Appendix A

Technical lemmas on the Brownian plane

We now recall the definition of the Brownian plane from [40], then show that it is equivalent to the one we gave in Section 5.3, and we finally prove Proposition 5.11.

A.1 Equivalence of definitions of the Brownian plane

The original definition goes as follows. Let $(\mathfrak{X}_t, t \in \mathbb{R})$ be such that $(\mathfrak{X}_t, t \geq 0)$ and $(\mathfrak{X}_{-t}, t \geq 0)$ are two independent three-dimensional Bessel processes. Since the overall minimum of $\mathfrak{X}$ is reached at 0, the maximum in the definition of $\widetilde{d}_{\mathfrak{X}}$ – given in (6.1) – is equal to

$$\max(\underline{\mathfrak{X}}(s,t), \underline{\mathfrak{X}}(t,s)) = \begin{cases} \inf\{\mathfrak{X}_u, u \in [s \wedge t, s \vee t]\} & \text{if } st \geq 0, \\ \inf\{\mathfrak{X}_u, u \notin [s \wedge t, s \vee t]\} & \text{if } st < 0. \end{cases}$$

Next, define $(\mathfrak{W}_t, t \in \mathbb{R})$ to be a centered Gaussian process conditionally given $\mathfrak{X}$, with covariance function specified by

$$\mathbb{E}[|\mathfrak{W}_s - \mathfrak{W}_t|^2 \mid \mathfrak{X}] = \widetilde{d}_{\mathfrak{X}}(s,t).$$

The Brownian plane was defined in [40] as

$$(\widetilde{M}_{\mathfrak{X},\mathfrak{W}}, \widetilde{D}_{\mathfrak{X},\mathfrak{W}}) = (\mathbb{R}/\{\widetilde{D}_{\mathfrak{X},\mathfrak{W}} = 0\}, \widetilde{D}_{\mathfrak{X},\mathfrak{W}}), \quad \text{where } \widetilde{D}_{\mathfrak{X},\mathfrak{W}} = d_{\mathfrak{W}}/\{\widetilde{d}_{\mathfrak{X}} = 0\}.$$

The following proposition shows that the definition given in Section 5.3 is equivalent to the one above. Recall the piece of notation $\underline{\underline{X}}_t = \underline{X}(0 \wedge t, 0 \vee t)$ and define the process $(\Pi_t = X_t - 2\underline{\underline{X}}_t, t \in \mathbb{R})$ by taking the Pitman transform of X on $\mathbb{R}_{\geq 0}$ and on $\mathbb{R}_{\leq 0}$.

Proposition A.1. *The process* (Π, W) *considered under* **Plane** *has same distribution as* $(\mathfrak{X}, \mathfrak{W})$ *defined above. Moreover, as metric spaces,* $(\widetilde{M}_{\Pi,W}, \widetilde{D}_{\Pi,W} = d_W/\{\widetilde{d}_\Pi = 0\})$ *and* $(M_{X,W}, D_{X,W})$ *are almost surely equal.*

Proof. We claim that $\widetilde{d}_\Pi = d_X$. This entails that, conditionally given X, the process W is also Gaussian with $\mathbb{E}[(W_s - W_t)^2 \mid X] = \widetilde{d}_\Pi(s,t)$ and, since Π has same distribution as $\mathfrak{X}$ by Pitman's $2M - X$ theorem [87, Theorem 1.3], we see that (Π, W) and $(\mathfrak{X}, \mathfrak{W})$ have same distribution. We then have

$$\widetilde{D}_{\Pi,W} = d_W/\{\widetilde{d}_\Pi = 0\} = d_W/\{d_X = 0\} = D_{X,W}.$$

Checking that $\widetilde{d}_\Pi = d_X$ is a classical exercise, based on the fact that, for $0 \le s < t$,

$$\underline{\Pi}(s,t) = \underline{X}(s,t) - \underline{\underline{X}}_s - \underline{\underline{X}}_t \quad \text{and} \quad \inf_{u \ge s} \Pi_u = -\underline{\underline{X}}_s. \tag{A.1}$$

The right equation is obtained from the left one by letting $t \to \infty$, noting that, for t large enough, $\underline{X}(s,t) = \underline{\underline{X}}_t$. The left equation comes from a straightforward case analysis. If $\underline{\underline{X}}_s = \underline{\underline{X}}_t$, then, for all $u \in [s,t]$, $\underline{\underline{X}}_u = \underline{\underline{X}}_s = \underline{\underline{X}}_t$ and so $\Pi_u = X_u - \underline{\underline{X}}_s - \underline{\underline{X}}_t$; taking the infimum on $u \in [s,t]$ gives the result. If $\underline{\underline{X}}_s > \underline{\underline{X}}_t$, then $\underline{X}(s,t) = \underline{\underline{X}}_t$ so the right-hand side is $-\underline{\underline{X}}_s$. Let $r \in [s,t]$ be such that $X_r = \underline{\underline{X}}_r = \underline{\underline{X}}_s$. We have

$$\Pi_r = X_r - 2\underline{\underline{X}}_r = -\underline{\underline{X}}_s$$

and, for $u \ge s$,

$$\Pi_u = X_u - 2\underline{\underline{X}}_u \ge -\underline{\underline{X}}_u \ge -\underline{\underline{X}}_s.$$

For $0 \le s < t$, the left equation of (A.1) entails

$$\begin{aligned}\widetilde{d}_\Pi(t,s) &= \Pi_s + \Pi_t - 2\underline{\Pi}(s,t)\\ &= X_s - 2\underline{\underline{X}}_s + X_t - 2\underline{\underline{X}}_t - 2(\underline{X}(s,t) - \underline{\underline{X}}_s - \underline{\underline{X}}_t) = d_X(s,t).\end{aligned}$$

For $s < 0 < t$, we have that

$$\begin{aligned}\underline{\Pi}(t,s) &= \inf_{u \ge t} \Pi_u \wedge \inf_{u \le s} \Pi_u = -(\underline{X}(0,t) \vee \underline{X}(s,0))\\ &= \underline{X}(s,t) - \underline{X}(s,0) - \underline{X}(0,t)\end{aligned}$$

and we conclude as above. The remaining case $s < t < 0$ is treated similarly. ∎

A.2 Convergence of the uniform infinite planar quadrangulation to the Brownian plane

We use here the setting of Section 5.4. The proof of Proposition 5.11 will follow similar lines as that of Proposition 4.9, using the coupling results of [40]. As the law of $\mathfrak{X}$ is obtained from that of X by taking the Pitman transform on $\mathbb{R}_{\ge 0}$ and on $\mathbb{R}_{\le 0}$, the same should be done for the contour process C of the tree $\mathbf{T}_\infty$. We thus define the process $(\mathfrak{C}(t) = C(t) - 2\underline{\underline{C}}(t), t \in \mathbb{R})$.

Note that this gives an alternate natural contour process since, for $i \in \mathbb{Z}$, it holds that

$$\mathfrak{C}(i) = d_{\mathbf{T}^{\Upsilon(i)}}(v_i, \rho^{|\Upsilon(i)|}) + |\Upsilon(i)| = d_{\mathbf{T}_\infty}(v_i, \rho^0).$$

In this setting of discrete trees, the Pitman transform on the contour process is very visual: it merely consists of going from reading the trees while moving *down* between trees to reading them while moving *up* between trees; see Figure A.1 for an illustration. We may now proceed to the proof of the convergence of the UIPQ to the Brownian plane.

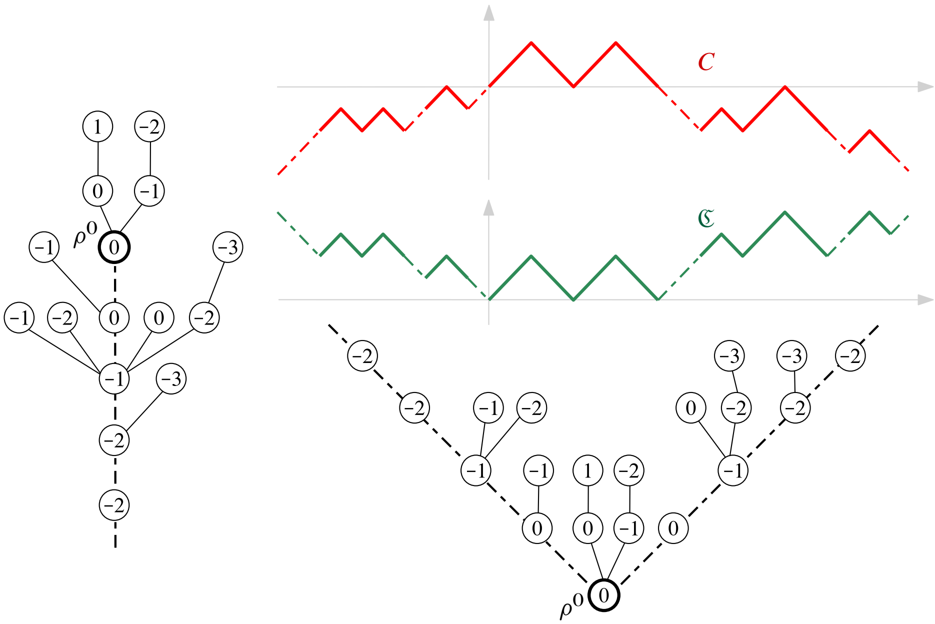

Figure A.1. *Left.* Representation of the infinite tree from Figure 5.2 after moving the trees in such a way that, for $k \geq 0$, ρ^k is located at $(0, -k)$ with $\mathbf{T}^k$ grafted on its right and $\mathbf{T}^{-k}$ on its left. *Top right.* Taking the Pitman transform of the contour process on $\mathbb{R}_{\geq 0}$ and on $\mathbb{R}_{\leq 0}$ yields an alternate contour process. The differing parts are represented with dot-dashed lines; they correspond to the edges of the infinite spine of the tree. Visually, the process C records the height of a particle moving at speed one around the tree when represented as on the left. *Bottom right.* Representation of the tree from Figure 5.2 where the root of $\mathbf{T}^k$ is now located at $(k, |k|)$ for each $k \in \mathbb{Z}$. Note that, in this representation, the roots of $\mathbf{T}^{-k}$ and $\mathbf{T}^k$ differ so that the spine is duplicated. The process $\mathfrak{C}$ records the height of a particle moving at speed one around this bi-infinite tree.

Proof of Proposition 5.11. Similarly to the proof of Proposition 4.9, we fix some number $K > 0$ and will sample a large plane quadrangulation such that its properly scaled version and its limit, the Brownian sphere, are indistinguishable from the rescaled UIPQ and the Brownian plane, in a neighborhood of 0 of amplitude K. We use again a superscript prime symbol $'$ for the objects related to the plane quadrangulation and its limit. Here, some care will also be needed when taking an inverse Pitman transform, since this operation a priori involves more than just a neighborhood of 0.

We fix $L > 0$ and $n \geq 1$, and consider a uniform random element (M'_n, λ'_n) of $\vec{\mathbf{M}}^{[0]}_{a_n,\varnothing}$, where $a_n = \lfloor nL \rfloor$, that is, M'_n is a uniform rooted plane tree with a_n edges, which we view as a map with a unique face f_*, and λ'_n is a labeling function uniformly distributed among those yielding a well-labeled tree.

We let $(\mathfrak{C}'_n, \mathfrak{L}'_n)$ be the contour and label function of this tree, we let

$$Q'_n = \mathrm{CVS}(M'_n, \lambda'_n; f_*)$$

be the quadrangulation encoded by (M'_n, λ'_n), and we set

$$\mathfrak{D}'_n(i, j) = d_{Q'_n}(v_i, v_j)$$

for $0 \le i, j \le 2a_n$, where v_i is the i-th visited vertex in M'_n in contour order, starting from the root corner, and viewed as a vertex of Q'_n. We extend $\mathfrak{D}'_n$ into a continuous function on $[0, 2a_n]^2$ by bilinearity, and all processes $\mathfrak{C}'_n$, $\mathfrak{L}'_n$, $\mathfrak{D}'_n$ to $[-2a_n, 2a_n]$ by the same formulas as (4.15) and (4.16) but with $l_n = 0$. We also define the rescaled versions $\mathfrak{C}'_{(n)}$, $\mathfrak{L}'_{(n)}$, $\mathfrak{D}'_{(n)}$ exactly as in (4.17). The joint convergence

$$(\mathfrak{C}'_{(n)}, \mathfrak{L}'_{(n)}, \mathfrak{D}'_{(n)}) \xrightarrow[n\to\infty]{(d)} (\mathfrak{X}', \mathfrak{W}', \mathfrak{D}')$$

on the space $\mathcal{C}([-L, L]) \times \mathcal{C}([-L, L]) \times \mathcal{C}([-L, L]^2)$ is then a consequence of [80, Theorem 3], where the limit is as follows. Restricted to $[0, L]$, the process $\mathfrak{X}'$ is a Brownian excursion of duration L and $\mathfrak{W}'$ is the random snake driven by $\mathfrak{X}'$, while $\mathfrak{D}'$ is a random pseudometric, which is an explicit function of $(\mathfrak{X}', \mathfrak{W}')$. Moreover, all these processes are extended to $[-L, L]$ by a simple translation of their argument by L.

Let us now recall the relevant aspects of the coupling results of [40], between the pairs $(\mathfrak{X}, \mathfrak{W})$ and $(\mathfrak{X}', \mathfrak{W}')$. It will be convenient to let

$$\overline{\mathfrak{T}}_x = \inf\{t \le 0 : \mathfrak{X}_t = x\}, \quad \mathfrak{T}_x = \sup\{t \ge 0 : \mathfrak{X}_t = x\}.$$

Fix $r > 0$ and $\varepsilon > 0$. Then by [40, Lemmas 5 and 6], it is possible to find $A > 1$ and then $\alpha > 0$ and $L_0 > 0$ large, such that for $L > L_0$ the two processes $(\mathfrak{X}, \mathfrak{W})$ and $(\mathfrak{X}', \mathfrak{W}')$ can be coupled in such a way that on some event $\mathcal{F}$ of probability $\mathbb{P}(\mathcal{F}) \ge 1 - \varepsilon$, the following properties hold:

- For every $s, t \in [-\alpha, \alpha]$, one has
$$\mathfrak{X}_t = \mathfrak{X}'_t, \quad \mathfrak{W}_t = \mathfrak{W}'_t. \tag{A.2}$$
- It holds that
$$-\alpha < \overline{\mathfrak{T}}_{A^4} \quad \text{and} \quad \mathfrak{T}_{A^4} < \alpha. \tag{A.3}$$
- For every $s, t \in [\overline{\mathfrak{T}}_A, \mathfrak{T}_A]$, the two conditions $\max(\tilde{D}_{\mathfrak{X},\mathfrak{W}}(0, t), \tilde{D}_{\mathfrak{X},\mathfrak{W}}(0, s)) \le r$ and $\max(\mathfrak{D}'(0, t), \mathfrak{D}'(0, s)) \le r$ are equivalent, and, if these are satisfied, one has
$$\mathfrak{D}'(s, t) = \tilde{D}_{\mathfrak{X},\mathfrak{W}}(s, t).$$

This choice of coupling being fixed, let us now define $(X_t, t \in \mathbb{R})$ as the unique process such that $\mathfrak{X}_t = X_t - 2\underline{\underline{X}}_t$ is the Pitman transform of X on $\mathbb{R}_{\geq 0}$ and on $\mathbb{R}_{\leq 0}$; more explicitly,

$$X_t = \begin{cases} \mathfrak{X}_t - 2 \inf_{s \geq t} \mathfrak{X}_s & \text{if } t \geq 0, \\ \mathfrak{X}_t - 2 \inf_{s \leq t} \mathfrak{X}_s & \text{if } t < 0. \end{cases}$$

Let us also define $W = \mathfrak{W}$. Then X indeed has the law of a two-sided Brownian motion, and W is the random snake driven by X, so that (X, W) has law **Plane**. Note that, in this particular coupling, we have $\overline{T}_x = \overline{\mathfrak{T}}_x$ and $T_x = \mathfrak{T}_x$ for every $x \geq 0$, and also $D_{X,W} = \widetilde{D}_{\mathfrak{X},\mathfrak{W}}$, by the observation in the proof of Proposition A.1. Moreover, on the event $\mathcal{F}$, the restriction $X|_{[\overline{T}_{A^2}, T_{A^2}]}$ is actually a function of $\mathfrak{X}'|_{[-\alpha,\alpha]}$. Indeed, by (A.2) and (A.3),

$$X_t = \begin{cases} \mathfrak{X}'_t - 2 \inf_{t \leq s \leq \alpha} \mathfrak{X}'_s & \text{if } 0 \leq t \leq T_{A^2}, \\ \mathfrak{X}'_t - 2 \inf_{-\alpha \leq s \leq t} \mathfrak{X}'_s & \text{if } \overline{T}_{A^2} \leq t < 0 \end{cases}$$

since, for $0 \leq t \leq T_{A^2}$, one has $\underline{\mathfrak{X}}(t, \infty) = \underline{\mathfrak{X}}(t, T_{A^2}) = \underline{\mathfrak{X}}(t, \alpha)$, and similarly in negative times.

By choosing appropriately the values of r, and enlarging the values of A and α if necessary, then, similarly to the proof of Proposition 4.9, we obtain that (A.2) holds on $[-K, K]$, and that the restrictions to $[-K, K]^2$ of $\mathfrak{D}'$ and $D_{X,W}$ coincide with probability at least $1 - \varepsilon$.

Next, keeping K, ε fixed, and possibly up to choosing L even larger, we need to couple the processes $(C_{(n)}, \Lambda_{(n)}, D_{(n)})$ and $(\mathfrak{C}'_{(n)}, \mathfrak{L}'_{(n)}, \mathfrak{D}'_{(n)})$ appropriately. To this end, we use the techniques of [40, Proposition 9]. The latter states that for $\varepsilon > 0$, there exists $\alpha > 0$ (independent of the choice of L arising in the definition of the scaling constant a_n) such that for every n large enough, one may couple the quadrangulations Q'_n and Q_∞ in such a way that, with probability at least $1 - \varepsilon$, the balls of radius $\alpha a_n^{1/4}$ around the root of Q'_n and Q_∞ are isometric. The proof proceeds by coupling the encoding labeled trees (M'_n, λ'_n) and $(\mathbf{T}_\infty, \lambda_\infty)$ in such a way that, with even larger probability, the first $\lfloor \delta a_n^{1/2} \rfloor$ generations of these trees coincide for some $\delta > 0$, and the minimal value of λ_∞ taken on the vertices $\rho^0, \rho^1, \ldots, \rho^{\lfloor \delta a_n^{1/2} \rfloor}$ of $\mathbf{T}_\infty$ is less than $-4\alpha a_n^{1/4}$. By choosing R and then L large enough in the first place, for our choice of K, we may also require that with probability at least $1 - \varepsilon$,

- the contour and label processes $\mathfrak{C}'_n$, $\mathfrak{L}'_n$ of (M'_n, λ'_n) and $\mathfrak{C}$, $\mathfrak{L}$ of $(\mathbf{T}_\infty, \lambda_\infty)$ on the interval $[-2nK, 2nK]$ involve only vertices of generations less than $\lfloor Rn^{1/2} \rfloor$, and
- the most recent common ancestor of the vertices at generation $\lfloor \delta a_n^{1/2} \rfloor$ has generation at least $\lfloor Rn^{1/2} \rfloor$.

In particular, on this event, the restriction of the process C_n' to $[-2nK, 2nK]$ is equal to the restriction of the process C on this same event – in words, the second itemized event means that the spine of $\mathbf{T}_\infty$ is determined by the data of M_n' up to generation $Rn^{1/2}$. Since the process C is the inverse Pitman transform of $\mathfrak{C}$, it is then a simple exercise to conclude that $(C_{(n)}', \Lambda_{(n)}', D_{(n)}')$, which coincides with $(C_{(n)}, \Lambda_{(n)}, D_{(n)})$ on $[-K, K]$ with high probability, converges to some (X', W', D'), which coincides with $(X, W, D_{X,W})$ on $[-K, K]$ with high probability. ■