Appendix B

Scaling limit of size parameters in labeled maps

B.1 Preliminaries

In this appendix, we prove Proposition 3.10, following the method of [19, Proposition 7] and [21, Proposition 7]. In the meantime, we obtain an asymptotic enumeration result for $\vec{\mathbf{Q}}^{[g]}_{n,\boldsymbol{l}_n}$ in Proposition B.1 below, which will also allow us to deduce Theorem 1.5 and Corollary 1.6 from Theorem 1.1.

Recall that $(g,k) \notin \{(0,0),(0,1)\}$, that $\boldsymbol{L} = (L^1,\dots,L^k)$ is a fixed k-tuple such that $L^1,\dots,L^b > 0$, while $L^{b+1},\dots,L^k = 0$, and that we consider a fixed sequence of k-tuples $\boldsymbol{l}_n = (l_n^1,\dots,l_n^k) \in (\mathbb{Z}_{\geq 0})^k$, $n \geq 1$, such that $l_n^i/\sqrt{2n} \to L^i$ as $n \to \infty$ for $1 \leq i \leq k$.

We furthermore assume that n is sufficiently large so that $l_n^i \geq 1$ for each $i \leq b$. We denote by $\vec{\mathbf{S}}$ the set of rooted genus g schemes with k holes, such that $h_1,\dots,h_b$ are faces. Note that our assumption on n ensures that $S_n \in \vec{\mathbf{S}}$.

"Free" parameters and notation. For every scheme $\mathbf{s} \in \vec{\mathbf{S}}$, not necessarily dominant, we arbitrarily fix, once and for all, half-edges $\epsilon_0 \in \vec{I}(\mathbf{s})$ and $\epsilon_i \in \vec{B}_i(\mathbf{s})$ for $1 \leq i \leq b$. We fix an orientation $I(\mathbf{s})$ of $\vec{I}(\mathbf{s})$ that contains ϵ_0, and we set $I'(\mathbf{s}) = I(\mathbf{s}) \setminus \{\epsilon_0\}$. We also let v_0 be the root vertex of $\mathbf{s}$, and $V'(\mathbf{s}) = V(\mathbf{s}) \setminus \{v_0\}$, as in Section 3.6. Finally, we set

$$\vec{B}_0(\mathbf{s}) = \bigsqcup_{b+1 \leq i \leq k} \vec{B}_i(\mathbf{s}), \qquad \vec{B}_+(\mathbf{s}) = \bigsqcup_{1 \leq i \leq b} \vec{B}_i(\mathbf{s}),$$
$$\vec{B}'_i(\mathbf{s}) = \vec{B}_i(\mathbf{s}) \setminus \{\epsilon_i\} \text{ for } 1 \leq i \leq b, \quad \vec{B}'_+(\mathbf{s}) = \bigsqcup_{1 \leq i \leq b} \vec{B}'_i(\mathbf{s}).$$

The motivation for introducing $\vec{B}_+$ and $\vec{B}_0$ is that we need a different treatment depending on whether the hole perimeters are in the scale $\sqrt{n}$ or $o(\sqrt{n})$. The sets with a prime symbol should be thought of as the sets containing the parameters on which there is a "degree of freedom." (*The reason for removing one element from I will become clear in a moment. We will not need a $\vec{B}'_0$ since the corresponding perimeters are all asymptotically null in the scale $\sqrt{n}$ of interest.*)

From now on, we use the shorthand piece of notation $\boldsymbol{x}^{\mathcal{E}}$ for a family $(x^j)_{j\in\mathcal{E}}$ indexed by a set $\mathcal{E}$. For any subset $\mathcal{F} \subseteq \mathcal{E}$, we also denote by $\boldsymbol{x}^{\mathcal{F}} = (x^j)_{j\in\mathcal{F}}$ the subfamily indexed by $\mathcal{F}$, and, in the case of real nonnegative numbers, by $\|\boldsymbol{x}\|_{\mathcal{F}} = \sum_{j\in\mathcal{F}} x^j$ (note in particular that $\|\boldsymbol{x}\|_{\varnothing} = 0$).

Counting scheme-rooted labeled maps with given size parameters. For the time being, we do not take the areas parameters into account. We fix a rooted scheme $\mathbf{s} \in \vec{\mathbf{S}}$, and size parameters $\boldsymbol{h}^{\vec{I}(\mathbf{s})}$, $\boldsymbol{l}^{\vec{B}(\mathbf{s})}$ and $\boldsymbol{\lambda}^{V(\mathbf{s})}$. We say that a labeled map is *scheme-rooted on* $\mathbf{s}$ if its scheme carries an extra root and the scheme rooted at this extra root is $\mathbf{s}$. We consider the elements of $\mathbf{M}_{n,\boldsymbol{l}}^{[g]}$ scheme-rooted on $\mathbf{s}$ whose size parameters are $\boldsymbol{h}^{\vec{I}(\mathbf{s})}$, $\boldsymbol{l}^{\vec{B}(\mathbf{s})}$ and $\boldsymbol{\lambda}^{V(\mathbf{s})}$. Reasoning as in Propositions 2.3 and 2.5, we can express the number of such elements as

$$12^{n-\|\boldsymbol{h}\|/2}2^{\|\boldsymbol{h}\|+\|\boldsymbol{l}\|}Q_{\|\boldsymbol{h}\|+\|\boldsymbol{l}\|}(2n+\|\boldsymbol{l}\|)\prod_{e\in I(\mathbf{s})}3^{h^e}M_{h^e}(\delta\lambda^e)\prod_{e\in\vec{B}(\mathbf{s})}2^{2l^e+\delta\lambda^e}P_{l^e}(\delta\lambda^e),$$

the products over $I(\mathbf{s})$ and $\vec{B}(\mathbf{s})$ respectively counting the number of ways to label the vertices along the edges of $I(\mathbf{s})$ and $\vec{B}(\mathbf{s})$, and the remaining term counting the labeled forests, which can be seen as one big labeled forest obtained by concatenating all the labeled forests indexed by the half-edges of $\vec{E}(\mathbf{s})$. After recalling that $\sum_{e\in\vec{B}_i(\mathbf{s})}\delta\lambda^e=0$ and that $\|\boldsymbol{l}\|_{\vec{B}_i(\mathbf{s})}=l^i$ for every $i\in\{1,2,\ldots,k\}$ corresponding to an external face of $\mathbf{s}$, we may recast this quantity as

$$12^n 8^{\|\boldsymbol{l}\|}Q_{\|\boldsymbol{h}\|+\|\boldsymbol{l}\|}(2n+\|\boldsymbol{l}\|)\prod_{e\in I(\mathbf{s})}M_{h^e}(\delta\lambda^e)\prod_{e\in\vec{B}(\mathbf{s})}P_{l^e}(\delta\lambda^e).$$

Consequently, the number of elements of $\vec{\mathbf{M}}_{n,\boldsymbol{l}}^{[g]}$ scheme-rooted on $\mathbf{s}$ (these labeled maps are thus rooted twice) whose size parameters are $\boldsymbol{h}^{\vec{I}(\mathbf{s})}$, $\boldsymbol{l}^{\vec{B}(\mathbf{s})}$ and $\boldsymbol{\lambda}^{V(\mathbf{s})}$ is equal to

$$\begin{aligned}\vec{\mathrm{S}}_n^{\mathbf{s}}(\boldsymbol{h},\boldsymbol{l},\boldsymbol{\lambda})&=(2n+\|\boldsymbol{l}\|)12^n 8^{\|\boldsymbol{l}\|}Q_{\|\boldsymbol{h}\|+\|\boldsymbol{l}\|}(2n+\|\boldsymbol{l}\|)\\&\quad\times\prod_{e\in I(\mathbf{s})}M_{h^e}(\delta\lambda^e)\prod_{e\in\vec{B}(\mathbf{s})}P_{l^e}(\delta\lambda^e),\end{aligned}\tag{B.1}$$

since there are $2n+\|\boldsymbol{l}\|$ possible rootings of the map.

Counting rooted labeled maps. Next, for $n,h\in\mathbb{N}$, $\mathbf{s}\in\vec{\mathbf{S}}$, we set

$$\mathcal{Z}_1^{\mathbf{s}}(h,n)=\sum_{\mathcal{T}_{\mathbf{s}}(h,n)}\prod_{e\in I(\mathbf{s})}M_{h^e}(\delta\lambda^e)\prod_{e\in\vec{B}(\mathbf{s})}P_{l^e}(\delta\lambda^e),\tag{B.2}$$

where the sum is taken over the set $\mathcal{T}_{\mathbf{s}}(h,n)$ of all size parameters from labeled maps in $\mathbf{M}_{n,\boldsymbol{l}_n}^{[g]}$ scheme-rooted on $\mathbf{s}$, having h edges in total on the internal edges of $\mathbf{s}$. More precisely, it is the set of tuples

$$(\boldsymbol{h}^{\vec{I}(\mathbf{s})},\boldsymbol{l}^{\vec{B}(\mathbf{s})},\boldsymbol{\lambda}^{V(\mathbf{s})})\in\mathbb{N}^{\vec{I}(\mathbf{s})}\times\mathbb{N}^{\vec{B}(\mathbf{s})}\times\mathbb{Z}^{V(\mathbf{s})}$$

such that $\|\boldsymbol{h}\|=2h$, $\|\boldsymbol{l}\|_{\vec{B}_i(\mathbf{s})}=l_n^i$ for $1\le i\le k$, $h^{\bar{e}}=h^e$ for all $e\in\vec{I}(\mathbf{s})$, $\lambda^{v_0}=0$. Note that the conditions

$$\|\boldsymbol{l}\|_{\vec{B}_i(\mathbf{s})}=l_n^i$$

may only be satisfied if $\boldsymbol{l}_n$ is *compatible* with $\mathbf{s}$ in the sense that $l_n^i=0\Leftrightarrow\vec{B}_i(\mathbf{s})=\varnothing$ for all i. As a result, $\mathcal{T}_{\mathbf{s}}(h,n)=\varnothing$ and thus $\mathcal{Z}_1^{\mathbf{s}}(h,n)=0$ whenever $\boldsymbol{l}_n$ is not compatible with $\mathbf{s}$.

By double counting the elements of $\vec{\mathbf{M}}^{[g]}_{n,\boldsymbol{l}_n}$ scheme-rooted on $\mathbf{s}$, we thus obtain that

$$|\vec{\mathbf{M}}^{[g]}_{n,\boldsymbol{l}_n}| = 12^n 8^{\|\boldsymbol{l}_n\|} \sum_{\mathbf{s}\in\vec{\mathbf{S}}} \frac{2n+\|\boldsymbol{l}_n\|}{2|E(\mathbf{s})|} \sum_{h\in\mathbb{N}} Q_{2h+\|\boldsymbol{l}_n\|}(2n+\|\boldsymbol{l}_n\|)\mathcal{Z}^{\mathbf{s}}_1(h,n), \qquad \text{(B.3)}$$

since we sum over $\bigcup_{\mathbf{s}\in\vec{\mathbf{S}},h\in\mathbb{N}}\{\mathbf{s}\}\times\mathcal{T}_{\mathbf{s}}(h,n)$ the number $\vec{\mathbb{S}}^{\mathbf{s}}_n(\boldsymbol{h},\boldsymbol{l}_n,\boldsymbol{\lambda})$ given by (B.1), divided by the number $2|E(\mathbf{s})|$ of possible extra rootings on the scheme.

B.2 Asymptotics of the scheme

When we work with a fixed scheme, which will be the case in all but the fourth paragraph, we drop the argument from the sets in the notation in order to ease the reading, thus writing I instead of $I(\mathbf{s})$, for instance.

Law of the scheme. Recall that the triple (M_n,λ_n,S_n) is a rooted, scheme-rooted, labeled map, where (M_n,λ_n) is uniformly distributed over $\vec{\mathbf{M}}^{[g]}_{n,\boldsymbol{l}_n}$, while, conditionally given it, S_n is rooted by uniformly choosing its root among $\{e,\overline{e}: e\in\vec{I}(S_n)\cup\vec{B}_+(S_n)\}$. Let us fix a rooted scheme $\mathbf{s}\in\vec{\mathbf{S}}$ whose root or its reverse belongs to $\vec{I}\cup\vec{B}_+$. Writing $\underline{\mathbf{s}}$ the nonrooted scheme corresponding to $\mathbf{s}$, observe that the set of rooted labeled maps in $\vec{\mathbf{M}}^{[g]}_{n,\boldsymbol{l}_n}$ with scheme $\underline{\mathbf{s}}$ is in bijection with the set of rooted labeled maps in $\vec{\mathbf{M}}^{[g]}_{n,\boldsymbol{l}_n}$ scheme-rooted on $\mathbf{s}$. Then,

$$\begin{aligned}
\mathbb{P}(S_n=\mathbf{s}) &= \sum_{\substack{(\mathbf{m},\lambda)\in\vec{\mathbf{M}}^{[g]}_{n,\boldsymbol{l}_n}\\ \text{with scheme } \underline{\mathbf{s}}}} \mathbb{P}((M_n,\lambda_n)=(\mathbf{m},\lambda),\ S_n=\mathbf{s})\\
&= \sum_{\substack{(\mathbf{m},\lambda)\in\vec{\mathbf{M}}^{[g]}_{n,\boldsymbol{l}_n}\\ \text{scheme-rooted on } \mathbf{s}}} \mathbb{P}((M_n,\lambda_n)=(\mathbf{m},\lambda))\mathbb{P}(S_n=\mathbf{s}\mid(M_n,\lambda_n)=(\mathbf{m},\lambda))\\
&= \sum_{h\in\mathbb{N}}\sum_{\mathcal{T}_{\mathbf{s}}(h,n)} \vec{\mathbb{S}}^{\mathbf{s}}_n(\boldsymbol{h},\boldsymbol{l}_n,\boldsymbol{\lambda})\frac{1}{|\vec{\mathbf{M}}^{[g]}_{n,\boldsymbol{l}_n}|}\frac{1}{|\vec{I}|+2|\vec{B}_+|} = \frac{\mathcal{Z}^{\mathbf{s}}_1(n)}{\mathcal{Z}_1(n)},
\end{aligned}$$

where

$$\mathcal{Z}^{\mathbf{s}}_1(n) = \frac{1}{|\vec{I}(\mathbf{s})|+2|\vec{B}_+(\mathbf{s})|}\sum_{h\in\mathbb{N}} Q_{2h+\|\boldsymbol{l}_n\|}(2n+\|\boldsymbol{l}_n\|)\mathcal{Z}^{\mathbf{s}}_1(h,n) \qquad \text{(B.4)}$$

and $\mathcal{Z}_1(n)=\sum_{\mathbf{s}\in\vec{\mathbf{S}}}\mathcal{Z}^{\mathbf{s}}_1(n)$ is the proper normalization constant.

Schemes with tadpoles. Here, we fix a scheme $\mathbf{s}$ whose external faces among $\mathfrak{h}_i$, $b+1\le i\le k$, are all tadpoles. Equivalently, each $\vec{B}_i$, $b+1\le i\le k$, is either empty or a singleton. In this case, by the Euler characteristic formula,

$$|V'|-|I|-|\vec{B}'_+| = -2g,$$

since $\mathbf{s}$ has $|I|+|\vec{B}'_+|+b+|\vec{B}_0|$ edges and $1+b+|\vec{B}_0|$ faces.

Assuming that $\boldsymbol{l}_n$ is compatible with $\mathbf{s}$, we write the sum over $\mathcal{T}_{\mathbf{s}}(h,n)$ in (B.2) as an integral under the Lebesgue measure

$$\mathrm{dL}_{\mathbf{s}} = \mathrm{d}\boldsymbol{h}^{I'} \otimes \mathrm{d}\boldsymbol{l}^{\vec{B}'_+} \otimes \mathrm{d}\boldsymbol{\lambda}^{V'} \quad \text{over } (\mathbb{R}_{\geq 0})^{I'} \times (\mathbb{R}_{\geq 0})^{\vec{B}'_+} \times \mathbb{R}^{V'}$$

and obtain

$$\mathcal{Z}_1^{\mathbf{s}}(h,n) = \prod_{e\in\vec{B}_0} P_{l^e}(0) \int \mathrm{dL}_{\mathbf{s}} \prod_{e\in I} M_{\underline{h}^e}(\delta\underline{\lambda}^e) \prod_{e\in\vec{B}_+} P_{\underline{l}^e}(\delta\underline{\lambda}^e),$$

where $l^e = l_n^i$ if e is the unique element of $\vec{B}_i$ for $i > b$, and

$$\begin{aligned}
\underline{h}^e &= \lceil h^e \rceil \text{ for } e \in I', \quad & \underline{h}^{\epsilon_0} &= h - \sum_{e\in I'} \underline{h}^e, \\
\underline{l}^e &= \lceil l^e \rceil \text{ for } e \in \vec{B}'_+, \quad & \underline{l}^{\epsilon_i} &= l_n^i - \sum_{e\in\vec{B}'_i} \underline{l}^e \text{ for } 1 \leq i \leq b, \\
\underline{\lambda}^v &= \lceil \lambda^v \rceil \text{ for } v \in V', \quad & \underline{\lambda}^{v_0} &= 0.
\end{aligned}$$

(*Note that the ceiling function is superfluous for integer parameters; we kept it for notational simplicity.*) In order to deal with the cases where $\underline{h}^{\epsilon_0} \leq 0$ or $\underline{l}^{\epsilon_i} \leq 0$, we simply declare[1]

$$M_\ell(j) = P_\ell(j) = 0$$

whenever $\ell \leq 0$.

Observe that $\boldsymbol{l}_n$ compatible with $\mathbf{s}$ means that $\vec{B}_0$ corresponds to $\{i > b : l_n^i \geq 1\}$. We then make the changes of variables in the natural scales to obtain

$$\begin{aligned}
\mathcal{Z}_1^{\mathbf{s}}(h,n) = 3^{b/2-g} 2^{|V'|/2-g/2-3b/4} n^{|V'|/2+g/2-b/4-1/2} \prod_{i>b: l_n^i \geq 1} P_{l_n^i}(0) \\
\times \int \mathrm{dL}_{\mathbf{s}} \prod_{e\in I} \left(\frac{8n}{9}\right)^{1/4} M_{\underline{\underline{h}}^e}(\delta\underline{\underline{\lambda}}^e) \prod_{e\in\vec{B}_+} \left(\frac{8n}{9}\right)^{1/4} P_{\underline{\underline{l}}^e}(\delta\underline{\underline{\lambda}}^e),
\end{aligned} \tag{B.5}$$

where

$$\begin{aligned}
\underline{\underline{h}}^e &= \lceil \sqrt{2n} h^e \rceil \text{ for } e \in I', \quad & \underline{\underline{h}}^{\epsilon_0} &= h - \sum_{e\in I'} \underline{\underline{h}}^e, \\
\underline{\underline{l}}^e &= \lceil \sqrt{2n} l^e \rceil \text{ for } e \in \vec{B}'_+, \quad & \underline{\underline{l}}^{\epsilon_i} &= l_n^i - \sum_{e\in\vec{B}'_i} \underline{\underline{l}}^e \text{ for } 1 \leq i \leq b, \\
\underline{\underline{\lambda}}^v &= \left\lceil \left(\frac{8n}{9}\right)^{1/4} \lambda^v \right\rceil \text{ for } v \in V', \quad & \underline{\underline{\lambda}}^{v_0} &= 0.
\end{aligned}$$

[1] This is just a convenience. Note that we set $M_0(0) = P_0(0) = 0$ here, although it would be more natural from a combinatorial point of view to set both these quantities to 1.

We finally use the same method to treat the summation over $h \in \mathbb{N}$ in (B.4), that is, we see it as an integral and do the proper change of variables. We write $\boldsymbol{l}_n \bowtie \mathbf{s}$ to mean "$\boldsymbol{l}_n$ compatible with $\mathbf{s}$":

$$\mathcal{Z}_1^{\mathbf{s}}(n) = \frac{\mathbf{1}_{\boldsymbol{l}_n \bowtie \mathbf{s}}}{|\vec{I}|+|\vec{B}_+|}\sqrt{\frac{2}{n}}\int_{\mathbb{R}_{\geq 0}} \mathrm{dh}\, n Q_{2\lceil\sqrt{2n}\mathrm{h}\rceil+\|\boldsymbol{l}_n\|}(2n+\|\boldsymbol{l}_n\|)\mathcal{Z}_1^{\mathbf{s}}(\lceil\sqrt{2n}\mathrm{h}\rceil, n). \quad \text{(B.6)}$$

Setting $h^{\epsilon_0} = \mathrm{h} - \sum_{e\in I'} h^e$, $l^{\epsilon_i} = L^i - \sum_{e\in\vec{B}_i'} l^e$ for $1 \leq i \leq b$, and $\lambda^{v_0} = 0$, by the local limit theorem [86, Theorem VII.1.6], it holds that, when $h \sim \sqrt{2n}\mathrm{h}$,

$$\left(\frac{8n}{9}\right)^{1/4} M_{\underline{\underline{h}}^e}(\delta\underline{\underline{\lambda}}^e) \underset{n\to\infty}{\longrightarrow} p_{\mathrm{h}^e}(\delta\lambda^e), \qquad \left(\frac{8n}{9}\right)^{1/4} P_{\underline{\underline{l}}^e}(\delta\underline{\underline{\lambda}}^e) \underset{n\to\infty}{\longrightarrow} p_{3\mathrm{l}^e}(\delta\lambda^e), \quad \text{(B.7)}$$

and

$$n Q_{2\lceil\sqrt{2n}\mathrm{h}\rceil+\|\boldsymbol{l}_n\|}(2n+\|\boldsymbol{l}_n\|) \underset{n\to\infty}{\longrightarrow} q_{2\mathrm{h}+\|\boldsymbol{L}\|}(1). \quad \text{(B.8)}$$

Consequently, provided the domination hypothesis obtained in the following paragraph, we get the following equivalent:

$$\mathcal{Z}_1^{\mathbf{s}}(n) \underset{n\to\infty}{\sim} c_1^{\mathbf{s}}(\boldsymbol{L})\, \mathbf{1}_{\boldsymbol{l}_n \bowtie \mathbf{s}}\, n^{|V'|/2+g/2-b/4-1} \prod_{i>b:\, l_n^i \geq 1} P_{l_n^i}(0), \quad \text{(B.9)}$$

where the constant $c_1^{\mathbf{s}}(\boldsymbol{L})$ is given by

$$\begin{aligned} c_1^{\mathbf{s}}(\boldsymbol{L}) = {} & \frac{1}{|\vec{I}|+2|\vec{B}_+|} 3^{b/2-g} 2^{|V'|/2-g/2-3b/4+1/2} \\ & \times \int_{\mathbb{R}_{\geq 0}} \mathrm{dh}\, q_{2\mathrm{h}+\|\boldsymbol{L}\|}(1) \int \mathrm{dL}_{\mathbf{s}} \prod_{e\in I} p_{\mathrm{h}^e}(\delta\lambda^e) \prod_{e\in\vec{B}_+} p_{3\mathrm{l}^e}(\delta\lambda^e). \end{aligned}$$

Domination hypothesis. In order to show that the convergence is dominated, we use the bounds of Petrov [86, Theorem VII.3.16], stating that there exists a constant C such that, for any $\ell \in \mathbb{N}$, $j \in \mathbb{Z}$, $i \in \mathbb{N}$, and $r \in \mathbb{N}$,

$$M_\ell(j) \vee P_\ell(j) \leq C\frac{1}{\sqrt{\ell}} \quad \text{and} \quad Q_i(\ell) \leq C\frac{i}{\ell^{3/2}}\frac{1}{1+(i^2/\ell)^r}. \quad \text{(B.10)}$$

We fix an arbitrary spanning tree of $\mathbf{s}$, that is, a tree with vertex-set V and edge-set a subset of E. We associate with any vertex $v \neq v_0$ the first edge of the unique path in the tree from v to v_0 and we denote by e_v the unique half-edge of $I \cup \vec{B}$ that corresponds to this edge.

We bound the integrand in (B.5) as follows. First, by (B.10), we have, for $e \in I'$,

$$\left(\frac{8n}{9}\right)^{1/4} M_{\underline{\underline{h}}^e}(\delta\underline{\underline{\lambda}}^e) \leq \left(\frac{8n}{9}\right)^{1/4}\frac{C}{\sqrt{\underline{\underline{h}}^e}} \leq \left(\frac{8n}{9}\right)^{1/4}\frac{C}{\sqrt{\sqrt{2n}\mathrm{h}^e}} = \sqrt{\frac{2}{3}}\frac{C}{\sqrt{\mathrm{h}^e}} \leq \frac{C}{\sqrt{\mathrm{h}^e}}.$$

For $h = \lceil \sqrt{2n}\mathsf{h} \rceil$ and $h^{\epsilon_0} = \mathsf{h} - \sum_{e \in I'} h^e$, a similar bound holds for $e = \epsilon_0$, up to possibly enlarging the constant. Indeed, it suffices to show that $\underline{\underline{h}}^{\epsilon_0}$ is bounded from below by a constant times $\sqrt{2n}h^{\epsilon_0}$ in order to complete the computation. We may assume that $\underline{\underline{h}}^{\epsilon_0} \geq 1$ as otherwise the left-hand side is null. Then, if $\sqrt{2n}h^{\epsilon_0} \leq 2|I'|$, it immediately holds that $\underline{\underline{h}}^{\epsilon_0} \geq \frac{1}{2|I'|}\sqrt{2n}h^{\epsilon_0}$. Otherwise,

$$\underline{\underline{h}}^{\epsilon_0} = h - \sum_{e \in I'} \underline{\underline{h}}^e \geq \sqrt{2n}h^{\epsilon_0} - |I'| \geq \frac{1}{2}\sqrt{2n}h^{\epsilon_0}.$$

In conclusion, up to changing the constant C, it holds that, for all $e \in I$,

$$\left(\frac{8n}{9}\right)^{1/4} M_{\underline{\underline{h}}^e}(\delta \underline{\underline{\lambda}}^e) \leq \frac{C}{\sqrt{h^e}}.$$

Similarly, up to enlarging the constant C even more, setting $l^{\epsilon_i} = L^i - \sum_{e \in \vec{B}'_i} l^e$ for $1 \leq i \leq b$, it holds that, for $e \in \vec{B}_+$,

$$\left(\frac{8n}{9}\right)^{1/4} P_{\underline{\underline{l}}^e}(\delta \underline{\underline{\lambda}}^e) \leq \frac{C}{\sqrt{l^e}}.$$

We use these bounds whenever $e \notin \vec{E}_V = \{e_v : v \in V \setminus \{v_0\}\}$ and then, we operate the integral with respect to $\mathrm{d}\boldsymbol{\lambda}^{V'}$ vertices by vertices, starting from a leaf of the fixed spanning tree, then from a leaf of the tree remaining after removing the first vertex, and so on until only v_0 remains. Since for any $\ell \in \mathbb{N}$,

$$\int \mathrm{d}x \left(\frac{8n}{9}\right)^{1/4} M_\ell\left(\left\lceil \left(\frac{8n}{9}\right)^{1/4} x \right\rceil\right) = 1,$$

and similarly with P_ℓ instead of M_ℓ, we obtain that, for n sufficiently large and after integration with respect to $\mathrm{d}\boldsymbol{\lambda}^{V'}$, the integrand in (B.5) is bounded by

$$\mathbf{1}_{\{\|\boldsymbol{h}\|_I \leq 2\mathsf{h}\}} \mathbf{1}_{\{\|\boldsymbol{l}\|_{\vec{B}'_+} \leq 2\|\boldsymbol{L}\|\}} \prod_{e \in I \setminus \vec{E}_V} \frac{C}{\sqrt{h^e}} \prod_{e \in \vec{B}_+ \setminus \vec{E}_V} \frac{C}{\sqrt{l^e}}. \tag{B.11}$$

This is integrable with respect to $\mathrm{d}\boldsymbol{h}^{I'} \otimes \mathrm{d}\boldsymbol{l}^{\vec{B}'_+}$ and is bounded, after integration, by some constant times some power of h. Taking r sufficiently large in (B.10) yields that this quantity multiplied by $nQ_{2\lceil\sqrt{2n}\mathsf{h}\rceil + \|\boldsymbol{l}_n\|}(2n + \|\boldsymbol{l}_n\|)$ is integrable with respect to $\mathrm{d}\mathsf{h}$. The claimed dominated convergence follows.

Dominant schemes. We will now see which schemes are such that $\mathcal{Z}_1^{\mathbf{s}}(n)$ has the highest possible order in n. The exponent of n in the equivalent (B.9) is maximal when $|V(\mathbf{s})|$ is the largest; in this case,

$$|V(\mathbf{s})| = 2(2g + k - 1).$$

This equality is obtained as in the proof of Lemma 2.1, since $|V(\mathbf{s})|$ being the largest means that the vertices have the lowest possible degrees, namely 3 for the internal vertices and 1 for the external vertices. More precisely, denoting by v, e, f the numbers of vertices, edges and faces of $\mathbf{s}$, as well as t the number of tadpoles among $\mathscr{h}_{b+1}, \dots, \mathscr{h}_k$, we obtain

$$f = b + t + 1, \quad 2e = 3(v - k + b + t) + k - b - t,$$

and the result from the Euler characteristic formula

$$v - e + f = 2 - 2g.$$

Next, the local limit theorem [86, Theorem VII.1.6] yields the existence of a compact set $K \subset (0, \infty)$ such that, for all $\ell \in \mathbb{N}$, $\sqrt{\ell} P_\ell(0) \in K$. Finally, for any $\mathbf{s} \in \vec{\mathbf{S}}$, we denote by $\mathbf{s}^\circ$ the scheme obtained by shrinking every tadpole among $\mathscr{h}_{b+1}, \dots, \mathscr{h}_k$ into a vertex. For any fixed dominant scheme $\mathbf{d} \in \vec{\mathbf{S}}^\star$, observe that there exists exactly one scheme among $\{\mathbf{s} \in \vec{\mathbf{S}} : \mathbf{s}^\circ = \mathbf{d}\}$ that is compatible with $\boldsymbol{l}_n$, namely the one whose tadpoles among $\mathscr{h}_{b+1}, \dots, \mathscr{h}_k$ are the holes indexed by $\{i > b : l_n^i \geq 1\}$. Furthermore, if $\mathbf{s} \in \vec{\mathbf{S}}$ is such that $\mathbf{s}^\circ \in \vec{\mathbf{S}}^\star$, then the external faces among $\mathscr{h}_i, b+1 \leq i \leq k$, of $\mathbf{s}$ are all tadpoles. We may thus use the equivalent (B.9) for these schemes. Consequently, as $n \to \infty$,

$$\sum_{\substack{\mathbf{s} \in \vec{\mathbf{S}} \\ \mathbf{s}^\circ = \mathbf{d}}} \mathcal{Z}_1^{\mathbf{s}}(n) = \Theta\Big(n^{5(g-1)/2 + k - b/4} \prod_{i > b : l_n^i \geq 1} (l_n^i)^{-1/2}\Big). \tag{B.12}$$

In particular, if $\mathbf{s}$ has only tadpoles among its external faces indexed by $b + 1 \leq i \leq k$ but is such that $\mathbf{s}^\circ$ is not dominant, then $\mathcal{Z}_1^{\mathbf{s}}(n)$ is negligible with respect to this sum.

Nondominant schemes. We will now see that the above is the highest order in n and that it is only obtained for the schemes that are dominant after the tadpoles shrinkage. To this end, we fix an arbitrary scheme $\mathbf{s} \in \vec{\mathbf{S}}$. As above, we consider an arbitrary spanning tree of $\mathbf{s}$ and still denote by $e_v \in I \cup \vec{B}$ the half-edge corresponding to $v \in V'$, as well as $\vec{E}_V = \{e_v : v \in V'\}$.

In (B.2), we bound $M_{h^e}(\delta\lambda^e)$ or $P_{l^e}(\delta\lambda^e)$ thanks to (B.10) if $e \notin \vec{E}_V$, and we operate the sum over $\boldsymbol{\lambda}^{V'}$ leaf by leaf as we did in the previous paragraph. Since for any $\ell \in \mathbb{N}$,

$$\sum_{j \in \mathbb{Z}} M_\ell(j) = \sum_{j \in \mathbb{Z}} P_\ell(j) = 1,$$

we obtain the bound

$$\mathcal{Z}_1^{\mathbf{s}}(h, n) \leq \sum_{\boldsymbol{h}^{\vec{I}}, \boldsymbol{l}^{\vec{B}}} \prod_{e \in I \setminus \vec{E}_V} \frac{C}{\sqrt{h^e}} \prod_{e \in \vec{B} \setminus \vec{E}_V} \frac{C}{\sqrt{l^e}},$$

where the sum is over the tuples $\boldsymbol{h}^{\vec{I}}, \boldsymbol{l}^{\vec{B}}$ satisfying the conditions of $\mathcal{T}_{\mathbf{s}}(h,n)$. Seeing the sums as integrals under the simplex Lebesgue measures $\Delta_I^h\, \Delta_{\vec{B}_i}^{l_n^i}$ whenever $\vec{B}_i \neq \varnothing$ yields integrals of Dirichlet distributions (with parameter vectors containing only $1/2$'s and 1's), after renormalization by h or l_n^i. As a result,

$$\begin{aligned} \mathcal{Z}_1^{\mathbf{s}}(h,n) &\lesssim h^{|I|-1-|I\setminus\vec{E}_V|/2} \prod_{\substack{1\le i\le k\\ \vec{B}_i\neq\varnothing}} (l_n^i)^{|\vec{B}_i|-1-|\vec{B}_i\setminus\vec{E}_V|/2} \\ &= h^{|I|/2+|I\cap\vec{E}_V|/2-1} \prod_{\substack{1\le i\le k\\ \vec{B}_i\neq\varnothing}} (l_n^i)^{|\vec{B}_i|/2+|\vec{B}_i\cap\vec{E}_V|/2-1}, \end{aligned}$$

where we used the symbol $\lesssim$ to mean bounded up to a constant independent[2] of $\mathbf{s}$, h, and n. Since l_n^i is in the scale $\sqrt{n}$ for $i \le b$, the part of the product concerning these indices is bounded by a constant times

$$n^{|\vec{B}'_+|/4+|\vec{B}_+\cap\vec{E}_V|/4-b/4}.$$

Recall that $B_i \neq \varnothing \Leftrightarrow l_n^i \geq 1$ when $\boldsymbol{l}_n$ is compatible with $\mathbf{s}$. Using (B.6), which is valid for any scheme, as well as the bound (B.10) as above to get integrability, we obtain

$$\begin{aligned} \mathcal{Z}_1^{\mathbf{s}}(n) &\lesssim \mathbf{1}_{\boldsymbol{l}_n \bowtie \mathbf{s}} n^{(|I|+|\vec{B}'_+|)/4+|(I\cup\vec{B}_+)\cap\vec{E}_V|/4-b/4-1} \prod_{i>b:l_n^i\geq 1} (l_n^i)^{|\vec{B}_i|/2+|\vec{B}_i\cap\vec{E}_V|/2-1} \\ &\lesssim \mathbf{1}_{\boldsymbol{l}_n \bowtie \mathbf{s}} n^{(|I|+|\vec{B}'_+|+|V'|)/4-b/4-1} \prod_{i>b:l_n^i\geq 1} (l_n^i)^{|\vec{B}_i|/2-1}, \end{aligned}$$

since $l_n^i = \mathcal{O}(\sqrt{n})$ for all i, and $|(I\cup\vec{B}_+\cup\vec{B}_0)\cap\vec{E}_V| = |V'|$. Using again the Euler characteristic formula, as well as the bound $|V| \le 2(2g+k-1)$, we obtain

$$\mathcal{Z}_1^{\mathbf{s}}(n) \lesssim \mathbf{1}_{\boldsymbol{l}_n \bowtie \mathbf{s}}\, n^{5(g-1)/2+k-b/4} \prod_{i>b:l_n^i\geq 1} (l_n^i)^{-1/2} \Big(\frac{l_n^i}{\sqrt{n}}\Big)^{(|\vec{B}_i|-1)/2},$$

which gives an order lower than that of (B.12) as soon as there exists $i > b$ such that $|\vec{B}_i| \geq 2$ since $l_n^i = o(\sqrt{n})$. As a result, the normalization constant $\mathcal{Z}_1(n)$ is of the order appearing in (B.12) and $\mathcal{Z}_1^{\mathbf{s}}(n) = o(\mathcal{Z}_1(n))$ whenever $\mathbf{s}^\circ \notin \vec{\mathbf{S}}^\star$. In particular, $\mathbb{P}(S_n^\circ \in \vec{\mathbf{S}}^\star) \to 1$ as $n \to \infty$ and we obtain the first statement of Proposition 3.10: with asymptotic probability 1, every vanishing face of M_n induces a tadpole in S_n.

[2]Recall from Lemma 2.1 that the number of edges in the schemes from $\vec{\mathbf{S}}$ is bounded.

B.3 Asymptotics of the size parameters

Limiting distribution of the size parameters. Given a bounded continuous function ϕ on the set

$$\bigcup_{\mathbf{s}\in\vec{\mathbf{S}}:\mathbf{s}^\circ=\mathbf{s}} \{\mathbf{s}\}\times(\mathbb{R}_{\geq 0})^{\vec{I}(\mathbf{s})}\times(\mathbb{R}_{\geq 0})^{\vec{B}(\mathbf{s})}\times\mathbb{R}^{V(\mathbf{s})},$$

we set, for $n,h\in\mathbb{N},\mathbf{s}\in\vec{\mathbf{S}}$,

$$\mathcal{Z}_\phi^{\mathbf{s}}(h,n)=\sum_{\mathcal{T}_{\mathbf{s}}(h,n)}\phi\Big(\mathbf{s}^\circ,\frac{\boldsymbol{h}^{\vec{I}(\mathbf{s}^\circ)}}{\sqrt{2n}},\frac{\boldsymbol{l}^{\vec{B}(\mathbf{s}^\circ)}}{\sqrt{2n}},\frac{\boldsymbol{\lambda}^{V(\mathbf{s}^\circ)}}{(8n/9)^{1/4}}\Big)\prod_{e\in I(\mathbf{s})}M_{h^e}(\delta\lambda^e)\prod_{e\in\vec{B}(\mathbf{s})}P_{l^e}(\delta\lambda^e),$$

and

$$\mathcal{Z}_\phi^{\mathbf{s}}(n)=\frac{1}{|\vec{I}(\mathbf{s})|+2|\vec{B}_+(\mathbf{s})|}\sum_{h\in\mathbb{N}}Q_{2h+\|\boldsymbol{l}_n\|}(2n+\|\boldsymbol{l}_n\|)\mathcal{Z}_\phi^{\mathbf{s}}(h,n),$$

so that

$$\mathbb{E}\Big[\phi\Big(S_n^\circ,\frac{\mathbf{H}_n^{\vec{I}(S_n^\circ)}}{\sqrt{2n}},\frac{\boldsymbol{L}_n^{\vec{B}(S_n^\circ)}}{\sqrt{2n}},\frac{\boldsymbol{\Lambda}_n^{V(S_n^\circ)}}{(8n/9)^{1/4}}\Big)\Big]=\frac{1}{\mathcal{Z}_1(n)}\sum_{\mathbf{s}\in\vec{\mathbf{S}}}\mathcal{Z}_\phi^{\mathbf{s}}(n).$$

Conducting with $\mathcal{Z}_\phi^{\mathbf{s}}(n)$ exactly the same computations as the ones we did with $\mathcal{Z}_1^{\mathbf{s}}(n)$, we obtain the same domination (up to $\sup|\phi|$) when $\mathbf{s}^\circ$ is not dominant and a similar equivalent when $\mathbf{s}^\circ$ is dominant, namely (B.9), where $c_1^{\mathbf{s}}(\boldsymbol{L})$ is replaced by

$$c_\phi^{\mathbf{s}}(\boldsymbol{L})=\frac{1}{|\vec{I}(\mathbf{s})|+2|\vec{B}_+(\mathbf{s})|}3^{b/2-g}2^{|V'(\mathbf{s})|/2-g/2-3b/4+1/2}\int_{\mathbb{R}_{\geq 0}}\mathrm{dh}\, q_{2\mathrm{h}+\|\boldsymbol{L}\|}(1)$$
$$\times\int \mathrm{dL}_{\mathbf{s}}\,\phi(\mathbf{s}^\circ,\boldsymbol{h}^{\vec{I}(\mathbf{s}^\circ)},\boldsymbol{l}^{\vec{B}(\mathbf{s}^\circ)},\boldsymbol{\lambda}^{V(\mathbf{s}^\circ)})\prod_{e\in I(\mathbf{s})}p_{h^e}(\delta\lambda^e)\prod_{e\in\vec{B}_+(\mathbf{s})}p_{3l^e}(\delta\lambda^e).$$

From the Euler characteristic formula, we obtain $|\vec{I}(\mathbf{s})|+2|\vec{B}_+(\mathbf{s})|=2|E(\mathbf{s}^\circ)|=2(6g+2p+b-3)$ does not depend on $\mathbf{s}$, and we remind that $|V'(\mathbf{s})|=4g+2p-3$ does not either. Let us consider a dominant scheme $\mathbf{d}\in\vec{\mathbf{S}}^\star$ and an integer $n\in\mathbb{N}$. We let $\mathbf{d}_n\in\vec{\mathbf{S}}$ be the unique scheme compatible with $\boldsymbol{l}_n$ and such that $\mathbf{d}_n^\circ=\mathbf{d}$. Recall that this is the scheme obtained from $\mathbf{d}$ by making into tadpoles the external vertices indexed by $\{i>b:l_n^i\geq 1\}$. Since the above integral only involves $\mathbf{s}^\circ$, we have $c_\phi^{\mathbf{d}_n}(\boldsymbol{L})=c_\phi^{\mathbf{d}}(\boldsymbol{L})$, and then

$$\mathcal{Z}_1(n)=\sum_{\mathbf{s}\in\vec{\mathbf{S}}}\mathcal{Z}_1^{\mathbf{s}}(n)\sim\sum_{\mathbf{d}\in\vec{\mathbf{S}}^\star}\mathcal{Z}_1^{\mathbf{d}_n}(n)\sim n^{5(g-1)/2+k-b/4}\prod_{i>b:l_n^i\geq 1}P_{l_n^i}(0)\sum_{\mathbf{d}\in\vec{\mathbf{S}}^\star}c_1^{\mathbf{d}}(\boldsymbol{L}),$$

and

$$\mathbb{E}\Big[\phi\Big(S_n^\circ, \frac{\mathbf{H}_n^{\vec{I}(S_n^\circ)}}{\sqrt{2n}}, \frac{\boldsymbol{L}_n^{\vec{B}(S_n^\circ)}}{\sqrt{2n}}, \frac{\boldsymbol{\Lambda}_n^{V(S_n^\circ)}}{(8n/9)^{1/4}}\Big)\Big] \underset{n\to\infty}{\longrightarrow} \frac{1}{\sum_{\mathbf{d}\in\vec{\mathbf{S}}^\star} c_1^{\mathbf{d}}(\boldsymbol{L})} \sum_{\mathbf{d}\in\vec{\mathbf{S}}^\star} c_\phi^{\mathbf{d}}(\boldsymbol{L}). \quad \text{(B.13)}$$

In passing, we obtain the following asymptotic formula for the cardinality of $\vec{\mathbf{Q}}^{[g]}_{n,\boldsymbol{l}_n}$, which readily yields Proposition 1.4 (corresponding to the case $k=b$), the unrooting giving a factor $1/4n$ coming from (1.2). We define the continuous function in $\boldsymbol{L}\in(0,\infty)^b$,

$$t_g(\boldsymbol{L}) = \sum_{\mathbf{d}\in\vec{\mathbf{S}}^\star} c_1^{\mathbf{d}}(\boldsymbol{L}),$$

and we add the excluded cases $(g,k)=(0,0)$ and $(g,k,b)=(0,1,1)$, which are needed in Section 1.5: we set

$$t_0(\varnothing) = \frac{1}{2\sqrt{\pi}} \quad \text{and} \quad t_0(L) = \frac{2^{-9/4}}{\pi\sqrt{L}}\exp\Big(-\frac{L^2}{2}\Big).$$

Proposition B.1. *As $n\to\infty$, it holds that*

$$|\vec{\mathbf{Q}}^{[g]}_{n,\boldsymbol{l}_n}| \sim 4t_g(\boldsymbol{L})12^n 8^{\|\boldsymbol{l}_n\|} n^{5(g-1)/2+k-b/4}\frac{e^\star}{e^\star+p_n^\diamondsuit}\prod_{i>b:l_n^i\ge 1} P_{l_n^i}(0),$$

where $e^\star = 6g+2p+b-3$ is the common number of edges of all dominant schemes, and $p_n^\diamondsuit = |\{i>b : l_n^i \ge 1\}|$ is the number of external faces among $\hbar_{b+1},\dots,\hbar_k$ in the maps of $\mathbf{M}^{[g]}_{n,\boldsymbol{l}_n}$.

In the excluded cases $(g,k)=(0,0)$ and $(g,k,b)=(0,1,1)$, a similar formula holds:

$$|\vec{\mathbf{Q}}^{[0]}_{n,\varnothing}| \sim 4t_0(\varnothing)12^n n^{-5/2} \quad \textit{and} \quad |\vec{\mathbf{Q}}^{[0]}_{n,(l_n)}| \sim 4t_0(L)12^n 8^{l_n} n^{-7/4}$$

for $L>0$ and $l_n\sim\sqrt{2n}L$ as $n\to\infty$.

Proof. Recall from Section 2.2 that $\vec{\mathbf{M}}^{[g]}_{n,\boldsymbol{l}_n}$ is in 1-to-2 correspondence with $\vec{\mathbf{Q}}^{[g]}_{n,\boldsymbol{l}_n 0}$, and that (B.3) gives its cardinality. Using (1.3) then (B.3) and (B.4), we obtain that

$$\begin{aligned}
|\vec{\mathbf{Q}}^{[g]}_{n,\boldsymbol{l}_n}| &= \frac{2}{n+\|\boldsymbol{l}_n\|+2-2g-k}|\vec{\mathbf{M}}^{[g]}_{n,\boldsymbol{l}_n}| \\
&= 2\frac{2n+\|\boldsymbol{l}_n\|}{n+\|\boldsymbol{l}_n\|+2-2g-k}12^n 8^{\|\boldsymbol{l}_n\|}\sum_{\mathbf{s}\in\vec{\mathbf{S}}}\frac{|\vec{I}(\mathbf{s})|+2|\vec{B}_+(\mathbf{s})|}{2|E(\mathbf{s})|}\mathcal{Z}_1^{\mathbf{s}}(n) \\
&\sim 4\times 12^n 8^{\|\boldsymbol{l}_n\|}\sum_{\mathbf{d}\in\vec{\mathbf{S}}^\star}\frac{|E(\mathbf{d})|}{|E(\mathbf{d}_n)|}\mathcal{Z}_1^{\mathbf{d}_n}(n) \\
&\sim 4\times 12^n 8^{\|\boldsymbol{l}_n\|} n^{5(g-1)/2+k-b/4}\frac{e^\star}{e^\star+p_n^\diamondsuit}\prod_{i>b:l_n^i\ge 1}P_{l_n^i}(0)\sum_{\mathbf{d}\in\vec{\mathbf{S}}^\star}c_1^{\mathbf{d}}(\boldsymbol{L}),
\end{aligned}$$

which gives the desired first statement.

The excluded cases $(g, k) = (0, 0)$ and $(g, k, b) = (0, 1, 1)$ are standard; they are obtained similarly, by computing $|\vec{\mathbf{M}}^{[0]}_{n,\varnothing}|$ and $|\vec{\mathbf{M}}^{[0]}_{n,(l_n)}|$. More precisely, it is well known that

$$|\vec{\mathbf{M}}^{[0]}_{n,\varnothing}| = 3^n \frac{(2n)!}{n!(n+1)!} \sim \frac{12^n}{\sqrt{\pi}} n^{-3/2}.$$

In order to compute the remaining cardinality, we proceed as in Appendix B.1 and obtain

$$|\vec{\mathbf{M}}^{[0]}_{n,(l_n)}| = \frac{2n + l_n}{l_n} 12^n 8^{l_n} Q_{l_n}(2n + l_n) P_{l_n}(0),$$

the division by l_n taking into account the fact that seeing an element of $\vec{\mathbf{M}}^{[0]}_{n,(l_n)}$ as a forest amounts to choose a first tree among l_n. From equivalents (B.7) and (B.8), this yields

$$|\vec{\mathbf{Q}}^{[0]}_{n,(l_n)}| \sim \frac{2}{n} \frac{2n}{\sqrt{2nL}} 12^n 8^{l_n} \frac{1}{n} q_L(1) \left(\frac{8n}{9}\right)^{-1/4} p_{3L}(0),$$

which gives the desired result. ∎

Limiting distribution of the areas. We finally take into account the areas. To this end, observe that, conditionally given

$$(S_n, \mathbf{H}_n^{\vec{I}(S_n)}, \boldsymbol{L}_n^{\vec{B}(S_n)}),$$

the area vector $\mathbf{A}_n^{\vec{E}(S_n)}$ is distributed as follows. We arrange the half-edges $e_1, \ldots, e_\kappa$ incident to the internal face of S_n according to the contour order, starting arbitrarily, and let $x_i = \sum_{j=1}^{i} \ell_j$, where $\ell_j = H_n^{e_j}$ if $e_j \in \vec{I}(S_n)$ or $\ell_j = L_n^{e_j}$ if $e_j \in \vec{B}(S_n)$. Then, $A_n^{e_1}$, $A_n^{e_1} + A_n^{e_2}$, $A_n^{e_1} + A_n^{e_2} + A_n^{e_3}, \ldots$ are distributed as the hitting times of the successive levels $-x_1, -x_1 - x_2, -x_1 - x_2 - x_3, \ldots$ by a simple random walk conditioned on hitting the final level $-\sum_{j=1}^{\kappa} x_j = -\|\mathbf{H}_n\| - \|\boldsymbol{L}_n\|$ at time $2n + \|\boldsymbol{l}_n\|$. The desired convergence (3.9) easily follows from this together with (B.13), as well as the fact that, for every $e \in \vec{B}_0(S_n)$, we have $A_n^e + L_n^e = \Theta((L_n^e)^2)$ in probability.

B.4 Boltzmann quadrangulations

We finally prove Theorem 1.5; in its setting,

$$\begin{aligned}
&\mathcal{W}(F(\Omega_{a^{-1}}(Q))\mathbf{1}_{\mathbf{Q}^{[g]}_{\boldsymbol{l}_a \mathbf{0}^p}}) \\
&\quad = \sum_{n \in [a^{-1}/K, a^{-1}K] \cap \mathbb{Z}_{\geq 0}} \mathcal{W}(\mathbf{Q}^{[g]}_{n,\boldsymbol{l}_a \mathbf{0}^p}) \mathcal{W}[F(\Omega_{a^{-1}}(Q)) \mid \mathbf{Q}^{[g]}_{n,\boldsymbol{l}_a \mathbf{0}^p}] \\
&\quad = a^{-1} \int_{a\lfloor a^{-1}/K \rfloor}^{a\lfloor a^{-1}K \rfloor} \mathrm{d}A \, \mathcal{W}(\mathbf{Q}^{[g]}_{\lfloor A/a \rfloor,\boldsymbol{l}_a \mathbf{0}^p}) \mathcal{W}[F(\Omega_{a^{-1}}(Q)) \mid \mathbf{Q}^{[g]}_{\lfloor A/a \rfloor,\boldsymbol{l}_a \mathbf{0}^p}].
\end{aligned}$$

By Theorem 1.1 and the definition of $\mathsf{S}^{[g]}_{A,\boldsymbol{L}}$, it holds that

$$\mathcal{W}[F(\Omega_{a^{-1}}(Q)) \mid \mathbf{Q}^{[g]}_{\lfloor A/a \rfloor, \boldsymbol{l}_a \mathbf{0}^p}] \underset{a\downarrow 0}{\longrightarrow} \mathbb{E}[F(\mathsf{S}^{[g]}_{A,\boldsymbol{L}\mathbf{0}^p})],$$

while Proposition 1.4 yields that

$$\mathcal{W}(\mathbf{Q}^{[g]}_{\lfloor A/a \rfloor, \boldsymbol{l}_a \mathbf{0}^p}) \underset{a\downarrow 0}{\sim} t_g\Big(\frac{\boldsymbol{L}}{\sqrt{A}}\Big)\Big(\frac{A}{a}\Big)^{(5g-7)/2+3b/4+p}.$$

Hence, Theorem 1.5 will be proved if we can show that the convergence in the last integral expression is dominated. However, this is a direct consequence of the discussion of the domination hypothesis around (B.11). Corollary 1.6 is proved in a very similar way, this time summing over all possible values of the perimeters, which results in the integral with respect to $\mathrm{d}\boldsymbol{L}$ on $(0,\infty)^b$.