Chapter 1

Introduction

The last 40 years have seen spectacular advances in symplectic topology. Most of them, especially those exhibiting aspects of symplectic rigidity, exploit algebraic structures that encode the behavior of moduli spaces of solutions of Cauchy–Riemann-type equations associated to (variants of) the symplectic action functional. Typical examples of such structures are Floer homology, Gromov–Witten invariants, and the Fukaya category. Given that these structures are, in essence, associated to a functional, they can be expected to admit refinements endowed with a finer structure reflective of an underlying filtration. Making this statement precise and incorporating this filtration in the respective algebraic structures is sometimes straightforward from the algebraic viewpoint. For instance, in favorable cases, the Floer complex – just like its more down-to-earth precursor, the Morse complex – is filtered, and its homology is a persistence module.

In other cases, such as that of the derived Fukaya category, which is the one that interests us here, this is far from immediate. In this memoir, we set up a new algebraic structure, called a *triangulated persistence category* (TPC), precisely to deal with this situation. This structure puts together persistence and triangulation and is a refinement of the notion of triangulated category. The construction is abstract and applicable to a variety of contexts unrelated to symplectic topology, as explained in more detail below, in Section 1.1.

The derived Fukaya category has a triangulated structure, and we show that, under certain constraints, it does admit a TPC refinement that is unique up to equivalence. The construction of this refinement and its uniqueness are delicate and require some novel geometric and algebraic steps. We describe in more detail the results and the constructions involved in Section 1.2. These constructions are also of independent interest.

A natural application of the construction of Fukaya-type triangulated persistence categories is a rigidity result for spaces of Lagrangian submanifolds. To fix ideas, let (X, ω) be a symplectic manifold. It is well known since the pioneering work of Gromov and Floer that closed Lagrangian submanifolds $L \subset X$ subject to certain *purely topological* constraints – the one used in this memoir is exactness – exhibit strong, and often surprising, rigidity properties that are intrinsically symplectic. Generally, this form of rigidity reflects individual properties of each of the Lagrangians in the fixed class. Two famous examples that have structured much of the modern work in the subject are the Arnold conjecture and the nearby Lagrangian conjecture, also due to Arnold.

In this work, we show that, in the same setting, a global form of rigidity is in effect. More precisely, let (X, ω) be a Liouville manifold that satisfies an algebraic finiteness condition that will be made explicit below. The set of closed, exact Lagrangians in X is endowed with a class of metrics, called *symplectic fragmentation* metrics, with some remarkable properties (see Corollary 3.7 for a more precise version and details):

- Up to a multiplicative constant, these metrics are dominated by the spectral metric (which is itself bounded above by the Hofer metric); thus, they carry symplectic content.
- The non-degeneracy of these metrics can be viewed as a form of Gromov's non-squeezing theorem, in the sense that the distance between two Lagrangians has a lower bound that can be expressed in terms of a purely geometric quantity: the supremum of the radii of standard symplectic balls that embed in a certain position relative to the two Lagrangians.
- The metrics are finite and thus they allow meaningful comparison of Lagrangians that are very different as smooth submanifolds (non-isotopic, or of different homotopy types), when classical metrics, such as the Hofer distance, are infinite.
- At the same time, they also satisfy a stability property for intersections, in the sense that, given two transverse Lagrangians L and N, if a third Lagrangian L' is sufficiently close to L in one of these metrics, then the number of intersections of L' with N cannot be smaller than the number of intersections of N with L.

The relation between this statement and the notion of TPC is that if a triangulated category admits a TPC refinement, then, by the main algebraic result in this memoir, its exact triangles are endowed with a so-called *persistence* triangular weight. The set of objects of a triangulated category, endowed with such a triangular weight, is easily seen to carry a family of natural pseudometrics called *fragmentation* pseudometrics. The symplectic fragmentation metric mentioned above is deduced from the fragmentation pseudometrics associated to the Fukaya TPC.

Remark 1.1. Precursors of the metrics introduced here have appeared in [10], based on Lagrangian cobordism machinery. However, the constructions in that paper lacked the proper algebraic setting, with the consequence that the finiteness of the distance between two Lagrangians depended on the existence of certain Lagrangian cobordisms. This issue was addressed, in part, in [9] through considerations involving immersed Lagrangians, which allow the construction of an abundance of immersed cobordisms. However, the immersed cobordism approach is technically very delicate, and less natural than the one proposed here, with the consequence that it leads to a family of metrics that are extremely hard to estimate.

1.1 Persistence and triangulation

Persistence theory, introduced in several pioneering works [14, 18, 23, 29, 33, 42, 62, 64], is an abstract framework that emerged from investigations in parts of data science as well as in topology, formalizing the structure and properties of a class of phenomena that are most easily seen in the homology of a chain complex (C, d) endowed with an increasing filtration $(C^{\leq \alpha}, d) \subset (C, d)$ of subcomplexes parametrized by $\alpha \in \mathbb{R}$. The homologies of the subcomplexes form a family $\{H(C^{\leq \alpha})\}_{\alpha \in \mathbb{R}}$ whose members are related by maps $i_{\alpha,\beta} : H(C^{\leq \alpha}) \to H(C^{\leq \beta})$, $\alpha \leq \beta$, subject to obvious compatibilities. This is an example of a persistence module. Given two filtered complexes (C, d) and (D, d) that are quasi-isomorphic, it is possible to compare them by the so-called interleaving distance. Its definition is based on the fact that the space of linear maps $v : C \to D$ is itself filtered by the "shift" of a map: v is of shift at most r if $v(C^{\leq \alpha}) \subset D^{\leq \alpha + r}$ for all $\alpha \in \mathbb{R}$. Using this, given two chain maps $\phi : C \to D$, $\psi : D \to C$ such that $\psi \circ \phi$ is chain homotopic to $\mathbb{1}_C$, there is a natural measurement for how far the composition $\psi \circ \phi$ is from the identity, namely the infimum of the "shifts" of chain homotopies $h : C \to C$ such that $dh + hd = \psi \circ \phi - \mathbb{1}_C$. The machinery of persistence modules is much more developed than the few elements mentioned here. For a survey of this topic and its applications in various branches of mathematics, see the monographs and papers by Edelsbrunner [28], Oudot [46], Chazal–de Silva–Glisse–Oudot [17], Polterovich–Shelukhin [48], Polterovich–Rosen–Samvelyan–Zhang [47], and Kislev–Shelukhin [41]. In particular, there is a beautiful interpretation of the bottleneck distance in terms of so-called barcodes [5, 59].

The main question that we address in the algebraic part of this memoir is independent of symplectic considerations:

How can one use a persistence-type structure on the morphisms of a category to compare not only (quasi-)isomorphic objects but rather define a pseudometric on the set of all objects?

We provide here a solution to this question based on mixing persistence with triangulation understood in the sense of triangulated categories as introduced by Puppe [50] and Verdier [60] in the early 1960s. Given a triangulated category $\mathcal{D}$, there is a simple notion of *triangular weight* w on $\mathcal{D}$ that we introduce in Section 2.1. This associates to each exact triangle Δ in $\mathcal{D}$ a non-negative number $w(\Delta)$ satisfying a couple of properties. The most relevant of them is a weighted variant of the octahedral axiom (we will give a more precise definition later). A basic example of a triangular weight is the flat one: it associates to each exact triangle the value 1. The interest of triangular weights is that they naturally lead to *fragmentation pseudometrics* on $\mathrm{Obj}(\mathcal{D})$ (we assume here that $\mathcal{D}$ is small) defined roughly as follows (see Section 2.1 for details). Such a pseudometric depends on a family of objects $\mathcal{F}$ of $\mathcal{D}$. With $\mathcal{F}$ fixed, and up to a certain normalization, the pseudodistance $d^{\mathcal{F}}(X, Y)$ between $X, Y \in \mathrm{Obj}(\mathcal{D})$ is (the symmetrization of) the infimum of the total weight of exact triangles needed to construct iteratively X out of Y by only attaching cones over morphisms with domain in $\mathcal{F}$. The weighted octahedral axiom implies that this $d^{\mathcal{F}}$ satisfies the triangle inequality. Using such pseudometrics one can analyze rigidity properties of various categories by exploring the induced topology on $\mathrm{Obj}(\mathcal{D})$.

The main algebraic part of the memoir is contained in Chapter 2, and its aim is to use persistence machinery to produce certain non-flat triangular weights. The main tool, as already mentioned above, is a refinement of triangulated categories, called *triangulated persistence categories* (TPCs). A triangulated persistence category has two main properties. First, it is a persistence category, a natural notion we introduce in Section 2.2. This is a category $\mathcal{C}$ whose morphisms $\hom_{\mathcal{C}}(A, B)$ are persistence modules, and whose composition is compatible with the persistence structure. (See [47] for a general introduction to persistence module theory.) The second main structural property of TPCs is that the objects of $\mathcal{C}$, together with the 0-persistence level morphisms $\hom^0_{\mathcal{C}}(A, B)$, have the structure of a triangulated category $\mathcal{C}_0$. The formal definition of TPCs is given in Section 2.3.

It is natural to associate to a persistence category $\mathcal{C}$ a limit category $\mathcal{C}_\infty$ that has the same objects as $\mathcal{C}$ and has as morphisms the ∞-limits of the morphisms in $\mathcal{C}$. In a different direction, a natural notion in a persistence category is that of an r-acyclic object: K is called *r-acyclic* if its identity morphism $\mathbb{1}_K \in \hom^0_{\mathcal{C}}(K, K)$ is 0 in $\hom^r_C(K, K)$. The acyclic objects for all $r \geq 0$ form a full subcategory $\mathcal{AC}$ of $\mathcal{C}$ that is also a persistence category, and in case $\mathcal{C}$ is a TPC, it is easy to see that $\mathcal{AC}$ is also a TPC. In particular, $\mathcal{AC}_0$ is triangulated.

These notions are tied together by the classical construction of Verdier localization. Indeed, assuming as above that $\mathcal{C}$ is a TPC, we will see that $\mathcal{C}_\infty$ coincides with the Verdier localization of $\mathcal{C}_0$ with respect to $\mathcal{AC}_0$. In particular, the category $\mathcal{C}_\infty$ is also triangulated.

We can now state the main result of the algebraic part of the memoir (restated more precisely in Theorem 2.65).

Theorem A. *If $\mathcal{C}$ is a triangulated persistence category, then, with the notation above, the Verdier localization $\mathcal{C}_\infty$ admits a non-flat triangular weight induced by the persistence structure of $\mathcal{C}$.*

The construction of this triangular weight is based on a definition of a class of weighted triangles in the category $\mathcal{C}$ itself. With this definition, the exact triangles in $\mathcal{C}_0$ have weight 0, but there are also other triangles in $\mathcal{C}$ of arbitrary positive weight. While the category $\mathcal{C}$, together with the class of finite-weight triangles, is not triangulated – even the formal expression of these triangles in $\mathcal{C}$ does not fit the axioms of triangulated categories – the properties of these triangles are sufficient to induce a triangular weight on the exact triangles of $\mathcal{C}_\infty$.

In summary, if a triangulated category $\mathcal{D}$ admits a TPC refinement – that is, a TPC $\mathcal{C}$ such that $\mathcal{C}_\infty = \mathcal{D}$ (as triangulated categories) – then $\mathcal{D}$ carries a non-flat triangular weight induced by the persistence structure of $\mathcal{C}$. As a result, this construction provides a technique to build non-discrete fragmentation pseudometrics on the objects of $\mathcal{D}$.

Some classes of examples are discussed in Section 2.5. Triangulated persistence categories are expected to be of use beyond the field of symplectic topology, and Chapter 2, which is essentially self-contained, can be read independently of the symplectic considerations that appear in Chapter 3.

Remark 1.2. Even the fragmentation pseudometrics associated to the flat weight are of interest. Many qualitative questions concerning numerical lower bounds for the complexity of certain geometric objects can be understood by means of inequalities involving such fragmentation pseudometrics. Classical examples are the Morse inequalities, the Lusternik–Schnirelmann inequality, as well as, in symplectic topology, the inequalities predicted by the Arnold conjectures. Remarkable results based on measurements using this flat weight and applied to the study of endofunctors have appeared recently in work of Orlov [45], as well as Dimitrov–Haiden–Katzarkov–Kontsevich [26] and Fan–Filip [30].

1.2 TPC refinements of the Fukaya derived category

Here is an overview of the geometric part of the memoir (Chapter 3). The main step here is to consider a finite family $\mathcal{X}$ of closed, exact Lagrangians in a symplectic manifold X, assumed in general position, and construct a TPC refinement of the derived Fukaya category of $\mathcal{X}$.

There are quite a few nuances here. First, this requires the construction of a filtered Fukaya-type category with objects the elements of $\mathcal{X}$, endowed with all possible primitives. A *weakly* filtered such category has been constructed in [10], but obtaining a genuinely filtered A_∞-structure is more delicate. It requires careful control of energy estimates (and the technique we use restricts us to finite families $\mathcal{X}$), but also the use of "cluster"-type moduli spaces, that mix J-holomorphic polygons and Morse trajectories. Fortunately, such moduli spaces have been studied and used frequently since [21], for instance in [15, 16].

The resulting filtered Fukaya category $\mathcal{F}uk(\mathcal{X})$ depends, of course, on choices of auxiliary structures such as perturbation data that we omit from the notation here. The next step is to pursue the construction of the derived version. As in the non-filtered version, this part is purely algebraic and applies to any filtered A_∞-category. Nonetheless, there are some significant differences with respect to the non-filtered case. Uniqueness up to equivalence is considerably more delicate to achieve because several basic algebraic A_∞-tools, such as the Hochschild complex and related constructions, require significant adjustment to adapt to the filtered setting. Moreover, at a more conceptual level, the two natural constructions of the derived category – one based on filtered twisted complexes and the other on the Yoneda embedding and A_∞ filtered modules – both lead to useful natural notions, but not to equivalent ones. Denote by $\mathcal{C}\mathcal{F}uk(\mathcal{X})$ the version based on filtered modules. Let $D\mathcal{F}uk(\mathcal{X})$ be the usual, unfiltered, derived Fukaya category of $\mathcal{X}$, and assume that $\mathcal{F} \subset \mathcal{X}$ is a family of triangular generators for $D\mathcal{F}uk(\mathcal{X})$. Fix also a second such family $\mathcal{F}'$ with each element being a small generic Hamiltonian deformation of a corresponding element in $\mathcal{F}$.

The main statement is the following – again in simplified form (the full statement is in Theorem 3.4):

Theorem B. *The category $\mathcal{C}\mathcal{F}uk(\mathcal{X})$ is a TPC and it is independent of the defining data up to TPC equivalence. Moreover, $\mathcal{C}\mathcal{F}uk(\mathcal{X})_\infty$ is triangulated equivalent to $D\mathcal{F}uk(\mathcal{X})$. Finally, there exists a fragmentation metric on $\mathcal{X}$ that is independent of the choices used in the construction of $\mathcal{C}\mathcal{F}uk(\mathcal{X})$ and is defined by*

$$D^{\mathcal{F},\mathcal{F}'} = \max\{D^{\mathcal{F}}, D^{\mathcal{F}'}\},$$

where $D^{\mathcal{F}}$ are the shift-invariant versions of the fragmentation pseudometrics $d^{\mathcal{F}}$ constructed as outlined in Section 1.1.

One delicate point worth emphasizing here is that while we expect $\mathcal{C}\mathcal{F}uk(\mathcal{X})$ to be unique up to *canonical* equivalence, the machinery in this memoir does not produce fully canonical equivalences (see Theorem 3.12).

Of course, as the set $\mathcal{X}$ is finite, this metric $D^{\mathcal{F},\mathcal{F}'}$ might appear to be uninteresting; however, the more precise result (Theorem 3.4) shows that the pseudometrics $D^{\mathcal{F}}$ satisfy some remarkable properties (see also Remark 3.5). These properties are then used to analyze how the pseudometrics change when the family $\mathcal{X}$ increases. Ultimately, this leads to the definition of the metric on the space of all closed exact Lagrangians that was claimed earlier in the introduction. This is stated more precisely in Corollary 3.7.

The construction of TPCs is inspired by recent constructions in symplectic topology and, in particular, by the shadow pseudometrics introduced in [10] and [9] in the study of Lagrangian cobordism. This aspect is discussed in Section 3.5.1. The construction of the filtered Fukaya category and the associated TPC are expected to be of independent interest.