the other hand. By the absolute continuity relation stated in Lemma 5.9, we see that the law of $\mathsf{Qd}^{(0,H/2)}$ is absolutely continuous with respect to that of a free quadrilateral with width $H/2$, and the same is true for $\mathsf{Qd}^{(H/2,H)}$. Using Proposition 5.7, we obtain that $\mathsf{Qd}^{(0,H)}$ is obtained by gluing two topological disks, both locally of Hausdorff dimension 4, along part of their boundaries, which allows us to conclude. ■

5.4 The uniform infinite planar quadrangulation

The uniform infinite planar quadrangulation (UIPQ) is the whole plane pendant of the UIHPQ defined in Section 4.4. It is simpler to describe and was introduced earlier [36, 42, 60]. Let $(\mathbf{T}^k, k \in \mathbb{Z})$ be a two-sided sequence of independent Bienaymé–Galton–Watson trees with a geometric offspring distribution of parameter $1/2$. We construct an infinite tree $\mathbf{T}_\infty$ embedded in the plane by mapping the roots of $\mathbf{T}^k$ and of $\mathbf{T}^{-k}$ to the point $\rho^k = (k, 0)$ for every $k \geq 0$, in such a way that, except for these roots, the trees $\mathbf{T}^k$, $k \geq 0$ are embedded in the open upper half-plane and the trees $\mathbf{T}^k$, $k < 0$ are embedded in the open lower half-plane, without intersection. Lastly, we link the roots ρ^k, ρ^{k+1} with a horizontal segment for every $k \geq 0$.

Conditionally on $\mathbf{T}_\infty$, we assign to the edges random numbers, independent and uniformly distributed in $\{-1, 0, 1\}$, and let $\lambda_\infty \colon V(\mathbf{T}_\infty) \to \mathbb{Z}$ be the labeling function whose increments along the edges are given by these numbers. Note that this uniquely defines λ_∞, up to the usual addition of a constant. We call $(\mathbf{T}_\infty, \lambda_\infty)$ the *infinite random well-labeled tree*. We then let $(c_i, i \in \mathbb{Z})$ be the sequence of corners of $\mathbf{T}_\infty$ in contour order, with origin the corner c_0 corresponding to the root of $\mathbf{T}^0$. The *uniform infinite planar quadrangulation* (*UIPQ* for short) is then the infinite map Q_∞ obtained by applying the CVS construction to $(\mathbf{T}_\infty, \lambda_\infty)$, that is, by linking every corner to its successor as defined in Section 2.1, and removing all edges of the tree afterward. The root of Q_∞ is defined as the corner preceding the arc from c_0 to its successor. As with the UIHPQ, there is no need to add an extra vertex with a corner c_∞.

As before, we denote by v_i the vertex of $\mathbf{T}_\infty$ incident to c_i and by $\Upsilon(i) \in \mathbb{Z}$ the index of the tree to which v_i belongs. We then define the *contour* and *label processes* on $\mathbb{R}$ by

$$C(i) = d_{\mathbf{T}^{\Upsilon(i)}}(v_i, \rho^{|\Upsilon(i)|}) - |\Upsilon(i)| \quad \text{and} \quad \Lambda(i) = \lambda_\infty(v_i) - \lambda_\infty(v_0), \quad i \in \mathbb{Z},$$

and by linear interpolation between integer values; see Figure 5.2. Observe that, in contrast with the definition of Section 4.4 for an infinite forest, there is an absolute

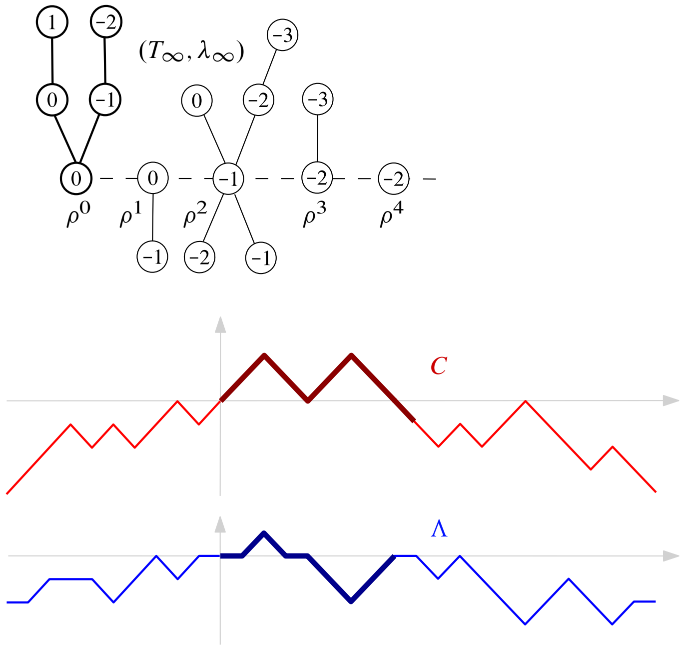

Figure 5.2. Contour and label processes associated with $(\mathbf{T}_\infty, \lambda_\infty)$. The infinite dashed line is the so-called *spine* of the tree. The tree $\mathbf{T}^0$ and the corresponding encoding processes are highlighted. Similarly to Figure 4.2, one might see the contour process as recording the height of a particle moving at speed one around the infinite tree obtained by now letting ρ^k be located at $(0, -k)$ with $\mathbf{T}^k$ grafted on its right and $\mathbf{T}^{-k}$ on its left (both upright) for $k \geq 0$; see the left of Figure A.1 for an illustration.

value in the definition of C. In fact, changing the $-$ into a $+$ amounts to taking the so-called *Pitman transform*, which is a one-to-one mapping, so this is just a matter of convention. We will come back to this in Appendix A. We can easily check that C is distributed as a two-sided random walk conditioned[2] on $C(-1) = -1$.

As before, we extend C and Λ to functions on $\mathbb{R}$ by linear interpolation between integer values. For $k \geq 0$, we set

$$\overline{\tau}_k = \max\{i \leq 0 : C(i) = -k\} \quad \text{and} \quad \tau_k = \min\{i \geq 0 : C(i) = -k\}.$$

Note that, for a fixed $k \geq 0$, the process $(k + C(s + \tau_k), 0 \leq s \leq \tau_{k+1} - \tau_k)$ is the contour process of $\mathbf{T}^k$, while, for $k \geq 1$, $(k + C(s + \overline{\tau}_{k+1} + 1), 0 \leq s \leq \overline{\tau}_k - \overline{\tau}_{k+1} - 1)$ is the contour process of $\mathbf{T}^{-k}$ without the last descending step. Therefore, in this notation, the forest composed of the k leftmost trees in the upper half-plane is coded

[2] See Remark 5.10 for the explanation of this conditioning.

by the interval $[0, \tau_k]$, while the forest composed of the k leftmost trees in the lower half-plane is coded by the interval $[\overline{\tau}_{k+1} + 1, 0]$. This slightly annoying shift will appear later on, in particular in the statement of Lemma 5.18.

Remark 5.10. As with the UIHPQ, the above definition gives a slight variant of the usual UIPQ, which is similarly defined by adding a further tree rooted at ρ^0 embedded in the lower half-plane, or equivalently, by removing the conditioning by $\{C(-1) = -1\}$. This bias is similar to the one we had for the UIHPQ. Here again, the reason for using this definition is that it will give the natural semigroup property for the discrete quadrilaterals.

We set

$$D_\infty(i, j) = d_{Q_\infty}(v_i, v_j), \quad i, j \in \mathbb{Z},$$

and extend it to a function on $\mathbb{R}^2$ by bilinear interpolation between integer values, as in (4.12). We define the renormalized versions $C_{(n)}$, $\Lambda_{(n)}$, $D_{(n)}$ of C, Λ, D_∞ by (4.13). The following proposition builds on the convergence obtained in [40] of the UIPQ to the Brownian plane. As was the case for the UIHPQ, it does not appear in this exact form in [40] and calls for a proof, which is postponed to Appendix A. Recall from Section 5.3 the definition of the distribution **Plane**.

Proposition 5.11. *The following convergence in distribution holds on* $\mathcal{C} \times \mathcal{C} \times \mathcal{C}^{(2)}$*:*

$$(C_{(n)}, \Lambda_{(n)}, D_{(n)}) \xrightarrow[n\to\infty]{(d)} (X, W, D_{X,W}),$$

where the limiting triple is understood under **Plane**.

5.5 Discrete quadrilaterals in the uniform infinite planar quadrangulation

We proceed as in the last paragraph of Section 4.4. But, here, the lack of an analog of Corollary 4.2 makes matter substantially more intricate. We consider a sequence $(h_n) \in \mathbb{N}^{\mathbb{N}}$ such that

$$\frac{h_n}{\sqrt{2n}} \xrightarrow[n\to\infty]{} H > 0.$$

For each n, we let F_n be the random forest consisting of the h_n trees $\mathbf{T}^0, \mathbf{T}^1, \ldots, \mathbf{T}^{h_n - 1}$, and ρ^{h_n}, as well as $\overline{F}_n$ be the random forest consisting of the h_n trees $\mathbf{T}^{-h_n}$, $\mathbf{T}^{-h_n+1}, \ldots, \mathbf{T}^{-1}$ and ρ^0. The pair $(F_n, \overline{F}_n)$ is a double forest in the terminology of Section 2.4 and the map $F_n \cup \overline{F}_n$ is well labeled by the restriction of λ_∞. We denote by Q_n the corresponding quadrilateral and by v_*, $\overline{v}_*$ its apexes; similarly to the previous section, we see it as part of the UIPQ Q_∞.

For each $i \in \mathbb{Z}$, the vertex v_i of $\mathbf{T}_\infty$ incident to c_i can still be seen as a vertex of Q_n when $\overline{\tau}_{h_n+1} + 1 \le i \le \tau_{h_n}$. We set

$$\widehat{D}_n(i,j) = d_{Q_n}(v_i, v_j), \quad \overline{\tau}_{h_n+1} + 1 \le i, j \le \tau_{h_n},$$

extend it to a function on $[\overline{\tau}_{h_n+1} + 1, \tau_{h_n}]^2$ by bilinear interpolation between integer values as in (4.12), and define its renormalized version

$$\widehat{D}_{(n)}(s,t) = \frac{\widehat{D}_n(2ns, 2nt)}{(8n/9)^{1/4}}, \quad \frac{\overline{\tau}_{h_n+1} + 1}{2n} \le s, t \le \frac{\tau_{h_n}}{2n}. \tag{5.5}$$

This section is devoted to the proof of the following result, which essentially amounts to stating that, jointly with the convergence of $\Omega_n(Q_\infty)$ to the Brownian plane, the properly rescaled quadrilateral $\Omega_n(Q_n)$ converges to $\mathsf{Qd}^{(0,H)}$.

Theorem 5.12. *The following convergence in distribution holds in $\mathcal{C} \times \mathcal{C} \times \mathcal{C}^{(2)} \times \mathcal{C}^{(2)}$:*

$$(C_{(n)}, \Lambda_{(n)}, D_{(n)}, \widehat{D}_{(n)}) \xrightarrow[n\to\infty]{(d)} (X, W, D_{X,W}, \widehat{D}^{(0,H)}),$$

where the limiting quadruple is understood under **Plane**.

The first step in the proof is the following tightness statement.

Lemma 5.13. *From every increasing sequence of integers, one may extract a subsequence along which the following convergence holds in $\mathcal{C}^{(2)}$, jointly with the convergence of Proposition 5.11:*

$$\widehat{D}_{(n)} \xrightarrow[n\to\infty]{(d)} \widetilde{D}, \tag{5.6}$$

where $\widetilde{D}$ is a random pseudometric on $[\overline{T}_H, T_H]$.

Proof. The classical tightness argument from [65, Proposition 3.2] implies that the laws of $\widehat{D}_{(n)}$, $n \ge 1$, are tight in $\mathcal{C}^{(2)}$. Together with Proposition 5.11, this yields the tightness of the laws of the sequence of the quadruples $(C_{(n)}, \Lambda_{(n)}, D_{(n)}, \widehat{D}_{(n)})$, and therefore, by Prokhorov's theorem, their joint convergence in distribution, at least along some subsequence, to a limiting process $(X, W, D_{X,W}, \widetilde{D})$, where the law of the first three components is determined by Proposition 5.11. Since $\widehat{D}_{(n)}$ is a pseudometric on $[(\overline{\tau}_{h_n+1} + 1)/2n, \tau_{h_n}/2n]$, and because of the convergence of $C_{(n)}$ to X implying the joint convergence of the bounds of this interval to $\overline{T}_H, T_H$, it is straightforward to check that all subsequential limits of these laws are carried by functions that are pseudometrics on the interval $[\overline{T}_H, T_H]$. ■

From now on, we fix a subsequence along which (5.6) holds, and only consider for the time being values of n that belong to this particular subsequence. By the Skorokhod representation theorem, we may and will assume that this convergence furthermore holds almost surely.

We define $\widetilde{\mathsf{Qd}}$ as the set $[\overline{T}_H, T_H]/\{\widetilde{D} = 0\}$, endowed with the metric $\widetilde{D}$. Beware that it is not clear at all that $\widetilde{\mathsf{Qd}} = \mathsf{Qd}^{(0,H)}$, and this is precisely what we want to prove. More precisely, we aim at showing that, almost surely, for every $s, t \in [\overline{T}_H, T_H]$, it holds that $\widetilde{D}(s,t) = \widehat{D}^{(0,H)}(s,t)$, which will entail Theorem 5.12.

Since the real number H is fixed once and for all, we will use in the remainder of this section the shorthand pieces of notation

$$\widehat{D} = \widehat{D}^{(0,H)} \quad \text{as well as } D = D_{X,W}.$$

We let $\mathbf{p}\colon \mathbb{R} \to \mathsf{BP}$ and $\widetilde{\mathbf{p}}\colon [\overline{T}_H, T_H] \to \widetilde{\mathsf{Qd}}$ be the canonical projections, which are continuous since D and $\widetilde{D}$ are continuous functions. Note that, clearly, for every n, it holds that $D_\infty \leq \widehat{D}_n$ on $[\overline{\tau}_{h_n+1} + 1, \tau_{h_n}]^2$, so that $D \leq \widetilde{D}$ on $[\overline{T}_H, T_H]$. As a result, there exists a unique continuous (even 1-Lipschitz) projection $\pi\colon \widetilde{\mathsf{Qd}} \to \mathbf{p}([\overline{T}_H, T_H])$ such that $\mathbf{p} = \pi \circ \widetilde{\mathbf{p}}$ on $[\overline{T}_H, T_H]$.

The inequality $\widetilde{D} \leq \widehat{D}$ follows from the usual following arguments. First we come back to discrete maps and observe that, for integers $i, j \in [\overline{\tau}_{h_n+1} + 1, \tau_{h_n}]$, we have $d_C(i,j) = 0$ if and only if v_i and v_j are the same vertex of $F_n \cup \overline{F}_n$, which implies that $\widehat{D}_n(i,j) = 0$. Next, by considering the so-called *maximal wedge path* consisting of the concatenation of the two geodesics from c_i and from c_j obtained by following subsequent successors up to the point where they coalesce, we obtain the classical upper bound similar to (2.1):

$$\widehat{D}_n(i,j) \leq d_\Lambda(i,j) + 2, \quad i, j \in [\overline{\tau}_{h_n+1} + 1, \tau_{h_n}] \text{ with } ij \geq 0. \tag{5.7}$$

Passing to the limit yields that $\{d_X = 0\} \subseteq \{\widetilde{D} = 0\}$ and that $\widetilde{D} \leq \hat{d}_W$, which imply the inequality $\widetilde{D} \leq \widehat{D}$. The converse inequality is harder and is the focus of what follows.

Let us start with some key properties of the pseudometrics D, $\widetilde{D}$, and $\widehat{D}$. The following lemma is proved in the exact same way as [25, Lemma 14].

Lemma 5.14. *The spaces $\widetilde{\mathsf{Qd}}$ and $\mathsf{Qd}^{(0,H)}$ are compact geodesic metric spaces.*

We will need the following identification of the set $\{\widetilde{D} = 0\}$, analog to Lemma 5.8.

Lemma 5.15. *The following holds almost surely. For every $s, t \in [\overline{T}_H, T_H]$ with $s \neq t$, it holds that $\widetilde{D}(s,t) = 0$ if and only if either $d_X(s,t) = 0$ or $\hat{d}_W(s,t) = 0$, these two cases being mutually exclusive.*

Proof. It follows very similar lines to that of [67, Proposition 3.1], and we will only sketch the main arguments. The fact that $d_X(s,t) = 0$ or $\hat{d}_W(s,t) = 0$ implies $\widetilde{D}(s,t) = 0$ is immediate from the inequality $\widetilde{D} \leq \widehat{D}$. Conversely, assume that $\widetilde{D}(s,t) = 0$ for some $s \neq t$ in $[\overline{T}_H, T_H]$. Then, in particular, since $D \leq \widetilde{D}$, it holds that $D(s,t) = 0$, so that either $d_W(s,t) = 0$ or $d_X(s,t) = 0$, and these two cases are exclusive. If we are in the case that s, t are of the same sign and that $d_W(s,t) = 0$, this trivially implies $\widehat{d}_W(s,t) = 0$, as wanted. And since $\widehat{d}_W \geq d_W$, it cannot hold that $\widehat{d}_W(s,t) = d_X(s,t) = 0$ at the same time. Hence, the proof will be complete if we can show that the situation where $d_W(s,t) = 0$ necessarily implies that s and t are of the same sign.

For this, we argue by contradiction, assuming that $t < 0 < s$ and $d_W(s,t) = 0$. Note that this implies in particular that s, t lie on some point of the geodesics Γ_0 and Ξ_0, respectively, meaning that $W_s = \inf_{u\in[0,s]} W_u$ and $W_t = \inf_{u\in[t,0]} W_u$. Then, by the convergence of $\Lambda_{(n)}$ to W, there exist $i_n \in [0, \tau_{h_n}]$ and $j_n \in [\overline{\tau}_{h_n+1} + 1, 0]$ such that $i_n/2n \to s$ and $j_n/2n \to t$, with the property that $\Lambda_n(i_n) = \min_{k\in[0,i_n]} \Lambda_n(k)$ and $\Lambda_n(j_n) = \min_{k\in[j_n,0]} \Lambda_n(k)$. This means that v_{i_n} lies on the maximal geodesic γ_n of Q_n, and v_{j_n} lies at distance 1 from the shuttle $\overline{\xi}_n$ of Q_n.

Now any geodesic path in Q_n from v_{i_n} to v_{j_n} will necessarily intersect the spine of the tree $\mathbf{T}_\infty$ at some tree root ρ^{l_n} with $0 \leq l_n \leq h_n$. Let $k_n \in [\overline{\tau}_{h_n+1} + 1, \tau_{h_n}]$ be an integer such that $v_{k_n} = \rho^{l_n}$. In terms of the contour process C_n, this means that $C_n(k_n) \leq C_n(l)$ for every $l \in [0 \wedge k_n, 0 \vee k_n]$. Up to extracting along a further subsequence, we may assume that $k_n/2n \to u \in [\overline{T}_H, T_H]$ as $n \to \infty$, and we observe that u must be such that $X_u \leq X_t$ for every $t \in [0 \wedge u, 0 \vee u]$, and in particular, we observe that $d_X(u, T_{H'}) = d_X(u, \overline{T}_{H'}) = 0$, where $H' = -X_u$. We may exclude the case where $H' = 0$ by noting that, necessarily, $W_s = W_t = W_u < 0$.

On the other hand, since v_{k_n} lies on a geodesic path from v_{i_n} to v_{j_n}, which has length $o(n^{1/4})$ because of our assumption that $\widetilde{D}(s,t) = 0$, it holds that $\widetilde{D}(s,u) = \widetilde{D}(u,t) = 0$. We arrive at the wanted contradiction since we have found four points $s \neq t$, $T_{H'} \neq \overline{T}_{H'}$ that are all identified by D but such that $d_W(s,t) = 0$ and $d_X(T_{H'}, \overline{T}_{H'}) = 0$. ■

As $\widetilde{D} \leq \widehat{D} \leq \widehat{d}_W$ and $\{d_X = 0\} \subseteq \{\widehat{D} = 0\}$, Lemma 5.15 implies that the equivalence relations $\{\widetilde{D} = 0\}$ and $\{\widehat{D} = 0\}$ coincide, and that $\widetilde{\mathbf{p}} = \widehat{\mathbf{p}}^{(0,H)}$. For this reason, we may, and will, systematically identify points of $\widetilde{\mathsf{Qd}}$ with points of $\mathsf{Qd}^{(0,H)}$. Moreover, the identity mapping $\mathsf{Qd}^{(0,H)} \to \widetilde{\mathsf{Qd}}$ is continuous, and by compactness of these spaces, we conclude that $\widetilde{\mathsf{Qd}}$ is homeomorphic to $\mathsf{Qd}^{(0,H)}$.

Theorem 5.12 will be obtained by compactness and continuity arguments from the following local version, stating that, locally and away either from both maximal geodesics or from both shuttles, the three distances under consideration are equal. The proof of the following lemma can straightforwardly be adapted from [25, Lemma 15], so that we only sketch it and refer the reader to the latter reference for the details. In an arbitrary pseudometric space (M,d), we denote by

$$d(x, A) = \inf\{d(x,y) : y \in A\}$$

the distance from a point $x \in M$ to a subset $A \subseteq M$.

Lemma 5.16. *The following holds almost surely. Fix* $\varepsilon > 0$*, and let* $s, t \in [\overline{T}_H, T_H]$ *be such that* $\widetilde{D}(s,t) < \varepsilon$ *and*

- *either* $\widetilde{D}(s, \Gamma_0 \cup \Gamma_{\overline{T}_H}) \wedge \widetilde{D}(t, \Gamma_0 \cup \Gamma_{\overline{T}_H}) > \varepsilon$*,*
- *or* $\widetilde{D}(s, \Xi_0 \cup \Xi_{T_H}) \wedge \widetilde{D}(t, \Xi_0 \cup \Xi_{T_H}) > \varepsilon$*.*

Then, it holds that $D(s,t) = \widetilde{D}(s,t) = \widehat{D}(s,t)$*.*

Proof. Let i_n, j_n be integers in $[\overline{\tau}_{h_n+1} + 1, \tau_{h_n}]$ such that $i_n/2n \to s$ and $j_n/2n \to t$ as $n \to \infty$. From the assumption that $\widetilde{D}(s,t) < \varepsilon$ and the convergence of $D_{(n)}$ toward $\widetilde{D}$, we deduce that $d_{Q_n}(v_{i_n}, v_{j_n}) < \varepsilon(8n/9)^{1/4}$ for every n large enough.

Next, keeping the same notation, assume that we are in the first alternative of the statement. Then we claim that for every n large enough, v_{i_n} and v_{j_n} must be at d_{Q_n}-distance at least $\varepsilon(8n/9)^{1/4}$ from the maximal geodesics γ_n and $\overline{\gamma}_n$ of Q_n. Indeed, if we assume otherwise, then up to taking an extraction along a further subsequence, we would find a point $k_n \in [\overline{\tau}_{h_n+1} + 1, \tau_{h_n}]$ such that for every n, v_{k_n} belongs to (the same) one of these maximal geodesics, and is at d_{Q_n}-distance at most $(8n/9)^{1/4}$ from (the same) one of two points v_{i_n} or v_{j_n}. To fix the ideas, assume that v_{k_n} is on γ_n and is close to v_{i_n} in the latter sense, the discussion being similar in the other cases. Up to taking yet another subsequence if necessary, we may assume that $k_n/2n$ converges to some $u \in [\overline{T}_H, T_H]$. Note that k_n, being a time of visit of the maximal geodesic γ_n, must be a left-minimum for the label process Λ_n restricted to nonnegative times, and, by passing to the limit, u must be a left-minimum of W restricted to nonnegative times, entailing that $\widetilde{D}(u, \Gamma_0) = 0$. Therefore, by passing to the limit in the inequality $D_{(n)}(i_n/2n, k_n/2n) \leq \varepsilon$, we would obtain that $\widetilde{D}(s, \Gamma_0) \leq \varepsilon$, a contradiction with our assumption.

Now observe that Q_∞ is obtained by the following two gluing operations, from Q_n and the infinite quadrangulation Q_n^{c}, encoded by the labeled double forest with trees grafted above ρ^{h_n+i}, $i \geq 0$, and below ρ^{h_n+i}, $i \geq 1$.

- First, by gluing the geodesic sides ξ_n and $\overline{\gamma}_n$ of Q_n to the (unique) maximal geodesic and shuttle of Q_n^{c}. Note that the resulting infinite quadrangulation is also obtained by performing the interval CVS construction on $\mathbf{T}_\infty$ with the intervals $\{c_i, i \leq 0\}$ and $\{c_i, i \geq 0\}$.
- Second, by gluing together the (unique) maximal geodesic and shuttle of the infinite quadrangulation obtained at the first step. Note that the geodesic sides of this infinite map are prolongations of γ_n and $\overline{\xi}_n$.

Therefore, Lemma 3.7 (ii) applied twice (once for each gluing operation) shows that if $v, w \in V(Q_n)$ are such that $d_{Q_n}(v,w) < K$ and

- either $d_{Q_n}(v, \gamma_n) \wedge d_{Q_n}(w, \gamma_n) > K$,
- or $d_{Q_n}(v, \overline{\gamma}_n) \wedge d_{Q_n}(w, \overline{\gamma}_n) > K$,

then $d_{Q_n}(v,w) = d_{Q_\infty}(v,w)$. Applying this to $v = v_{i_n}$, $w = v_{j_n}$, and $K = (8n/9)^{1/4}\varepsilon$ yields, after passing to the limit, that $\widetilde{D}(s,t) = D(s,t)$. Since $\widetilde{D} \le \widehat{D} \le D$, this yields the result in the first alternative of the statement. The second case, with shuttles instead of maximal geodesics, is similar. ■

We may finally prove Theorem 5.12.

Proof of Theorem 5.12. We follow the same lines as in the proof of [25, Theorem 11]. As we observed before, the metric spaces $\widetilde{\mathsf{Qd}}$ and $\mathsf{Qd}^{(0,H)}$ are homeomorphic. Therefore, since the geodesics γ and $\overline{\gamma}$ do not intersect in $\mathsf{Qd}^{(0,H)}$, the same is true in $\widetilde{\mathsf{Qd}}$, and similarly, the geodesics ξ and $\overline{\xi}$ do not intersect in these spaces. Moreover, as we know, these four geodesics intersect only at $\gamma(0) = \overline{\xi}(0)$, $\xi(0) = \overline{\gamma}(0)$, $x_*^{(0,H)}$ and $\overline{x}_*^{(0,H)}$. Therefore, for every $x \in \widetilde{\mathsf{Qd}} \setminus \{\gamma(0), \overline{\gamma}(0), x_*^{(0,H)}, \overline{x}_*^{(0,H)}\}$, there exists $\varepsilon > 0$ such that the open ball $B_{\widetilde{D}}(x,\varepsilon)$ of radius ε around x for the metric $\widetilde{D}$ intersects neither $\gamma \cup \overline{\gamma}$ nor $\xi \cup \overline{\xi}$. By Lemma 5.16, this implies that the balls $B_{\widetilde{D}}(x,\varepsilon)$, and $B_{\widehat{D}}(x,\varepsilon)$ are isometric. Hence, $\widetilde{\mathsf{Qd}}$ and $\mathsf{Qd}^{(0,H)}$ are two compact geodesic metric spaces that are locally isometric except possibly around four points. Therefore, the lengths of paths that do not go through these four points must be the same in both spaces. It is then easy to see that the same is true for all paths that visit each of these four points at most once, by splitting into subpaths, and by standard properties of lengths of paths. One concludes by observing that, given a path in $\widetilde{\mathsf{Qd}}$, one may construct another path of length smaller than or equal to that of the initial path, and that visits each of the four distinguished points at most once. Since a geodesic space is a length space [32], the distance between two points is given by the infimum of length of paths between these points. Therefore, $\widetilde{\mathsf{Qd}}$ and $\mathsf{Qd}^{(0,H)}$ are isometric. ■

5.6 Scaling limit of conditioned quadrilaterals

In this section, we finally prove Theorem 2.8. As a preliminary result, we will need a simple estimate on distances in quadrilaterals. We invite the reader to recall the combinatorial setting of Section 2.4 and to consult Figure 5.3. Let $((\mathbf{f},\overline{\mathbf{f}}),\lambda)$ be a well-labeled double forest and let $\mathbf{q}$ be the corresponding quadrilateral. For $k \in \{1, 2, \dots, h-1\}$, keeping only the *first* k trees in $\mathbf{f}$ and the *last* k trees in $\overline{\mathbf{f}}$ yields a submap of $\mathbf{f} \cup \overline{\mathbf{f}}$, well labeled by the restriction of λ. We let $\mathbf{q}_k$ be the corresponding quadrilateral, which we naturally see as a submap of $\mathbf{q}$. We will need the following coarse comparison between distances in $\mathbf{q}$ and $\mathbf{q}_k$.

Lemma 5.17. *Let* $\varpi = 2 + \max\{\lambda(u) : u \in V(\mathbf{q}) \setminus V(\mathbf{q}_k)\} - \min\{\lambda(u) : u \in V(\mathbf{q}) \setminus V(\mathbf{q}_k)\}$. *Then, for any* $v, w \in V(\mathbf{q}_k)$, *one has*

$$d_{\mathbf{q}}(v,w) \le d_{\mathbf{q}_k}(v,w) \le d_{\mathbf{q}}(v,w) + \varpi.$$

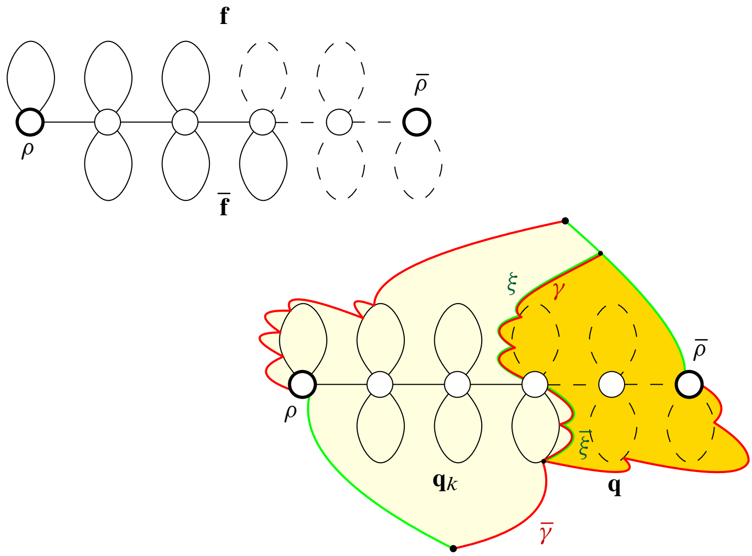

Figure 5.3. Here, $h = 5$ and $k = 3$. On the top left, a schematic picture of a double forest $(\mathbf{f}, \overline{\mathbf{f}})$, assumed to be well labeled, and its "truncation" obtained after removing the dashed elements. On the bottom right, a schematic picture of the corresponding quadrilaterals: the quadrilateral $\mathbf{q}$ is obtained by gluing $\mathbf{q}_k$ (in light yellow) along its sides ξ and $\overline{\gamma}$ with the quadrilateral (in dark yellow) coded by the dashed elements along its sides γ and $\overline{\xi}$ (only these four geodesic sides of interest are named in the picture).

Proof. Observe that $\mathbf{q}$ may be obtained by gluing $\mathbf{q}_k$ along its sides ξ and $\overline{\gamma}$ with the quadrilateral coded by the double forest obtained by taking the last $h - k$ trees in $\mathbf{f}$ and the first $h - k$ trees in $\overline{\mathbf{f}}$, well labeled by the restriction of λ, along its sides γ and $\overline{\xi}$. The lemma is then a straightforward consequence of Lemma 3.7 (i) since the lengths of the glued geodesics are bounded by the quantity ϖ. ■

We now prove Theorem 2.8 by proceeding similarly to Section 4.5. Recall the notation $(C(t), \Lambda(t), t \in \mathbb{R})$, τ_k, $\overline{\tau}_k$ from Section 5.4. Let $\mathbb{P}_\infty$ be the law of (C, Λ) and assume without loss of generality that the latter is the canonical process. Although we use the same notation $\mathbb{P}_\infty$ as in Section 4.5, we believe that there is little risk of confusion. For $j \geq 1$, let $\mathcal{F}_j$ be the σ-algebra generated by $(C(i), \Lambda(i), 0 \leq i \leq j)$, and let $\mathcal{G}_j$ be the one generated by $(C(i), \Lambda(i+1), -j \leq i \leq -1)$. Note that $\mathcal{F}_{\tau_h}$ is the σ-algebra generated by the h leftmost trees of $\mathbf{T}_\infty$ in the upper half-plane, together with their labels, as well as the label of the root ρ^h. Similarly, $\mathcal{G}_{-\overline{\tau}_{h+1}}$ is the σ-algebra generated by the h leftmost trees of $\mathbf{T}_\infty$ in the lower half-plane, together with their labels, as well as the label of the root ρ^0: the meaning of the shift by $+1$ in the process Λ is that we do not want to incorporate the information of the label of the root ρ^{h+1} in $\mathcal{G}_{-\overline{\tau}_{h+1}}$.

Next, for $a, \overline{a}, h \in \mathbb{N}$ and $\delta \in \mathbb{Z}$, we denote by $\mathbb{P}_{a,\overline{a},h,\delta}$ the distribution of

$$(C|_{[-2\overline{a}-h,2a+h]}, \Lambda|_{[-2\overline{a}-h,2a+h]})$$

under

$$\mathbb{P}_\infty[\,\cdot \mid \tau_h = 2a + h, \overline{\tau}_{h+1} + 1 = -2\overline{a} - h, \Lambda(\tau_h) = \delta].$$

The corresponding double forest encoded by this process is thus composed of a spine $\rho^0, \ldots, \rho^h$ of length h, onto which are grafted $2h$ Bienaymé–Galton–Watson trees with Geometric$(1/2)$ offspring distribution and uniform admissible labels, conditioned on the fact that the total number of edges in the upper half-plane h trees is a, the number of edges in the lower half-plane h trees is $\overline{a}$, and the label of the last root ρ^h is δ. The following lemma gives an absolute continuity relation between the laws $\mathbb{P}_{a,\overline{a},h,\delta}$ and $\mathbb{P}_\infty$. Its proof, which we omit, is similar to that of Lemma 4.11, using the enumeration results of Proposition 2.5. Recall from Propositions 2.3 and 2.5 the definitions of Q_ℓ and M_ℓ.

Lemma 5.18. *Fix the integers $0 < k < h$, as well as positive integers $a, \overline{a} \in \mathbb{N}$ and $\delta \in \mathbb{Z}$. For every nonnegative functional G that is $\mathcal{F}_{\tau_k} \vee \mathcal{G}_{-\overline{\tau}_{k+1}}$-measurable, we have*

$$\mathbb{E}_{a,\overline{a},h,\delta}[G] = \mathbb{E}_\infty[\Psi_{a,\overline{a},h,\delta}(\tau_k, -(\overline{\tau}_{k+1} + 1), k, \Lambda(\tau_k)) \cdot G],$$

where

$$\Psi_{a,\overline{a},h,\delta}(s,t,h',j) = \frac{Q_{h-h'}(2a+h-s)}{Q_h(2a+h)} \frac{Q_{h-h'}(2\overline{a}+h-t)}{Q_h(2\overline{a}+h)} \frac{M_{h-h'}(\delta - j)}{M_h(\delta)}.$$

From now on, in addition to the sequence (h_n), we fix three sequences (a_n), $(\overline{a}_n)$, (δ_n) as in (2.5). The following is a tedious but straightforward consequence of the local limit theorem [26, Theorem 8.4.1].

Lemma 5.19. *If the integer-valued sequence (h'_n) satisfies $h'_n/\sqrt{2n} \to H' \in (0, H)$, then*

$$\sup_{\substack{0 \le s \le a_n \\ 0 \le t \le \overline{a}_n}, j \in \mathbb{Z}} \left| \Psi_{a_n,\overline{a}_n,h_n,\delta_n}(s,t,h'_n,j) - \psi_{A,\bar{A},H',\Delta}\left(\frac{s}{n}, \frac{t}{n}, H', \left(\frac{9}{8n}\right)^{1/4} j\right) \right| \underset{n\to\infty}{\longrightarrow} 0.$$

We proceed to the conditioned version of Theorem 5.12. Recall the definition of $\widehat{D}_{(n)}$ given in (5.5).

Proposition 5.20. *On $\mathcal{C} \times \mathcal{C} \times \mathcal{C}^{(2)}$, the triple $(C_{(n)}, \Lambda_{(n)}, \widehat{D}_{(n)})$ considered under the distribution $\mathbb{P}_{a_n,\overline{a}_n,h_n,\delta_n}$ converges in distribution to $(X, W, \widehat{D}_{X,W} = \widehat{D}^{(0,H)})$, considered under $\mathbf{Quad}_{A,\bar{A},H,\Delta}$.*

Proof. The arguments are very close to those used in the proof of Proposition 4.13 in Section 4.5, adding Lemma 5.17 and Proposition 5.2 to cover the additional difficulty. The joint convergence of the first two coordinates is also standard. Then, fix $\varepsilon \in (0, H)$ and set

$$h_n^\varepsilon = h_n - \lfloor \varepsilon\sqrt{2n} \rfloor.$$

Let $\widehat{D}_n^\varepsilon$ and $\widehat{D}_{(n)}^\varepsilon$ be defined as in (5.5) and above, but with h_n^ε instead of h_n. For simplicity, for every $i \in \mathbb{R}$, let

$$i^\varepsilon = (\overline{\tau}_{h_n^\varepsilon+1} + 1) \vee i \wedge \tau_{h_n^\varepsilon}$$

and define j^ε similarly for any $j \in \mathbb{R}$. Define also

$$\kappa_n^\varepsilon = (\tau_{h_n} - \tau_{h_n^\varepsilon}) + (\overline{\tau}_{h_n^\varepsilon+1} - \overline{\tau}_{h_n+1}).$$

From (5.7), we obtain

$$\begin{aligned}|\widehat{D}_n(i,j) - \widehat{D}_n(i^\varepsilon, j^\varepsilon)| &\le \widehat{D}_n(i, i^\varepsilon) + \widehat{D}_n(j, j^\varepsilon) \\ &\le 4(\omega(\Lambda_n; \kappa_n^\varepsilon) + 1).\end{aligned}$$

Using Lemma 5.17, we have for every $i, j \in [\overline{\tau}_{h_n^\varepsilon+1} + 1, \tau_{h_n^\varepsilon}]$,

$$|\widehat{D}_n(i,j) - \widehat{D}_n^\varepsilon(i,j)| \le \omega(\Lambda_n; \kappa_n^\varepsilon) + 2.$$

These two facts together then imply that

$$\mathrm{dist}_{\mathcal{C}^{(2)}}(\widehat{D}_{(n)}^\varepsilon, \widehat{D}_{(n)}) \le \frac{\kappa_n^\varepsilon}{2n} + 5\omega\Big(\Lambda_{(n)}, \frac{\kappa_n^\varepsilon}{2n}\Big) + \mathcal{O}(n^{-1/4}).$$

We now use the convergence of the first two coordinates, implying, for every $\eta > 0$,

$$\begin{aligned}\limsup_{n\to\infty} \mathbb{P}_{a_n,\bar{a}_n,h_n,\delta_n}(\mathrm{dist}_{\mathcal{C}^{(2)}}(\widehat{D}_{(n)}^\varepsilon, \widehat{D}_{(n)}) \ge \eta) \\ \le \mathbf{Quad}_{A,\bar{A},H,\Delta}(\kappa^\varepsilon + 5\omega(W; \kappa^\varepsilon) \ge \eta),\end{aligned} \tag{5.8}$$

where

$$\kappa^\varepsilon = A - T_{H-\varepsilon} + \overline{T}_{H-\varepsilon} + \bar{A}.$$

Since almost surely under $\mathbf{Quad}_{A,\bar{A},H,\Delta}$, the quantity κ^ε tends to 0 as $\varepsilon \to 0$, we deduce that the left-hand side in (5.8) also converges to 0. It remains to show that $\widehat{D}_{(n)}^\varepsilon$ under $\mathbb{P}_{a_n,\bar{a}_n,h_n,\delta_n}$ converges toward $\widehat{D}^{(0,H-\varepsilon)}$ under $\mathbf{Quad}_{A,\bar{A},H,\Delta}$ to conclude, by the principle of accompanying laws, that $\widehat{D}_{(n)}$ converges to the distributional limit of $\widehat{D}^{(0,H-\varepsilon)}$ as $\varepsilon \to 0$, which is $\widehat{D}^{(0,H)}$ by Proposition 5.2. To this end, we consider the restrictions $C_{(n)}^\varepsilon$, $\Lambda_{(n)}^\varepsilon$ of $C_{(n)}$, $\Lambda_{(n)}$ to the intervals $[(\overline{\tau}_{h_n^\varepsilon+1} + 1)/2n, \tau_{h_n^\varepsilon}/2n]$ and, letting F be a nonnegative bounded continuous function, we observe that, using

Lemma 5.18, then Theorem 5.12 (for the choice of $H - \varepsilon$ instead of H) and Lemma 5.19, and finally Lemma 5.9, we have

$$\begin{aligned}
&\mathbb{E}_{a_n,\bar{a}_n,h_n,\delta_n}[F(C^{\varepsilon}_{(n)}, \Lambda^{\varepsilon}_{(n)}, D^{\varepsilon}_{(n)})] \\
&\quad = \mathbb{E}_{\infty}[\Psi_{a_n,\bar{a}_n,h_n,\delta_n}(\tau_{h_n^{\varepsilon}}, -1 - \overline{\tau}_{h_n^{\varepsilon}+1}, \Lambda(\tau_{h_n^{\varepsilon}}))G(C^{\varepsilon}_{(n)}, \Lambda^{\varepsilon}_{(n)}, D^{\varepsilon}_{(n)})] \\
&\quad \underset{n\to\infty}{\longrightarrow} \mathbf{Plane}\big[\psi_{A,\bar{A},H,\Delta}(T_{H-\varepsilon}, -\overline{T}_{H-\varepsilon}, H-\varepsilon, W_{T_{H-\varepsilon}}) \\
&\qquad\qquad\qquad \times G(X^{(0,H-\varepsilon)}, W^{(0,H-\varepsilon)}, \widehat{D}^{(0,H-\varepsilon)})\big] \\
&\quad = \mathbf{Quad}_{A,\bar{A},H,\Delta}[G(X^{(0,H-\varepsilon)}, W^{(0,H-\varepsilon)}, \widehat{D}^{(0,H-\varepsilon)})].
\end{aligned}$$

This concludes the proof. ∎

From there, we easily obtain the wanted GHP convergence by arguments similar to those developed in the proof of Theorem 2.6 at the end of Section 4.5.