Preface

The main topic of the book is how geometric–isoperimetric-type inequalities intertwine with functional inequalities such as the Prékopa–Leindler inequality, the Sobolev inequality, Poincaré inequality, on the one hand, and also with the uniqueness of the solution of certain Minkowski-type Monge–Ampère equations on the sphere, on the other hand. The classical setup, due to Minkowski around 1900, is that the characterization of the equality case in the Brunn–Minkowski inequality for convex bodies (cf. Theorem 1.12.3) yields the uniqueness result about the Minkowski problem (cf. Section 9.2), and the similar pattern is followed in the case of Firey's L_p-Brunn–Minkowski inequality and Lutwak's L_p-Minkowski problem for $p > 1$ (cf. Section 9.4), or in the case of capacity and torsional rigidity (cf. Section 9.5.7). On the other hand, the pattern is reversed in the case of the L_p-Brunn–Minkowski and L_p-Minkowski conjectures when $p < 1$ is close to 1. In this case, first the uniqueness of the solution of the even L_p-Minkowski problem had been recently established using spectral gap estimates à la Hilbert and the continuity method from PDE, and this uniqueness result in turn yields the L_p-Brunn–Minkowski inequality (cf. Section 9.4). In some sense, the first eight chapters provide the necessary background for Chapter 9 – and this fact explains the length of this book.

It is a mission of this work to clarify the relation between two frequently but sometimes carelessly used notions of surface area of bodies in $\mathbb{R}^n$: on the one hand, the $(n-1)$-dimensional Hausdorff measure of the boundary, and on the other hand, the outer Minkowski content based on the parallel domain (see Chapter 4 and Section 5.4). The advantage of the first approach is that it fits nicely the coarea formula, while the second approach suits the use of measure concentration and the Brunn–Minkowski inequality.

We introduce various tools useful in related problems. For example, we discuss proofs of the Brunn–Minkowski inequality in $\mathbb{R}^n$ (a generalization of the isoperimetric inequality) using symmetrization (à la Steiner) in Section 1.10, parametrization of "height" (à la Brunn and Minkowski) in Section 1.12, combinatorial ideas (à la Hadwiger and Ohmann) in Section 3.2, optimal transport (à la Knothe and Gromov) as a special case of the Prékopa–Leindler inequality in Section 3.4, and spectral theory (à la Hilbert and Aleksandrov) in Section 8.5.2. A close to optimal stability version of the Brunn–Minkowski inequality for convex bodies in terms of volume difference is also presented in Section 8.6.

The book has various readings. It discusses

- a basic introduction into convexity in Chapters 1 and 2, which is appended by the more advanced theory of associated ellipsoids and the Blaschke–Santaló inequality in Chapter 6;

- a survey into the theory of sets of finite perimeter and functions of bounded variation in $\mathbb{R}^n$ in Chapter 5;
- the isoperimetric and anisotropic isoperimetric inequality from various angles, including the case of convex bodies in Section 2.4 together with a stability version in Section 8.6, the case of bodies with Lipschitz boundary in the Euclidean, hyperbolic, and spherical spaces, or with respect to a log-concave density in Chapter 4, and the most general case of sets of finite perimeter in $\mathbb{R}^n$ in Chapter 5;
- functional versions of geometric inequalities, including the Prékopa–Leindler inequality in Section 3.4 and the anisotropic Sobolev inequality in Sections 4.2, 4.3 and 5.3 as versions of the Brunn–Minkowski inequality, the log-Sobolev inequality in Section 4.6.1 derived from the Gaussian isoperimetric inequality, or the functional Santaló inequality and its reverse forms as functional versions of the Blaschke–Santaló inequality and its reverse forms in Chapter 6;
- the classical theory of mixed volumes and the corresponding Minkowski and Aleksandrov–Fenchel inequalities (stemming from the Brunn–Minkowski inequality) in Chapter 7 (in terms of polytopes) and in Chapter 8 (in terms of smooth convex bodies);
- the interplay between Brunn–Minkowski-type inequalities in $\mathbb{R}^n$ and Monge–Ampère equations on S^{n-1} within Lutwak's L_p-Minkowski theory in Sections 7.6, 8.7 and 8.8, and in Chapter 9.

We have made an effort to ensure that the chapters can be read independently, quoting the relevant results from the other chapters where necessary. The diagram below explains the direct interrelations among chapters.

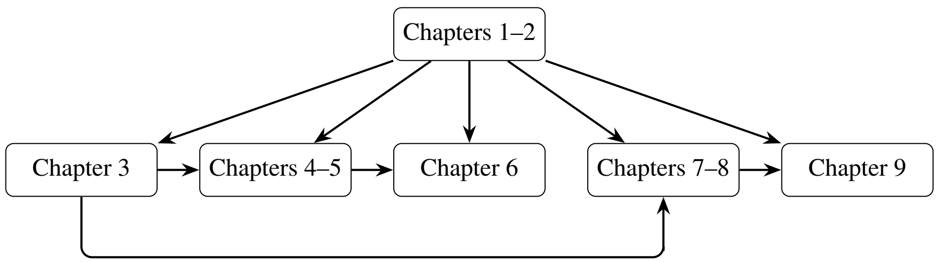

Figure 1. Chapter flow.

As we explained above, Chapters 1 and 2 contain the fundamental material about convex sets in $\mathbb{R}^n$, but these chapters can be skipped if somebody is already familiar with the basics.

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