Chapter 6

Construction from a continuous unicellular map

Our proof of Theorem 1.1 gives a description of the limiting Brownian surfaces as gluings of elementary pieces, which appear either in the Brownian plane or in the Brownian half-plane. Although this construction has a clear geometric content, it can be arguably cumbersome to work with, having in mind, for instance, the universal character that the spaces $\mathsf{S}_{\boldsymbol{L}}^{[g]}$ are expected to bear.

Indeed, we believe that Brownian surfaces arise as universal limits for many more classes of maps satisfying mild conditions (for instance, uniformly distributed maps) and a more direct description seems to be useful in order to show such results. In particular, we believe that the Brownian torus is the scaling limit of essentially simple triangulations, as considered in [15]. In fact, most of the known results of convergence toward the Brownian sphere use a re-rooting technique due to Le Gall [67], which, very roughly speaking, says that if maps in a given class are properly encoded by discrete objects converging to the random snake driven by a normalized Brownian excursion and if these maps and the limiting object exhibit a property of invariance under uniform re-rooting, then the limiting space is the Brownian sphere. We expect this approach to be generalizable to our context and we now give a description of Brownian surfaces that is a direct generalization of the classical definition of the Brownian sphere. This can be thought of a continuum version of the Cori–Vauquelin–Schaeffer bijection, building on a continuum version of a unicellular map (a map with only one internal face).

For a function $f \in \mathcal{C}$ and $s, t \in I(f)$ with $t < s$, we extend (4.1) by setting

$$\underline{f}(s,t) = \inf_{I(f)\setminus[t,s]} f$$

and we set, for $s, t \in I(f)$,

$$\widetilde{d}_f(s,t) = f(s) + f(t) - 2\max\{\underline{f}(s,t), \underline{f}(t,s)\}. \tag{6.1}$$

The difference with (4.2) is that we now take into account the minimum of f on the "interval" from $s \vee t$ to $s \wedge t$ on the "circle" $I(f)/\{\overline{\tau}(f) = \tau(f)\}$.

The Brownian sphere. As a warm-up, let us first recall the definition of the Brownian sphere. It is the metric space $\mathsf{S}_{\varnothing}^{[0]} = ([0,1], \widetilde{d}_Z)/\{d_{\mathbf{e}} = 0\}$, where Z is the random snake driven by a normalized Brownian excursion $\mathbf{e}$.

Recall that the continuum random tree (CRT) introduced by Aldous [7, 8] is the $\mathbb{R}$-tree[1] $\mathcal{T}_{\mathbf{e}} = ([0,1]/\{d_{\mathbf{e}} = 0\}, d_{\mathbf{e}})$, hence the Brownian sphere $\mathsf{S}_{\varnothing}^{[0]}$ may actually be

[1] See Section 4.1.1.

seen as a quotient of the CRT. In fact, Le Gall [65] showed that the pseudometric $\widetilde{d}_Z/\{d_{\mathbf{e}} = 0\}(s,t) = 0$ if and only if $\widetilde{d}_Z(s,t) = 0$ or $d_{\mathbf{e}}(s,t) = 0$, so the topological space $\mathsf{S}^{[0]}_{\varnothing}$ is obtained by a continuous analog to the Cori–Vauquelin–Schaeffer bijection.

The Brownian disk. Let us turn to the Brownian disk with perimeter $L \in (0,\infty)$. It is the metric space $\mathsf{S}^{[0]}_{(L)} = ([0,1], \widetilde{d}_W)/\{d_X = 0\}$, where (X, W) is the pair encoding a slice with area 1, width L and tilt 0, that is, distributed according to $\mathbf{Slice}_{1,L,0}$ (defined in Section 4.2).

The most natural continuous object generalizing the CRT in the case of the disk is the gluing

$$\mathcal{M}^{[0]}_{(L)} = ([0,1], d_X)/R,$$

where R is the coarsest equivalence relation containing $\{d_X = 0\}$ and $\{(0,1)\}$. Since $\widetilde{d}_W(0,1) = 0$, the Brownian disk is also $([0,1], \widetilde{d}_W)/R$ and can be seen as a quotient of $\mathcal{M}^{[0]}_{(L)}$. Visually, $\mathcal{M}^{[0]}_{(L)}$ is obtained by taking a circle of length L and gluing a Brownian forest of mass 1 and length L on it. The random snake W then assigns Brownian labels to it (with a Brownian bridge multiplied by $\sqrt{3}$ on the circle and standard Brownian motions everywhere else).

The general case. The CRT and the structure $\mathcal{M}^{[0]}_{(L)}$ are the continuous equivalent to the encoding objects of Section 2.2 in the particular cases of the sphere and the disk. In general, we have a similar yet even more intricate construction, which we now describe. Let $g \geq 0$ be fixed and $\boldsymbol{L} = (L^1, \dots, L^b)$ be a b-tuple of positive real numbers. Let then $(S, (A^e)_{e\in\vec{E}(S)}, (H^e)_{e\in\vec{I}(S)}, (L^e)_{e\in\vec{B}(S)}, (\Lambda^v)_{v\in V(S)})$ be a random vector distributed according to the distribution $\mathrm{Param}_{\boldsymbol{L}}$, defined around (3.8). Conditionally given this vector, we consider the following collection of processes. For each $e \in \vec{E}(S)$,

- the process X^e is a first-passage bridge of standard Brownian motion from 0 to $-H^e$ with duration A^e,
- the process Z^e is a random snake driven by the reflected process $X^e - \underline{X}^e$,

the processes (X^e, Z^e), $e \in \vec{E}(S)$, being independent. Independently, the process ζ^e is a Brownian bridge

- of duration H^e from Λ^{e^-} to Λ^{e^+}, with variance 1 if $e \in \vec{I}(S)$,
- of duration L^e from Λ^{e^-} to Λ^{e^+}, with variance 3 if $e \in \vec{B}(S)$.

Furthermore, for $e \in \vec{I}(S)$, the bridges are linked through the relation

$$\zeta^{\bar{e}}(s) = \zeta^e(H^e - s), \quad 0 \leq s \leq H^e,$$

and, except for these relations, are independent. We then set, for each $e \in \vec{E}(S)$,

$$W^e_t = Z^e_t + \zeta^e_{-\underline{X}^e_t}, \quad 0 \leq t \leq A^e.$$

In the end, we obtain a collection of processes (X^e, W^e), $e \in \vec{E}(S)$, which are linked through the relations linking ζ^e with $\zeta^{\overline{e}}$, translating the fact that the labels of the floors of forests grafted on both sides of the same internal edge of the scheme should correspond.

We arrange the half-edges $e_1, \dots, e_\kappa$ incident to the internal face of S according to the contour order, starting from the root, and we define the concatenation

$$(W_s)_{0 \le s \le 1} = W^{e_1} \bullet \cdots \bullet W^{e_\kappa},$$

which is a continuous process. We define $\widetilde{d}_W$ by (6.1) as above and now define the equivalence relation along which to glue.

Roughly speaking, we glue together Brownian forests coded by the X^e's according to the scheme structure. For $s \in [0, 1)$, we denote by $[s]$ the integer in $\{1, \dots, \kappa\}$ such that

$$\sum_{i=1}^{[s]-1} A^{e_i} \le s < \sum_{i=1}^{[s]} A^{e_i} \quad \text{and} \quad \langle s \rangle = s - \sum_{i=1}^{[s]-1} A^{e_i} \in [0, A^{e_{[s]}}).$$

By convention, we also set $[1] = 1$ and $\langle 1 \rangle = 0$. We define the relation R on $[0, 1]$ as the coarsest equivalence relation for which $s\,R\,t$ if one of the following occurs:

$$[s] = [t], \quad d_{X^{e_{[s]}}}(\langle s \rangle, \langle t \rangle) = 0, \tag{6.2a}$$

$$\begin{aligned} e_{[s]} = \overline{e}_{[t]}, \quad X^{[s]}\langle s \rangle = \underline{X}^{[s]}\langle s \rangle, \quad X^{[t]}\langle t \rangle = \underline{X}^{[t]}\langle t \rangle, \\ X^{[s]}\langle s \rangle = H^{e_{[t]}} - X^{[t]}\langle t \rangle, \end{aligned} \tag{6.2b}$$

where we wrote $X^{[s]}\langle s \rangle$ instead of $X^{e_{[s]}}(\langle s \rangle)$ for short. Formulas (6.2a) identify numbers coding the same point in one of the Brownian forests, while formulas (6.2b) identify the floors of forests "facing each other": the numbers s and t should code floor points (second and third equalities) of forests facing each other (first equality) and correspond to the same point (fourth equality).

Proposition 6.1. *The Brownian surface* $\mathsf{S}_{\boldsymbol{L}}^{[g]}$ *has same distribution as* $([0, 1], \widetilde{d}_W)/R$.

Let us give a similar interpretation as in the case of the disk. Let first $(X_s)_{0 \le s \le 1}$ be the continuous process obtained by shifting and concatenating $X^{e_1}, \dots, X^{e_\kappa}$. Then $\mathsf{S}_{\boldsymbol{L}}^{[g]}$ may be seen as a quotient of

$$\mathcal{M}_{\boldsymbol{L}}^{[g]} = ([0, 1], d_X)/R,$$

which can be pictured as follows. Starting from the random vector $(S, (A^e)_{e \in \vec{E}(S)}, (H^e)_{e \in \vec{I}(S)}, (L^e)_{e \in \vec{B}(S)})$, we first construct the metric graph obtained from S by assigning either the length H^e or L^e to the edge corresponding to e. For every half-edge e incident to the internal face of S, we then glue a Brownian forest of mass A^e

and length H^e or L^e on e. We equip this space $\mathcal{M}_{\boldsymbol{L}}^{[g]}$ with Brownian labels (with variance $\sqrt{3}$ on the boundary edges) and define $\mathsf{S}_{\boldsymbol{L}}^{[g]}$ from there by the same process as in the case of the Brownian disk.

Proof of Proposition 6.1. First of all, recall from Section 3.6 that the Brownian surface $\mathsf{S}_{\boldsymbol{L}}^{[g]}$ is defined as the gluing along geodesic sides of a collection of continuum elementary pieces distributed as follows. Conditionally given

$$(S, (A^e)_{e\in\vec{E}(S)}, (H^e)_{e\in\vec{I}(S)}, (L^e)_{e\in\vec{B}(S)}, (\Lambda^v)_{v\in V(S)}),$$

the elementary pieces EP^e, $e \in \vec{E}(S)$, are only dependent through the relation linking EP^e with $\mathsf{EP}^{\overline{e}}$ and, setting $\Delta^e = \Lambda^{e^+} - \Lambda^{e^-}$,

- if $e \in \vec{B}(S)$, then EP^e is a slice with area A^e, width L^e and tilt Δ^e,
- if $e \in \vec{I}(S)$, then EP^e is a quadrilateral with half-areas A^e and $A^{\overline{e}}$, width H^e and tilt Δ^e.

Furthermore, it is straightforward from the definition of the pairs (X^e, W^e), $e \in \vec{E}(S)$, that if $e \in \vec{B}(S)$, then the pair $(X^e, W^e - \Lambda^{e^-})$ is distributed as $\mathbf{Slice}_{A^e,L^e,\Delta^e}$. When $e \in \vec{I}(S)$, we denote by

$$(\overline{X}^e, \overline{W}^e) = (X^e_{s+A^e} - 2\underline{X}^e_{s+A^e} - H^e, W^e_{s+A^e})_{-A^e\leq s\leq 0}$$

the process obtained by shifting the Pitman transform of X^e in order to obtain a process from $-H^e$ to 0, as well as changing the time range to $[-A^e, 0]$. By standard results on Brownian motion and random snakes, the pair obtained by concatenating $(\overline{X}^{\overline{e}}, \overline{W}^{\overline{e}} - \Lambda^{\overline{e}^-})$ with $(X^e, W^e - \Lambda^{e^-})$ has the law of a process distributed as $\mathbf{Quad}_{A^e,A^{\overline{e}},H^e,\Delta^e}$.

As a result, we may assume that the elementary piece EP^e is encoded by

- the pair $(X^e, W^e - \Lambda^{e^-})$ if $e \in \vec{B}(S)$,
- the concatenation of $(\overline{X}^{\overline{e}}, \overline{W}^{\overline{e}} - \Lambda^{\overline{e}^-})$ with $(X^e, W^e - \Lambda^{e^-})$ if $e \in \vec{I}(S)$.

This yields a collection of elementary pieces with the proper laws and dependencies; the fact that, for $e \in \vec{I}(S)$, EP^e and $\mathsf{EP}^{\overline{e}}$ are the same with exchanged shuttles and maximal geodesics is a simple application of the Pitman transform.

For $s \in [0, 1]$, we denote by $\boldsymbol{\pi}(s)$ the projection in the gluing $\mathsf{S}_{\boldsymbol{L}}^{[g]}$ of the point $\langle s\rangle$ of the elementary piece $\mathsf{EP}^{e_{[s]}}$. We claim that $\boldsymbol{\pi}\colon [0, 1] \to \mathsf{S}_{\boldsymbol{L}}^{[g]}$ is onto. Indeed, for each half-edge $\epsilon \in \vec{E}(S)$, recall that the elementary piece EP^ϵ is defined as a quotient of $[0, A^\epsilon]$ and observe that $\{\langle s\rangle : s \text{ such that } e_{[s]} = \epsilon\} = [0, A^\epsilon)$; furthermore, the "missing point" A^ϵ of EP^ϵ is glued to a point 0 of some elementary piece, which is $\boldsymbol{\pi}(s)$ for some s satisfying $\langle s\rangle = 0$. Writing d_{S} the distance in $\mathsf{S}_{\boldsymbol{L}}^{[g]}$ and $\mathrm{d}_R = \widetilde{d}_W/R$, it is sufficient to show that, for $s, t \in [0, 1]$,

$$\mathrm{d}_R(s, t) = \mathrm{d}_{\mathsf{S}}(\boldsymbol{\pi}(s), \boldsymbol{\pi}(t)).$$

As the pseudometric d_f defined in (4.2) is unchanged by the addition of an additive constant, setting

$$\mathrm{d}_\epsilon = \begin{cases} d_{W^\epsilon} & \text{if } \epsilon \in \vec{B}(S), \\ \widehat{d}_{\overline{W}^{\bar{\epsilon}} \bullet W^\epsilon} & \text{if } \epsilon \in \vec{I}(S), \end{cases}$$

the quantity $\mathrm{d}_{\mathrm{S}}(\boldsymbol{\pi}(s), \boldsymbol{\pi}(t))$ is the infimum of sums of the form $\sum_{i=1}^{\ell} \mathrm{d}_{\epsilon_i}(s_i, t_i)$, where

- $\epsilon_1 = e_{[s]}$, $s_1 = \langle s \rangle$, $\epsilon_\ell = e_{[t]}$, $s_\ell = \langle t \rangle$,
- for all i, it holds that

$$s_i, t_i \in \begin{cases} [0, A^{\epsilon_i}] & \text{if } \epsilon_i \in \vec{B}(S), \\ [-A^{\bar{\epsilon}_i}, A^{\epsilon_i}] & \text{if } \epsilon_i \in \vec{I}(S), \end{cases}$$

- for all i,
  - (a) either $\epsilon_i = \epsilon_{i+1} \in \vec{B}(S)$ and $d_{X^{\epsilon_i}}(t_i, s_{i+1}) = 0$,
  - (b) or $\epsilon_i = \epsilon_{i+1} \in \vec{I}(S)$ and $d_{\overline{X}^{\bar{\epsilon}_i} \bullet X^{\epsilon_i}}(t_i, s_{i+1}) = 0$,
  - (c) or the point t_i of EP^{ϵ_i} is glued to the point s_{i+1} of $\mathsf{EP}^{\epsilon_{i+1}}$.

As $\mathrm{d}_\epsilon(u, v) = \infty$ whenever $uv < 0$, we may furthermore assume that, for all i, $s_i t_i \geq 0$. Now, for each i, we set

$$\widetilde{s}_i = \sum_{j=1}^{[\epsilon_i]-1} A^{e_j} + s_i \ \text{ if } s_i \geq 0, \quad \widetilde{s}_i = \sum_{j=1}^{[\bar{\epsilon}_i]} A^{e_j} + s_i \ \text{ if } s_i < 0,$$

where we wrote $[\epsilon]$ the index of the half-edge ϵ in the ordering $e_1, \ldots, e_\kappa$ of $\vec{E}(S)$. We define $\widetilde{t}_i$ similarly. It is easy to check that $\widetilde{s}_1 = s$, $\widetilde{t}_\ell = t$ and that, for each i, we have $\mathrm{d}_{\epsilon_i}(s_i, t_i) = d_W(\widetilde{s}_i, \widetilde{t}_i)$. Furthermore, for each i, we have the following:

(a) If $\epsilon_i = \epsilon_{i+1} \in \vec{B}(S)$ and $d_{X^{\epsilon_i}}(t_i, s_{i+1}) = 0$, then, unless $t_i = s_{i+1} = A^{\epsilon_i}$ (in which case $\widetilde{t}_i = \widetilde{s}_{i+1}$), it holds that $s_i < A^{\epsilon_i}$ and $t_i < A^{\epsilon_i}$, which yields that $\widetilde{t}_i \, R \, \widetilde{s}_{i+1}$ by (6.2a).

(b) If $\epsilon_i = \epsilon_{i+1} \in \vec{I}(S)$ and $d_{\overline{X}^{\bar{\epsilon}_i} \bullet X^{\epsilon_i}}(t_i, s_{i+1}) = 0$, then
  - if $t_i s_{i+1} \geq 0$, then $\widetilde{t}_i \, R \, \widetilde{s}_{i+1}$ by (6.2a) as above,
  - if $t_i s_{i+1} < 0$, then $\widetilde{t}_i \, R \, \widetilde{s}_{i+1}$ by (6.2b).

(c) If the point t_i of EP^{ϵ_i} is glued to the point s_{i+1} of $\mathsf{EP}^{\epsilon_{i+1}}$, then it implies that $\widetilde{d}_W(\widetilde{t}_i, \widetilde{s}_{i+1}) = 0$ (recall the situation depicted in Figure 3.4).

As a result, since $\widetilde{d}_W \leq d_W$, it holds that

$$\mathrm{d}_R(s, t) \leq \mathrm{d}_{\mathrm{S}}(\boldsymbol{\pi}(s), \boldsymbol{\pi}(t)).$$

The converse inequality is very similar, noting that R identifies points in the elementary piece EP^ϵ as does $d_{X^\epsilon} = 0$ or $d_{\overline{X}^{\bar{\epsilon}} \bullet X^\epsilon} = 0$, and that $\widetilde{d}_W$ encodes all the functions d_ϵ, together with the gluings of the elementary pieces. The use of $\widetilde{d}_W$ and not d_W takes into account the gluings of shuttles with maximal geodesics of elementary pieces "overflying" the root, as, for instance, in Figure 3.4, the shuttle of $\mathsf{EP}^{e_{14}}$ with part of the maximal geodesic of EP^{e_1}, or part of the shuttle of $\mathsf{EP}^{e_{12}}$ with part of the maximal geodesic of EP^{e_7}. The details are left to the reader. ■