Chapter 2

Triangulation persistence categories: Algebra 101

This chapter contains the main algebraic machinery introduced in the memoir and is self-contained, except for some basic elements of homological algebra, as can be found in [61].[1]

In Section 2.1 we briefly introduce the notion of triangular weight and discuss its application to measuring the complexity of cone decompositions in triangulated categories. In Section 2.2 we introduce persistence categories, which are, in short, categories enriched by persistence modules. Triangulated persistence categories are introduced in Section 2.3. In Section 2.4 we prove the main algebraic result of the chapter, namely that the ∞-level of a TPC carries a specific triangular weight induced by the persistence structure. Finally, in Section 2.5 we discuss some classes of natural TPC examples that are not symplectic in nature (the symplectic examples are deferred to Chapter 3).

2.1 Triangular weights

In this subsection we introduce triangular weights associated to a triangulated category $\mathcal{D}$. Using such a triangular weight w on $\mathcal{D}$ we define a class of so-called fragmentation pseudometrics $d^{\mathcal{F}}_w$ on $\mathrm{Obj}(\mathcal{D})$. All categories used in this memoir (in particular, $\mathcal{D}$) are assumed to be small unless otherwise indicated.

Definition 2.1. Let $\mathcal{D}$ be a triangulated category and denote by $\mathcal{T}_{\mathcal{D}}$ its class of exact triangles. A *triangular weight* w on $\mathcal{D}$ is a function

$$w : \mathcal{T}_{\mathcal{D}} \to [0, \infty)$$

that satisfies properties (i) and (ii) below:

(i) [Weighted octahedral axiom] Assume that the triangles $\Delta_1 : A \to B \to C \to TA$ and $\Delta_2 : C \to D \to E \to TC$ are both exact. There are exact triangles $\Delta_3 : B \to D \to F \to TB$ and $\Delta_4 : TA \to F \to E \to T^2 A$ making the diagram below

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[1] A version of this chapter appeared earlier as an independent preprint [11]. The only changes compared to [11], besides minor corrections of imprecisions, concern the relations to Verdier localization in Section 2.3.2.

commute, except for the bottom-right square, which anti-commutes:

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & TA \arrow[r] \arrow[d] & TA \arrow[d] \\
B \arrow[r] \arrow[d] & D \arrow[r] \arrow[d] & F \arrow[r] \arrow[d] & TB \arrow[d] \\
C \arrow[r] \arrow[d] & D \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & TC \arrow[d] \\
TA \arrow[r] & 0 \arrow[r] & T^2A \arrow[r] & T^2A
\end{tikzcd}$$

and such that

$$w(\Delta_3) + w(\Delta_4) \leq w(\Delta_1) + w(\Delta_2).$$

(ii) [Normalization] There exists some $w_0 \in [0, \infty)$ such that $w(\Delta) \geq w_0$ for all $\Delta \in \mathcal{T}_{\mathcal{D}}$ and $w(\Delta') = w_0$ for all triangles Δ' of the form $0 \to X \xrightarrow{\mathbb{1}_X} X \to 0$, $X \in \mathrm{Obj}(\mathcal{D})$, and their rotations. Moreover, in the diagram in (i), if $B = 0$ we may take Δ_3 to be

$$\Delta_3 : 0 \to D \to D \to 0. \tag{2.1}$$

Remark 2.2. (a) Neglecting the weight constraints, given the triangles Δ_1, Δ_2, Δ_3 as in point (i), the octahedral axiom is easily seen to imply the existence of Δ_4 making the diagram commutative, as in the definition.

(b) The condition in point (ii), above equation (2.1), can be reformulated as a replacement property for exact triangles in the following sense: if $\Delta_2 : C \to D \to E \to TC$ is exact and C is isomorphic to A' $(= TA)$, then there is an exact triangle $A' \to D \to E \to TA'$ of weight at most $w(\Delta_2) + w(\Delta_1) - w_0$, where Δ_1 is the exact triangle $T^{-1}A' \to 0 \to C \to A'$.

Given an exact triangle $\Delta : A \to B \xrightarrow{f} C \to TA$ in $\mathcal{D}$ and any $X \in \mathrm{Obj}(\mathcal{D})$, there is an associated exact triangle $X \oplus \Delta : A \to X \oplus B \xrightarrow{\mathbb{1}_X \oplus f} X \oplus C \to TA$ and a similar one, $\Delta \oplus X$. We say that a triangular weight w on $\mathcal{D}$ is *subadditive* if, for any exact triangle $\Delta \in \mathcal{T}_{\mathcal{D}}$ and any object X of $\mathcal{D}$, we have

$$w(X \oplus \Delta) \leq w(\Delta)$$

and similarly for $\Delta \oplus X$.

The simplest example of a triangular weight on a triangulated category $\mathcal{D}$ is the flat one, $w_{fl}(\Delta) = 1$, for all triangles $\Delta \in \mathcal{T}_{\mathcal{D}}$. This weight is obviously subadditive. A weight that is not proportional to the flat one is called *non-flat*.

The interest of triangular weights comes from the next definition, which provides a measure of the complexity of cone decompositions in $\mathcal{D}$; this leads in turn to the definition of corresponding pseudometrics on the set $\mathrm{Obj}(\mathcal{D})$.

Definition 2.3. Fix a triangulated category $\mathcal{D}$ together with a triangular weight w on $\mathcal{D}$. Let X be an object of $\mathcal{D}$. An *iterated cone decomposition* D of X with *linearization* $\ell(D) = (X_1, X_2, \ldots, X_n)$ consists of a family of exact triangles in $\mathcal{D}$:

$$\begin{cases} \Delta_1 : X_1 \to 0 \to Y_1 \to TX_1 \\ \Delta_2 : X_2 \to Y_1 \to Y_2 \to TX_2 \\ \Delta_3 : X_3 \to Y_2 \to Y_3 \to TX_3 \\ \qquad\qquad \vdots \\ \Delta_n : X_n \to Y_{n-1} \to X \to TX_n. \end{cases}$$

To accommodate the case $n = 1$ we set $Y_0 = 0$. The *weight* of such a cone decomposition is defined by

$$w(D) = \sum_{i=1}^{n} w(\Delta_i) - w_0.$$

This weight of cone decompositions naturally leads to a class of pseudometrics on the objects of $\mathcal{D}$, as follows.

Let $\mathcal{F} \subset \mathrm{Obj}(\mathcal{D})$. For two objects X, X' of $\mathcal{D}$, define

$$\begin{aligned} \delta^{\mathcal{F}}(X, X') = \inf\bigl\{w(D) \mid\ & D \text{ is an iterated cone decomposition of } X \\ & \text{with linearization } (F_1, \ldots, T^{-1}X', \ldots, F_k), \\ & \text{where } F_i \in \mathcal{F}, k \geq 0\bigr\}. \end{aligned} \tag{2.2}$$

Note that we allow here $k = 0$, i.e., the linearization of D is allowed to consist of only one element, $T^{-1}X'$, without using any elements F_i from the family $\mathcal{F}$. Fragmentation pseudometrics are obtained by symmetrizing $\delta^{\mathcal{F}}$, as below.

Proposition 2.4. *Let $\mathcal{D}$ be a triangulated category and let w be a triangular weight on $\mathcal{D}$. Fix $\mathcal{F} \subset \mathrm{Obj}(\mathcal{C})$ and define*

$$d^{\mathcal{F}} : \mathrm{Obj}(\mathcal{C}) \times \mathrm{Obj}(\mathcal{C}) \to [0, \infty) \cup \{+\infty\}$$

by

$$d^{\mathcal{F}}(X, X') = \max\{\delta^{\mathcal{F}}(X, X'), \delta^{\mathcal{F}}(X', X)\}.$$

(i) *The map $d^{\mathcal{F}}$ is a pseudometric, called* the fragmentation pseudometric *associated to w and $\mathcal{F}$.*

(ii) *If w is subadditive, then*

$$d^{\mathcal{F}}(A \oplus B, A' \oplus B') \leq d^{\mathcal{F}}(A, A') + d^{\mathcal{F}}(B, B') + w_0. \tag{2.3}$$

In particular, if $w_0 = 0$, then $\mathrm{Obj}(\mathcal{D})$ *with the operation given by $\oplus$ and the topology induced by $d^{\mathcal{F}}$ is an H-space. (Recall that a topological space* *is called an* H-space *if there exists a continuous map* $\mu : X \times X \to X$ *with an identity element* e *such that* $\mu(e, x) = \mu(x, e) = x$ *for any* $x \in X$*.)*

The proof of Proposition 2.4 is based on simple manipulations with exact triangles. We will prove a similar statement in Section 2.3.4 in a more complicated setting and we will then briefly discuss in Section 2.3.4.1 how the arguments given in that case also imply Proposition 2.4.

Remark 2.5. (a) If $\mathcal{F}$ is invariant under translation, in the sense that $T\mathcal{F} \subset \mathcal{F}$ and, moreover, $\mathcal{F}$ is a family of triangular generators for $\mathcal{C}$, then the metric $d^{\mathcal{F}}$ admits finite values. This is not difficult to show by first proving that $\delta^{\mathcal{F}}(0, X)$ is finite for all $X \in \mathrm{Obj}(\mathcal{C})$ (it is immediate that $\delta^{\mathcal{F}}(X, 0)$ is finite).

(b) It is sometimes useful to view an iterated cone decomposition as in Definition 2.3 as a sequence of objects and maps forming the successive triangles Δ_i below

$$\begin{tikzcd}
Y_0 \arrow[rr] & & Y_1 \arrow[r] \arrow[ddl, dashed] & \cdots \arrow[r] & Y_i \arrow[rr] & & Y_{i+1} \arrow[r] \arrow[ddl, dashed] & \cdots \arrow[r] & Y_{n-1} \arrow[rr] & & Y_n \arrow[ddl, dashed] \\
& \Delta_1 & & & & \Delta_{i+1} & & & & \Delta_n & \\
& X_1 \arrow[uul] & & & & X_{i+1} \arrow[uul] & & & & X_n \arrow[uul] &
\end{tikzcd} \tag{2.4}$$

where the dotted arrows represent maps $Y_i \to TX_i$ and $Y_0 = 0$, $Y_n = X$.

(c) The definition of fragmentation pseudometrics is quite flexible and there are a number of possible variants. One of them will be useful later. Instead of $\delta^{\mathcal{F}}$ as given in (2.2), we may use

$$\underline{\delta}^{\mathcal{F}}(X, X') = \inf\Bigg\{\sum_{i=1}^{n} w(\Delta_i) \mid \Delta_i \text{ are successive exact triangles as in (2.4)} \\ \text{with } Y_1 = X',\ X = Y_n, \text{ and } X_i \in \mathcal{F}, n \in \mathbb{N}\Bigg\}. \tag{2.5}$$

For this to be coherent we need to assume here $0 \in \mathcal{F}$. Comparing with the definition of $\delta^{\mathcal{F}}$ in (2.2), $\underline{\delta}^{\mathcal{F}}$ corresponds to only taking into account cone decompositions with linearization $(T^{-1}X', F_1, \ldots, F_n)$ and with the first triangle $\Delta_1 : T^{-1}X' \to 0 \to X' \to X'$. There are two advantages to this expression: the first is that it is trivial to see in this case that $\underline{\delta}^{\mathcal{F}}$ satisfies the triangle inequality, which does not even require the weighted octahedral axiom. The other advantage is that one starts the sequence of triangles from X', and thus the negative translate $T^{-1}X'$ is not needed to define $\underline{\delta}^{\mathcal{F}}$. There is an associated fragmentation pseudometric $\underline{d}^{\mathcal{F}}$ obtained by symmetrizing $\underline{\delta}^{\mathcal{F}}$, and this satisfies a formula similar to (2.3). Of course, the disadvantage of this fragmentation pseudometric is that it is larger than $d^{\mathcal{F}}$ and thus more often infinite.

2.2 Persistence categories

We introduce in this section the notion of a persistence category – a category whose morphisms are persistence modules and such that composition respects the persistence structure – and then proceed with a number of related structures and immediate properties.

2.2.1 Basic definitions

View the real axis $\mathbb{R}$ as a category with $\mathrm{Obj}(\mathbb{R}) = \{x \mid x \in \mathbb{R}\}$ and, for any $x, y \in \mathrm{Obj}(\mathbb{R})$, the hom-set

$$\hom_{\mathbb{R}}(x, y) = \begin{cases} i_{x,y} & \text{if } x \leq y, \\ \emptyset & \text{if } x > y. \end{cases}$$

By definition, for any $x \leq y \leq z$ in $\mathbb{R}$, $i_{y,z} \circ i_{x,y} = i_{x,z}$. We denote this category by $(\mathbb{R}, \leq)$. It admits an additive structure. Explicitly, consider the bifunctor $\oplus : (\mathbb{R}, \leq) \times (\mathbb{R}, \leq) \to (\mathbb{R}, \leq)$ defined by $\oplus(r, s) := r + s$, where $0 \in \mathbb{R}$ is the zero object and, for any two pairs $(r, s), (r', s') \in \mathrm{Obj}((\mathbb{R}, \leq) \times (\mathbb{R}, \leq))$,

$$\hom_{(\mathbb{R},\leq)\times(\mathbb{R},\leq)}((r, s), (r', s')) = \begin{cases} (i_{r,r'}, i_{s,s'}) & \text{if } r \leq r' \text{ and } s \leq s', \\ \emptyset & \text{otherwise,} \end{cases}$$

and further $\oplus(i_{r,r'}, i_{s,s'}) := i_{r+s,r'+s'} \in \hom_{(\mathbb{R},\leq)}(r + s, r' + s')$. Fix a ground field $\mathbf{k}$ and denote by $\mathrm{Vect}_{\mathbf{k}}$ the category of $\mathbf{k}$-vector spaces.

Definition 2.6. A category $\mathcal{C}$ is called a *persistence category* if it is endowed with the following additional structure. For any $A, B \in \mathrm{Obj}(\mathcal{C})$, we are given a functor $E_{A,B} : (\mathbb{R}, \leq) \to \mathrm{Vect}_{\mathbf{k}}$ such that the following two conditions are satisfied:

(i) The hom-set in $\mathcal{C}$ is $\hom_{\mathcal{C}}(A, B) = \{(f, r) \mid f \in E_{A,B}(r)\}$. We write $\hom^r_{\mathcal{C}}(A, B) := E_{A,B}(r)$, or simply $\hom^r(A, B)$, when the ambient category $\mathcal{C}$ is not emphasized.

(ii) The composition $\circ : \hom^r_{\mathcal{C}}(A, B) \times \hom^s_{\mathcal{C}}(B, C) \to \hom^{r+s}_{\mathcal{C}}(A, C)$ in $\mathcal{C}$ is a natural transformation from $E_{A,B} \times E_{B,C}$ to $E_{A,C} \circ \oplus$ (with $\oplus$ the product $(\mathbb{R}, \leq) \times (\mathbb{R}, \leq) \to (\mathbb{R}, \leq)$). Explicitly, the following diagram commutes:

$$\begin{tikzcd}
\hom^r_{\mathcal{C}}(A, B) \times \hom^s_{\mathcal{C}}(B, C) \arrow[r, "\circ_{(r,s)}"] \arrow[d, "E_{A,B}(i_{r,r'}) \times E_{B,C}(i_{s,s'})"'] & \hom^{r+s}_{\mathcal{C}}(A, C) \arrow[d, "E_{A,C}(i_{r+s,r'+s'})"] \\
\hom^{r'}_{\mathcal{C}}(A, B) \times \hom^{s'}_{\mathcal{C}}(B, C) \arrow[r, "\circ_{(r',s')}"] & \hom^{r'+s'}_{\mathcal{C}}(A, C)
\end{tikzcd}$$

Remark 2.7. Item (i) means that each hom-set $\hom_{\mathcal{C}}(A, B)$ is a persistence $\mathbf{k}$-module with persistence structure morphisms $E_{A,B}(i_{r,s})$ for any $r \leq s$ in $\mathbb{R}$. Here, we use the weakest possible definition of a persistence $\mathbf{k}$-module, in the sense that no regularities, such as the finiteness of the dimension of $\hom^r_{\mathcal{C}}(A, B)$ or the semi-continuity when changing the parameter r, are required (see [47, Subsection 1.1]).

We will often denote an element in $\hom_{\mathcal{C}}(A, B)$ by a single symbol $\bar{f}$ instead of a pair (f, r). We will use the notation $\lceil \bar{f} \rceil = r$ to denote the real number r and refer to this number as the *shift* (or persistence level) of $\bar{f}$. For each $A \in \mathrm{Obj}(\mathcal{C})$, the identity $\bar{\mathbb{1}}_A := (\mathbb{1}_A, 0) \in \hom^0_{\mathcal{C}}(A, A)$ is of shift 0. If one of the objects A or B is the zero object, then $\hom_{\mathcal{C}}(A, B)$ contains only the zero morphism, denoted by 0, and it lies in $\hom^r_{\mathcal{C}}(A, B)$ for any $r \in \mathbb{R}$. For brevity, we will denote from now on the structural morphisms $E_{A,B}(i_{r,s})$ by $i_{r,s}$.

A persistence structure allows us to consider morphisms that are identified up to r-shift and, similarly, objects that are negligible up to a shift by r.

Definition 2.8. Fix a persistence category $\mathcal{C}$.

(i) For $f, g \in \hom^\alpha_{\mathcal{C}}(A, B)$, we say that f and g are *r-equivalent* for some $r \geq 0$ if

$$i_{\alpha,\alpha+r}(f - g) = 0.$$

We write $f \simeq_r g$ if f and g are r-equivalent.

(ii) Two morphisms $f \in \hom^\alpha_{\mathcal{C}}(A, B)$ and $g \in \hom^\beta_{\mathcal{C}}(A, B)$ are *∞-equivalent*, written $f \simeq_\infty g$, if there exist $r, r' \geq 0$ with $\alpha + r = \beta + r'$ such that $i_{\alpha,\alpha+r}(f) = i_{\beta,\beta+r'}(g)$.

(iii) An object $K \in \mathrm{Obj}(\mathcal{C})$ is called *r-acyclic* for some $r \geq 0$ if its identity morphism $\mathbb{1}_K \in \hom^0_{\mathcal{C}}(K, K)$ has the property that $\mathbb{1}_K \simeq_r 0$.

Obviously, if $f \simeq_r g$ then $f \simeq_s g$ for all $s \geq r$. Notice also that $\simeq_r$ is indeed an equivalence relation. Indeed, for $r \neq \infty$ this follows immediately from the fact that $i_{\alpha,\beta} : \hom^\alpha_{\mathcal{C}}(A, B) \to \hom^\beta_{\mathcal{C}}(A, B)$ is a linear map, and it is an easy exercise for $r = \infty$.

Definition 2.9. Given a persistence category $\mathcal{C}$, there are two categories naturally associated to it, as follows:

(i) the *0-level* of $\mathcal{C}$, denoted by $\mathcal{C}_0$, which is the category with the same objects as $\mathcal{C}$ and, for any $A, B \in \mathrm{Obj}(\mathcal{C})$, with $\hom_{\mathcal{C}_0}(A, B) := \hom^0_{\mathcal{C}}(A, B)$.

(ii) the *limit category* (or *∞-level*) of $\mathcal{C}$, denoted by $\mathcal{C}_\infty$, which again has the same objects as $\mathcal{C}$ but, for any $A, B \in \mathrm{Obj}(\mathcal{C})$,

$$\hom_{\mathcal{C}_\infty}(A, B) := \varinjlim_{\alpha \to \infty} \hom^\alpha_{\mathcal{C}}(A, B),$$

where the direct limit is taken with respect to the morphisms

$$i_{\alpha,\beta} : \hom^{\alpha}_{\mathcal{C}}(A, B) \to \hom^{\beta}_{\mathcal{C}}(A, B)$$

for any $\alpha \leq \beta$.

Remark 2.10. (a) In general, a persistence category is not pre-additive as the hom-sets $\hom_{\mathcal{C}}(A, B)$ are generally not abelian groups. However, it is easy to see that both $\mathcal{C}_0$ and $\mathcal{C}_\infty$ are pre-additive (the proof is immediate in the first case and a simple exercise in the second).

(b) The limit category $\mathcal{C}_\infty$ can be equivalently defined as the quotient category $\mathcal{C}/\simeq_\infty$, which is defined by

$$\mathrm{Obj}(\mathcal{C}/\simeq_\infty) = \mathrm{Obj}(\mathcal{C}) \quad \text{and} \quad \hom_{\mathcal{C}/\simeq_\infty}(A, B) = \hom_{\mathcal{C}}(A, B)/\simeq_\infty .$$

Two objects $A, B \in \mathrm{Obj}(\mathcal{C})$ are said to be *0-isomorphic*, and we write $A \equiv B$, if they are isomorphic in the category $\mathcal{C}_0$. This is obviously an equivalence relation, and it preserves r-acyclics in the sense that if $K \simeq_r 0$ and $K \equiv K'$, then $K' \simeq_r 0$.

2.2.2 Persistence functors

Persistence categories come with associated notions of persistence functors and natural transformations relating them, as described below.

Definition 2.11. Given two persistence categories $\mathcal{C}$ and $\mathcal{C}'$, a *persistence functor* $\mathcal{F} : \mathcal{C} \to \mathcal{C}'$ is a functor that is compatible with the persistence structures. More explicitly, the action of $\mathcal{F}$ on morphisms restricts to maps $(\mathcal{F}_{A,B})_r : \hom^r_{\mathcal{C}}(A, B) \to \hom^r_{\mathcal{C}'}(\mathcal{F}(A), \mathcal{F}(B))$ defined for any $A, B \in \mathrm{Obj}(\mathcal{C})$ and $r \in \mathbb{R}$. Moreover, for every $r \leq s$ we have the following commutative diagram:

$$\begin{tikzcd}
\hom^r_{\mathcal{C}}(A, B) \arrow[r, "(\mathcal{F}_{A,B})_r"] \arrow[d, "i^{\mathcal{C}}_{r,s}"'] & \hom^r_{\mathcal{C}'}(\mathcal{F}(A), \mathcal{F}(B)) \arrow[d, "i^{\mathcal{C}'}_{r,s}"] \\
\hom^s_{\mathcal{C}}(A, B) \arrow[r, "(\mathcal{F}_{A,B})_s"] & \hom^s_{\mathcal{C}'}(\mathcal{F}(A), \mathcal{F}(B))
\end{tikzcd}$$

where $i^{\mathcal{C}}_{r,s}$ and $i^{\mathcal{C}'}_{r,s}$ are persistence structure maps in $\mathcal{C}$ and $\mathcal{C}'$, respectively. In particular, for each $\bar{f} \in \hom_{\mathcal{C}}(A, B)$ with $\lceil \bar{f} \rceil = r$, we have $\lceil \mathcal{F}_{A,B}(\bar{f}) \rceil = r$.

For any functor $E : (\mathbb{R}, \leq) \to \mathrm{Vect}_{\mathbf{k}}$ and $\alpha \in \mathbb{R}$, we denote by $\Sigma^\alpha E : (\mathbb{R}, \leq) \to \mathrm{Vect}_{\mathbf{k}}$ the α-shift of E, defined by $\Sigma^\alpha E(r) = E(r + \alpha)$ and $\Sigma^\alpha E(i_{r,s}) = E(i_{r+\alpha,s+\alpha})$ for any $i_{r,s} : r \to s$, $r \leq s$.

Definition 2.12. Given two persistence functors between two persistence categories $\mathcal{F}, \mathcal{G} : \mathcal{C} \to \mathcal{C}'$, a *persistence natural transformation* $\eta : \mathcal{F} \to \mathcal{G}$ is a natural transformation for which there exists $r \in \mathbb{R}$ such that for any $A \in \mathrm{Obj}(\mathcal{C})$, the morphism

$\eta_A : \mathcal{F}(A) \to \mathcal{G}(A)$ belongs to $\hom^r_{\mathcal{C}'}(\mathcal{F}(A), \mathcal{G}(A))$. We say that η is a natural transformation of shift r.

Remark 2.13. (a) The morphisms $\eta_A : \mathcal{F}(A) \to \mathcal{G}(A)$, $A \in \mathrm{Obj}(\mathcal{C})$, give rise to the following commutative diagrams for all $X \in \mathrm{Obj}(\mathcal{C})$ and for any $\alpha \leq \beta \in \mathbb{R}$:

$$\begin{tikzcd}
\hom^{\alpha}_{\mathcal{C}'}(X, \mathcal{F}(A)) \arrow[r, "\eta_A \circ"] \arrow[d, "i_{\alpha,\beta}"'] & \hom^{\alpha+r}_{\mathcal{C}'}(X, \mathcal{G}(A)) \arrow[d, "i_{\alpha+r,\beta+r}"] \\
\hom^{\beta}_{\mathcal{C}'}(X, \mathcal{F}(A)) \arrow[r, "\eta_A \circ"] & \mathrm{Mor}^{\beta+r}_{\mathcal{C}'}(X, \mathcal{G}(A))
\end{tikzcd} \tag{2.6}$$

and

$$\begin{tikzcd}
\hom^{\alpha}_{\mathcal{C}'}(\mathcal{G}(A), X) \arrow[r, "\circ \eta_A"] \arrow[d, "i_{\alpha,\beta}"'] & \hom^{\alpha+r}_{\mathcal{C}'}(\mathcal{F}(A), X) \arrow[d, "i_{\alpha+r,\beta+r}"] \\
\hom^{\beta}_{\mathcal{C}'}(\mathcal{G}(A), X) \arrow[r, "\circ \eta_A"] & \hom^{\beta+r}_{\mathcal{C}'}(\mathcal{F}(A), X)
\end{tikzcd} \tag{2.7}$$

(b) Given two persistence categories $\mathcal{C}, \mathcal{C}'$, the persistence functors themselves form a persistence category denoted by $\mathcal{P}\mathrm{Fun}(\mathcal{C}, \mathcal{C}')$, where

$$\hom_{\mathcal{P}\mathrm{Fun}(\mathcal{C},\mathcal{C}')}(\mathcal{F}, \mathcal{G}) = \big\{(\eta, r) \mid \eta \text{ is a natural transformation from } \mathcal{F} \text{ to } \mathcal{G} \text{ of shift } r\big\}.$$

When $\mathcal{C} = \mathcal{C}'$, simply denote $\mathcal{P}\mathrm{Fun}(\mathcal{C}, \mathcal{C}')$ by $\mathcal{P}\mathrm{End}(\mathcal{C})$. It is easy to verify that $\mathcal{P}\mathrm{End}(\mathcal{C})$ admits a strict monoidal structure.

Definition 2.14. Let $\mathcal{C}', \mathcal{C}''$ be two persistence categories. A persistence functor $\mathcal{F} : \mathcal{C}' \to \mathcal{C}''$ is called an *equivalence of persistence categories* (or *persistence equivalence*) if there exists a persistence functor $\mathcal{G} : \mathcal{C}'' \to \mathcal{C}'$ such that $\mathcal{G} \circ \mathcal{F}$ is isomorphic to $\mathbb{1}_{\mathcal{C}'}$ via a persistence natural transformation of shift 0, whose inverse also has shift 0, and the analogous condition holds for $\mathcal{F} \circ \mathcal{G}$ too. We will say that $\mathcal{C}'$ and $\mathcal{C}''$ are persistence equivalent or equivalent as persistence categories.

Standard arguments show that a persistence functor $\mathcal{F} : \mathcal{C}' \to \mathcal{C}''$ is an equivalence of persistence categories if and only if it is full and faithful (in the obvious persistence sense) and for every object $Y \in \mathrm{Obj}(\mathcal{C}'')$ there exists $X \in \mathrm{Obj}(\mathcal{C}')$ such that Y is 0-isomorphic to $\mathcal{F}(X)$ (i.e., the latter two objects are isomorphic in the 0-level subcategory $\mathcal{C}''_0$ of $\mathcal{C}''$).

2.2.3 Shift functors

The role of shift functors, to be introduced below, is to allow morphisms of arbitrary shift (as well as r-equivalences) to be represented as morphisms of shift 0, at the cost of “shifting” the domain (or the target); see Remark 2.19. This turns out to be very helpful in the study of triangulation for persistence categories.

View the real axis $\mathbb{R}$ as a strict monoidal category $(\mathbb{R}, +)$ induced by the additive group structure of $\mathbb{R}$. In other words, $\mathrm{Obj}(\mathbb{R}) = \{x \mid x \in \mathbb{R}\}$ and, for any $x, y \in \mathrm{Obj}(\mathbb{R})$, $\hom_{\mathbb{R}}(x, y) = \{\eta_{x,y}\}$, such that $\eta_{x,x} = \mathbb{1}_x$ and, for any $x, y, z \in \mathbb{R}$, $\eta_{y,z} \circ \eta_{x,y} = \eta_{x,z}$. In particular, $\eta_{x,y} \circ \eta_{y,x} = \mathbb{1}_y$ and $\eta_{y,x} \circ \eta_{x,y} = \mathbb{1}_x$; hence each morphism $\eta_{x,y}$ is an isomorphism. The monoidal structure is defined by $\oplus(x, y) := x + y$ on objects and, for any two morphisms $(\eta_{r,r'}, \eta_{s,s'})$, we have $\oplus(\eta_{r,r'}, \eta_{s,s'}) := \eta_{r+s,r'+s'}$.

Definition 2.15. Let $\mathscr{C}$ be a persistence category. A *shift functor* on $\mathscr{C}$ is a strict monoidal functor $\Sigma : (\mathbb{R}, +) \to \mathcal{P}\mathrm{End}(\mathscr{C})$ such that $\Sigma(\eta_{x,y}) : \Sigma(x) \to \Sigma(y)$ is a natural transformation of shift $\lceil \Sigma(\eta_{x,y}) \rceil = y - x$ for any $x, y \in \mathbb{R}$ and $\eta_{x,y} \in \hom_{\mathbb{R}}(x, y)$.

For later use, we write $\Sigma^r := \Sigma(r) \in \mathcal{P}\mathrm{End}(\mathscr{C})$ and, for brevity, we denote $\Sigma(\eta_{r,s})$ by $\eta_{r,s}$ for $r, s \in \mathbb{R}$. We let $(\eta_{r,s})_A$ be the respective morphism $\Sigma^r A \to \Sigma^s A$.

Remark 2.16. Since Σ is a strict monoidal functor, it preserves the monoidal product. Therefore, $\Sigma^s \circ \Sigma^r = \Sigma^{r+s}$ and $\Sigma^0 = \mathbb{1}$. Moreover, since each $\eta_{r,s}$ is an isomorphism in $(\mathbb{R}, +)$, the corresponding natural transformation $\eta_{r,s}$ is a natural isomorphism. We also have $\Sigma^r(\eta_{s,s'})_A = (\eta_{s+r,s'+r})_A$ for each object A in $\mathscr{C}$ and all $r, s, s' \in \mathbb{R}$.

In particular, this implies that for any $Y, A \in \mathrm{Obj}(\mathscr{C})$ and $\alpha \in \mathbb{R}$, we have an isomorphism

$$\hom_{\mathscr{C}}^{\alpha}(Y, A) \xrightarrow{(\eta_{0,r})_A \circ} \hom_{\mathscr{C}}^{\alpha+r}(Y, \Sigma^r A). \tag{2.8}$$

Similarly, for any $A, X \in \mathrm{Obj}(\mathscr{C})$ and $\alpha \in \mathbb{R}$, we have an isomorphism

$$\hom_{\mathscr{C}}^{\alpha-r}(\Sigma^r A, X) \xrightarrow{\circ(\eta_{0,r})_A} \hom_{\mathscr{C}}^{\alpha}(A, X). \tag{2.9}$$

Further, for any $A, B \in \mathrm{Obj}(\mathscr{C})$, the isomorphisms (2.8) and (2.9) imply the existence of an isomorphism

$$\hom_{\mathscr{C}}^{\alpha+s-r}(\Sigma^r A, \Sigma^s B) \simeq \hom_{\mathscr{C}}^{\alpha}(A, B). \tag{2.10}$$

In particular, when $r = s$, we get a canonical isomorphism

$$\Sigma^r : \hom_{\mathscr{C}}^{\alpha}(A, B) \to \hom_{\mathscr{C}}^{\alpha}(\Sigma^r A, \Sigma^r B). \tag{2.11}$$

Finally, the diagrams (2.6) and (2.7) imply that the following diagrams, obtained by setting $\mathscr{F} = \Sigma^0$ and $\mathscr{G} = \Sigma^r$, are commutative for any $\alpha \leq \beta$:

$$\begin{tikzcd}
\hom_{\mathcal{C}}^{\alpha}(X, A) \arrow[r, "(\eta_{0,r})_A \circ"] \arrow[d, "i_{\alpha,\beta}"'] & \hom_{\mathcal{C}}^{\alpha+r}(X, \Sigma^r A) \arrow[d, "i_{\alpha+r,\beta+r}"] \\
\hom_{\mathcal{C}}^{\beta}(X, A) \arrow[r, "(\eta_{0,r})_A \circ"] & \mathrm{Mor}_{\mathcal{C}}^{\beta+r}(X, \Sigma^r A)
\end{tikzcd} \tag{2.12}$$

and

$$\begin{tikzcd}
\hom_{\mathcal{C}}^{\alpha}(\Sigma^r A, X) \arrow[r, "\circ(\eta_{0,r})_A"] \arrow[d, "i_{\alpha,\beta}"'] & \hom_{\mathcal{C}}^{\alpha+r}(A, X) \arrow[d, "i_{\alpha+r,\beta+r}"] \\
\hom_{\mathcal{C}}^{\beta}(\Sigma^r A, X) \arrow[r, "\circ(\eta_{0,r})_A"] & \hom_{\mathcal{C}}^{\beta+r}(A, X)
\end{tikzcd} \tag{2.13}$$

All the horizontal morphisms in (2.12) and (2.13) are isomorphisms, but the vertical morphisms (which are the persistence structure morphisms) are not necessarily so.

Assume that $\mathcal{C}$ is a persistence category (with persistence structure morphisms denoted by $i_{r,s}$) endowed with a shift functor Σ. To ease notation, for $A \in \mathrm{Obj}(\mathcal{C})$, $r \geq 0$, we consider $(\eta_{r,0})_A \in \hom_{\mathcal{C}}^{-r}(\Sigma^r A, A)$ and $(\eta_{0,-r})_A \in \hom_{\mathcal{C}}^{-r}(A, \Sigma^{-r} A)$, and we will denote below by η_r^A the maps

$$\eta_r^A = i_{-r,0}((\eta_{r,0})_A) \quad \text{or} \quad \eta_r^A = i_{-r,0}((\eta_{0,-r})_A). \tag{2.14}$$

Thus $\eta_r^A \in \hom_{\mathcal{C}}^0(\Sigma^r A, A)$ or $\eta_r^A \in \hom_{\mathcal{C}}^0(A, \Sigma^{-r} A)$, depending on the context. Note that there is no ambiguity of the notation η_r^A due to the canonical identification via Σ^r in (2.11). The notions discussed before, r-acyclicity, r-equivalence, and so forth, can be reformulated in terms of compositions with appropriate shift morphisms η_r^A.

The next lemma is a characterization of r-equivalence that follows easily from the diagrams (2.12) and (2.13).

Lemma 2.17. *Suppose that* $f \in \hom^{\alpha}(A, B)$. *Then* $i_{\alpha,\alpha+r}(f) = 0$ *for some* $r \geq 0$ *if and only if* $f \circ \eta_r^A = 0$ *in* $\hom^{\alpha}(\Sigma^r A, B)$ *and (equivalently) if and only if* $\eta_r^B \circ f = 0$ *in* $\hom^{\alpha}(A, \Sigma^{-r} B)$.

In particular, we easily see that for two morphisms $f, g \in \hom^{\alpha}(A, B)$, $f \simeq_r g$ if and only if $f \circ \eta_r^A = g \circ \eta_r^A$. Moreover, r-equivalence is preserved under shifts. Further, it is immediate to check that $f \in \hom^{\alpha}(A, B)$ and $g \in \hom^{\beta}(A, B)$ are ∞-equivalent if and only if there exist $r, r' \geq 0$ with $\alpha + r = \beta + r'$ such that

$$f \circ \eta_r^A = g \circ \eta_{r'}^A \quad \text{in } \hom^{\alpha+r}(A, B),$$

where we identify both $\hom^{\alpha}(\Sigma^r A, B)$ and $\hom^{\beta}(\Sigma^{r'} A, B)$ with $\hom^{\alpha+r}(A, B)$ through the canonical isomorphisms in Remark 2.16.

Here is a similar characterization of r-acyclicity.

Lemma 2.18. *$K \simeq_r 0$ is equivalent to each of the following:*

(i) $\eta_r^K = 0$.

(ii) $i_{\alpha,\alpha+r} : \hom^{\alpha}(A, K) \to \hom^{\alpha+r}(A, K)$ *vanishes for any* $\alpha \in \mathbb{R}$ *and* A.

(iii) $i_{\alpha,\alpha+r} : \hom^{\alpha}(K, A) \to \hom^{\alpha+r}(K, A)$ *vanishes for any* $\alpha \in \mathbb{R}$ *and* A.

Proof. Point (i) is an immediate consequence of the definition of r-acylics in Definition 2.8 and of Lemma 2.17 applied for $A, B = K$, $f = \mathbb{1}_K$. We now prove (ii). The proof of (iii) is similar and will be omitted. It is obvious that (ii) implies $K \simeq_r 0$ by specializing to $A = K$, $\alpha = 0$ and applying $i_{0,r}$ to $\mathbb{1}_K$. To prove the converse, we first use the diagram (2.12) to deduce that the map $i_{\alpha,\alpha+r}$ factors as

$$\begin{tikzcd}
\hom^{\alpha}(A, K) \arrow[r, "(i_{-r,0}(\eta_{0,-r})_K)\circ"] \arrow[rr, bend right, "i_{\alpha,\alpha+r}"'] & \hom^{\alpha}(A, \Sigma^{-r} K) \arrow[r, "(\eta_{-r,0})_K\circ"] & \hom^{\alpha+r}(A, K)\ .
\end{tikzcd}$$

Therefore, since $(\eta_{-r,0})_K\circ$ is an isomorphism, for any $f \in \hom^{\alpha}(A, K)$ we have $i_{\alpha,\alpha+r}(f) = 0$ if and only if $i_{-r,0}(\eta_{0,-r})_K \circ f = 0$. From point (i) we know that this relation is true for $f = \mathbb{1}_K$. Now, for any $f \in \hom^{\alpha}(A, K)$, we write $f = \mathbb{1}_K \circ f$ and conclude that $i_{-r,0}(\eta_{0,-r})_K \circ f = i_{-r,0}(\eta_{0,-r} \circ \mathbb{1}_K) \circ f = 0$. ■

In particular, we see that K is r-acyclic if and only if any of its shifts $\Sigma^{\alpha} K$ is so.

Remark 2.19. (a) Assume that $\mathscr{C}$ is a persistence category endowed with a shift functor Σ and that $f_1, f_2 \in \hom^{\alpha}_{\mathscr{C}}(A, B)$. Then, for all practical purposes, we may replace f_i with the $\mathscr{C}_0$ morphisms $\widetilde{f_i} \in \hom^{0}_{\mathscr{C}}(\Sigma^{\alpha} A, B)$, where $\widetilde{f_i} = f_i \circ (\eta_{\alpha,0})_A$. The property $f_1 \simeq_r f_2$ is equivalent to

$$\widetilde{f_1} \circ \eta_r^{\Sigma^{\alpha} A} \simeq_0 \widetilde{f_2} \circ \eta_r^{\Sigma^{\alpha} A},$$

which is a relation in $\mathscr{C}_0$.

(b) Shift functors are natural in many geometric examples. Nonetheless, for a given persistence category $\mathscr{C}$, the existence of a shift functor Σ on $\mathscr{C}$ is a constraining additional structure. In particular, a persistence category $\mathscr{C}$ endowed with a shift functor Σ contains considerable redundant information. Indeed, the isomorphism (2.10) implies that all the morphisms in $\mathscr{C}$ are determined by the morphisms in $\mathscr{C}_0$ together with Σ. In other words, given a category $\mathscr{C}_0$ endowed with a shift functor Σ (appropriately defined), one can define a persistence category $\mathscr{C}$ with the same objects as $\mathscr{C}_0$ by using (2.10) to define morphisms of arbitrary shifts out of the morphisms in $\mathscr{C}_0$. We will see such an example in Section 2.2.4.

(c) There is an obvious way to formally complete any persistence category $\mathscr{C}$ to a larger persistence category $\widetilde{\mathscr{C}}$ that is endowed with a canonical shift functor.

This is achieved by formally adding objects $\Sigma^r X$ for each $r \in \mathbb{R}$ and $X \in \mathrm{Obj}(\mathcal{C})$, and defining morphisms such that the relations in Remark 2.16 are satisfied. In view of this and of the redundancy at point (b), one could prefer to replace the notion of a persistence category with a structure consisting of a category – corresponding to $\mathcal{C}_0$ – and a shift functor. This leads to an equivalent formalism. We stick in this memoir with the formalism of persistence categories as introduced in Definition 2.6, as we found it easiest to handle in algebraic manipulations and because it corresponds naturally to most of our geometric examples.

2.2.4 An example of a persistence category

We give an example of a persistence category that is constructed from persistence **k**-modules. To some extent, this is the motivation of the definition of a persistence category. Recall that for a persistence **k**-module $\mathbb{V}$, $\mathbb{V}[r]$ denotes another persistence **k**-module which comes from an r-shift from $\mathbb{V}$ in the sense that

$$\mathbb{V}[r]_t = \mathbb{V}_{r+t} \quad \text{and} \quad \iota^{\mathbb{V}[r]}_{s,t} = \iota^{\mathbb{V}}_{s+r,t+r}.$$

A persistence morphism $f : \mathbb{V} \to \mathbb{W}$ is an $\mathbb{R}$-family of morphisms $f = \{f_t\}$ that commutes with the persistence structure maps of $\mathbb{V}$ and $\mathbb{W}$, i.e., $f_t \circ \iota^{\mathbb{V}}_{s,t} = \iota^{\mathbb{W}}_{s,t} \circ f_s$. Similarly, one can define an r-shifted persistence morphism $f[r]$, where $(f[r])_t = f_{r+t}$.

Let $\mathcal{P}\mathrm{Mod}_{\mathbf{k}}$ be the category of persistence **k**-modules; then we claim that $\mathcal{P}\mathrm{Mod}_{\mathbf{k}}$ can be enriched to be a persistence category $\mathcal{C}^{\mathcal{P}\mathrm{Mod}_{\mathbf{k}}}$. Indeed, let $\mathrm{Obj}(\mathcal{C}^{\mathcal{P}}\mathrm{Mod}_{\mathbf{k}}) = \mathrm{Obj}(\mathcal{P}\mathrm{Mod}_{\mathbf{k}})$, and for objects $\mathbb{V}$, $\mathbb{W}$ in $\mathcal{P}\mathrm{Mod}_{\mathbf{k}}$, define

$$\hom_{\mathcal{C}^{\mathcal{P}\mathrm{Mod}_{\mathbf{k}}}}(\mathbb{V}, \mathbb{W}) := \left\{\{\hom_{\mathcal{P}\mathrm{Mod}_{\mathbf{k}}}(\mathbb{V}, \mathbb{W}[r])\}_{r\in\mathbb{R}}; \{i_{r,s}\}_{r,s\in\mathbb{R},\, r\leq s}\right\}. \tag{2.15}$$

Here, $\hom^r_{\mathcal{C}^{\mathcal{P}\mathrm{Mod}_{\mathbf{k}}}}(\mathbb{V}, \mathbb{W}) = \hom_{\mathcal{P}\mathrm{Mod}_{\mathbf{k}}}(\mathbb{V}, \mathbb{W}[r])$, and $\hom_{\mathcal{P}\mathrm{Mod}_{\mathbf{k}}}(\cdot, \cdot)$ consists of persistence morphisms. For any $r \leq s$, the well-defined persistence morphism $\iota^{\mathbb{W}}_{\cdot+r,\cdot+s} : \mathbb{W}[r] \to \mathbb{W}[s]$ induces structure maps $i_{r,s} := \iota^{\mathbb{W}}_{\cdot+r,\cdot+s}\circ$ in (2.15). Moreover, the composition $\circ_{(r,s)} : \hom^r_{\mathcal{C}^{\mathcal{P}\mathrm{Mod}_{\mathbf{k}}}}(\mathbb{U}, \mathbb{V}) \times \hom^s_{\mathcal{C}^{\mathcal{P}\mathrm{Mod}_{\mathbf{k}}}}(\mathbb{V}, \mathbb{W}) \to \hom^{r+s}_{\mathcal{C}^{\mathcal{P}\mathrm{Mod}_{\mathbf{k}}}}(\mathbb{U}, \mathbb{W})$ is defined by

$$(f, g) \mapsto g[r] \circ f,$$

where we use the identification $\hom_{\mathcal{P}\mathrm{Mod}_{\mathbf{k}}}(\mathbb{V}, \mathbb{W})[r] = \hom_{\mathcal{P}\mathrm{Mod}_{\mathbf{k}}}(\mathbb{V}[r], \mathbb{W}[r])$ for any $r \in \mathbb{R}$. Moreover, for the following diagram, where $r \leq r'$ and $s \leq s'$:

$$\begin{tikzcd}
\hom_{\mathcal{P}\mathrm{Mod}_{\mathbf{k}}}(\mathbb{U}, \mathbb{V}[r]) \times \hom_{\mathcal{P}\mathrm{Mod}_{\mathbf{k}}}(\mathbb{V}, \mathbb{W}[s]) \arrow[r, "\circ_{(r,s)}"] \arrow[d, "i_{r,r'} \times i_{s,s'}"'] & \hom_{\mathcal{P}}(\mathbb{U}, \mathbb{W}[r+s]) \arrow[d, "i_{r+s,r'+s'}"] \\
\hom_{\mathcal{P}\mathrm{Mod}_{\mathbf{k}}}(\mathbb{U}, \mathbb{V}[r']) \times \mathrm{Hom}_{\mathcal{P}\mathrm{Mod}_{\mathbf{k}}}(\mathbb{V}, \mathbb{W}[s']) \arrow[r, "\circ_{(r',s')}"] & \hom_{\mathcal{P}}(\mathbb{U}, \mathbb{W}[r'+s'])
\end{tikzcd} \tag{2.16}$$

we have

$$\begin{aligned}(\circ_{(r',s')} \circ (i_{r,r'} \times i_{s,s'}))(f,g) &= \circ_{(r',s')}(i_{r,r'}(f), i_{s,s'}(g)) \\ &= i_{s,s'}(g)[r'] \circ i_{r,r'}(f) \\ &= (\iota^{\mathbb{W}}_{\cdot+s,\cdot+s'} \circ g)[r'] \circ (\iota^{\mathbb{W}}_{\cdot+r,\cdot+r'} \circ f) \\ &= \iota^{\mathbb{W}}_{\cdot+r'+s,\cdot+r'+s'} \circ g[r'] \circ \iota^{\mathbb{V}}_{\cdot+r,\cdot+r'} \circ f \\ &= \iota^{\mathbb{W}}_{\cdot+r'+s,\cdot+r'+s'} \circ \iota^{\mathbb{W}}_{\cdot+r+s,\cdot+r'+s} \circ g[r] \circ f \\ &= \iota^{\mathbb{W}}_{\cdot+r+s,\cdot+r'+s'} \circ (g[r] \circ f) \\ &= (\iota_{r+s,r'+s'} \circ \circ_{(s,t)})(f,g),\end{aligned}$$

where the fifth equality is due to the fact that g is a persistence morphism (so, in particular, it commutes with the persistence structure maps). Therefore, the diagram (2.16) is commutative and $\mathcal{C}^{\mathcal{P}\mathrm{Mod}_{\mathbf{k}}}$ is a persistence category in the sense of Definition 2.6.

Since $\hom^0_{\mathcal{C}^{\mathcal{P}\mathrm{Mod}_{\mathbf{k}}}}(\mathbb{V},\mathbb{W}) = \hom_{\mathcal{P}\mathrm{Mod}_{\mathbf{k}}}(\mathbb{V},\mathbb{W})$, we have $\mathcal{C}_0^{\mathcal{P}\mathrm{Mod}_{\mathbf{k}}} = \mathcal{P}\mathrm{Mod}_{\mathbf{k}}$. An example of a persistence endofunctor on $\mathcal{C}^{\mathcal{P}\mathrm{Mod}_{\mathbf{k}}}$, denoted by $\Sigma^\alpha : \mathcal{C}^{\mathcal{P}\mathrm{Mod}_{\mathbf{k}}} \to \mathcal{C}^{\mathcal{P}\mathrm{Mod}_{\mathbf{k}}}$, is defined by

$$\Sigma^\alpha(\mathbb{V}) := \mathbb{V}[-\alpha] \quad \text{and} \quad \Sigma^\alpha(f) = f[-\alpha] \tag{2.17}$$

for any $\alpha \in \mathbb{R}$. It is immediate to see that Σ^α is a persistence endofunctor on $\mathcal{C}^{\mathcal{P}\mathrm{Mod}_{\mathbf{k}}}$ for any $\alpha \in \mathbb{R}$ in the sense of Definition 2.11.

We now define a shift functor on $\mathcal{C}^{\mathcal{P}\mathrm{Mod}_{\mathbf{k}}}$, denoted by

$$\Sigma : (\mathbb{R},+) \to \mathcal{P}\mathrm{Fun}(\mathcal{C}^{\mathcal{P}\mathrm{Mod}_{\mathbf{k}}}),$$

by

$$\Sigma(\alpha) := \Sigma^\alpha \text{ as defined in (2.17)} \quad \text{and} \quad \eta_{\alpha,\beta} = \mathbb{1}_{\cdot[-\alpha]}$$

for any $\alpha,\beta \in \mathbb{R}$. Indeed, evaluate $\eta_{\alpha,\beta}$ on any object $\mathbb{V}$:

$$\begin{aligned}(\eta_{\alpha,\beta})_{\mathbb{V}} = \mathbb{1}_{\mathbb{V}[-\alpha]} &\in \hom_{\mathcal{P}\mathrm{Mod}_{\mathbf{k}}}(\mathbb{V}[-\alpha], \mathbb{V}[-\alpha]) \\ &= \hom_{\mathcal{P}\mathrm{Mod}_{\mathbf{k}}}(\mathbb{V}[-\alpha], \mathbb{V}[-\beta+\beta-\alpha]) \\ &= \hom^{\beta-\alpha}_{\mathcal{C}^{\mathcal{P}\mathrm{Mod}_{\mathbf{k}}}}(\mathbb{V}[-\alpha], \mathbb{V}[-\beta]) \\ &= \hom^{\beta-\alpha}_{\mathcal{C}^{\mathcal{P}\mathrm{Mod}_{\mathbf{k}}}}(\Sigma^\alpha\mathbb{V}, \Sigma^\beta\mathbb{V}).\end{aligned}$$

In other words, $\eta_{\alpha,\beta}$ is a persistence natural transformation of shift $\beta-\alpha$ as in Definition 2.12. Therefore, Σ defines a shift functor on $\mathcal{C}^{\mathcal{P}\mathrm{Mod}_{\mathbf{k}}}$.

Finally, for each $r \geq 0$, recall that the notation $\eta_r^{\mathbb{V}}$ in (2.14) denotes the composition $i_{-r,0} \circ (\eta_{r,0})_{\mathbb{V}}$. In particular, $\eta_r^{\mathbb{V}} \in \hom^0_{\mathcal{C}^{\mathcal{P}\mathrm{Mod}_{\mathbf{k}}}}(\Sigma^r\mathbb{V},\mathbb{V}) = \hom_{\mathcal{P}\mathrm{Mod}_{\mathbf{k}}}(\mathbb{V}[-r],\mathbb{V})$ equals the composition

$$\mathbb{V}[-r] \xrightarrow{\mathbb{1}_{\mathbb{V}[-r]}} \mathbb{V}[-r] \xrightarrow{\iota^{\mathbb{V}}_{\cdot+r,\cdot}\circ} \mathbb{V},$$

which is just $\iota^{\mathbb{V}}_{\cdot - r, \cdot}$, the persistence structure maps of $\mathbb{V}$. Assume that objects in $\mathscr{P}\mathrm{Mod}_{\mathbf{k}}$ admit sufficient regularities so that they can be equivalently described via barcodes (see [25]). In this case, by Lemma 2.18 (i), the r-acyclic objects in $\mathcal{C}^{\mathscr{P}\mathrm{Mod}_{\mathbf{k}}}$ are precisely those persistence $\mathbf{k}$-modules with only bars of length at most r in their barcodes (see [57–59]).

Remark 2.20. (a) The way that the category $\mathscr{P}\mathrm{Mod}_{\mathbf{k}}$ is enriched to $\mathcal{C}^{\mathscr{P}\mathrm{Mod}_{\mathbf{k}}}$ above is also investigated in the recent work [13, Section 10]. In particular, the morphism set defined in (2.15) coincides with the enriched morphism set in [13, Proposition 10.2], and $\mathcal{C}^{\mathscr{P}\mathrm{Mod}_{\mathbf{k}}}$ is similar to $\underline{\mathrm{Mod}}_R^P$ in [13, Proposition 10.3] (when taking $R = \mathbf{k}$ and $P = \mathbb{R}$).

(b) The notion of a persistence category endowed with shift functors is very natural in persistence considerations, as already mentioned in point (a) above. The same notion appears in [51] under the name of a *locally persistent category*, and the 0-level (from Definition 2.9) appears in that work, where it is called the *underlying category* of the respective locally persistent category. The definition of interleaving of persistence modules adapts trivially to this context – of persistence categories or, equivalently, locally persistent categories – to provide a (pseudo)distance, possibly infinite, on the objects of such a category, as in Definition 2.84. The work [51] analyzes and establishes key properties – for instance, completeness – for interleaving distances of this sort under certain assumptions – such as existence of products, or co-products, or a model structure, or existence of limits – on the 0-level category.

Starting from Section 2.3, we focus on properties of persistence categories with 0-levels that have the structure of triangulated categories. In practice, this means that they are often homotopy categories of other categories. From this perspective, while we work, in some sense, at the homotopy level, [51] is geared towards considering 0-levels that directly have a model category structure. In our case, the triangulation is tied to the persistence structure by some simple axioms. If $\mathcal{C}$ is such a category, called a *triangulated persistence category (TPC)*, then we will see that the ∞-level is endowed with a categorical weight induced by the persistence structure, and the general machinery in Section 2.1 leads to a class of fragmentation (pseudo)metrics on the objects of $\mathcal{C}$. These fragmentation metrics extend, on the one hand, interleaving-type metrics and, on the other hand, complexity measurements such as those mentioned in Remark 1.2; they are also similar to classical notions in topology such as cone-length [19]. In Section 3 we show that certain derived Fukaya categories admit TPC refinements and thus their objects are endowed with persistence fragmentation metrics. The interest of this class of fragmentation metrics in this symplectic context is that, under favourable geometric assumptions, these metrics are both non-degenerate and finite while interleaving-type distances take infinite values.

Some elementary relations between interleaving and the rest of the algebraic machinery in the memoir appear in Section 2.4.3.4. Moreover, it is likely that, in some cases, some of the deeper properties of the interleaving distances discussed in [51] can be related in more substantial ways to our fragmentation metrics; however, we will not pursue these questions here.

2.3 Triangulated persistence categories

This section is central for the rest of the memoir. It investigates triangulation properties in the context of persistence categories. We start with two key definitions in Section 2.3.1: Definition 2.21, which introduces the notion of a triangulated persistence category (TPC) – a persistence category $\mathcal{C}$ with a shift functor whose 0-level $\mathcal{C}_0$ is triangulated – and Definition 2.26, which introduces the notion of an r-isomorphism. We then discuss a number of useful properties of r-isomorphisms. These properties are, in some sense, "shift"-controlled analogs of properties that appear when defining the Verdier localization of a triangulated category. Indeed, in Section 2.3.2 we see that the acyclics of finite order in $\mathcal{C}$ form a triangulated subcategory $\mathcal{A}\mathcal{C}_0$ of $\mathcal{C}_0$, and that the Verdier localization of $\mathcal{C}_0$ with respect to this subcategory is the ∞-level category $\mathcal{C}_\infty$ of $\mathcal{C}$, which is therefore itself triangulated. The main aim of our algebraic formalism is to construct a notion of weighted exact triangles in $\mathcal{C}$ – and this is pursued in Section 2.3.3, in particular in Definition 2.3.3. We then discuss in Section 2.3.4 associated fragmentation pseudometrics.

2.3.1 Main definitions

We will consistently use below the characterization of r-equivalence in Lemma 2.17 as well as that of r-acyclics in Lemma 2.18.

Definition 2.21. A *triangulated persistence category* is a persistence category $\mathcal{C}$ endowed with a shift functor Σ such that the following three conditions are satisfied:

(i) The 0-level category $\mathcal{C}_0$ is triangulated with a translation automorphism denoted by T. Note that, in particular, $\mathcal{C}_0$ is additive, and we further assume that the restriction of the persistence structure of $\mathcal{C}$ to $\mathcal{C}_0$ is compatible with the additive structure on $\mathcal{C}_0$ in the obvious way. Specifically, this means that $\hom^r_{\mathcal{C}}(A \oplus B, C) = \hom^r_{\mathcal{C}}(A, C) \oplus \hom^r_{\mathcal{C}}(B, C)$ for all $r \leq 0$, and the persistence maps $i_{r,s}$, $r \leq s \leq 0$, are compatible with this splitting. The same also holds for $\hom^r_{\mathcal{C}}(A, B \oplus C)$.

(ii) The restriction of Σ^r to $\mathrm{End}(\mathcal{C}_0)$ is a triangulated endofunctor of $\mathcal{C}_0$ for each $r \in \mathbb{R}$. Note that each of the functors Σ^r, being a triangulated functor, is also assumed to be additive. We further assume that all the natural transformations $\eta_{r,s} : \Sigma^r \to \Sigma^s$, $s, r \in \mathbb{R}$, are compatible with the additive structure on $\mathcal{C}_0$.

(iii) For any $r \geq 0$ and any $A \in \mathrm{Obj}(\mathcal{C})$, the morphism $\eta_r^A : \Sigma^r A \to A$ defined in (2.14) embeds into an exact triangle of $\mathcal{C}_0$,

$$\Sigma^r A \xrightarrow{\eta_r^A} A \to K \to T\Sigma^r A,$$

such that K is r-acyclic.

Example 2.22. The fundamental example of a triangulated persistence category is provided by the homotopy category of filtered (co)-chain complexes over a field $\mathbf{k}$, $H^0\mathcal{FK}_{\mathbf{k}}$. The objects are filtered cochain complexes (C, ∂) over $\mathbf{k}$, with

$$C : \cdots \subset C^{\leq \alpha} \subset C^{\leq \beta} \subset \cdots \quad (\alpha \leq \beta \in \mathbb{R}),$$

and ∂ does not increase filtration; hence each $C^{\leq \alpha}$ is itself a cochain complex – a more complete description is given in Section 2.5.2. The morphisms are homotopy classes of filtered chain maps

$$\hom^r(C, C') = \{f : C \to C' \mid f \text{ is a chain map, } f(C^{\leq \alpha}) \subset (C')^{\leq \alpha + r}\}/\simeq_r,$$

where the relation $\simeq_r$ is cochain homotopy via a homotopy $h : C^* \to C^{*-1}$ such that $h(C^{\leq \alpha}) \subset (C')^{\leq \alpha + r}$. The translation functor is defined as usual by translating degree (and keeping the filtration unchanged), namely $(TC)^i = C^{i+1}$, and with the obvious action on morphisms. The shift functor acts on objects by $[\Sigma^r C]^{\leq \alpha} = C^{\leq \alpha - r}$ with the obvious differential and the obvious action on morphisms. The 0-level of $H^0\mathcal{FK}_{\mathbf{k}}$, $[H^0\mathcal{FK}_{\mathbf{k}}]_0$, is the subcategory with the same objects but whose morphisms come only from filtration-preserving chain maps. This is a triangulated category because, for chain-preserving maps, the mapping-cone construction is filtration preserving. The r-acyclics in this case are filtered complexes C such that $\mathbb{1}_C$ is chain homotopic to 0 through a chain homotopy that shifts filtration by at most r.

Note that we also have the (full) subcategory $\mathcal{FK}_k^{\mathrm{fg}} \subset \mathcal{FK}_{\mathbf{k}}$ of *finitely generated* filtered cochain complexes, which is also a TPC. The category $[H^0\mathcal{FK}_{\mathbf{k}}^{\mathrm{fg}}]_\infty$ is equivalent to the usual homotopy category of finitely generated cochain complexes.

Remark 2.23. (a) Given that $\mathcal{C}_0$ is triangulated, the functors $\hom_{\mathcal{C}}^s(X, -)$ and $\hom_{\mathcal{C}}^s(-, X)$ are exact for $s = 0$. This property, together with the fact that Σ^s is a triangulated functor for all s and the relations in Remark 2.16, implies that these functors are exact *for all* $s \in \mathbb{R}$.

(b) Condition (ii) requires in particular that Σ and T commute. Thus, $T\Sigma^r X = \Sigma^r TX$ for each object X, and for any $f \in \hom_{\mathcal{C}}^0(A, B)$ we have $\Sigma^r Tf = T\Sigma^r f$. Additionally, each Σ^r preserves the additive structure of $\mathcal{C}_0$, and it takes each exact triangle in $\mathcal{C}_0$ to an exact triangle. Moreover, the assumptions above imply that we have canonical isomorphisms

$$\hom_{\mathcal{C}}^r(A \oplus B, C) \cong \hom_{\mathcal{C}}^r(A, C) \oplus \hom_{\mathcal{C}}^r(B, C) \quad \forall r \in \mathbb{R},$$

and the persistence maps $i_{r,s}$ are compatible with these isomorphisms. The same holds also for $\hom_{\mathcal{C}}^{r}(A, B \oplus C)$. Finally, the maps η_r^A, $A \in \mathrm{Obj}(\mathcal{C}_0)$, $r \geq 0$, are compatible with the additive structure on $\mathcal{C}_0$.

Notice also that the functor T extends from $\mathcal{C}_0$ to a functor on $\mathcal{C}$. Indeed, T is already defined on all the objects of $\mathcal{C}$ as well as on all the morphisms of shift 0. For $f \in \hom^r(A, B)$, we define $Tf = T(f \circ (\eta_{r,0})_A) \circ (\eta_{0,r})_{TA}$. It is easily seen that with this definition T is indeed a functor and it immediately follows that $T((\eta_{r,s})_A) = (\eta_{r,s})_{TA}$ for all objects A in $\mathcal{C}$ and $r, s \in \mathbb{R}$. Further, by using the identifications in Remark 2.16, it also follows that T is a persistence functor. In particular, we have $\eta_r^{TA} = T\eta_r^A$ for each object A in $\mathcal{C}$.

(c) Given that 0-isomorphisms preserve r-acyclicity, as noted in Section 2.2.1, condition (iii) in Definition 2.21 does not depend on the specific extension of η_r^A to an exact triangle.

(d) In a way similar to Remark 2.19 (b), the data encoded in a triangulated persistence category is determined by the triangulated category $\mathcal{C}_0$ together with an appropriate shift functor $\Sigma : (\mathbb{R}, +) \to \mathrm{End}(\mathcal{C}_0)$. From this data it is easy to define a triangulated persistence category $\mathcal{C}$ with the same objects as $\mathcal{C}_0$, that has $\mathcal{C}_0$ as its 0-level and with morphisms endowed with a persistence structure such that (2.12) and (2.13) are satisfied with respect to the given shift functor Σ. We do not give further details here, but we will see such an example in Section 2.5.4.

It is clear that TPCs form a category with respect to persistence functors that respect the additional structure. The appropriate notion is formalized below.

Definition 2.24. Let $\mathcal{C}'$ and $\mathcal{C}''$ be two TPCs. A persistence functor $\mathcal{F} : \mathcal{C}' \to \mathcal{C}''$ is called a *TPC functor* if it satisfies the following conditions:

(i) $\mathcal{F}$ is compatible with the shift functors $\Sigma_{\mathcal{C}'}$, $\Sigma_{\mathcal{C}''}$ of the two categories, namely $\mathcal{F} \circ \Sigma_{\mathcal{C}'}^r = \Sigma_{\mathcal{C}''}^r \circ \mathcal{F}$ for all $r \in \mathbb{R}$, and $(\eta_{r,s})_{\mathcal{F}(A)} = \mathcal{F}((\eta_{r,s})_A)$ for all $A \in \mathrm{Obj}(\mathcal{C}')$ and all r, s.

(ii) The 0-level $\mathcal{F}|_{\mathcal{C}_0'} : \mathcal{C}_0' \to \mathcal{C}_0''$ of the functor $\mathcal{F}$ is triangulated.

In the definition above, the fact that $\mathcal{F}|_{\mathcal{C}_0'}$ maps $\mathcal{C}_0'$ to $\mathcal{C}_0''$ follows from the assumption that $\mathcal{F}$ is a persistence functor. Modifying Definition 2.14, we now give the definition of an equivalence between TPCs.

Definition 2.25. Let $\mathcal{C}'$ and $\mathcal{C}''$ be two TPCs. A TPC functor $\mathcal{F} : \mathcal{C}' \to \mathcal{C}''$ is called a *TPC equivalence* if there exists a TPC functor $\mathcal{G} : \mathcal{C}'' \to \mathcal{C}'$ such that both $\mathcal{F} \circ \mathcal{G}$ and $\mathcal{G} \circ \mathcal{F}$ are isomorphic to the respective identity functors via persistence natural transformations of shift 0.

Standard results in triangulated categories (e.g., [39, Section 1.2]) imply that a TPC functor $\mathcal{F} : \mathcal{C}' \to \mathcal{C}''$ is a TPC equivalence if and only if it is an equivalence of persistence categories.

Definition 2.26. Let $\mathcal{C}$ be a triangulated persistence category. A map $f \in \hom^0_{\mathcal{C}}(A,B)$ is said to be an r*-isomorphism* (from A to B) if it embeds into an exact triangle in $\mathcal{C}_0$

$$A \xrightarrow{f} B \to K \to TA$$

such that $K \simeq_r 0$. In this case, we write $f : A \simeq_r B$.

Remark 2.27. (a) If f is an r-isomorphism, then f is an s-isomorphism for any $s \geq r$. It is not difficult to check, and we will see this explicitly in Remark 2.30, that for $r = 0$ this definition is equivalent to the notion of 0-isomorphism introduced before (namely, isomorphism in the category $\mathcal{C}_0$).

(b) The relation $T(\eta^K_r) = \eta^{TK}_r$ implies that TK is r-acyclic if and only if K is r-acyclic and, therefore, f is an r-isomorphism if and only if Tf is one.

(c) From Definition 2.21 (iii) we see that for any $r \geq 0$ and $A \in \mathrm{Obj}(\mathcal{C})$ we have

$$\eta^A_r : \Sigma^r A \simeq_r A.$$

Proposition 2.28. *Any triangulated persistence category $\mathcal{C}$ has the following properties:*

(i) *If $f : A \to B$ is an r-isomorphism, then there exist $\phi \in \hom^0_{\mathcal{C}}(B, \Sigma^{-r} A)$ and $\psi \in \hom^0_{\mathcal{C}}(\Sigma^r B, A)$ such that*

$$\phi \circ f = \eta^A_r \text{ in } \hom^0_{\mathcal{C}}(A, \Sigma^{-r} A) \quad \text{and} \quad f \circ \psi = \eta^B_r \text{ in } \hom^0_{\mathcal{C}}(\Sigma^r B, B).$$

The map ψ is called a right r-inverse *of f and ϕ is a* left r-inverse *of f. They satisfy $\Sigma^r \phi \simeq_r \psi$.*

(ii) *If f is an r-isomorphism, then any two left r-inverses ϕ, ϕ' of f are themselves r-equivalent, and the same conclusion holds for right r-inverses.*

(iii) *If $f : A \simeq_r B$ and $g : B \simeq_s C$, then $g \circ f : A \simeq_{r+s} C$.*

Proof. (i) We first construct ϕ. In $\mathcal{C}_0$, the morphism $f : A \to B$ embeds into an exact triangle $A \xrightarrow{f} B \xrightarrow{g} K \xrightarrow{h} TA$ with $K \simeq_r 0$. Using the fact that Σ and T commute, the following diagram is easily seen to be commutative:

$$\begin{tikzcd}
K \arrow[r, "h"] \arrow[d, "\eta^K_r"'] & TA \arrow[d, "\eta^{TA}_r"] \\
\Sigma^{-r} K \arrow[r, "\Sigma^{-r} h"] & \Sigma^{-r} TA
\end{tikzcd}$$

Thus $\eta^{TA}_r \circ h = \Sigma^{-r} h \circ \eta^K_r = 0$, since K is r-acyclic (and so $\eta^K_r = 0$). By rotating exact triangles in $\mathcal{C}_0$ we obtain a new $\mathcal{C}_0$-exact triangle $K \xrightarrow{h} TA \xrightarrow{Tf} TB \xrightarrow{Tg} TK$

and consider the diagram below (in $\mathcal{C}_0$):

$$\begin{tikzcd}
K \arrow[r, "h"] \arrow[d] & TA \arrow[r, "Tf"] \arrow[d, "\eta_r^{TA}"] & TB \arrow[r, "Tg"] \arrow[d, dashed, "\widetilde{\phi}"] & TK \arrow[d] \\
0 \arrow[r] & \Sigma^{-r} TA \arrow[r, "\mathbb{1}"] & \Sigma^{-r} TA \arrow[r] & 0
\end{tikzcd}$$

The left square commutes, so we deduce the existence of a map

$$\widetilde{\phi} \in \hom^0_{\mathcal{C}}(TB, \Sigma^{-r} TA)$$

that makes the middle and right squares commutative. The desired left inverse of f is $\phi = T^{-1}\widetilde{\phi}$. A similar argument leads to the existence of ψ. We postpone the identity $\Sigma^r \phi \simeq_r \psi$ after the proof of (ii).

(ii) If ϕ, ϕ' are two left inverses of f then $(\phi - \phi') \circ f = 0$. Therefore

$$(\phi - \phi') \circ \eta_r^B = (\phi - \phi') \circ (f \circ \psi) = ((\phi - \phi') \circ f) \circ \psi = 0.$$

Lemma 2.17 implies that $\phi \simeq_r \phi'$. The same argument works for right inverses. We now return to the identity $\Sigma^r \phi \simeq_r \psi$ (using the notation from (i)). We have the following commutative diagram:

$$\begin{tikzcd}
\Sigma^r B \arrow[r, "\psi"] \arrow[rr, bend right, "\eta_r^B"'] & A \arrow[r, "f"] \arrow[rr, bend left, "\eta_r^A"] & B \arrow[r, "\phi"] & \Sigma^{-r} A\,.
\end{tikzcd}$$

Therefore, $\eta_r^A \circ \psi = \phi \circ \eta_r^B$. By the naturality properties of η we also have $\phi \circ \eta_r^B = \eta_r^A \circ \Sigma^r \phi$. Thus, by Lemma 2.17, $\Sigma^r \phi \simeq_r \psi$.

(iii) We will make use of the following lemma.

Lemma 2.29. *If $K \to K'' \to K' \to TK$ is an exact triangle in $\mathcal{C}_0$, $K \simeq_r 0$ and $K' \simeq_s 0$, then $K'' \simeq_{r+s} 0$.*

Proof of Lemma 2.29. We associate the following commutative diagram to the exact triangle in the statement:

$$\begin{tikzcd}
\hom^{\alpha}(K'', K) \arrow[r] \arrow[d] & \hom^{\alpha}(K'', K'') \arrow[r] \arrow[d, "v"] & \hom^{\alpha}(K'', K') \arrow[d, "0"] \\
\hom^{\alpha+s}(K'', K) \arrow[r, "k"] \arrow[d, "0"'] & \hom^{\alpha+s}(K'', K'') \arrow[r, "h"] \arrow[d, "t"] & \hom^{\alpha+s}(K'', K') \\
\hom^{\alpha+r+s}(K'', K) \arrow[r, "n"] & \hom^{\alpha+r+s}(K'', K'') &
\end{tikzcd}$$

Here, the vertical morphisms are the persistence structure maps. The rightmost vertical map and the lower-leftmost vertical map are both 0 due to our hypothesis together with Lemma 2.18. The functor $\hom^{\alpha}(K'', -)$ is exact which implies that $t \circ v = 0$ and, again by Lemma 2.18, we deduce $K'' \simeq_{r+s} 0$. ■

Returning to the proof of Proposition 2.28, point (iii) now follows immediately by using the octahedral axiom to construct the following commutative diagram in $\mathcal{C}_0$:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d] & B \arrow[r] \arrow[d, "g"] & K \arrow[d] \\
A \arrow[r, "g \circ f"] \arrow[d] & C \arrow[r] \arrow[d] & K'' \arrow[d] \\
0 \arrow[r] & K' \arrow[r] & K'
\end{tikzcd}$$

with exact rows and columns, and applying Lemma 2.29 to the rightmost column. ■

Remark 2.30. (a) Points (i) and (ii) in Proposition 2.28 imply that the notion of 0-isomorphism $f : A \to B$, as given in Definition 2.26 for $r = 0$, is equivalent to an isomorphism in $\mathcal{C}_0$. In particular, for $r = 0$, f admits a unique inverse in $\hom^0_{\mathcal{C}}(B, A)$.

(b) Point (iii) in Proposition 2.28 shows that being r-isomorphic (for a fixed r) cannot be expected to be an equivalence relation on $\mathrm{Obj}(\mathcal{C})$ (unless $r = 0$).

Here are several useful additional results and corollaries.

The first is a version of the five-lemma in the TPC context.

Proposition 2.31. *Consider the following commutative diagram in $\mathcal{C}_0$:*

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "u"'] & B \arrow[r] \arrow[d, "v"'] & C \arrow[r] \arrow[d, dashed, "w"'] & TA \arrow[d, "Tu"'] \\
A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & TA'
\end{tikzcd} \tag{2.18}$$

such that the two rows are exact triangles. If u is an r-isomorphism and v is an s-isomorphism, then:

(i) *There exists an $(r+s)$-isomorphism w making the diagram commutative.*

(ii) *Any w making the diagram commutative is a $3(r+s)$-isomorphism.*

Proof. Part (i) of the proposition is an easy consequence of Lemma 2.29 and the octahedral axiom in $\mathcal{C}_0$.

To show part (ii) we will use the following notation. For an object $Z \in \mathrm{Obj}(\mathcal{C})$, we denote by H_Z the functor $\hom_{\mathcal{C}}(Z, -) : \mathcal{C} \to \mathcal{C}^{\mathcal{P}\mathrm{Mod}_{\Bbbk}}$ (the target of this functor is an obvious enrichment of the category of persistence modules; see Section 2.2.4).

Similarly, denote by H^*_Z the (contravariant) functor $\hom_{\mathcal{C}}(-, Z)$. Further, if U and V are persistence modules and $F : U \to V$ is a morphism in $\mathcal{C}^{\mathcal{P}\mathrm{Mod}_{\mathbf{k}}}$, we say that F is:

- r-epi if for all $y \in V$ there exists $x \in U$ such that $F(x) = i_r(y)$. (Here and in what follows, i_r stands for the persistence structural map on $\hom_C(A, B)$ whose restriction to $\hom^\alpha(A, B)$ is the map $i_{\alpha,\alpha+r} : \hom^\alpha_C(A, B) \to \hom^{\alpha+r}_C(A, B)$, $\alpha \in \mathbb{R}$.)
- r-mono if for all $x \in U$ with $F(x) = 0$ we have $i_r(x) = 0$.

The proof is based on properties of right and left inverses that are contained in the following statement.

Lemma 2.32. *Let* $\phi \in \hom_{\mathcal{C}_0}(M, N)$.

(i) *ϕ admits a right r-inverse if and only if $H_Z(\phi)$ is r-epi for all $Z \in \mathrm{Obj}(\mathcal{C})$. The existence of a right r-inverse implies that $H^*_Z(\phi)$ is r-mono for all Z.*

(ii) *ϕ admits a left r-inverse if and only if $H^*_Z(\phi)$ is r-epi for all $Z \in \mathrm{Obj}(\mathcal{C})$. If such a left r-inverse exists, then $H_Z(\phi)$ is r-mono for all Z.*

(iii) *Consider a morphism of exact triangles as in (2.18). Assume that for every $Z \in \mathrm{Obj}(\mathcal{C})$, $H_Z(u)$ is r-epi and s-mono and that $H_Z(v)$ is r'-epi and s'-mono. Then $H_Z(w)$ is $(r' + r + s')$-epi and $(s + s' + r)$-mono for all Z.*

(iv) *If ϕ admits a right r-inverse and a left s-inverse, then ϕ is a $(r + s)$-isomorphism.*

Proof of Lemma 2.32. We start with (i). Assume that ϕ admits a right r-inverse $\psi \in \hom_{\mathcal{C}_0}(\Sigma^r N, M)$. Let $h \in \hom^\alpha_{\mathcal{C}}(Z, N)$. We have $h \circ \eta^Z_r = \eta^N_r \circ \Sigma^r h = \phi \circ (\psi \circ \Sigma^r h) = H_Z(\phi)(\psi \circ \Sigma^r h)$, with $\psi \circ \Sigma^r h \in \hom^\alpha_{\mathcal{C}}(\Sigma^r Z, M) = \hom^{r+\alpha}_{\mathcal{C}}(Z, M)$. Therefore

$$i_{\alpha,\alpha+r}(h) = (h \circ \eta^Z_r) \circ (\eta_{0,r})_Z = \phi \circ \big(\psi \circ \Sigma^r h \circ (\eta_{0,r})_Z\big).$$

It follows that $H_Z(\phi)$ is r-epi. Conversely, assume that $H_N(\phi)$ is r-epi. Then η^N_r is in the image of $H_N(\phi)$, which means that ϕ admits a right r-inverse. To finish with (i), let $k \in \hom^\alpha_{\mathcal{C}}(N, Z)$ be such that $k \circ \phi = 0$. We write $0 = k \circ \phi = k \circ \phi \circ \psi = k \circ \eta^N_r = \eta^Z_r \circ \Sigma^r k$, which means $i_r(k) = 0 \in \hom^{r+\alpha}_{\mathcal{C}}(N, Z)$. Thus $H^*_Z(\phi)$ is r-mono.

Point (ii) is entirely similar to (i).

For point (iii), we first notice that, by assumption, the maps $H_Z(u)$ and $H_Z(v)$ satisfy the epi and mono conditions with constants that are the same for all objects Z in $\mathcal{C}$. It is immediate to see that $H_Z(Tu)$ is r-epi (respectively, s-mono) if and only if $H_{T^{-1}Z}(u)$ is r-epi (and, respectively, s-mono). This implies that $H_Z(T^l u)$ is r-epi and s-mono, and that $H_Z(T^l v)$ is r'-epi and s'-mono for all $l \in \mathbb{Z}$ (and all Z). We now apply the exact functor H_Z to the diagram (2.18), and we obtain two long exact sequences of persistence modules related by comparison morphisms. The desired conclusion follows by direct diagram chasing, as in the proof of the classical five-lemma.

For point (iv), we use the triangulated structure of $\mathscr{C}_0$ to obtain an object K and the following commutative diagram in $\mathscr{C}_0$:

$$\begin{tikzcd}
M \arrow[r, "id"] \arrow[d, "id"'] & M \arrow[r] \arrow[d, "\phi"'] & 0 \arrow[r] \arrow[d, "p"'] & TM \arrow[d, "id"'] \\
M \arrow[r, "\phi"] & N \arrow[r] & K \arrow[r] & TM
\end{tikzcd} \tag{2.19}$$

whose rows are exact triangles. Given that ϕ admits a right r-inverse, we deduce from (i) that $H_Z(\phi)$ is r-epi for all objects Z in $\mathscr{C}$. The existence of a left s-inverse implies, by (ii), that $H_Z(\phi)$ is also s-mono for all Z. We now use (iii) to deduce that $H_K(p)$ is $(r+s)$-epi. This implies that $i_{r+s}(id_K)$ is in the image of $H_K(p)$. But p is the null map, so $i_{r+s}(id_K) = 0$. Hence $K \simeq_{r+s} 0$. It follows that ϕ is an $(r+s)$-isomorphism. ∎

We now return to the proof of the second point of Proposition 2.31 with the notation and the assumptions there. We denote by K any object that completes the map w to an exact triangle

$$C \xrightarrow{w} C' \to K \to TC.$$

An r-isomorphism admits both right and left r-inverses. Thus, points (i) and (ii) of the lemma show that $H_Z(u)$ is r-epi and r-mono and that $H_Z(v)$ is s-epi and s-mono for all objects Z in $\mathscr{C}$. Point (iii) of the lemma then implies that $H_Z(w)$ is $(2s+r)$-epi and $(2r+s)$-mono for all Z. We now consider a diagram just as (2.19) but with $M = C$, $N = C'$, $\phi = w$, and we use point (iii) of the lemma to deduce that the map $H_K(p)$ is $3(r+s)$-epi, which means that K is $3(r+s)$-acyclic. ∎

Corollary 2.33. *If $f : A \to B$ is an r-isomorphism, then any right inverse $\psi \in$* $\hom^0(\Sigma^r B, A)$ *(given by* (i) *in Proposition* 2.28*) is a $2r$-isomorphism. The same conclusion holds for any left inverse.*

Proof. By the octahedral axiom (in $\mathscr{C}_0$), we have the commutative diagram

$$\begin{tikzcd}
\Sigma^r B \arrow[r] \arrow[d, "\psi"'] & \Sigma^r B \arrow[r] \arrow[d, "\eta_r^B"] & 0 \arrow[d] \\
A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & K \arrow[d] \\
K'' \arrow[r] & K' \arrow[r] & K
\end{tikzcd}$$

where $K'' \to K' \to K \to TK''$ is exact. By (iii) in Definition 2.21, $K' \simeq_r 0$. Therefore, by Lemma 2.29, $K'' \simeq_{2r} 0$ and thus ψ is $2r$-isomorphism. A similar argument applies to the left inverse of f. ∎

Remark 2.34. The fact that left and right inverses of r-isomorphisms are only $2r$-isomorphisms has significant impact on the various algebraic properties of TPCs. However, this seems unavoidable. For example, it is easy to construct examples of r-isomorphisms in the (homotopy) category of filtered cochain complexes that admit a unique right inverse which cannot be better than a $2r$-isomorphism.

The next consequence is immediate but useful, so we state it separately.

Corollary 2.35. *If $f : A \to B$ is an r-isomorphism, then for any $u, u' \in \hom^0_{\mathcal{C}}(B, C)$ such that $u \circ f = u' \circ f$, we have $u \simeq_r u'$, i.e., u and u' are r-equivalent. Similarly, if $v, v' \in \hom^0_{\mathcal{C}}(D, A)$ and $f \circ v = f \circ v'$, then $v \simeq_r v'$.*

Corollary 2.36. *Assume that the following diagram in $\mathcal{C}_0$:*

$$\begin{tikzcd}
K \arrow[r] \arrow[d, "\mathbb{1}_K"'] & A \arrow[r, "\phi"] \arrow[d, "f"'] & A' \arrow[r] \arrow[d, "f'"'] & TK \arrow[d, "\mathbb{1}_{TK}"'] \\
K \arrow[r] & B \arrow[r, "\psi"] & B' \arrow[r] & TK
\end{tikzcd}$$

is commutative, that the two rows are exact, and that $K \simeq_r 0$. Then the induced morphism f' is unique up to r-equivalence.

Proof. Since $K \simeq_r 0$, by definition, ϕ is an r-isomorphism. For any two induced morphisms $f'_1, f'_2 \in \hom^0_{\mathcal{C}}(A', B')$, we have $f'_1 \circ \phi = f'_2 \circ \phi = \psi \circ f$ and the conclusion follows from Corollary 2.35. ∎

Corollary 2.37. *Let $\phi : A \to A'$ be an r-isomorphism. Then for any $f \in \hom^0_{\mathcal{C}}(A, B)$, there exists $f' \in \hom^0_{\mathcal{C}}(A', \Sigma^{-r} B)$ such that the following diagram commutes in $\mathcal{C}_0$:*

$$\begin{tikzcd}
A \arrow[r, "\phi"] \arrow[d, "f"'] & A' \arrow[d, "f'"] \\
B \arrow[r, "\eta_r^B"] & \Sigma^{-r} B
\end{tikzcd}$$

Proof. Since $\phi : A \to A'$ is an r-isomorphism, there exists a left r-inverse denoted by $\psi : A' \to \Sigma^{-r} A$ such that $\psi \circ \phi = \eta_r^A$. Set $f' := \Sigma^{-r} f \circ \psi \in \hom^0_{\mathcal{C}}(A', \Sigma^{-r} B)$. ∎

Similar direct arguments lead to the next consequence.

Corollary 2.38. *Consider the following commutative diagram in $\mathcal{C}_0$:*

$$\begin{tikzcd}
A \arrow[r, "\phi"] \arrow[d, "f"'] & A' \arrow[d, "f'"] \\
B \arrow[r, "\phi'"] & B'
\end{tikzcd}$$

where $f \in \hom^0_{\mathcal{C}}(A, B)$, $f' \in \hom^0_{\mathcal{C}}(A'.B')$, *and* ϕ, ϕ' *are* r*-isomorphisms. Let* ψ, ψ' *be any left inverses of* ϕ, ϕ'*, respectively. Then the diagram*

$$\begin{tikzcd}
A' \arrow[r, "\psi"] \arrow[d, "f'"'] & \Sigma^{-r} A \arrow[d, "\Sigma^{-r} f"] \\
B' \arrow[r, "\psi'"] & \Sigma^{-r} B
\end{tikzcd}$$

is r*-commutative, in the sense that* $\Sigma^{-r} f \circ \psi \simeq_r \psi' \circ f'$*. A similar conclusion holds for right inverses.*

2.3.2 Relation to Verdier localization

Proposition 2.39. *Let* $\mathcal{C}$ *be a triangulated persistence category and let* $\mathcal{AC}$ *be the full subcategory of* $\mathcal{C}$ *with objects the* r*-acyclic objects of* $\mathcal{C}$ *(for all* $r \geq 0$*).*

(i) *The category* $\mathcal{AC}$ *is a triangulated persistence category on its own, with* 0*-level denoted by* $\mathcal{AC}_0$ $(= (\mathcal{AC})_0)$*, the full subcategory of* $\mathcal{C}_0$ *whose objects are those of* $\mathcal{AC}$*.*

(ii) *The infinity level* $\mathcal{C}_\infty$ *of* $\mathcal{C}$ *coincides with the Verdier quotient (a.k.a. localization)* $\mathcal{C}_0/\mathcal{AC}_0$ *of* $\mathcal{C}_0$ *by* $\mathcal{AC}_0$*. In particular,* $\mathcal{C}_\infty$ *is triangulated.*

Remark 2.40. (a) The collection S of all r-isomorphisms (for every $r \geq 0$) forms a multiplicative system in $\mathcal{C}_0$. The Verdier quotient above is the same as the localization $S^{-1}\mathcal{C}_0$ of $\mathcal{C}_0$ by S. For a definition of the localization, see [40, Subsection 1.6].

(b) A result somewhat similar to Proposition 2.39, established for Tamarkin categories, appears in [37, Proposition 6.7].

Proof. For the first point of the proposition we first notice that the subcategory of acyclics, $\mathcal{AC}$, is a persistence category. It is obviously endowed with a shift functor by restricting the shift functor of $\mathcal{C}$. Moreover, its 0-level is clearly a full subcategory of $\mathcal{C}_0$. Finally, Lemma 2.29 implies that $\mathcal{AC}_0$ is a triangulated subcategory of $\mathcal{C}_0$, which implies that it is a TPC.

We now turn to the second point of the proposition. By inspecting the definition of Verdier localization (for instance in [43, Chapter 2]), we see that the localization of $\mathcal{C}_0$ at $\mathcal{AC}_0$, denoted by $\mathcal{C}_0/\mathcal{AC}_0$, is a category with the same objects as $\mathcal{C}_0$ and having as morphisms $A \to B$ equivalence classes of roof diagrams:

$$A \xleftarrow{u} A' \xrightarrow{f'} B$$

with u an r-isomorphism for some $r \geq 0$, and $f' \in \hom_{\mathcal{C}_0}(A', B)$. Two roof diagrams $A \xleftarrow{u} A' \xrightarrow{f'} B$ and $A \xleftarrow{u_1} A'_1 \xrightarrow{f'_1} B$ are equivalent if they are related by a third roof diagram $A \xleftarrow{u_2} A'_2 \xrightarrow{f'_2} B$ in the sense that there are maps a', a'_1 in $\mathcal{C}_0$ making the following diagram commutative:

$$\begin{tikzcd}
A & A' \arrow[l, "u"'] \arrow[r, "f'"] & B \\
A \arrow[u, "\mathbb{1}_A"'] \arrow[d, "\mathbb{1}_A"] & A'_2 \arrow[l, "u_2"'] \arrow[r, "f'_2"] \arrow[u, "a'"'] \arrow[d, "a'_1"] & B \arrow[u, "\mathbb{1}_B"'] \arrow[d, "\mathbb{1}_B"] \\
A & A'_1 \arrow[l, "u_1"'] \arrow[r, "f'_1"] & B
\end{tikzcd}$$

The category $\mathcal{C}_\infty$ appears in Definition 2.9. Its objects are the same as those of $\mathcal{C}_0$ and its morphisms are

$$\hom_{\mathcal{C}_\infty}(A, B) = \varinjlim_{r\to\infty} \hom^r_{\mathcal{C}}(A, B) = \varinjlim_{r\to\infty} \hom_{\mathcal{C}_0}(\Sigma^r A, B),$$

where the second equality comes from (2.9). Given a morphism $f \in \hom_{\mathcal{C}_\infty}(A, B)$, this means that $\bar{f}$ is represented by $\bar{f} : \Sigma^r A \to B$ for some $r \geq 0$ as well as by all compositions $\Sigma^s A \xrightarrow{\eta_{s-r}} \Sigma^r A \xrightarrow{\bar{f}} B$. We now define a functor

$$\Phi : \mathcal{C}_\infty \to \mathcal{C}_0/\mathcal{A}\mathcal{C}_0.$$

It is the identity on objects and for a morphism $f \in \hom_{\mathcal{C}_\infty}(A, B)$ we let $\Phi(f)$ be the equivalence class of a roof diagram $A \xleftarrow{\eta_r} \Sigma^r A \xrightarrow{\bar{f}} B$, where $\bar{f} : \Sigma^r A \to B$ represents f. Any two roof diagrams that are associated to two representatives of f are immediately seen to be equivalent, and as a result Φ is well defined.

It remains to show that Φ is an isomorphism. Surjectivity is immediate. Fix H a roof diagram $A \xleftarrow{a} A' \xrightarrow{b} B$. As a is an r-isomorphism (for some r), we deduce from Proposition 2.28 the existence of a right r-inverse $\phi : \Sigma^r A \to A'$ of a such that $a \circ \phi = \eta^A_r$. Therefore, we may define a new roof diagram H', $A \xleftarrow{\eta_r} \Sigma^r A \xrightarrow{b\circ\phi} B$. The roof diagrams H' and H are clearly equivalent, and thus their equivalence class belongs to the image of Φ.

We now show that Φ is injective. For this we consider the commutative diagram

$$\begin{tikzcd}
A & \Sigma^{r'} A \arrow[l, "\eta^A_{r'}"'] \arrow[r, "f''"] & B \\
A \arrow[u, "\mathbb{1}_A"'] \arrow[d, "\mathbb{1}_A"] & A' \arrow[l, "u"'] \arrow[r, "f"] \arrow[u, "a''"'] \arrow[d, "a'"] & B \arrow[u, "\mathbb{1}_B"'] \arrow[d, "\mathbb{1}_B"] \\
A & \Sigma^r A \arrow[l, "\eta^A_r"'] \arrow[r, "f'"] & B
\end{tikzcd}$$

with each row a roof diagram. We need to show that f'' and f' represent the same element in $\mathcal{C}_\infty$. In fact, u is an s-isomorphism for some $s \geq 0$. Let $v : \Sigma^s A \to A'$ be a right inverse of u. Now consider the commutative diagram

$$\begin{tikzcd}
A & \Sigma^{r'} A \arrow[l, "\eta^A_{r'}"'] \arrow[r, "f''"] & B \\
A \arrow[u, "\mathbb{1}_A"] \arrow[d, "\mathbb{1}_A"'] & \Sigma^s A \arrow[l, "\eta_s"'] \arrow[r, "\bar{f}"] \arrow[u, "\bar{a}''"] \arrow[d, "\bar{a}'"'] & B \arrow[u, "\mathbb{1}_B"] \arrow[d, "\mathbb{1}_B"'] \\
A & \Sigma^r A \arrow[l, "\eta^A_r"'] \arrow[r, "f'"] & B
\end{tikzcd}$$

where $\bar{a}' = a' \circ v$, $\bar{a}'' = a'' \circ v$, $\bar{f} = f \circ v$. Notice that we have $\eta^A_r \circ \bar{a}' = \eta^A_s = \eta_r \circ \eta^A_{s-r}$. This means, by Corollary 2.35, that $\bar{a}' \simeq_r \eta^A_{s-r}$. Similarly, we have $\bar{a}'' \simeq_{r'} \eta^A_{s-r'}$. For $r'' \geq \max\{r, r'\}$ we deduce $\bar{a}' \circ \eta^A_{r''} = \eta^A_{s-r+r''}$ and $\bar{a}'' \circ \eta^A_{r''} = \eta^A_{s-r'+r''}$. Thus, by composing on the middle node with $\eta^A_{r''} : \Sigma^{s+r''} A \to \Sigma^s A$, we get a new commutative diagram similar to the one above but with $\bar{a}''$ and $\bar{a}'$ being replaced with $\eta^A_{s-r+r''}$ and $\eta^A_{s-r+r'}$, respectively. We deduce that f'' and f' give in $\mathcal{C}_\infty$ the same element as $\bar{f} \circ \eta^A_{r''}$, which concludes the proof. ■

Remark 2.41. By the properties of Verdier localization, the category $\mathcal{C}_\infty$ is triangulated in such a way that, by definition, a triangle in $\mathcal{C}_\infty$ is exact if it is isomorphic with the image (in $\mathcal{C}_\infty$) of an exact triangle from $\mathcal{C}_0$.

2.3.3 Weighted exact triangles

The key feature of a triangulated persistence category $\mathcal{C}$ is that there is a natural way to associate weights to a class of triangles larger than the exact triangles in $\mathcal{C}_0$.

Definition 2.42. A *strict exact triangle in* $\mathcal{C}$ is a pair $\widetilde{\Delta} = (\Delta, r)$ where $r \in [0, +\infty)$ and Δ is a diagram

$$\Delta : A \xrightarrow{\bar{u}} B \xrightarrow{\bar{v}} C \xrightarrow{\bar{w}} \Sigma^{-r} TA$$

in $\mathcal{C}_0$ with $\bar{u} \in \hom^0_{\mathcal{C}}(A, B)$, $\bar{v} \in \hom^0_{\mathcal{C}}(B, C)$, and $\bar{w} \in \hom^0_{\mathcal{C}}(C, \Sigma^{-r} TA)$, such that the following holds. There exists an exact triangle $A \xrightarrow{u} B \xrightarrow{v} C' \xrightarrow{w} TA$ in $\mathcal{C}_0$, with $u = \bar{u}$, an r-isomorphism $\phi : C' \to C$, and a right r-inverse of ϕ, denoted by

$\psi : \Sigma^r C \to C'$, such that the following diagram commutes:

$$\begin{tikzcd}
 & & \Sigma^r C \arrow[d, "\psi"'] \arrow[dr, "\Sigma^r \bar{w}"] & \\
A \arrow[r, "u"] & B \arrow[r, "v"] \arrow[dr, "\bar{v}"'] & C' \arrow[r, "w"] \arrow[d, "\phi"] & TA \\
 & & C &
\end{tikzcd} \tag{2.20}$$

The weight of the strict exact triangle $\widetilde{\Delta}$ is the number r, and we denote it by $w(\widetilde{\Delta})$.

Remark 2.43. (a) To simplify terminology, we will often denote strict exact triangles by the diagram Δ, with the weight identified implicitly by the amount of down "shift" of the last term. Notice that if $\Sigma^s A \neq A$ for all s, then the diagram Δ determines the weight of the triangle. However, when this is not the case, it is necessary to indicate the weight explicitly. For example, for any $r \geq 0$, the pair $(0 \to X \xrightarrow{\mathbb{1}_X} X \to 0, r)$ is a strict exact triangle of weight r because $\Sigma^s 0 = 0$ in $\mathscr{C}_0$ and two such triangles are different as soon as the corresponding weights are different. In what follows, we will not always write strict exact triangles as pairs. We will often simply write that a diagram Δ as above is strict exact of weight $w(\Delta) = r$. Although there is a slight imprecision in writing $w(\Delta) = r$ (since Δ does not determine r), the meaning of this should be clear: (Δ, r) is a strict triangle of weight r.

(b) Any exact triangle in $\mathscr{C}_0$ is a strict exact triangle of weight 0. Conversely, it is a simple exercise to see that a strict exact triangle of weight 0 is exact as a triangle in $\mathscr{C}_0$.

(c) Consider the following diagram:

$$\begin{tikzcd}
 & \Sigma^r B \arrow[r, "\Sigma^r \bar{v}"] \arrow[d, "\eta_r^B"'] & \Sigma^r C \arrow[d, "\psi"'] \arrow[dr, "\Sigma^r \bar{w}"] & \\
A \arrow[r, "u"] & B \arrow[r, "v"] \arrow[dr, "\bar{v}"'] & C' \arrow[r, "w"] \arrow[d, "\phi"] & TA \arrow[d, "\eta_r^{TA}"] \\
 & & C \arrow[r, "\bar{w}"] & \Sigma^{-r} TA
\end{tikzcd}$$

which is derived from the commutative diagram (2.20). The two squares in the diagram are not commutative in general, but they are r-commutative. Indeed, since ϕ is an r-isomorphism, let $\widetilde{\psi} : C \to \Sigma^{-r} C'$ be a left r-inverse of ϕ. As ψ is a right r-inverse of ϕ, we deduce from Proposition 2.28 (i) that $\Sigma^{-r}\psi \simeq_r \widetilde{\psi}$. Therefore, $\bar{w} \circ \phi = \Sigma^{-r} w \circ \Sigma^{-r} \psi \circ \phi \simeq_r \Sigma^{-r} w \circ \widetilde{\psi} \circ \phi = \Sigma^{-r} w \circ \eta_r^{C'} = \eta_r^{TA} \circ w$. Using Corollary 2.35, we also see that $\psi \circ \Sigma^r \bar{v} \simeq_r v \circ \eta_r^B$, because $\phi \circ \psi \circ \Sigma^r \bar{v} = \phi \circ v \circ \eta_r^B$.

(d) Because T commutes with Σ and with the natural transformations η, it immediately follows that this functor preserves strict exact triangles as well as their weight.

Example 2.44. Recall that the map $\Sigma^r A \xrightarrow{\eta_r^A} A$ embeds into an exact triangle in $\mathscr{C}_0$, $\Sigma^r A \xrightarrow{\eta_r^A} A \xrightarrow{v} K \xrightarrow{w} T\Sigma^r A$, where K is r-acyclic. We claim that the diagram

$$\Sigma^r A \xrightarrow{\eta_r^A} A \xrightarrow{v} K \xrightarrow{0} TA$$

is a strict exact triangle of weight r. Indeed, we have the commutative diagram

$$\begin{tikzcd}
 & & \Sigma^r K \arrow[d, "\eta_r^K"'] \arrow[dr, "0"] & \\
\Sigma^r A \arrow[r, "\eta_r^A"] & A \arrow[r, "v"] \arrow[dr, "v"'] & K \arrow[r, "w"] \arrow[d, "\mathbb{1}_K"] & T\Sigma^r A \\
 & & K &
\end{tikzcd}$$

where the upper-right triangle is commutative since K is r-acyclic (so $\eta_r^K = 0$ by Lemma 2.18 (i)). Moreover, $\mathbb{1}_K$ is an r-isomorphism (recall that $T\Sigma^r A = \Sigma^r TA$).

Note that the diagram

$$\Sigma^r A \xrightarrow{\eta_r^A} A \to 0 \to TA$$

is also a strict exact triangle of weight r.

Proposition 2.45 (Weight invariance). *Strict exact triangles satisfy the following two properties:*

(i) *Suppose the two diagrams $A \xrightarrow{\bar{u}} B \xrightarrow{\bar{v}} C$ and $A' \xrightarrow{\bar{u}'} B' \xrightarrow{\bar{v}'} C'$ are isomorphic in $\mathscr{C}_0$, i.e., we have the following commutative diagram in $\mathscr{C}_0$:*

$$\begin{tikzcd}
A \arrow[r, "\bar{u}"] \arrow[d, "f"] & B \arrow[r, "\bar{v}"] \arrow[d, "g"] & C \arrow[d, "h"] \\
A' \arrow[r, "\bar{u}'"] & B' \arrow[r, "\bar{v}'"] & C'
\end{tikzcd} \tag{2.21}$$

Then $A \xrightarrow{\bar{u}} B \xrightarrow{\bar{v}} C$ completes to a strict exact triangle of weight r, denoted by $\Delta : A \xrightarrow{\bar{u}} B \xrightarrow{\bar{v}} C \xrightarrow{\bar{w}} \Sigma^{-r}TA$ if and only if $A' \xrightarrow{\bar{u}'} B' \xrightarrow{\bar{v}'} C'$ completes to a strict exact triangle of weight r, denoted by $\Delta' : A' \xrightarrow{\bar{u}'} B' \xrightarrow{\bar{v}'} C' \xrightarrow{\bar{w}'} \Sigma^{-r}TA'$. Moreover, Δ and Δ' are isomorphic in $\mathscr{C}_0$.

(ii) *If $\Delta : A \xrightarrow{\bar{u}} B \xrightarrow{\bar{v}} C \xrightarrow{\bar{w}} \Sigma^{-r}TA$ satisfies $w(\Delta) = r$, then $\Delta' : A \xrightarrow{\bar{u}} B \xrightarrow{\bar{v}} C \xrightarrow{\bar{w}'} \Sigma^{-r-s}TA$ satisfies $w(\Delta') = r + s$ for $s \geq 0$, where $\bar{w}'$ is the composition*

$$C \xrightarrow{\bar{w}} \Sigma^{-r}TA \xrightarrow{\eta_s^{\Sigma^{-r}TA}} \Sigma^{-r-s}TA.$$

Proof. (i) The property claimed here immediately follows from the fact that, in $\mathscr{C}_0$, all 0-isomorphisms admit inverses. Therefore, if $A \xrightarrow{\bar{u}} B \xrightarrow{\bar{v}} C$ completes to a strict exact triangle Δ, then the desired map $\bar{w}'$ for Δ' can be chosen as $\bar{w}' = T(f) \circ \bar{w} \circ h^{-1}$, where h^{-1} is the inverse of map h in (2.21). The weight can be easily deduced from Definition 2.42.

(ii) By definition, there exists a commutative diagram

$$\begin{tikzcd}
 & & \Sigma^r C \arrow[d, "\psi"'] \arrow[dr, "\Sigma^r \bar{w}"] & \\
A \arrow[r, "\bar{u}"] & B \arrow[r, "v"] \arrow[dr, "\bar{v}"'] & C' \arrow[r, "w"] \arrow[d, "\phi"] & TA \\
 & & C &
\end{tikzcd}$$

Set $\psi' = \psi \circ \eta_s^{\Sigma^r C} \in \hom^r(\Sigma^{r+s} C, C')$. Then ψ' is an $(r+s)$-right inverse of ϕ. Consider the diagram

$$\begin{tikzcd}
 & & \Sigma^{r+s} C \arrow[d, "\psi'"'] \arrow[dr, "\Sigma^{r+s} \bar{w}'"] & \\
A \arrow[r, "\bar{u}"] & B \arrow[r, "v"] \arrow[dr, "\bar{v}"'] & C' \arrow[r, "w"] \arrow[d, "\phi"] & TA \\
 & & C &
\end{tikzcd}$$

where $\bar{w}' : C \to \Sigma^{-r-s} TA$ is defined by $\bar{w}' = \eta_s^{\Sigma^{-r} TA} \circ \bar{w}$, and notice that the upper-right triangle is commutative. ■

Proposition 2.46 (Weighted rotation property). *Given a strict exact triangle*

$$\Delta : A \xrightarrow{\bar{u}} B \xrightarrow{\bar{v}} C \xrightarrow{\bar{w}} \Sigma^{-r} TA$$

satisfying $w(\Delta) = r$*, there exists a triangle*

$$R(\Delta) : B \xrightarrow{\bar{v}} C \xrightarrow{\bar{w}'} \Sigma^{-r} TA \xrightarrow{-\bar{u}'} \Sigma^{-2r} TB$$

satisfying $w(R(\Delta)) = 2r$*, where* $\bar{w}' \simeq_r \bar{w}$ *and* $\bar{u}'$ *is the composition*

$$\bar{u}' : \Sigma^{-r} TA \xrightarrow{\Sigma^{-r} T\bar{u}} \Sigma^{-r} TB \xrightarrow{\eta_r^{\Sigma^{-r} TB}} \Sigma^{-2r} TB.$$

We call $R(\Delta)$ the (first) positive rotation of Δ.

Proof. By definition, there exists a commutative diagram

$$\begin{tikzcd}
& & \Sigma^r C \arrow[d, "\psi"'] \arrow[dr, "\Sigma^r \bar{w}"] & \\
A \arrow[r, "\bar{u}"] & B \arrow[r, "v"] \arrow[dr, "\bar{v}"'] & C' \arrow[r, "w"] \arrow[d, "\phi"] & TA \\
& & C &
\end{tikzcd}$$

where $A \xrightarrow{\bar{u}} B \xrightarrow{v} C' \xrightarrow{w} TA$ is an exact triangle in $\mathcal{C}_0$ and ψ is a right r-inverse of ϕ. By the rotation property of $\mathcal{C}_0$, $B \xrightarrow{v} C' \xrightarrow{w} TA \xrightarrow{-T\bar{u}} TB$ is an exact triangle in $\mathcal{C}_0$. We now construct the following diagram in $\mathcal{C}_0$ in which the upper squares will be commutative and the lower square r-commutative:

$$\begin{tikzcd}
B \arrow[r, "v"] \arrow[d, "\mathbb{1}_B"'] & C' \arrow[r, "w"] \arrow[d, "\phi"'] & TA \arrow[r, "-T\bar{u}"] \arrow[d, "\bar{\phi}"'] & TB \arrow[d, "\mathbb{1}_{TB}"'] \\
B \arrow[r, "\bar{v}"] & C \arrow[r, "w'"] \arrow[d, "\Sigma^{-r}\psi"'] & TA' \arrow[r, "u'"] \arrow[d, "\bar{\psi}"'] & TB \\
& \Sigma^{-r}C' \arrow[r, "\Sigma^{-r}w"] & \Sigma^{-r}TA &
\end{tikzcd} \tag{2.22}$$

Here the second row of maps comes from embedding $B \xrightarrow{\bar{v}} C$ into an exact triangle $B \xrightarrow{\bar{v}} C \xrightarrow{w'} A'' \to TB$ for some A'' in $\mathcal{C}_0$ and $A' = T^{-1}A''$. The map $\bar{\phi}$ is then induced by the functoriality of triangles in $\mathcal{C}_0$ and is an r-isomorphism by Proposition 2.31 (i). So far this gives the three upper squares of the diagram and their commutativity. To construct the lower square, let $\bar{\psi}$ be a left r-inverse of $\bar{\phi}$ (i.e., $\bar{\psi} \circ \bar{\phi} = \eta_r^{TA}$). By Corollary 2.33, $\bar{\psi}$ is a $2r$-isomorphism.

We claim that the lower square in the diagram (2.22) is r-commutative, and therefore we have $\bar{\psi} \circ w' \simeq_r \Sigma^{-r} w \circ \Sigma^{-r}\psi = \bar{w}$.

Indeed, let ψ' be a left r-inverse of ϕ. By using the commutativity of the upper-middle square in the diagram (2.22), we deduce $\bar{\psi} \circ w' \circ \phi = \Sigma^{-r} w \circ \psi' \circ \phi$. As ϕ is an r-isomorphism, we obtain

$$\bar{\psi} \circ w' \simeq_r \Sigma^{-r} w \circ \psi' \simeq_r \Sigma^{-r} w \circ \Sigma^{-r}\psi = \bar{w},$$

because, by Proposition 2.28, we have $\Sigma^{-r}\psi \simeq_r \psi'$. This shows that the lower square is r-commutative and that the related r-identity holds.

We next consider the diagram

$$\begin{tikzcd}
& & \Sigma^r TA \arrow[d, "\bar{\phi}\circ\eta_r^{TA}"'] \arrow[dr, "\Sigma^{2r}\bar{u}'"] & \\
B \arrow[r, "\bar{v}"] & C \arrow[r, "w'"] \arrow[dr, "\bar{w}'"'] & TA' \arrow[r, "u'"] \arrow[d, "\bar{\psi}"'] & TB \\
& & \Sigma^{-r} TA &
\end{tikzcd}$$

where $\bar{u}' = \Sigma^{-2r}(u' \circ \bar{\phi} \circ \eta_r^{TA})$ and $\bar{w}' = \bar{\psi} \circ w'$. Given that $\bar{\psi}$ is a $2r$-isomorphism, this means that we have a strict exact triangle of weight $2r$ of the form

$$B \xrightarrow{\bar{v}} C \xrightarrow{\bar{w}'} TA' \xrightarrow{\bar{u}'} \Sigma^{-2r} TB.$$

We already know that $\bar{w}' = \bar{\psi} \circ w' \simeq_r \bar{w}$. On the other hand,

$$\Sigma^{2r}\bar{u}' = u' \circ (\bar{\phi} \circ \eta_r^{TA}) = -T\bar{u} \circ \eta_r^{TA} = -\eta_r^{\Sigma^r TB} \circ \Sigma^r T\bar{u},$$

which concludes the proof. ■

Remark 2.47. An entirely similar argument also shows that there exists a strict exact triangle of weight $2r$ and of the form

$$R^{-1}(\Delta) : T^{-1}\Sigma^r C \to A \to B \to \Sigma^{-r} C,$$

which is the (first) negative rotation of Δ. Note that $R^{-1}(R(\Delta)) \neq R(R^{-1}(\Delta)) \neq \Delta$.

Remark 2.48. Proposition 2.46 describes a rotation of weighted exact triangles that does *not* preserve weights. Indeed, the rotation of the weight-r triangle Δ considered there has weight $2r$. It is not clear to what extent one can improve this. Ideally, one would like to be able to rotate Δ into a weighted exact triangle $B \to C \to \Sigma^{-r} TA \to \Sigma^{-r} TB$ of the *same* weight r. There is some evidence, coming from symplectic topology, indicating that in certain circumstances this might be possible (see Section 3.5.1.9). However, the algebraic setting in this memoir, in particular the definition of weighted exact triangles, might be too general to render this feasible, at least without additional assumptions on Δ.

Proposition 2.49 (Weighted octahedral formula). *Given two strict exact triangles*

$$\Delta_1 : E \xrightarrow{\beta} F \xrightarrow{\alpha} X \xrightarrow{k} \Sigma^{-r} TE$$

and

$$\Delta_2 : X \xrightarrow{u} A \xrightarrow{\gamma} B \xrightarrow{b} \Sigma^{-s} TX$$

with $w(\Delta_1) = r$ and $w(\Delta_2) = s$, there exists a diagram

$$\begin{tikzcd}
E \arrow[r] \arrow[d, "\beta"'] & 0 \arrow[r] \arrow[d] & TE \arrow[r] \arrow[d] & TE \arrow[d, "T\beta"] \\
F \arrow[r, "u\circ\alpha"] \arrow[d, "\alpha"'] & A \arrow[r] \arrow[d, "\mathbb{1}"'] & C \arrow[r] \arrow[d] & TF \arrow[d, "(\Sigma^{-s}T\alpha)\circ\eta_s^{TF}"] \\
X \arrow[r, "u"] \arrow[d, "k"'] & A \arrow[r, "\gamma"] \arrow[d] & B \arrow[r, "b"] \arrow[d] & \Sigma^{-s}TX \arrow[d, "\Sigma^{-s}(Tk)"] \\
\Sigma^{-r}TE \arrow[r] & 0 \arrow[r] & \Sigma^{-r-s}T^2E \arrow[r] & \Sigma^{-r-s}T^2E
\end{tikzcd}\tag{2.23}$$

with all squares commutative except for the bottom-right one, which is r-anti-commutative, such that the triangles

$$\Delta_3 : F \to A \to C \to TF \quad \textit{and} \quad \Delta_4 : TE \to C \to B \to \Sigma^{-r-s}T^2E$$

are strict exact with $w(\Delta_3) = 0$ and $w(\Delta_4) = r + s$.

By forgetting the Σ's (or assuming that $r = s = 0$), this is equivalent to the usual octahedral axiom in a triangulated category (namely $\mathcal{C}_0$) and the bottom-right square is commutative up to sign (or anti-commutative).

Proof of Proposition 2.49. By definition, there are two commutative diagrams

$$\begin{tikzcd}
 & E \arrow[d] & \\
 & F \arrow[d, "\alpha'"'] \arrow[dr, "\alpha"] & \\
\Sigma^r X \arrow[r, "\psi_1"] \arrow[dr, "\Sigma^r k"'] & X' \arrow[r, "\phi"] \arrow[d, "\delta"'] & X \\
 & TE &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & \Sigma^s B \arrow[d, "\psi_3"'] \arrow[dr, "\Sigma^s b"] & \\
X \arrow[r, "u"] & A \arrow[r, "v'"] \arrow[dr, "\gamma"'] & B' \arrow[r, "g"] \arrow[d, "\bar{\phi}"'] & TX \\
 & & B &
\end{tikzcd}\tag{2.24}$$

with ϕ an r-isomorphism and $\bar{\phi}$ an s-isomorphism, and ψ_1 and ψ_3 are their right r- and s-inverses, respectively. By the octahedral axiom in $\mathcal{C}_0$, we construct the following diagram, with all squares commutative except for the bottom-right one, which is anti-commutative:

$$\begin{tikzcd}
E \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & TE \arrow[r] \arrow[d] & TE \arrow[d] \\
F \arrow[r, "\alpha''"] \arrow[d, "\alpha'"'] & A \arrow[r] \arrow[d, "\mathbb{1}_A"'] & C \arrow[r] \arrow[d, "w"'] & TF \arrow[d, "T\alpha'"'] \\
X' \arrow[r, "u\circ\phi"] \arrow[d, "\delta"'] & A \arrow[r, "v"] \arrow[d] & B'' \arrow[r, "\theta"] \arrow[d, "t"'] & TX' \arrow[d, "T\delta"'] \\
TE \arrow[r] & 0 \arrow[r] & T^2E \arrow[r, "\mathbb{1}_{T^2E}"] & T^2E
\end{tikzcd}$$

Thus $\alpha'' = u \circ \phi \circ \alpha'$, $t = -T\delta \circ \theta$. We denote by

$$\Delta_3 : F \xrightarrow{\alpha''} A \to C \to TF$$

the respective exact triangle in $\mathcal{C}_0$ so that, as in Remark 2.43, $w(\Delta_3) = 0$. The map $w \in \hom^0(C, B'')$ is induced from the commutativity of the middle-left triangle. We now consider the following diagram:

$$\begin{tikzcd}
F \arrow[r] \arrow[d, "\alpha'"'] & A \arrow[r] \arrow[d, "\mathbb{1}_A"'] & C \arrow[r] \arrow[d, "w"'] & TF \arrow[d, "T\alpha'"'] \\
X' \arrow[r, "u\circ\phi"] \arrow[d, "\phi"'] & A \arrow[r, "v"] \arrow[d, "\mathbb{1}_A"'] & B'' \arrow[r, "\theta"] \arrow[d, "\phi'"'] & TX' \arrow[d, "T\phi"'] \\
X \arrow[r, "u"] & A \arrow[r, "v'"] \arrow[d, "\mathbb{1}_A"'] & B' \arrow[r, "g"] \arrow[d, "\bar{\phi}"'] & TX \\
 & A \arrow[r, "\gamma"] & B \arrow[r, "b"] & \Sigma^{-s}TX
\end{tikzcd} \tag{2.25}$$

The three long rows are exact triangles in $\mathcal{C}_0$, and we deduce the existence of $\phi' \in \hom^0(B'', B')$ making the adjacent squares commutative. This is an r-isomorphism by Proposition 2.31 (i). We fix a right r-inverse $\psi_2 \in \hom^0(\Sigma^r B', B'')$ of ϕ'. The composition $\phi'' = \bar{\phi} \circ \phi' \in \hom^0(B'', B)$ is an $(r+s)$-isomorphism by Proposition 2.28 (iii). Let $\psi'' = \psi_2 \circ \Sigma^r \psi_3$ (recall ψ_3 from (2.24)) and notice that ψ'' is a right $(r+s)$-inverse of ϕ''.

We are now able to define the triangle Δ_4:

$$\Delta_4 : TE \to C \xrightarrow{\phi''\circ w} B \xrightarrow{\Sigma^{-r-s}(t\circ\psi'')} \Sigma^{-r-s}T^2E.$$

The following commutative diagram shows that Δ_4 is strict exact and $w(\Delta_4) = r + s$:

$$\begin{tikzcd}
 & & \Sigma^{r+s} B \arrow[d, "\psi''"'] \arrow[dr, "t \circ \psi''"] & \\
TE \arrow[r] & C \arrow[r, "w"] \arrow[dr, "\phi'' \circ w"'] & B'' \arrow[r, "t"] \arrow[d, "\phi''"] & T^2 E \\
 & & B &
\end{tikzcd}$$

It is easy to check that all the squares in (2.23), except the bottom-right one, are commutative.

We now check the r-anti-commutativity of the bottom-right square. We need to show that $\Sigma^{-s}(Tk) \circ b \simeq_r -\Sigma^{-r-s}(t \circ \psi'')$, which is equivalent to $\Sigma^r(Tk) \circ \Sigma^{r+s} b \simeq_r -t \circ \psi''$. Given that the $B''(TX')(TX)B'$ square in (2.25) commutes, Corollary 2.38 yields the following r-commutative diagram:

$$\begin{tikzcd}
\Sigma^r B' \arrow[r, "\Sigma^r g"] \arrow[d, "\psi_2"'] & \Sigma^r TX \arrow[d, "T\psi_1"] \\
B'' \arrow[r, "\theta"] & TX'
\end{tikzcd}$$

Now consider the following diagram, commutative except for the middle square, which is r-commutative:

$$\begin{tikzcd}
\Sigma^{r+s} B \arrow[d, "\Sigma^r \psi_3"'] \arrow[dr, "\Sigma^{r+s} b"] \arrow[dd, bend right=60, "\psi''"'] & & \\
\Sigma^r B' \arrow[r, "\Sigma^r g"] \arrow[d, "\psi_2"'] & \Sigma^r TX \arrow[d, "T\psi_1"'] \arrow[dr, "\Sigma^r(Tk)"] & \\
B'' \arrow[r, "\theta"] \arrow[rr, bend right, "-t"'] & TX' \arrow[r, "T\delta"] & T^2 E
\end{tikzcd}$$

Write

$$\begin{aligned}
-t \circ \psi'' &= (T\delta) \circ (\theta \circ \psi_2) \circ \Sigma^r \psi_3 \\
&\simeq_r (T\delta) \circ ((T\psi_1) \circ \Sigma^r g) \circ \Sigma^r \psi_3 = \Sigma^r(Tk) \circ \Sigma^{r+s} b,
\end{aligned}$$

which completes the proof. ■

Given a triple of maps $\Delta : A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} D$ with shifts $\lceil u \rceil, \lceil v \rceil, \lceil w \rceil \in \mathbb{R}$, it is useful to introduce a special notation for an associated triple in $\mathscr{C}_0$, denoted by

$\Sigma^{s_1,s_2,s_3,s_4}\Delta$, for $s_1, s_2, s_3, s_4 \in \mathbb{R}$ satisfying the following relations:

$$\begin{aligned} -s_1 + s_2 + \lceil u \rceil &\leq 0, \\ -s_2 + s_3 + \lceil v \rceil &\leq 0, \\ -s_3 + s_4 + \lceil w \rceil &\leq 0. \end{aligned}$$

The triple $\Sigma^{s_1,s_2,s_3,s_4}\Delta$ has the form

$$\Sigma^{s_1} A \xrightarrow{\bar{u}} \Sigma^{s_2} B \xrightarrow{\bar{v}} \Sigma^{s_3} \xrightarrow{\bar{w}} \Sigma^{s_4} D, \tag{2.26}$$

where $\bar{u}$ is the composition of the composition $\Sigma^{s_1} A \xrightarrow{(\eta_{s_1,0})_A} A \xrightarrow{u} B \xrightarrow{(\eta_{0,s_2})_B} \Sigma^{s_2} B$ and the persistence structure map $i_{-s_1+s_2+\lceil u \rceil,0}$, i.e.,

$$\bar{u} = i_{-s_1+s_2+\lceil u \rceil,0}\big((\eta_{0,s_2})_B \circ u \circ (\eta_{s_1,0})_A\big).$$

The definitions of $\bar{v}$ and $\bar{w}$ are similar and, in particular, $\lceil \bar{u} \rceil = \lceil \bar{v} \rceil = \lceil \bar{w} \rceil = 0$. The inequalities above ensure that the resulting triangle (2.26) has all morphisms in $\mathscr{C}_0$.

For $s_1 = s_2 = s_3 = s_4 = k$ (which implies that $\lceil u \rceil, \lceil v \rceil, \lceil w \rceil \leq 0$) we write, for brevity,

$$\Sigma^k \Delta := \Sigma^{s_1,s_2,s_3,s_4}\Delta.$$

Remark 2.50. Assume that $\Delta : A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} \Sigma^{-r} TA$ is strict exact of weight $w(\Delta) = r$.

(a) It is a simple exercise to show that the triangle $\Sigma^k \Delta : \Sigma^k A \to \Sigma^k B \to \Sigma^k C \to \Sigma^{-r+k} TA$ is strict exact and $w\big(\Sigma^k \Delta\big) = r$.

(b) For $s \geq 0$, Proposition 2.45 (ii) claims that $\Sigma^{0,0,0,-s}\Delta$ is strict exact of weight $r + s$. It is again an easy exercise to see that $\Sigma^{0,0,-s,-s}\Delta$ is strict exact of weight $r + s$.

Proposition 2.51 (Functoriality of triangles). *Consider two strict exact triangles as below, with* $f \in \hom^0(A_1, A_2)$ *and* $g \in \hom^0(B_1, B_2)$*:*

$$\begin{tikzcd}
\Delta_1 : & A_1 \arrow[r, "\bar{u}_1"] \arrow[d, "f"'] & B_1 \arrow[r, "\bar{v}_1"] \arrow[d, "g"'] & C_1 \arrow[r, "\bar{w}_1"] & \Sigma^{-r} TA_1 \\
\Delta_2 : & A_2 \arrow[r, "\bar{u}_2"] & B_2 \arrow[r, "\bar{v}_2"] & C_2 \arrow[r, "\bar{w}_2"] & \Sigma^{-s} TA_2
\end{tikzcd}$$

and assume that $w(\Delta_1) = r$ *and* $w(\Delta_2) = s$*. Then there exists a morphism* $h \in \hom^0(C_1, \Sigma^{-r} C_2)$ *inducing maps relating the triangles* $\Delta_1 \to \Sigma^{0,0,-r,-r}\Delta_2$*, as in* *the following diagram:*

$$\begin{tikzcd}
A_1 \arrow[r] \arrow[d, "f"'] & B_1 \arrow[r] \arrow[d, "g"'] & C_1 \arrow[r] \arrow[d, "h"'] & \Sigma^{-r}TA_1 \arrow[d, "\eta_s^{TA_2}\circ\Sigma^{-r}Tf"] \\
A_2 \arrow[r] & B_2 \arrow[r] & \Sigma^{-r}C_2 \arrow[r] & \Sigma^{-r-s}TA_2
\end{tikzcd}$$

where the middle square is r-commutative and the right square is s-commutative.

The proof is left as an exercise.

Proposition 2.52. *Let $\Delta : A \xrightarrow{\bar{u}} B \xrightarrow{\bar{v}} C \xrightarrow{\bar{w}} \Sigma^{-r}TA$ be a strict exact triangle of weight r in $\mathcal{C}$ and let $\Delta' : A \xrightarrow{\bar{u}} B \xrightarrow{v} C' \xrightarrow{w} TA$ be the exact triangle in $\mathcal{C}_0$ associated to Δ as in Definition 2.42. There are morphisms of triangles $h : \Sigma^{2r}\Delta \to \Delta'$ and $h' : \Delta' \to \Sigma^{-r}\Delta$ such that the compositions $h' \circ h$ and $h \circ \Sigma^{3r}h'$ have as vertical maps the shifted natural transformations $\eta_{3r}^{(-)}$ defined in (2.14).*

Proof. We use the notation in Definition 2.42 and consider the diagram below:

$$\begin{tikzcd}
\Sigma^r A \arrow[r, "\Sigma^r\bar{u}"] \arrow[d, "\eta_r^A"'] & \Sigma^r B \arrow[r, "\Sigma^r\bar{v}"] \arrow[d, "\eta_r^B"'] & \Sigma^r C \arrow[r, "\Sigma^r\bar{w}"] \arrow[d, "\psi"'] & TA \arrow[d, "\mathbb{1}_{TA}"] \\
A \arrow[r, "\bar{u}"] \arrow[d, "\mathbb{1}_A"'] & B \arrow[r, "v"] \arrow[d, "\mathbb{1}_B"'] & C' \arrow[r, "w"] \arrow[d, "\phi"'] & TA \arrow[d, "\eta_r^{TA}"] \\
A \arrow[r, "\bar{u}"] & B \arrow[r, "\bar{v}"] & C \arrow[r, "\bar{w}"] & \Sigma^{-r}TA
\end{tikzcd}$$

Set $h_1 = (\eta_r^A, \eta_r^B, \psi, \mathbb{1}_{TA})$ and $h_1' = (\mathbb{1}_A, \mathbb{1}_B, \phi, \eta_r^{TA})$. Notice that h_1 as well as h_1' are not morphisms of triangles because the bottom-right square is only r-commutative, and the same is true for the top-middle square, as discussed in Remark 2.43 (c). Let $h = h_1 \circ \eta_r$ and $h' = \eta_r \circ h_1'$ (where we view η_r as a quadruple of morphisms of the form η_r^-). It follows that both h and h' are morphisms of triangles. Moreover, given that $\phi \circ \psi = \eta_r^C$, it is clear that $h' \circ h = \eta_{3r}$. The other composition, $h \circ \Sigma^{3r}h'$, has a term of the form $\psi \circ \eta_r \circ \eta_r \circ \Sigma^{3r}$; thus, by Proposition 2.28, this coincides with $\eta_{3r}^{C'}$, as claimed. ■

Remark 2.53. Proposition 2.52 shows that a strict exact triangle of weight r is approximately isomorphic, in a sense similar to interleaving, to an exact triangle in $\mathcal{C}_0$.

2.3.4 Fragmentation pseudometrics on Obj($\mathcal{C}$)

In a triangulated persistence category there is a natural notion of iterated cone decomposition, similar to the corresponding notion in the triangulated setting from Section 2.1.

Definition 2.54. Let $\mathcal{C}$ be a triangulated persistence category, and let $X \in \mathrm{Obj}(\mathcal{C})$. An *iterated cone decomposition* D of X with *linearization* $(X_1, X_2, \ldots, X_n)$, where $X_i \in \mathrm{Obj}(\mathcal{C})$, consists of a family of strict exact triangles in $\mathcal{C}$,

$$\begin{cases} \Delta_1 : X_1 \to 0 \to Y_1 \to \Sigma^{-r_1} T X_1 \\ \Delta_2 : X_2 \to Y_1 \to Y_2 \to \Sigma^{-r_2} T X_2 \\ \Delta_3 : X_3 \to Y_2 \to Y_3 \to \Sigma^{-r_3} T X_3 \\ \qquad\qquad \vdots \\ \Delta_n : X_n \to Y_{n-1} \to X \to \Sigma^{-r_n} T X_n. \end{cases}$$

The weight of such a cone decomposition is defined by

$$w(D) = \sum_{i=1}^{n} w(\Delta_i).$$

The linearization of D is denoted by $\ell(D) = (X_1, \ldots, X_n)$.

Proposition 2.55. *Assume that X admits an iterated cone decomposition D with linearization $(X_1, \ldots, X_n)$ and, for some $i \in \{1, \ldots, n\}$, X_i admits an iterated cone decomposition D' with linearization $(A_1, \ldots, A_k)$. Then X admits an iterated cone decomposition D'' with linearization*

$$(X_1, \ldots, X_{i-1}, TA_1, \ldots, TA_k, X_{i+1}, \ldots, X_n).$$

Moreover, the weights of these cone decompositions satisfy $w(D'') = w(D) + w(D')$.

A cone decomposition D'' as in the statement of Proposition 2.55 is called a *refinement* of the cone decomposition D with respect to D'.

Example 2.56. A single strict exact triangle $A \to B \to X \to \Sigma^{-r} TA$ can be regarded as a cone decomposition D of X with linearization $(T^{-1}B, A)$ such that $w(D) = r$. Assume that A fits into a second strict exact triangle $E \to F \to A \to \Sigma^{-s} TE$ of weight s. Thus we have a cone decomposition D' of A, with linearization $(T^{-1}F, E)$ and $w(D') = s$. Diagram (2.23) from Proposition 2.49 yields the commutative diagram

$$\begin{tikzcd} E \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & TE \arrow[d] \\ F \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & Y \arrow[d] \\ A \arrow[r] & B \arrow[r] & X \end{tikzcd}$$

for some object $Y \in \mathrm{Obj}(\mathcal{C})$. In particular, we obtain a strict exact triangle $TE \to Y \to X \to \Sigma^{-r-s} T^2 E$ of weight $r + s$. Thus, we have a refinement of D with respect to D' as follows:

$$D'' := \begin{cases} T^{-1}B \to 0 \to B \to B \\ F \to B \to Y \to TF \\ TE \to Y \to X \to \Sigma^{-r-s}TE. \end{cases}$$

Moreover, $w(D'') = r + s = w(D) + w(D')$.

Proof of Proposition 2.55. By definition, the cone decomposition D consists of a family of strict exact triangles in $\mathscr{C}$ as follows:

$$\begin{cases} \qquad\qquad \vdots \\ \Delta_{i-1} : X_{i-1} \to Y_{i-2} \to Y_{i-1} \to \Sigma^{-r_{i-1}} X_{i-1} \\ \Delta_i : X_i \quad \to Y_{i-1} \to Y_i \quad \to \Sigma^{-r_i} X_i \\ \Delta_{i+1} : X_{i+1} \to Y_i \quad \to Y_{i+1} \to \Sigma^{-r_{i+1}} X_{i+1} \\ \qquad\qquad \vdots \end{cases}$$

We aim to replace the triangle Δ_i by a sequence of strict exact triangles

$$\bar{\Delta}_j : TA_j \to B_{j-1} \to B_j \to \Sigma^{-s_j} T^2 A_j$$

for $j \in \{1, \ldots, k\}$, with $B_0 = Y_{i-1}$, $B_k = Y_i$, and such that

$$\sum_{j=1}^{k} w(\bar{\Delta}_j) = w(\Delta_i) + w(D'). \tag{2.27}$$

In this case, the ordered family of strict exact triangles $(\Delta_1, \ldots, \Delta_{i-1}, \bar{\Delta}_1, \ldots, \bar{\Delta}_k, \Delta_{i+1}, \ldots, \Delta_n)$ forms the refinement D'', and (2.27) implies that

$$w(D'') = \sum_{j \in \{1,\ldots,n\} \setminus \{i\}} w(\Delta_j) + \sum_{j=1}^{k} w(\bar{\Delta}_j) = \sum_{j=1}^{n} w(\Delta_j) + w(D') = w(D) + w(D'),$$

as claimed.

In order to obtain the desired sequence of strict exact triangles, we focus on Δ_i and, to shorten notation, we rename its terms by $A = X_i$, $B = Y_{i-1}$, $C = Y_i$, and $r = r_i$ so that, with this notation, Δ_i is a strict exact triangle $A \to B \to C \to \Sigma^{-r} TA$.

We now fix notation for the cone decomposition D' of $A = X_i$. It consists of the following family of strict exact triangles:

$$\begin{cases} \Delta'_1 : A_1 \to 0 \to Z_1 \to \Sigma^{-s_1} TA_1 \\ \Delta'_2 : A_2 \to Z_1 \to Z_2 \to \Sigma^{-s_2} TA_2 \\ \qquad\qquad \vdots \\ \Delta'_{k-1} : A_{k-1} \to Z_{k-2} \to Z_{k-1} \to \Sigma^{-s_{k-1}} TA_{k-1} \\ \Delta'_k : A_k \to Z_{k-1} \to A \to \Sigma^{-s_k} TA_k. \end{cases}$$

We will apply Proposition 2.49 iteratively. The first step is the following commutative diagram, obtained from (2.23),

$$\begin{tikzcd}
A_k \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & TA_k \arrow[d] \\
Z_{k-1} \arrow[r] \arrow[d] & B \arrow[r] \arrow[d, "\mathbb{1}_B"'] & B_{k-1} \arrow[d] \\
A \arrow[r] & B \arrow[r] & C
\end{tikzcd}$$

for some $B_{k-1} \in \mathrm{Obj}(\mathscr{C})$. Define

$$\bar{\Delta}_k : TA_k \to B_{k-1} \to C \to \Sigma^{-r-s_k} T^2 A_k.$$

We have

$$w(\bar{\Delta}_k) = r + s_k = w(\Delta_i) + w(\Delta'_k). \tag{2.28}$$

We then consider the following commutative diagram, again obtained from (2.23),

$$\begin{tikzcd}
A_{k-1} \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & TA_{k-1} \arrow[d] \\
Z_{k-2} \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & B_{k-2} \arrow[d] \\
Z_{k-1} \arrow[r] & B \arrow[r] & B_{k-1}
\end{tikzcd}$$

for some $B_{k-2} \in \mathrm{Obj}(\mathscr{C})$. Define $\bar{\Delta}_{k-1}$ to be the strict exact triangle

$$\bar{\Delta}_{k-1} : TA_{k-1} \to B_{k-2} \to B_{k-1} \to \Sigma^{-s_{k-1}} T^2 A_{k-1}.$$

Then

$$w(\bar{\Delta}_{k-1}) = s_{k-1} = w(\Delta'_{k-1}).$$

Inductively, we obtain $B_i \in \mathrm{Obj}(\mathscr{C})$ and strict exact triangles

$$\bar{\Delta}_i : TA_i \to B_{i-1} \to B_i \to \Sigma^{-s_i} T^2 A_i$$

for $2 \le i \le k-1$ such that

$$w(\bar{\Delta}_i) = s_i = w(\Delta'_i). \tag{2.29}$$

The final step lies in considering the diagram

$$\begin{tikzcd}
A_1 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & TA_1 \arrow[d] \\
0 \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & B \arrow[d] \\
Z_1 \arrow[r] & B \arrow[r] & B_1
\end{tikzcd} \tag{2.30}$$

for some $B \in \mathrm{Obj}(\mathscr{C})$. Define $\bar{\Delta}_1$ to be the strict triangle

$$\bar{\Delta}_1 : TA_1 \to B \to B_1 \to \Sigma^{-s_1} TA_1.$$

Then

$$w(\bar{\Delta}_1) = w(\Delta'_1) = s_1. \tag{2.31}$$

Together, the ordered family $(\bar{\Delta}_1, \dots, \bar{\Delta}_k)$ forms the desired sequence of strict exact triangles. Finally, the equalities (2.28), (2.29), and (2.31) yield

$$\sum_{j=1}^{k} w(\bar{\Delta}_j) = s_1 + s_2 + \cdots + s_k + r = \sum_{j=1}^{k} w(\Delta'_j) + w(\Delta_i) = w(D') + w(\Delta_i),$$

as claimed in (2.27). ■

Let $\mathcal{F} \subset \mathrm{Obj}(\mathscr{C})$ be a family of objects of $\mathscr{C}$. For two objects $X, X' \in \mathrm{Obj}(\mathscr{C})$, define, just as in Section 2.1,

$$\begin{aligned} \delta^{\mathcal{F}}(X, X') = \inf\bigl\{w(D) \mid\ & D \text{ is an iterated cone decomposition of } X \\ & \text{with linearization } (F_1, \dots, T^{-1}X', \dots, F_k), \\ & \text{where } F_i \in \mathcal{F}, k \in \mathbb{N}\bigr\}. \end{aligned} \tag{2.32}$$

Corollary 2.57. *With the definition of $\delta^{\mathcal{F}}$ in (2.32), we have the inequality*

$$\delta^{\mathcal{F}}(X, X') \leq \delta^{\mathcal{F}}(X, X'') + \delta^{\mathcal{F}}(X'', X')$$

for any $X, X', X'' \in \mathrm{Obj}(\mathscr{C})$.

Proof. For any $\epsilon > 0$, there are cone decompositions D and D' of X and X'', respectively, such that

$$w(D) \leq \delta^{\mathcal{F}}(X, X'') + \epsilon \quad \text{and} \quad w(D') \leq \delta^{\mathcal{F}}(X'', X') + \epsilon$$

with linearizations

$$\ell(D) = (F_1, \dots, T^{-1}X'', \dots, F_s) \quad \text{and} \quad \ell(D') = (F'_1, \dots, T^{-1}X', \dots, F'_k),$$

respectively, $F_i, F'_j \in \mathcal{F}$. This means that $T^{-1}X''$ has a corresponding cone decomposition $T^{-1}D'$ with linearization $\ell(T^{-1}D') = (T^{-1}F'_1, \ldots, T^{-2}X', \ldots, T^{-1}F'_k)$. Proposition 2.55 implies that there exists a cone decomposition D'' of X that is a refinement of D with respect to $T^{-1}D'$ such that

$$\ell(D'') = (F_1, \ldots, F'_1, \ldots, T^{-1}X', \ldots, F'_k, \ldots, F_s)$$

and

$$w(D'') = w(D) + w(D') \leq \delta^{\mathcal{F}}(X, X'') + \delta^{\mathcal{F}}(X'', X') + 2\epsilon,$$

which implies the claim. ■

Finally, there are also fragmentation pseudometrics specific to this situation with properties similar to those in Proposition 2.4.

Definition 2.58. Let $\mathcal{C}$ be a triangulated persistence category and let $\mathcal{F} \subset \mathrm{Obj}(\mathcal{C})$. The *fragmentation pseudometric*

$$d^{\mathcal{F}} : \mathrm{Obj}(\mathcal{C}) \times \mathrm{Obj}(\mathcal{C}) \to [0, \infty) \cup \{+\infty\}$$

associated to $\mathcal{F}$ is defined by

$$d^{\mathcal{F}}(X, X') = \max\{\delta^{\mathcal{F}}(X, X'), \delta^{\mathcal{F}}(X', X)\}.$$

Remark 2.59. (a) It is clear from Corollary 2.57 that $d^{\mathcal{F}}$ satisfies the triangle inequality and, by definition, it is symmetric. It is immediate to see that $d^{\mathcal{F}}(X, X) = 0$ for all objects X (this is because of the existence of the exact triangle in $\mathcal{C}_0$, $T^{-1}X \to 0 \to X \to X$). It is of course possible that this pseudometric is degenerate, and it is also possible that it is not finite.

(b) If $X \in \mathcal{F}$, then $\delta^{\mathcal{F}}(TX, 0) = 0$ because of the exact triangle $X \to 0 \to TX \to TX$. On the other hand, $\delta^{\mathcal{F}}(0, TX)$ is not generally trivial. However, the exact triangle $TX \to TX \to 0 \to T^2X$ shows that, if $TX \in \mathcal{F}$, then $\delta^{\mathcal{F}}(0, TX) = 0$.

(c) It follows from the previous point that if $\mathcal{F} = \mathrm{Obj}(\mathcal{C})$, then $d^{\mathcal{F}}(X, X') = 0$ (in other words, the pseudometric $d^{\mathcal{F}}(-,-)$ is completely degenerate). More generally, if the family $\mathcal{F}$ is T invariant (in the sense that if $X \in \mathcal{F}$, then $TX, T^{-1}X \in \mathcal{F}$), then T is an isometry with respect to the pseudometric $d^{\mathcal{F}}$ and $d^{\mathcal{F}}(X, X') = 0$ for all $X, X' \in \mathcal{F}$.

(d) Remark 2.5 (c) applies also in this setting, in the sense that we may define at this triangulated persistence level fragmentation pseudometrics $\underline{d}^{\mathcal{F}}$ given by (the symmetrization of) formula (2.5) but making use of weighted triangles in $\mathcal{C}$ instead of the exact triangles in the triangulated category $\mathcal{D}$.

Recall that by assumption $\mathcal{C}_0$ is triangulated and thus additive. Therefore, for any two objects $X, X' \in \mathrm{Obj}(\mathcal{C}_0) = \mathrm{Obj}(\mathcal{C})$, the direct sum $X \oplus X'$ is a well-defined object in $\mathrm{Obj}(\mathcal{C})$.

Proposition 2.60. *For any* $A, B, A', B' \in \mathrm{Obj}(\mathscr{C})$*, we have*

$$d^{\mathcal{F}}(A \oplus B, A' \oplus B') \leq d^{\mathcal{F}}(A, A') + d^{\mathcal{F}}(B, B').$$

Proof. The proof follows easily from the following lemma.

Lemma 2.61. *Let* $\Delta : A \to B \to C \to \Sigma^{-r}TA$ *and* $\bar{\Delta} : \bar{A} \to \bar{B} \to \bar{C} \to \Sigma^{-s}T\bar{A}$ *be two strict exact triangles with* $w(\Delta) = r$ *and* $w(\Delta') = s$*. Then*

$$\Delta'' : A \oplus \bar{A} \to B \oplus \bar{B} \to C \oplus \bar{C} \to \Sigma^{-\max\{r,s\}}\big(TA \oplus T\bar{A}\big)$$

is a strict exact triangle with $w(\Delta'') = \max\{r, s\}$*.*

Proof of Lemma 2.61. By definition, there are two commutative diagrams

$$\begin{tikzcd}
& & \Sigma^r C \arrow[d, "\psi"'] \arrow[rd] & \\
A \arrow[r] & B \arrow[r] \arrow[rd] & C' \arrow[r] \arrow[d, "\phi"] & TA \\
& & C &
\end{tikzcd}
\qquad
\begin{tikzcd}
& & \Sigma^s \bar{C} \arrow[d, "\bar{\psi}"'] \arrow[rd] & \\
\bar{A} \arrow[r] & \bar{B} \arrow[r] \arrow[rd] & \bar{C}' \arrow[r] \arrow[d, "\bar{\phi}"] & T\bar{A} \\
& & \bar{C} &
\end{tikzcd}$$

This yields the commutative diagram

$$\begin{tikzcd}
& & \Sigma^{\max\{r,s\}} C \oplus \bar{C} \arrow[d, "\psi \oplus \bar{\psi}"'] \arrow[rd] & \\
A \oplus \bar{A} \arrow[r] & B \oplus \bar{B} \arrow[r] \arrow[rd] & C' \oplus \bar{C}' \arrow[r] \arrow[d, "\phi \oplus \bar{\phi}"] & TA \oplus T\bar{A} \\
& & C \oplus \bar{C} &
\end{tikzcd}$$

and it is easy to check that $\phi \oplus \bar{\phi}$ is a $\max\{r, s\}$-isomorphism. ■

Returning to the proof of the proposition, it suffices to prove that $\delta^{\mathcal{F}}(A \oplus B, A' \oplus B') \leq \delta^{\mathcal{F}}(A, A') + \delta^{\mathcal{F}}(B, B')$. For any $\epsilon > 0$, by definition there exist cone decompositions D and D' of A and B, respectively, with $\ell(D) = (F_1, \ldots, F_{i-1}, T^{-1}A', F_{i+1}, \ldots, F_s)$ and $\ell(D') = (F_1', \ldots, T^{-1}B', \ldots, F_{s'}')$ such that

$$w(D) \leq \delta^{\mathcal{F}}(A, A') + \epsilon \quad \text{and} \quad w(D') \leq \delta^{\mathcal{F}}(B, B') + \epsilon.$$

The desired cone decomposition of $A \oplus B$ is defined as follows:

$$D'' := \begin{cases} F_1 \to 0 \to E_1 \to \Sigma^{-r_1} T F_1 \\ F_2 \to E_1 \to E_2 \to \Sigma^{-r_2} T F_2 \\ \qquad\qquad \vdots \\ F_s \to E_{s-1} \to A \to \Sigma^{-r_s} T F_s \\ F'_1 \to A \oplus 0 \to A \oplus E'_1 \to \Sigma^{-r'_1} T F'_1 \\ F'_2 \to A \oplus E'_1 \to A \oplus E'_2 \to \Sigma^{-r'_2} T F'_{s+2-i} \\ \qquad\qquad \vdots \\ T^{-1} B' \to A \oplus E'_j \to A \oplus E'_{j+1} \to \Sigma^{-r'_{B'}} B' \\ \qquad\qquad \vdots \\ F'_{s'} \to A \oplus E'_{s'-1} \to A \oplus B \to \Sigma^{-r'_{s'}} T F'_{s'}. \end{cases} \tag{2.33}$$

Here we identify $0 \oplus F'_i = F'_i$. The first s triangles come from the decomposition of A, and the following s' triangles are associated, using Lemma 2.61, to the respective triangles in the decomposition of B and to the triangle $0 \to A \to A \to 0$ (of weight 0). It is obvious that $w(D'') = w(D) + w(D')$ and thus $\delta^{\mathcal{F}}(A \oplus B, A' \oplus B') \leq \delta^{\mathcal{F}}(A, A') + \delta^{\mathcal{F}}(B, B')$. ■

The next statement is an immediate consequence of Proposition 2.60.

Corollary 2.62. *The set* $\mathrm{Obj}(\mathcal{C})$ *with the topology induced by the fragmentation pseudometric* $d^{\mathcal{F}}$ *is an* H*-space relative to the operation*

$$(A, B) \in \mathrm{Obj}(\mathcal{C}) \times \mathrm{Obj}(\mathcal{C}) \to A \oplus B \in \mathrm{Obj}(\mathcal{C}).$$

2.3.4.1 Proof of Proposition 2.4. We now return to the setting in Section 2.1. Thus, $\mathcal{D}$ is a triangulated category, w is a triangular weight on $\mathcal{D}$ in the sense of Definition 2.1, and the quantities $w(D)$ (associated to an iterated cone decomposition D), $\delta^{\mathcal{F}}$, and $d^{\mathcal{F}}$ are defined as in Section 2.1.

The first (and main) step is to establish a result similar to Proposition 2.55. Namely, if X admits an iterated cone decomposition D with linearization $(X_1, \ldots, X_n)$ and some X_i admits a decomposition D' with linearization $(A_1, \ldots, A_k)$, then X admits an iterated cone decomposition D'' with linearization $(X_1, \ldots, X_{i-1}, TA_1, \ldots, TA_k, X_{i+1}, \ldots, X_n)$ and

$$w(D'') \leq w(D) + w(D'). \tag{2.34}$$

For convenience, recall from Section 2.1 that the expression of $w(D)$ is

$$w(D) = \sum_{i=1}^{n} w(\Delta_i) - w_0, \tag{2.35}$$

where $w_0 = w(0 \to X \xrightarrow{\mathbb{1}_X} X \to 0)$ for all X. To show (2.34) we go through exactly the same construction of the refinement D'' of the decomposition D with respect to D', as in the proof of Proposition 2.55, assuming now that all shifts are trivial along the way. The analog of the diagram (2.30) that appears in the last step of the construction of D'' remains possible in this context due to point (ii) of Definition 2.1. By tracking the respective weights along the construction and using Remark 2.2 (b) to estimate the weight of $TA_1 \to B \to B_1 \to T^2 A_1$ from (2.30), we deduce (with the notation in the proof of Proposition 2.55) that

$$\sum_{j=1}^{k} w(\bar{\Delta}_j) \leq w(D') + w(\Delta_i) - w_0,$$

which implies (2.34). Once formula (2.34) is established, it immediately follows that $\delta^{\mathcal{F}}(-,-)$ satisfies the triangle inequality. Further, because the weight of a cone decomposition is given by (2.35), it follows that the cone decomposition D of X with linearization $(T^{-1}X)$ given by the single exact triangle $T^{-1}X \to 0 \to X \to X$ is of weight $w(D) = 0$. As a consequence, $\delta^{\mathcal{F}}(X, X) = 0$. It follows that $d^{\mathcal{F}}(-,-)$ is a pseudometric, as claimed in point (i) of Proposition 2.4.

Assuming now that w is subadditive, the same type of decomposition as in equation (2.33) can be constructed to show that $\delta^{\mathcal{F}}(A \oplus B, A' \oplus B') \leq \delta^{\mathcal{F}}(A, A') + \delta^{\mathcal{F}}(B, B') + w_0$, which implies the claim. ■

2.4 A persistence triangular weight on $\mathcal{C}_\infty$

The purpose of this section is to further explore the structure of the limit category $\mathcal{C}_\infty$ associated to a triangulated persistence category $\mathcal{C}$. We already know from Section 2.3.2 that the category $\mathcal{C}_\infty$ is triangulated. The main aim here is to use the properties of the weighted exact triangles introduced in Section 2.3.3 to endow $\mathcal{C}_\infty$ with a triangular weight, in the sense of Section 2.1.

2.4.1 Weight of exact triangles in $\mathcal{C}_\infty$

In this section we will use the weighted strict exact triangles in $\mathcal{C}$ to associate weights to the exact triangles in $\mathcal{C}_\infty$.

Assume that $\mathcal{C}$ is a persistence category and recall its ∞-level $\mathcal{C}_\infty$ from Definition 2.9: its objects are the same as those of $\mathcal{C}$ and its morphisms are

$$\hom_{\mathcal{C}_\infty}(A,B) = \varinjlim_{\alpha\to\infty} \hom^{\alpha}_{\mathcal{C}}(A,B)$$

for any two objects A, B of $\mathcal{C}$. For a morphism $\bar{f}$ in $\mathcal{C}$, we denote by $[\bar{f}]$ the corresponding morphism in $\mathcal{C}_\infty$, and if $f = [\bar{f}]$ for $f \in \hom_{\mathcal{C}_\infty}$ and $\bar{f} \in \hom_{\mathcal{C}}$, we say that $\bar{f}$ represents f. We use the same terminology for diagrams (including triangles) in $\mathcal{C}$ in relation to corresponding diagrams in $\mathcal{C}_\infty$, in the sense that a diagram in $\mathcal{C}$ represents one in $\mathcal{C}_\infty$ if the objects in the two cases are the same and the morphisms in the diagram in $\mathcal{C}$ represent the corresponding ones in the $\mathcal{C}_\infty$ diagram. Clearly, all r-commutativities and r-isomorphisms in $\mathcal{C}$ become, respectively, commutativities and isomorphisms in $\mathcal{C}_\infty$. For instance, if K is r-acyclic, then K is isomorphic to 0 in $\mathcal{C}_\infty$.

For further reference, notice also that the hom-sets of $\mathcal{C}_\infty$ admit a natural filtration as follows. For any $A, B \in \mathrm{Obj}(\mathcal{C}_\infty)$ and $f \in \hom_{\mathcal{C}_\infty}(A,B)$, let the *spectral invariant* of f be given by

$$\sigma(f) := \inf\bigl\{k \in \mathbb{R}\cup\{-\infty\} \mid f = \bigl[\bar{f}\bigr] \text{ for some } \bar{f} \in \hom^{k}_{\mathcal{C}}(A,B)\bigr\} \tag{2.36}$$

and

$$\hom^{\le\alpha}_{\mathcal{C}_\infty}(A,B) = \{f \in \hom_{\mathcal{C}_\infty}(A,B) \mid \sigma(f) \le \alpha\}.$$

Assume from now on that $\mathcal{C}$ is a triangulated persistence category. In this case, we have already seen in Section 2.3.2 that $\mathcal{C}_\infty$ is identified with $\mathcal{C}_0/\mathcal{A}\mathcal{C}_0$, the Verdier quotient of $\mathcal{C}_0$ by the subcategory of acyclics. Thus $\mathcal{C}_\infty$ is triangulated with its exact triangles defined through isomorphism with the image in $\mathcal{C}_\infty$ of the exact triangles in $\mathcal{C}_0$; see Remark 2.41.

Before proceeding, we notice that the shift functor Σ associated to $\mathcal{C}$ (see Definition 2.21) induces a similar functor $\Sigma : (\mathbb{R},+) \to \mathrm{End}(\mathcal{C}_\infty)$. We continue to use the same notation for the r-shifts Σ^r and the natural transformations $\eta_{r,s} : \Sigma^r \to \Sigma^s$. At the same time, in contrast to morphisms in $\mathcal{C}$, there is no meaning to the "amount of shift" for a morphism in $\mathcal{C}_\infty$ (though one can associate to such a morphism its spectral invariant as above). Similarly, the functor T (which is defined as in Remark 2.23 (b) on all of $\mathcal{C}$) also induces a similar functor on $\mathcal{C}_\infty$.

Given a triple of maps $\Delta : A \to B \to C \to D$ in $\mathcal{C}$, the shifted triple $\Sigma^{s_1,s_2,s_3,s_4}\Delta$ was defined in (2.26), and we will use the same notation for similar triples in $\mathcal{C}_\infty$. Note however that in $\mathcal{C}_\infty$ the inequalities relating the s_i's and the shifts of u, v, w are no longer relevant (in fact, do not make sense) and the shift Σ^{s_1,s_2,s_3,s_4} will be used in $\mathcal{C}_\infty$ without these constraints.

Definition 2.63. The *unstable weight* $w_\infty(\Delta)$ of an exact triangle $\Delta : A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} TA$ in $\mathcal{C}_\infty$ is the infimum of the weights of the strict exact triangles in $\mathcal{C}$ of the form $(\widetilde{\Delta}, r)$, where

$$\widetilde{\Delta} : A \xrightarrow{u'} \Sigma^{-q} B \xrightarrow{v'} \Sigma^{-s} C \xrightarrow{w'} \Sigma^{-r} TA,$$

$0 \leq q \leq s \leq r$, and $\widetilde{\Delta}$ represents Δ in the following sense: the class of the composition $A \xrightarrow{u'} \Sigma^{-q} B \xrightarrow{\eta_{-q,0}} B$ in $\mathcal{C}_\infty$ equals u, and similarly for v' and w'. The *weight* of Δ, $\overline{w}(\Delta)$, is given by

$$\overline{w}(\Delta) = \inf\{w_\infty(\Sigma^{s,0,0,s} \Delta) \mid s \geq 0\}.$$

Remark 2.64. By definition, $\overline{w}(\Delta) \leq w_\infty(\Delta)$, and Example 2.72 below shows that this inequality can be strict.

For the weight $\overline{w}$ of exact triangles in $\mathcal{C}_\infty$ defined as above, recall that $\overline{w}_0$ denotes the normalization constant in Definition 2.1 (ii). The main result is the following.

Theorem 2.65. *Let $\mathcal{C}$ be a triangulated persistence category. The limit category $\mathcal{C}_\infty$ with the triangular structure coming from the identification $\mathcal{C}_\infty \simeq \mathcal{C}_0 / \mathcal{AC}_0$ in Proposition* 2.39 *admits $\overline{w}$ as a triangular, subadditive weight with $\overline{w}_0 = 0$.*

A persistence refinement of a triangulated category $\mathcal{D}$ is a TPC, denoted by $\widetilde{\mathcal{D}}$, such that $\widetilde{\mathcal{D}}_\infty = \mathcal{D}$. The triangular weight $\overline{w}$ as in Theorem 2.65 is called the persistence weight induced by the respective refinement. The following consequence of Theorem 2.65 is immediate from the general constructions in Section 2.1.

Corollary 2.66. *If a small triangulated category $\mathcal{D}$ admits a TPC refinement $\widetilde{\mathcal{D}}$, then* Obj$(\mathcal{D})$ *is endowed with a family of fragmentation pseudometrics $\bar{d}^{\mathcal{F}}$, defined as in Section* 2.1*, associated to the persistence weight $\overline{w}$ induced by the refinement $\widetilde{\mathcal{D}}$, and it has an H-space structure with respect to the topologies induced by these metrics.*

Remark 2.67. (a) We have seen in Section 2.3.4, in particular Definition 2.58, that for a TPC $\mathcal{C}$ there are fragmentation pseudometrics $d^{\mathcal{F}}$ defined on Obj$(\mathcal{C})$. The metrics $\bar{d}^{\mathcal{F}}$ associated to the persistence weight $\overline{w}$ on $\mathcal{C}_\infty$ through the construction in Section 2.1 are defined on the same underlying set, Obj$(\mathcal{C})$. The relation between them is

$$\bar{d}^{\mathcal{F}} \leq d^{\mathcal{F}}$$

for any family of objects $\mathcal{F}$. The interest in working with $\bar{d}^{\mathcal{F}}$ rather than with $d^{\mathcal{F}}$ is that if $\mathcal{F}$ is a family of triangular generators of $\mathcal{C}_\infty$ and is closed under the action of T, then $\bar{d}^{\mathcal{F}}$ is finite (see Remark 2.5).

(b) As discussed in Remark 2.5 (c) (and also in Remark 2.59 (d)), in this setting too we may define a simpler (but generally larger) fragmentation pseudometric of the type $\underline{\bar{d}}^{\mathcal{F}}$.

We postpone the proof of Theorem 2.65 to Section 2.4.2. Here, we present a few examples shedding some light on Definition 2.63.

Example 2.68. Assume that $\Delta : A \xrightarrow{\bar{u}} B \xrightarrow{\bar{v}} C \xrightarrow{\bar{w}} \Sigma^{-r}TA$ is a strict exact triangle of weight r in $\mathcal{C}$. Consider the following triangle in $\mathcal{C}_\infty$,

$$\Delta_\infty : A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} TA,$$

where $u = [\bar{u}]$, $v = [\bar{v}]$, and $w = \big[C \xrightarrow{\bar{w}} \Sigma^{-r}TA \xrightarrow{(\eta_{-r,0})_{TA}} TA\big]$. We claim that Δ_∞ is an exact triangle in $\mathcal{C}_\infty$ and $w_\infty(\Delta_\infty) \leq r$. Indeed, by construction, Δ_∞ is represented by

$$\overline{\Delta}_\infty : A \xrightarrow{\bar{u}} B \xrightarrow{\bar{v}} C \xrightarrow{(\eta_{-r,0})_A \circ \bar{w}} TA$$

and $\lceil \bar{u} \rceil = \lceil \bar{v} \rceil = 0$, $\lceil (\eta_{-r,0})_A \circ \bar{w} \rceil = r + 0 = r$. Now, the shifted triangle $\widetilde{\Delta}_\infty = \Sigma^{0,0,0,-r}\overline{\Delta}_\infty$ obviously equals Δ, the initial strict exact triangle of weight r. Thus, $[\widetilde{\Delta}_\infty] = \Delta_\infty$ and $w(\widetilde{\Delta}_\infty) = r$. Therefore, $w_\infty(\Delta_\infty) \leq r$.

Remark 2.69. A special case of the situation in Example 2.68 is worth emphasizing. Any exact triangle in $\mathcal{C}_0$ induces an exact triangle in $\mathcal{C}_\infty$ with unstable weight equal to 0. This implies that any morphism $f \in \hom_{\mathcal{C}_\infty}(A, B)$ with $\sigma(f) \leq 0$ can be completed to an exact triangle of unstable weight 0 in $\mathcal{C}_\infty$. Indeed, we first represent f by a morphism $\bar{f} \in \hom^{\alpha}_{\mathcal{C}}(A, B)$ with $\alpha \leq 0$. If $\alpha \neq 0$, we shift $\bar{f}$ up using the persistence structure maps and set $\bar{f}' = i_{\alpha,0}(\bar{f})$. We obviously have $[\bar{f}'] = f$. We then complete $\bar{f}'$ to an exact triangle in $\mathcal{C}_0$. The image of this triangle in $\mathcal{C}_\infty$ is exact, of unstable weight 0, and has f as the first morphism in the triple.

Example 2.70. Consider the strict exact triangle $\Delta : A \to 0 \to \Sigma^{-r}TA \xrightarrow{\mathbb{1}} \Sigma^{-r}TA$ in $\mathcal{C}$, which is of weight $r \geq 0$ (see Remark 2.50 (b)). Let Δ_∞ be the following triangle in $\mathcal{C}_\infty$:

$$\Delta_\infty : A \to 0 \to \Sigma^{-r}TA \xrightarrow{\eta_{-r,0}} TA.$$

We claim that if $\sigma(\mathbb{1}_A) = 0$, then $\bar{w}(\Delta_\infty) = w_\infty(\Delta_\infty) = r$. Indeed, assume that $w_\infty(\Delta_\infty) = s < r$. Then there exists a strict exact triangle in $\mathcal{C}$ of the form

$$A \to 0 \to \Sigma^{-r'}TA \xrightarrow{w'} \Sigma^{-s}TA,$$

with $r' \geq r$, of weight s and such that $\lceil w' \rceil = 0$, $[\eta_{-s,0} \circ w' \circ \eta_{-r,-r'}] = [\eta_{-r,0}]$. Notice that $\lceil (\eta_{-s,0} \circ w' \circ \eta_{-r,-r'}) \rceil = r - r' + s$. Thus, as $r' \geq r > s$, we deduce $\sigma([\eta_{-r,0}]) < r$. By writing $\mathbb{1}_A = \eta_{0,-r} \circ \eta_{-r,0}$ we deduce $\sigma(\mathbb{1}_A) < 0$. We conclude that $w_\infty(\Delta_\infty) = r$. We next consider a triangle $\Sigma^{k,0,0,k}\Delta_\infty$:

$$\Sigma^k A \to 0 \to \Sigma^{-r}TA \to \Sigma^k TA$$

and rewrite it as

$$\Delta' : B \to 0 \to \Sigma^{-r-k}TB \to TB,$$

with $B = \Sigma^k A$. Suppose that $w_\infty(\Delta') < r + k$; then $\sigma(\mathbb{1}_B) < 0$, by the previous argument. This implies $\sigma(\mathbb{1}_A) < 0$ and again contradicts our assumption. Thus $w_\infty(\Delta') = r + k$ and $\bar{w}(\Delta_\infty) = r$.

Remark 2.71. For an object $A \in \mathrm{Obj}(\mathcal{C})$ we always have $\sigma(\mathbb{1}_A) \leq 0$. We have seen just above that if $\sigma([(\eta_{-r,0})_A]) < r$, then $\sigma(\mathbb{1}_A) < 0$. The same conclusion remains true if $\sigma([(\eta_{0,-r})_A]) < -r$ by the same argument. Another useful observation is that if the map $\eta_s : \Sigma^s A \to A$ is an r-isomorphism, with $r < s$, then $\sigma(\mathbb{1}_A) \leq r - s$. Indeed, if η_s is an r-isomorphism, then it has a left r-inverse $\psi : \Sigma^r A \to \Sigma^s A$ with $\eta_s \circ \psi = \eta_r$, which implies $\sigma([\eta_r]) \leq \sigma([\eta_s]) \leq -s$ and thus $\sigma(\mathbb{1}_A) \leq r - s$.

Example 2.72. Let $f \in \hom_{\mathcal{C}_\infty}(A, B)$ and suppose that $\sigma(f) > 0$. We will see here that we can extend f to an exact triangle in $\mathcal{C}_\infty$, of unstable weight at most $\sigma(f) + \epsilon$ (for any $\epsilon > 0$) but, at the same time, no triangle extending f has unstable weight less than $\sigma(f)$. Fix $\sigma(f) < t \leq \sigma(f) + \epsilon$ and let $\bar{f} \in \hom^t_{\mathcal{C}}(A, B)$ be a representative of f. Consider the composition $f' : A \xrightarrow{\bar{f}} B \xrightarrow{(\eta_{0,-t})_B} \Sigma^{-t} B$. Then $f' \in \hom^0_{\mathcal{C}}(A, \Sigma^{-t} B)$. There exists an exact triangle in $\mathcal{C}_0$

$$\Delta : A \xrightarrow{f'} \Sigma^{-t} B \xrightarrow{v'} C \xrightarrow{w} TA$$

for some $C \in \mathrm{Obj}(\mathcal{C}_0)$. In particular, $w \in \hom^0_{\mathcal{C}}(C, TA)$. Next, consider the triangle

$$\Sigma^{0,0,0,-t} \Delta : A \xrightarrow{f'} \Sigma^{-t} B \xrightarrow{v'} C \xrightarrow{w'} \Sigma^{-t} TA, \quad \text{where } w' = \eta_t^{TA} \circ w.$$

By Remark 2.50 (b), $\Sigma^{0,0,0,-t}\Delta$ is a strict exact triangle in $\mathcal{C}$ of weight t. Finally, consider the following triangle in $\mathcal{C}$, obtained by shifting up the last three terms of $\Sigma^{0,0,0,-t}\Delta$:

$$\bar{\Delta} : A \xrightarrow{\bar{f} = (\eta_{-t,0})_B \circ f'} B \xrightarrow{v := \Sigma^t v'} \Sigma^t C \xrightarrow{w := \Sigma^t w'} TA.$$

Its image in $\mathcal{C}_\infty$ is the triangle

$$\Delta_\infty : A \xrightarrow{f = [\bar{f}]} B \xrightarrow{[v]} \Sigma^t C \xrightarrow{[w]} TA$$

and is exact. In the terminology of Definition 2.63, the representative $\tilde{\Delta}$ of Δ_∞ is the strict exact triangle $\Sigma^{0,0,0,-t}\Delta$. In particular, $w_\infty(\Delta_\infty) \leq t$. Notice that Definition 2.63 immediately implies that any triangle $\Delta'' : A \xrightarrow{f} B \to D \to TA$ in $\mathcal{C}_\infty$ satisfies $w_\infty(\Delta'') \geq \sigma(f)$ (because, with the notation of the definition, the weight of the triangle $\tilde{\Delta}$ in that definition is at least q). At the same time, $\bar{w}(\Delta_\infty) = 0$ because $\Sigma^{t,0,0,t}(\Delta_\infty)$ has as representative

$$\Sigma^t \Delta : \Sigma^t A \to B \to \Sigma^t C \to \Sigma^t TA,$$

which is exact in $\mathcal{C}_0$.

Remark 2.73. (a) The definition of the weights of the exact triangles in $\mathcal{C}_\infty$ is designed precisely to allow for the construction in Example 2.72. This is quite different compared to the case when the spectral invariant of f is non-positive (compare with Remark 2.69) because the persistence structure maps can be used to "shift" up but not down. It also follows from Example 2.72 that for $f \in \hom_{\mathcal{C}_\infty}(A, B)$ with $\sigma(f) \geq 0$ we have

$$\sigma(f) = \inf\{w_\infty(\Delta) \mid \Delta : A \xrightarrow{f} B \to C \to TA\}.$$

(b) It is not difficult to see that if $w_\infty(\Sigma^{s,0,0,s}\Delta) \leq r$ for a triangle Δ in $\mathcal{C}_\infty$ and $s \geq 0$, then $w_\infty(\Delta) \leq r + s$.

Example 2.74. Let $\Delta : A \to B \to C \to TA$ be an exact triangle in $\mathcal{C}_0$. It is clear that, in $\mathcal{C}_\infty$ and for $s \leq 0$, the corresponding triangles of the form $\Delta_s : \Sigma^s A \to B \to C \to \Sigma^s TA$ (defined using the (pre)-composition of the maps in Δ with the appropriate maps η's on A) have the property $\bar{w}(\Delta_s) = 0$. On the other hand, if $s > 0$ this is no longer the case, in general. Indeed, assuming $s > 0$ and $\bar{w}(\Delta_s) = 0$, it follows that for some possibly even larger $r > 0$ we have $w_\infty(\Delta_r) = 0$. This means that for any sufficiently small $\epsilon > 0$ there is an exact triangle in $\mathcal{C}_0$ of the form $\Sigma^{r'}A \to \Sigma^{-\epsilon'}B \to C' \to \Sigma^{r'}TA$, where $r' \geq r$, together with an ϵ-isomorphism $\phi : C' \to \Sigma^{-\epsilon''}C$ satisfying the conditions in Definition 2.42, with $0 \leq \epsilon' \leq \epsilon'' \leq \epsilon$ (see Definition 2.63). This triangle can be compared to a shift $\Sigma^{-2\epsilon}\Delta$ of the initial Δ (the constant 2 is necessary to ensure the commutativity of the rectangles in the comparison diagram; see Remark 2.43 (c)). The resulting commutative diagram is

$$\begin{tikzcd}
\Sigma^{r'}A \arrow[r] \arrow[d, "\eta^A"] & \Sigma^{-\epsilon'}B' \arrow[r] \arrow[d, "\eta^B"] & C' \arrow[r] \arrow[d, "\eta^C_{2\epsilon}\circ\phi"] & \Sigma^{r'}TA \arrow[d, "\eta^A"] \\
\Sigma^{-2\epsilon}A \arrow[r] & \Sigma^{-2\epsilon}B \arrow[r] & \Sigma^{-2\epsilon}C \arrow[r] & \Sigma^{-2\epsilon}TA
\end{tikzcd}$$

The map out of C' in this diagram is a 3ϵ-isomorphism and, by Proposition 2.31 (ii), we deduce that $\eta^A_{r'+2\epsilon}$ is a 15ϵ-isomorphism. We can take ϵ as small as needed and we deduce $\sigma(\mathbb{1}_A) \leq -r' \leq -s$.

Example 2.75. Let $\Delta : A \to 0 \to \Sigma^r TA \to TA$ be a triangle in $\mathcal{C}_\infty$ with the last map (the class of) η^{TA}_r and with $r \geq 0$. Assuming $\sigma(\mathbb{1}_A) = 0$, we claim that $w_\infty(\Delta) = r$ and $\bar{w}(\Delta) = 0$. Indeed, the fact that $\bar{w}(\Delta) = 0$ is obvious because $\Sigma^{r,0,0,r}\Delta : \Sigma^r A \to 0 \to \Sigma^r TA \to \Sigma^r TA$ is exact in $\mathcal{C}_0$. Now assume that $w_\infty(\Delta) < r$. Then there is a triangle $A \to 0 \to \Sigma^{r-s'}TA \xrightarrow{u} \Sigma^{-s}TA$ which is strict exact in $\mathcal{C}$ and with $s \geq s'$ and $w_\infty(\Delta) \leq s < r$. Using the definition of strict exact triangles and the existence of the exact triangle $A \to 0 \to TA \to TA$ in $\mathcal{C}_0$, we deduce that there exists an s-isomorphism $\phi : TA \to \Sigma^{r-s'}TA$ with a right inverse $\psi : \Sigma^{s+r-s'}TA \to TA$ that coincides with η^{TA} in $\mathcal{C}_\infty$. As a result, ϕ coincides with $\eta^{TA}_{0,r-s'}$ in $\mathcal{C}_\infty$ and thus $0 = \lceil\phi\rceil \geq r - s'$ ($\sigma(\mathbb{1}_A) = 0$ implies that $\sigma\big([\eta^{TA}_{0,r-s'}]\big) = r - s'$). Therefore, $s' \geq r$, which contradicts our assumption $s < r$.

Example 2.76. Another example of interest is given by the triangle in $\mathcal{C}_\infty$:

$$\Delta : 0 \to X \to \Sigma^r X \to 0,$$

where the map $X \to \Sigma^r X$ is the class of $(\eta_{0,r})_X$. We claim that $w_\infty(\Delta) \leq |r|$. We start with the case $r > 0$. In that case, by shifting down the last two objects by r, we obtain a strict exact triangle in $\mathcal{C}$: $(0 \to X \to X \to 0, r)$ (here it is important to view strict exact triangles as pairs (triangle, weight) as in Definition 2.42). In the case $r \leq 0$, there is a strict exact triangle in $\mathcal{C}$: $(0 \to X \to \Sigma^r X \to 0, -r)$ that can be reached from Δ by shifting down the last object by $|r|$. This shows the claim. Additionally, it is easy to see that if $\sigma(\mathbb{1}_X) = 0$, then $w_\infty(\Delta) = r$.

Example 2.77. In this example we consider an exact triangle in $\mathcal{C}_\infty$,

$$\Delta : 0 \to X \to Y \to 0.$$

We claim that $\overline{w}(\Delta) = \inf\{r \geq 0 \mid \exists \text{ an } r\text{-isomorphism } \phi : \Sigma^s X \to Y \text{ with } 0 \leq s \leq r\}$. By the definition of the weight of triangles in $\mathcal{C}_\infty$, we are looking for strict exact triangles in $\mathcal{C}$ of the form

$$(0 \to X \to \Sigma^{-s} Y \to \Sigma^{-r} 0, r)$$

with $s \leq r$. The unstable weight, in this case equal to the stable weight, is obtained by infimizing r. The existence of such a strict exact triangle is equivalent to the existence of an r-isomorphism $\phi : X \to \Sigma^{-r} Y$, which shows the claim.

Example 2.78. Consider an exact triangle in $\mathcal{C}_\infty$ of the form

$$\Delta : T^{-1} X \to 0 \to Y \to X.$$

We claim that $w_\infty(\Delta) = \inf\{r \geq 0 \mid \exists \text{ an } r\text{-isomorphism } \phi : \Sigma^s X \to Y \text{ with } 0 \leq s \leq r\}$ and $\overline{w}(\Delta) = \inf\{r \geq 0 \mid \exists \text{ an } r\text{-isomorphism } \phi : \Sigma^s X \to Y \text{ with } s \geq 0\}$. The relevant strict exact triangles (of weight r) in this case are

$$T^{-1} X \to 0 \to \Sigma^{-s} Y \to \Sigma^{-r} X$$

with $s \leq r$. Again, the existence of such a triangle is equivalent to the existence of an r-isomorphism $\phi : X \to \Sigma^{-s} Y$, which provides the estimate for w_∞. To estimate $\overline{w}$, we apply the same argument but replace X by $\Sigma^k X$ with k positive. The condition then becomes that $\phi : \Sigma^k X \to \Sigma^{-s} Y$ is an r-isomorphism and that $s \leq r$.

Example 2.79. Consider an exact triangle in $\mathcal{C}_\infty$ of the form

$$\Delta : T^{-1} X \to K \to Y \to X,$$

where K is k-acyclic. We claim that

$$\bar{w}(\Delta) \geq \inf\{r \geq 0 \mid \exists \text{ an } (r+2k)\text{-isomorphism } \phi : \Sigma^s X \to Y \text{ with } s \geq 0\}.$$

We thus assume that there exists a strict exact triangle $\Delta' : T^{-1}X \to \Sigma^{-s_1}K \to \Sigma^{-s_2}Y \to \Sigma^{-r}X$ of weight r, with $s_1 \leq s_2 \leq r$, that represents Δ. As this is strict exact, there is an exact triangle in $\mathcal{C}_0$ of the form $T^{-1}X \to \Sigma^{-s_1}K \to C \to X$ that maps to Δ as in Definition 2.42. In particular, there is an r-isomorphism $C \to \Sigma^{-s_2}Y$. The fact that $K \simeq_k 0$ immediately implies that there is a k-isomorphism $C \to X$. We consider a right k-inverse of it, $\Sigma^k X \to C$. This a $2k$-isomorphism. By composition we get a $(2k+r)$-isomorphism $\Sigma^k X \to \Sigma^{-s_2}Y$. The constraint $s_2 \leq r$ is eliminated by applying the same argument as in Example 2.78, replacing X with $\Sigma^s X$ for some non-negative s.

Remark 2.80. It is useful to know how the weights of triangles in $\mathcal{C}_\infty$ behave with respect to rotation. Thus let $\Delta : A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} TA$ be an exact triangle in $\mathcal{C}_\infty$ of unstable weight r. Then its first positive rotation $R(\Delta) : B \xrightarrow{v} C \xrightarrow{w} TA \xrightarrow{-Tu} TB$ is of unstable weight at most $2r$. Indeed, by definition, there exists a triangle $\bar{\Delta} : A \xrightarrow{\bar{u}} B \xrightarrow{\bar{v}} C \xrightarrow{\bar{w}} TA$ representing Δ such that the shifted triangle $\tilde{\Delta} : A \xrightarrow{u'} \Sigma^{-t_1}B \xrightarrow{v'} \Sigma^{-t_1-t_2}C \xrightarrow{w'} \Sigma^{-r}TA$ (where we put $t_1 = \lceil \bar{u} \rceil$, $t_2 = \lceil \bar{v} \rceil$, $t_3 = \lceil \bar{w} \rceil$, and $r = t_1 + t_2 + t_3$) is a strict exact triangle in $\mathcal{C}$ of weight r. By Proposition 2.46, the first positive rotation

$$R(\tilde{\Delta}) : \Sigma^{-t_1}B \xrightarrow{v'} \Sigma^{-t_1-t_2}C \xrightarrow{w''} \Sigma^{-r}TA \xrightarrow{u''} \Sigma^{-t_1-2r}TB$$

is a strict exact triangle in $\mathcal{C}$ of weight $2r$. We have

$$\begin{aligned} u'' &= -(\eta_{0,-t_1-2r})_B \circ \bar{u} \circ (\eta_{-r,0})_A \\ v' &= (\eta_{0,-t_1-t_2})_C \circ \bar{v} \circ (\eta_{-t_1,0})_B \\ w'' &\simeq_r w = (\eta_{0,-r})_A \circ \bar{w} \circ (\eta_{-t_1-t_2,0})_C. \end{aligned}$$

Consider a new triangle:

$$\Sigma^{t_1}R(\tilde{\Delta}) : B \xrightarrow{\Sigma^{t_1}v'} \Sigma^{-t_2}C \xrightarrow{\Sigma^{t_1}w''} \Sigma^{-r+t_1}TA \xrightarrow{\Sigma^{t_1}u''} \Sigma^{-2r}TB.$$

By Remark 2.50 (a), $\Sigma^{t_1}R(\tilde{\Delta})$ is also a strict exact triangle in $\mathcal{C}$ of weight $2r$. By shifting this triangle up we get to $B \to C \to TA \to TB$, which represents $R(\Delta)$; thus $w_\infty(R(\Delta)) \leq 2r$. In a similar way, using Remark 2.47, one can treat the negative rotation $R^{-1}(\Delta)$ of Δ.

2.4.2 Proof of Theorem 2.65

There are two steps. The first is to show that each exact triangle in $\mathcal{C}_\infty$ has finite unstable weight. The second step is to show that $\bar{w}$ satisfies Definition 2.1 and is subadditive with $\bar{w}_0 = 0$.

2.4.2.1 Every triangle in $\mathcal{C}_\infty$ has finite unstable weight. Let $\Delta : A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} TA$ be an exact triangle in $\mathcal{C}_\infty$. Thus, there exist a triangle $\bar{\Delta} : A \xrightarrow{\bar{u}} B \xrightarrow{\bar{v}} C \xrightarrow{\bar{w}} TA$ in $\mathcal{C}$ and an exact triangle in $\mathcal{C}_0$, $\Delta' : A' \xrightarrow{u'} B' \xrightarrow{v'} C' \xrightarrow{w'} TA'$, together with isomorphisms in $\mathcal{C}_\infty$, $a : A' \to A$, $b : B' \to B$, $c : C' \to C$, such that the resulting map $(a, b, c, -Ta)$ of triangles in $\mathcal{C}_\infty$ is an isomorphism of triangles. We may assume that the shifts x of $\bar{u}$, y of $\bar{v}$, and z of $\bar{w}$ are all non-negative.

We represent the maps a, b, c by s-isomorphisms $\bar{a} : \Sigma^r A' \to A$, $\bar{b} : \Sigma^r B' \to B$, $\bar{c} : \Sigma^r C' \to C$, which is possible by taking s and r sufficiently large. We now consider the diagram in $\mathcal{C}_0$:

$$\begin{tikzcd}
\Sigma^r A' \arrow[r, "\Sigma^r u'"] \arrow[d, "\bar{a}"'] & \Sigma^r B' \arrow[d, "\eta_x \circ \bar{b}"] \\
A \arrow[r, "\eta_x \circ \bar{u}"'] & \Sigma^{-x} B
\end{tikzcd}$$

This diagram r'-commutes for r' sufficiently large. Lemma 2.17 shows that, by taking r sufficiently large, we may assume that the above square commutes in $\mathcal{C}_0$ (at the cost of making both r and s very large). We complete the diagram to the right, thus obtaining a new diagram in $\mathcal{C}_0$:

$$\begin{tikzcd}
\Sigma^r A' \arrow[r, "\Sigma^r u'"] \arrow[d, "\bar{a}"'] & \Sigma^r B' \arrow[r, "\Sigma^r v'"] \arrow[d, "\eta_x \circ \bar{b}"] & \Sigma^r C' \arrow[r, "\Sigma^r w'"] \arrow[d, "c''"] & \Sigma^r TA' \arrow[d, "a''"] \\
A \arrow[r, "\eta_x \circ \bar{u}"'] & \Sigma^{-x} B \arrow[r, "v''"'] & \Sigma^{-x-y-t} C \arrow[r, "w''"'] & \Sigma^{-x-y-t-t'-w} TA
\end{tikzcd}$$

with each square commutative and with the arrows marked with $(-)''$ being compositions of $(-)$ with the appropriate η's. The commutativity in $\mathcal{C}_0$ for the second square requires possibly a shift by $-t$ of the third term in the bottom row. Similarly, the commutativity of the third square requires a shift by $-t'$ for the last term of the row (this is a variant of Proposition 2.51). Obviously, the top row is exact in $\mathcal{C}_0$ and the vertical maps are k-isomorphisms, each for a different value of k. At the cost of yet again increasing t and t', we can intercalate between the two rows a new exact triangle in $\mathcal{C}_0$:

$$\begin{tikzcd}
\Sigma^r A' \arrow[r, "\Sigma^r u'"] \arrow[d, "\bar{a}"'] & \Sigma^r B' \arrow[r, "\Sigma^r v'"] \arrow[d, "\eta_x \circ \bar{b}"] & \Sigma^r C' \arrow[r, "\Sigma^r w'"] \arrow[d, "c'''"] & \Sigma^r TA' \arrow[d, "\bar{a}"] \\
A \arrow[r, "\eta_x \circ \bar{u}"'] \arrow[d] & \Sigma^{-x} B \arrow[r, "v''"'] \arrow[d] & C'' \arrow[r] \arrow[d, "d"] & TA \arrow[d, "e"] \\
A \arrow[r, "\eta_x \circ \bar{u}"'] & \Sigma^{-x} B \arrow[r, "v''"'] & \Sigma^{-x-y-t} C \arrow[r, "w''"'] & \Sigma^{-x-y-t-t'-w} TA
\end{tikzcd}$$

The map c''' is induced by the first two vertical maps between the top rows, and thus it is a k-isomorphism for some sufficiently large k. Given that c'' is such an isomorphism too, by possibly increasing t we obtain the existence of a k'-isomorphism d with k' very large. By possibly increasing t' we can also get the commutativity of the bottom-right square. Again by possibly increasing t' we may ensure that $k' = x + y + t + t' + w$ (recall that any k'-isomorphism is also a k''-isomorphism for $k'' \geq k'$). This means that the bottom row is a strict exact triangle in $\mathcal{C}$ of weight k'. To end the proof, we notice that this triangle is of the form $\widetilde{\Delta}$ as in Definition 2.63.

2.4.2.2 The weighted octahedral axiom in $\mathcal{C}_\infty$. To finish the proof of Theorem 2.65 we need to show that the weighted octahedral axiom is satisfied by $\overline{w}$, that $\overline{w}$ satisfies the normalization in Definition 2.1 (ii) with $\overline{w}_0 = 0$, and that it is subadditive. We start below with the weighted octahedral axiom and will end with the other properties.

Lemma 2.81. *The weight $\overline{w}$ as defined in Definition 2.63 satisfies the weighted octahedral axiom from Definition 2.1* (i).

Proof. Recall that given the exact triangles $\Delta_1 : A \to B \to C \to TA$ and $\Delta_2 : C \to D \to E \to TC$ in $\mathcal{C}_\infty$ we need to show that there are exact triangles $\Delta_3 : B \to D \to F \to TB$ and $\Delta_4 : TA \to F \to E \to T^2 A$ making the diagram below commute, except for the bottom-right square, which anti-commutes:

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & TA \arrow[r] \arrow[d] & TA \arrow[d] \\
B \arrow[r] \arrow[d] & D \arrow[r] \arrow[d] & F \arrow[r] \arrow[d] & TB \arrow[d] \\
C \arrow[r] \arrow[d] & D \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & TC \arrow[d] \\
TA \arrow[r] & 0 \arrow[r] & T^2 A \arrow[r] & T^2 A
\end{tikzcd}$$

and such that $\overline{w}(\Delta_3) + \overline{w}(\Delta_4) \leq \overline{w}(\Delta_1) + \overline{w}(\Delta_2)$.

In $\mathcal{C}$, there are triangles $\bar{\Delta}_1 : A \to B \to C \to TE$ with non-negative morphism shifts t_1, t_2, t_3, and $\bar{\Delta}_2 : C \to D \to E \to TC$ with non-negative morphism shifts k_1, k_2, k_3, representing Δ_1 and Δ_2, respectively, and such that the associated triangles

$$\tilde{\Delta}_1 : A \to \Sigma^{-t_1} B \to \Sigma^{-t_1-t_2} C \to \Sigma^{-r} TA, \quad \text{where } r = t_1 + t_2 + t_3,$$

and

$$\tilde{\Delta}_2 : C \to \Sigma^{-k_1} D \to \Sigma^{-k_1-k_2} E \to \Sigma^{-s} TC, \quad \text{where } s = k_1 + k_2 + k_3,$$

are strict exact triangles in $\mathcal{C}$ of weight r and s, respectively. Consider $\Sigma^{-t_1-t_2}\tilde{\Delta}_2$:

$$\Sigma^{-t_1-t_2}\tilde{\Delta}_2 : \Sigma^{-t_1-t_2} C \to \Sigma^{-k_1-t_1-t_2} D \to \Sigma^{-k_1-k_2-t_1-t_2} E \to \Sigma^{-s-t_1-t_2} TC.$$

The weighted octahedral property for strict exact triangles in $\mathcal{C}$, stated in Proposition 2.49, implies that we can construct the following diagram in $\mathcal{C}$, commutative except for the bottom-right square, which is r-anti-commutative:

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & TA \arrow[r] \arrow[d] & TA \arrow[d] \\
\Sigma^{-t_1} B \arrow[r] \arrow[d] & \Sigma^{-k_1-t_1-t_2} D \arrow[r] \arrow[d] & D' \arrow[r] \arrow[d] & \Sigma^{-t_1} TB \arrow[d] \\
\Sigma^{-t_1-t_2} C \arrow[r] \arrow[d] & \Sigma^{-k_1-t_1-t_2} D \arrow[r] \arrow[d] & \Sigma^{-k_1-k_2-t_1-t_2} E \arrow[r] \arrow[d] & \Sigma^{-t_1-t_2-s} TC \arrow[d] \\
\Sigma^{-r} TA \arrow[r] & 0 \arrow[r] & \Sigma^{-r-s} T^2 A \arrow[r] & \Sigma^{-r-s} T^2 A
\end{tikzcd} \tag{2.37}$$

Here the triangle $\Delta_3' : \Sigma^{-t_1} B \to \Sigma^{-k_1-t_1-t_2} D \to D' \to \Sigma^{-t_1} TB$ is an exact triangle in $\mathcal{C}_0$. The triangle $\Delta_3'' : \Sigma^{k_1+t_2} B \to D \to \Sigma^{k_1+t_1+t_2} D' \to \Sigma^{k_1+t_2} TB$, obtained by shifting Δ_3' up by $k_1 + t_1 + t_2$, is also exact in $\mathcal{C}_0$. Let $[\Delta_3'']$ be the image of this triangle in $\mathcal{C}_\infty$. We put $F = \Sigma^{k_1+t_1+t_2} D'$ and take Δ_3 to be the triangle in $\mathcal{C}_\infty$

$$\Delta_3 : B \to D \to F \to TB$$

obtained by applying $\Sigma^{-k_1-t_2,0,0,-k_1-t_2}$ to $[\Delta_3'']$. We obviously have $w_\infty([\Delta_3'']) = 0$ and thus $\bar{w}(\Delta_3) = 0$.

The next step is to identify the triangle Δ_4. Proposition 2.49 implies that the third column in (2.37),

$$\Delta_4' : TA \to \Sigma^{-k_1-t_1-t_2} F \to \Sigma^{-k_1-k_2-t_1-t_2} E \to \Sigma^{-r-s} T^2 A,$$

is a strict exact triangle in $\mathcal{C}$. Let $[\Delta_4']$ be the image of this triangle in $\mathcal{C}_\infty$, and let Δ_4 be given by applying $\Sigma^{0,k_1+t_1+t_2,k_1+k_2+t_1+t_2,r+s}$ to $[\Delta_4']$:

$$\Delta_4 : TA \to F \to E \to T^2 A.$$

We deduce from Definition 2.63 that $w_\infty(\Delta_4) \leq r + s$ and thus also $\overline{w}(\Delta_4) \leq r + s$.

The commutativity required in the statement follows from that provided by Proposition 2.49 for (2.37). ∎

Remark 2.82. For the triangle Δ_3 produced in the proof above, it is easy to see that $w_\infty(\Delta_3) \leq k_1 + t_2$. Therefore we have

$$w_\infty(\Delta_3) + w_\infty(\Delta_4) \leq (k_1 + t_2) + (r + s) \leq 2(r + s) = 2(w_\infty(\Delta_1) + w_\infty(\Delta_2)).$$

Thus the weight w_∞ satisfies a weak form of the weighted octahedral axiom.

The next step in proving Theorem 2.65 is to show the normalization property in Definition 2.1 (ii). This property is satisfied with the constant $\overline{w}_0 = 0$. Indeed, any triangle $0 \to X \to X \to 0$ and all its rotations are exact in $\mathcal{C}_0$, and thus they are of unstable weight equal to 0. The last verification needed is to see that, if $B = 0$ in the diagram of the weighted octahedral axiom, then the triangle Δ_3 (constructed in the proof of Lemma 2.81) can be of the form $\Delta_3 : 0 \to D \to D \to 0$. This is trivially satisfied in our construction because if $B = 0$ we may take $t_1 = t_2 = 0$ and the triangle $\Delta_3' : 0 \to \Sigma^{-k_1} D \xrightarrow{\mathbb{1}} \Sigma^{-k_1} D \to 0$.

Finally, to finish the proof of Theorem 2.65 we need to show that $\overline{w}$ is subadditive. Thus, assuming that $\Delta : A \to B \to C \to TA$ is exact in $\mathcal{C}_\infty$ and that X is an object in $\mathcal{C}$, we have $\overline{w}(X \oplus \Delta) \leq \overline{w}(\Delta)$, where the triangle $X \oplus \Delta$ has the form $X \oplus \Delta : A \to X \oplus B \to X \oplus C \to TA$. We consider the strict exact triangle in $\mathcal{C}$

$$\widetilde{\Delta} : A \to \Sigma^{-s_1} B \to \Sigma^{-s_1-s_2} C \to \Sigma^{-s_1-s_2-s_3} TA$$

associated to Δ as in Definition 2.63, where $s_i \geq 0$, $1 \leq i \leq 3$. Consider the triangle

$$\Delta' : 0 \to \Sigma^{-s_1} X \xrightarrow{\eta_{s_2}^X} \Sigma^{-s_1-s_2} X \to 0.$$

This triangle is obtained from the exact triangle in $\mathcal{C}_0$ $0 \to \Sigma^{-s_1} X \xrightarrow{\mathbb{1}} \Sigma^{-s_1} X \to 0$ by applying $\Sigma^{0,0,-s_2,-s_2}$, and it is of weight at most s_2. By Lemma 2.61 we have $w(\Delta' \oplus \widetilde{\Delta}) \leq w(\widetilde{\Delta})$. We now note that $\Delta' \oplus \widetilde{\Delta}$ can be viewed as being obtained from $X \oplus \Delta$ by applying $\Sigma^{0,-s_1,-s_1-s_2,-s_1-s_2-s_3}$, and thus $w_\infty(X \oplus \Delta) \leq w(\widetilde{\Delta})$, which implies the claim. The proof for $\Delta \oplus X$ is similar.

2.4.3 Some properties of fragmentation pseudometrics

The purpose of this section is to rapidly review some of the properties of the persistence fragmentation pseudometrics. We start by recalling the main definitions; we then discuss some algebraic properties and relations to standard notions such as the interleaving distance.

2.4.3.1 Persistence fragmentation pseudometrics: Review of main definitions. Assume that $\mathcal{C}$ is a TPC. We have defined in Sections 2.3.4 and 2.4.1 three similarly defined types of measurements on the objects of $\mathcal{C}$ which, after symmetrization, define fragmentation pseudometrics on $\mathrm{Obj}(\mathcal{C})$. In Section 2.3.4, this construction uses directly the weight w of the strict exact triangles in $\mathcal{C}$ (making use of Proposition 2.49), and it gives rise to the pseudometrics $d^{\mathcal{F}}$ as in Definition 2.58, as well as to a simplified version $\underline{d}^{\mathcal{F}}$ mentioned in Remark 2.59 (d).

In Section 2.4.1 we have endowed the triangles of the category $\mathcal{C}_\infty$ with weights: an unstable weight w_∞ as well as a (smaller) stable weight $\bar{w}$, as given in Definition 2.63. Working in the category $\mathcal{C}_\infty$ has a significant advantage over working in the category $\mathcal{C}$ because, in contrast to the latter, in $\mathcal{C}_\infty$ any morphism can be completed to an exact triangle of finite weight. Moreover, when we have a (translation-invariant) family of triangular generators $\mathcal{F}$, the fragmentation pseudodistance relative to $\mathcal{F}$ is finite; see Remark 2.5 (a). Finally, in $\mathcal{C}_\infty$ exact triangles have the standard form expected in a triangulated category and do not involve shifts. Therefore, we will focus here on the fragmentation pseudometrics defined using w_∞ and, mainly, $\bar{w}$.

An important remark at this point is that the unstable weight w_∞ does not satisfy the weighted octahedral axiom (but only its weak form, as discussed in Remark 2.82), and thus only the pseudometrics of the form $\underline{\bar{d}}^{\mathcal{F}}$ can be defined using it. By contrast, $\bar{w}$ does satisfy the weighted octahedral axiom and there is a pseudometric $\bar{d}^{\mathcal{F}}$ associated to it through the construction in Section 2.1. Both $\bar{w}$ and w_∞ are subadditive and satisfy the normalization property in Definition 2.1 with $w_0 = 0$.

To eliminate possible ambiguities, we recall the definitions of the two relevant pseudometrics here. Both of them are based on considering a sequence of exact triangles in $\mathcal{C}_\infty$ as below:

$$\begin{tikzcd}
Y_0 \arrow[rr] & & Y_1 \arrow[r] \arrow[ddl, dashed] & \dots \arrow[r] & Y_i \arrow[rr] & & Y_{i+1} \arrow[r] \arrow[ddl, dashed] & \dots \arrow[r] & Y_{n-1} \arrow[rr] & & Y_n \arrow[ddl, dashed] \\
& \Delta_1 & & & & \Delta_{i+1} & & & & \Delta_n & \\
& X_1 \arrow[uul] & & & & X_{i+1} \arrow[uul] & & & & X_n \arrow[uul] &
\end{tikzcd} \tag{2.38}$$

where the dotted arrows represent maps $Y_i \to TX_i$. We fix a family of objects $\mathcal{F}$ in $\mathcal{C}$ with $0 \in \mathcal{F}$ and define

$$\underline{\bar{\delta}}^{\mathcal{F}}(X, X') = \inf\Big\{\sum_{i=1}^{n} w_\infty(\Delta_i) \mid \Delta_i \text{ are successive exact triangles as in (2.38)} \\ \text{with } X' = Y_0,\ X = Y_n, \text{ and } X_i \in \mathcal{F}, n \in \mathbb{N}\Big\}, \tag{2.39}$$

and

$$\bar{\delta}^{\mathcal{F}}(X,X') = \inf\Big\{\sum_{i=1}^{n} \bar{w}(\Delta_i) \mid \Delta_i \text{ are as in (2.38) with } Y_0 = 0,\, X = Y_n,\, X_i \in \mathcal{F}, \\ n \in \mathbb{N} \text{ except for some } j \text{ such that } X_j = T^{-1}X'\Big\}. \tag{2.40}$$

Finally, the pseudometrics $\underline{\bar{d}}^{\mathcal{F}}$ and $\bar{d}^{\mathcal{F}}$ are obtained by symmetrizing $\underline{\bar{\delta}}^{\mathcal{F}}$ and $\bar{\delta}^{\mathcal{F}}$, respectively:

$$\underline{\bar{d}}^{\mathcal{F}}(X,X') = \max\{\underline{\bar{\delta}}^{\mathcal{F}}(X,X'), \underline{\bar{\delta}}^{\mathcal{F}}(X',X)\},$$
$$\bar{d}^{\mathcal{F}}(X,X') = \max\{\bar{\delta}^{\mathcal{F}}(X,X'), \bar{\delta}^{\mathcal{F}}(X',X)\}.$$

2.4.3.2 Algebraic properties. There are many fragmentation pseudometrics of persistence type associated to the same weight, depending on the choice of family $\mathcal{F}$. In fact, the choices available are even more abundant for the following two reasons:

(i) Triangular weights themselves can be mixed. For instance, if $\mathcal{C}$ is a TPC, there is a triangular weight of the form $\bar{w}^{+} = \bar{w} + w_{fl}$ that is defined on $\mathcal{C}_\infty$ (where w_{fl} is the flat weight defined in Section 2.1).

(ii) Fragmentation metrics themselves can also be mixed. If $d^{\mathcal{F}}$ and $d^{\mathcal{F}'}$ are two fragmentation pseudometrics (whether defined with respect to the same weight or not), then the expressions $\alpha d^{\mathcal{F}} + \beta d^{\mathcal{F}'}$ with $\alpha, \beta \geq 0$, as well as $\max\{d^{\mathcal{F}}, d^{\mathcal{F}'}\}$, are also pseudometrics.

In essence, although it is not easy to produce interesting subadditive triangular weights on a triangulated category, once such a weight is constructed – as in the case of the persistence weight $\bar{w}$ defined on $\mathcal{C}_\infty$ (where $\mathcal{C}$ is a TPC) – one can associate to it a large class of pseudometrics, either by combining the weight with the flat one and/or by "mixing" the pseudometrics associated to different families $\mathcal{F}$. Another useful (and obvious) property relating the pseudometrics $d^{\mathcal{F}}$ and $d^{\mathcal{F}'}$ associated to the same triangular weight is the following:

(iii) If $\mathcal{F} \subset \mathcal{F}'$, then $d^{\mathcal{F}'} \leq d^{\mathcal{F}}$.

The last useful construction has to do with making the metrics invariant with respect to the action of the shift functor.

(iv) For a given fragmentation metric $d^{\mathcal{F}}$ we define its shift-invariant version

$$\widehat{d^{\mathcal{F}}}(X,Y) = \max\{\widehat{\delta^{\mathcal{F}}}(X,Y), \widehat{\delta^{\mathcal{F}}}(Y,X)\}. \tag{2.41}$$

Here

$$\widehat{\delta^{\mathcal{F}}}(X,Y) = \inf_{r,s \in \mathbb{R}} \delta^{\mathcal{F}}(\Sigma^r X, \Sigma^s Y)$$

is the shift-invariant version of the semi (pseudo)metrics $\delta^{\mathcal{F}}$ as in (2.39) and (2.40). It is immediate to see that $\widehat{\delta^{\mathcal{F}}}$ satisfies the triangle inequality. By symmetrizing, we obtain indeed a pseudometric that is obviously bounded above by $d^{\mathcal{F}}$. In case the family $\mathcal{F}$ is closed under the action of Σ^r for all $r \in \mathbb{R}$, the metrics of type $d^{\mathcal{F}}$ have the property that $d^{\mathcal{F}}(X, Y) = d^{\mathcal{F}}(\Sigma^r X, \Sigma^r Y)$ for all $r \in \mathbb{R}$. In this case, the shift-invariant metric associated to $d^{\mathcal{F}}$ has a simpler form $\widehat{d^{\mathcal{F}}}(X, Y) = \inf_{r \in \mathbb{R}} d^{\mathcal{F}}(\Sigma^r X, Y)$. The interest of this type of shift-invariant pseudometric is that it compares the "shape" of objects, in contrast to a comparison of the objects themselves that is sensitive to translations (the spectral distance in symplectic topology is of this type). Thus, for instance, two Morse functions f and $f + k$ with $k \in \mathbb{R}$ are not distinguished by shift-invariant-type pseudometrics.

2.4.3.3 Vanishing and non-degeneracy of fragmentation metrics. We fix here a TPC denoted by $\mathcal{C}$ together with the associated weights and pseudometrics, as above. We will denote by $\bar{d}^{\{0\}}$ the pseudometric associated to the family consisting of only the element 0. In view of point (iii) above, $\bar{d}^{\{0\}}$ is an upper bound for all the pseudometrics $\bar{d}^{\mathcal{F}}$.

It is obvious, as noticed in Remark 2.59, that in general $\bar{d}^{\mathcal{F}}$ is degenerate. For instance, if $\mathcal{F} = \mathrm{Obj}(\mathcal{C})$ then $\bar{d}^{\mathcal{F}} \equiv 0$. The rest of Remark 2.59 also continues to apply to $\bar{d}^{\mathcal{F}}$. We list below some other easily proven properties. We assume for all the objects X involved here that $\sigma(\mathbb{1}_X) = 0$, and we will use the calculations in Examples 2.70, 2.72, 2.74, and 2.75. Recall the notion of r-isomorphism from Definition 2.26; in particular, this is a morphism in $\mathcal{C}_0$. A 0-isomorphism is simply an isomorphism in the category $\mathcal{C}_0$ and is denoted by $\equiv$.

(i) If $X \equiv X'$, then $\bar{d}^{\mathcal{F}}(X, X') = 0 = \underline{\bar{d}}^{\mathcal{F}}(X, X')$ for any family $\mathcal{F}$.

(ii) We have

$$\begin{aligned} &2\,\bar{\delta}^{\{0\}}(X, X') \\ &\qquad \geq \inf\{r \in \mathbb{R} \mid \exists \text{ an } r\text{-isomorphism } \phi : \Sigma^k X' \to X \text{ for some } k \geq 0\} \\ &\qquad \geq \bar{\delta}^{\{0\}}(X, X'). \end{aligned}$$

For the first inequality, consider a sequence of triangles (2.38) of total weight at most r. We intend to show that, for some $k \geq 0$, there exists a $2r$-isomorphism $\Sigma^k X' \to X$. We make use of Examples 2.77, 2.78, and 2.79. We assume, without loss of generality, that $T^{-1}X'$ appears in the jth triangle. The first triangles are of the form $0 \to Y_i \to Y_{i+1} \to 0$ for $i \leq j-1$ with $Y_0 = 0$. By Example 2.78 we deduce that Y_{j-1} is r_0-acyclic and that r_0 is at most the sum of the weights of the first $j-1$ triangles. The next triangle is of the form $T^{-1}X' \to Y_{j-1} \to Y_j \to X'$ and of weight r_j. Example 2.79 shows that there exists a $(2r_0 + r_j)$-isomorphism $X' \to \Sigma^{-k} Y_j$. The next triangles, of the form $0 \to Y_i \to Y_{i+1} \to 0$, have weights r_i, and there are r_i-isomorphisms $Y_i \to \Sigma^{-s_i} Y_{i+1}$ with $r_i \geq s_i$ (see Example 2.77). Putting things together, we have $2r \geq 2r' \geq 2r_0 + r_j + \cdots + r_n$, and there is a $2r'$-isomorphism $X' \to \Sigma^{-s_1-s_2-s_3-\cdots-s_n} X$. This implies that $2\,\bar{\delta}^{\{0\}} \geq \inf$. For the second inequality, assume that $\phi : X' \to \Sigma^{-k} X$ is an r-isomorphism. We need to construct a cone decomposition of $\bar{w}$-weight at most r. We first assume $k \leq r$. The first triangle is $T^{-1}X' \to 0 \to X'$; it is exact in $\mathcal{C}_0$ and of weight 0. The second triangle is $0 \to X' \to X \to 0$. The associated strict exact triangle is $(0 \to X' \to \Sigma^{-k} X \to 0, r)$, and it uses ϕ in an obvious way to compare with the exact triangle $0 \to X' \to X' \to 0$ in $\mathcal{C}_0$. So we are left with the case $k > r$. In this case, the first triangle is $T^{-1}X' \to 0 \to \Sigma^{k-r} X' \to X'$. Its $\bar{w}$-weight is null. The next triangle is $0 \to \Sigma^{k-r} X' \to X \to 0$, the associated strict exact triangle being $(0 \to \Sigma^{k-r} X' \to \Sigma^{-r} X \to 0, r)$, where ϕ is now used to compare with the exact triangle $0 \to \Sigma^{k-r} X' \to \Sigma^{k-r} X' \to 0$.

(iii) We have

$$\underline{\bar{\delta}}^{\{0\}}(X, X') = \inf\{r \in \mathbb{R} \mid \exists \text{ an } r\text{-isomorphism } \phi : \Sigma^k X' \to X \text{ with } r \geq k \geq 0\}.$$

This happens because the first triangle in the sequence (2.38) is $0 \to X' \to Y_1 \to 0$ and the next triangles are of the form $0 \to Y_i \to Y_{i+1} \to 0$. Each of them has a weight estimated by the numbers r_i for which there is an r_i-isomorphism $Y_i \to \Sigma^{-s_i} Y_{i+1}$ with $0 \leq s_i \leq r_i$, which shows the claim.

(iv) We have $\bar{d}^{\{0\}}(X, \Sigma^r X) = r$ for any $r \in \mathbb{R}$ (this follows from Examples 2.70 and 2.75).

(v) For the shift-invariant metric $\widehat{\bar{d}^{\{0\}}}$ induced by $\bar{d}^{\{0\}}$ through the formula (2.41) we have $\widehat{\bar{d}^{\{0\}}}(X, \Sigma^r X) = 0$ for all X and $r \in \mathbb{R}$.

Thus $\bar{d}^{\{0\}}$ is finite for pairs of objects that are isomorphic in $\mathcal{C}_\infty$ and $\bar{d}^{\{0\}}(X, X')$ is the optimal upper bound r such that there are s-isomorphisms in $\mathcal{C}$ with $s \leq r$, from some positive shift of X to X' and, similarly, from some positive shift of X' to X. To some extent, $\bar{d}^{\{0\}}$ can be viewed as an abstract analog of the interleaving distance in the theory of persistence modules (cf. [47, Section 1.3]). We explore the relation with interleaving in more detail in Section 2.4.3.4 (see also Proposition 2.105).

Remark 2.83. There is another pseudometric $\bar{d}^{\emptyset}$, which means that $\mathcal{F} = \emptyset$. In this case, by definition, this $\bar{d}^{\emptyset}$ is again an algebraic analog of the interleaving distance. We will not use this pseudometric later in the memoir, so we do not further discuss its properties here.

For $\underline{\bar{d}}^{\{0\}}$ there is the additional constraint that the respective shifts should also be bounded by r. As a consequence:

(vi) If $\bar{d}^{\{0\}}(X, X') = 0$, then X and X' are 0-isomorphic up to shift. Moreover, if X and X' are not 0-isomorphic, they are both periodic in the sense that there exist k and k' (not both null) and 0-isomorphisms $\Sigma^k X \to X$, $\Sigma^{k'} X' \to X'$.

(vii) If $\underline{\bar{d}}^{\{0\}}(X, X') = 0$, then $X \equiv X'$.

In summary, this means that the best we can expect from the fragmentation pseudometrics is that they should be non-degenerate on the space of 0-isomorphism types. From now on, we will say that a fragmentation pseudometric is non-degenerate if this is the case. Assuming no periodic objects exist, the metric $\bar{d}^{\{0\}}$ is non-degenerate in this sense. However, the distance it measures for two objects that are not isomorphic in $\mathcal{C}_\infty$ is infinite. On the other hand, a metric such as $\bar{d}^{\mathcal{F}}$ (as well as $\underline{\bar{d}}^{\mathcal{F}}$), where $\mathcal{F}$ is a family of triangular generators of $\mathcal{C}_\infty$, is finite but is in general degenerate.

The last point we want to raise in this section is that mixing fragmentation pseudometrics can sometimes produce non-degenerate ones. We will see an example of this sort in the symplectic Section 3.1, but we end here by describing a more general, abstract argument. Fix two families $\mathcal{F}_i$, $i = 1, 2$, of generators of $\mathcal{C}_\infty$. Consider the mixed pseudometric defined by

$$\bar{d}^{\mathcal{F}_1, \mathcal{F}_2} = \max\{\bar{d}^{\mathcal{F}_1}, \bar{d}^{\mathcal{F}_2}\}. \tag{2.42}$$

The idea is that if these two families are "separated" in a strong sense, then the mixed metric is non-degenerate. For instance, denote by $\mathcal{F}_i^{\Delta}$ the subcategory of $\mathcal{C}_0$ that is generated by $\mathcal{F}_i$. Now assume that $\mathrm{Obj}(\mathcal{F}_1^{\Delta}) \cap \mathrm{Obj}(\mathcal{F}_2^{\Delta}) = \{0\}$ (this is of course quite restrictive). We now claim that $\underline{\bar{d}}^{\mathcal{F}_1, \mathcal{F}_2}$ is non-degenerate and that $\bar{d}^{\mathcal{F}_1, \mathcal{F}_2}$ satisfies a weaker non-degeneracy condition, namely, that $\bar{d}^{\mathcal{F}_1, \mathcal{F}_2}(X, 0) = 0$ if and only if $X \equiv 0$. This latter fact follows immediately by noticing that $\bar{d}^{\mathcal{F}_i}(X, 0) = 0$ means that $X \in \mathcal{F}_i^{\Delta}$. We leave the former as an exercise.

2.4.3.4 Fragmentation pseudometrics, the interleaving pseudometric, and other algebraic measurements.

The aim of this section is to describe relations between the fragmentation pseudometrics introduced above and the interleaving distance, which is well known in persistence theory as well as in Morse and Floer theory. A discussion of the bottleneck distance, which is closely related to interleaving, is included in Section 2.5.2.

Adapting the definition of the interleaving distance [47] to TPCs is immediate.

Definition 2.84. Let $\mathcal{C}$ be a triangulated persistence category with shift functor Σ. Given two objects $X, Y \in \mathrm{Obj}(\mathcal{C})$, the *interleaving distance* between X and Y is defined by

$$\begin{aligned} d_{\mathrm{int}}(X, Y) = \inf\bigl\{r \geq 0 \mid \exists \phi \in \hom_{\mathcal{C}_0}(\Sigma^r X, Y),\ \psi \in \hom_{\mathcal{C}_0}(\Sigma^r Y, X) \\ \text{such that } \psi \circ \Sigma^r \phi = \eta_{2r}^X,\ \phi \circ \Sigma^r \psi = \eta_{2r}^Y\bigr\}. \end{aligned}$$

It is a simple exercise to check that this is indeed a pseudometric.

We will also make use of the shift-invariant version:

$$\widehat{d_{\mathrm{int}}}(X,Y)=\inf_{r\in\mathbb{R}} d_{\mathrm{int}}(\Sigma^r X,Y),$$

which we will refer to as the *shift-invariant interleaving pseudometric*.

Lemma 2.85. *Let $\mathcal{C}$ be a triangulated persistence category and let X, Y be two objects in $\mathcal{C}$. If $\phi : X \to Y$ is an r-isomorphism, then $d_{\mathrm{int}}(X,Y) \le r$. Conversely, if $d_{\mathrm{int}}(X,Y) = s$, then for any $r > s$ there exist $4r$-isomorphisms $\Sigma^r X \to Y$ and $\Sigma^r Y \to X$.*

Proof. We start with the first part of the lemma. By the results in Section 2.3.1, ϕ has a right r-inverse $\psi : \Sigma^r Y \to X$ such that $\phi \circ \psi = \eta_r^Y$. Let $\phi' = \phi \circ \eta_r^X$ be such that we have the diagram

$$\Sigma^{2r} Y \xrightarrow{\Sigma^r \psi} \Sigma^r X \xrightarrow{\phi'} Y$$

with $\phi' \circ \Sigma^r \psi = \eta_{2r}^Y$. We now consider the composition $\psi \circ \Sigma^r \phi'$:

$$\Sigma^{2r} X \xrightarrow{\eta_r^{\Sigma^r X}} \Sigma^r X \xrightarrow{\Sigma^r \phi} \Sigma^r Y \xrightarrow{\psi} X.$$

Proposition 2.28 claims that ψ is r-equivalent to a left inverse of ϕ. Thus we have $\psi \circ \Sigma^r \phi \simeq_r \eta_r^X$, and hence $\psi \circ \Sigma^r \phi' = \eta_{2r}^X$, which shows the claim.

We pass to the second part of the lemma and now assume that $d_{\mathrm{int}}(X,Y) = s < r$. We fix morphisms $f : \Sigma^r X \to Y$ and $g : \Sigma^r Y \to X$ such that $g \circ \Sigma^r f = \eta_{2r}^X$ and $f \circ \Sigma^r g = \eta_{2r}^Y$. This means that both f and g have right and left $2r$-inverses. Therefore, by Lemma 2.32 (iv), they are both $4r$-isomorphisms. ■

Recall now the largest of our metrics from Section 2.4.3.1, $\bar{d}^{\{0\}}$, and its shift-invariant version $\widehat{d^{\{0\}}}$; see also Sections 2.4.3.2 and 2.4.3.3.

Corollary 2.86. *In the setting above we have*

$$\frac{1}{2}\,\widehat{d_{\mathrm{int}}}(X,Y) \le \widehat{d^{\{0\}}}(X,Y) \le 4\,\widehat{d_{\mathrm{int}}}(X,Y).$$

In particular, all shift-invariant pseudometrics of the type $\widehat{\bar{d}^{\mathcal{F}}}$ have as upper bound

$4 \cdot$ *(the shift-invariant interleaving pseudometric).*

Proof. We start with the first inequality from the left. Assume that $\widehat{d^{\{0\}}}(X,Y) = s$ and fix $r > s$, $r < s + \epsilon$. This means that there exists $m \in \mathbb{R}$ such that $\bar{d}^{\{0\}}(\Sigma^m X, Y) < r$. In particular, $\bar{\delta}^{\{0\}}(\Sigma^m X, Y) < r$. Thus, by point (ii) in Section 2.4.3.3, we deduce that there exists some $k \ge 0$ and a $2r$-isomorphism $\Sigma^{m+k} X \to Y$. By the first point of Lemma 2.85 we deduce that $d_{\mathrm{int}}(\Sigma^{m+k} X, Y) \le 2r$, and thus $\widehat{d_{\mathrm{int}}}(X,Y) \le 2r$, which implies the desired inequality.

We now turn to the second inequality. Let $s = \widehat{d_{\text{int}}}(X, Y)$, and let r be such that $s < r < s + \epsilon$. There exists $m \in \mathbb{R}$ such that $d_{\text{int}}(\Sigma^m X, Y) < r$. From the second point of Lemma 2.85, we deduce that there are $4r$-isomorphisms $\Sigma^{r+m} X \to Y$ and $\Sigma^r Y \to \Sigma^m X$. This means, by point (ii) in Section 2.4.3.3, that $\bar{\delta}^{\{0\}}(\Sigma^m X, Y) \leq 4r$ and $\bar{\delta}^{\{0\}}(Y, \Sigma^m X) \leq 4r$. Thus $\bar{d}^{\{0\}}(\Sigma^m X, Y) \leq 4r$, and we conclude that $\widehat{d^{\{0\}}}(X, Y) \leq 4\widehat{d_{\text{int}}}(X, Y)$. ■

2.4.3.5 Other algebraic measurements. Other algebraic pseudometrics based on measuring the weight of cone decompositions – and not necessarily individual triangles – have appeared in [10]. The basic measurement introduced there can be viewed as a sort of extension of the interleaving distance and is easily formulated in the TPC setting (and in fact in any persistence category). To fix ideas, let $\mathcal{C}$ be a TPC and let $f : X \to Y$ be a morphism in $\mathcal{C}$. We define

$$\rho(f) = \inf_{g,s}\left\{\max(\lceil f \rceil + \lceil g \rceil, s, 0) \mid g : Y \to X,\ g \circ f \simeq_s \mathbb{1}_X\right\}.$$

The way this is used in [10] is the following. Consider a triple (η, Y_n, ϕ) consisting of an iterated cone decomposition η in $\mathcal{C}_0$ with final term Y_n, as below:

$$\begin{tikzcd}
Y_0 \arrow[rr] & & Y_1 \arrow[r] \arrow[ddl, dashed] & \cdots \arrow[r] & Y_i \arrow[rr] & & Y_{i+1} \arrow[r] \arrow[ddl, dashed] & \cdots \arrow[r] & Y_{n-1} \arrow[rr] & & Y_n \arrow[ddl, dashed] \\
 & \Delta_1 & & & & \Delta_{i+1} & & & & \Delta_n & \\
 & X_1 \arrow[uul] & & & & X_{i+1} \arrow[uul] & & & & X_n \arrow[uul] &
\end{tikzcd}$$

and with $\phi : Y \to Y_n$ a morphism in $\mathcal{C}$ that induces an isomorphism in $\mathcal{C}_\infty$. The weight W of such a triple is defined by $W(\eta, Y_n, \phi) = \rho(\phi)$. This can be used to compare objects X, Y in $\mathcal{C}$ relative to a family of objects $\mathcal{F}$, which we assume to be closed under the action of Σ, by defining, for two objects X, Y,

$$\begin{aligned} &Z^{\mathcal{F}}(Y, X) \\ &\quad = \inf\left\{W(\eta, Y_n, \phi) \mid \exists j, \forall i \neq j,\ X_i \in \mathcal{F},\ T^{-1}X = X_j,\ \phi : \Sigma^k Y \to Y_n,\ k \in \mathbb{R}\right\}. \end{aligned}$$

Such a $Z^{\mathcal{F}}$ can be obviously symmetrized and it is shift invariant (because $\mathcal{F}$ is closed under the action of Σ and we included the parameter k in the infimum). However, the fact that the triangle inequality is satisfied is non-trivial, and it is not clear whether this is true for general TPCs. As we will see later in Remark 2.103, the triangle inequality holds in important examples such as the homotopy category of a filtered pre-triangulated dg-category, and similarly for filtered modules over a filtered A_∞-category (this is the case treated in [10]). This subtlety is related to the degree of precision in constructing maps induced on cones. Nonetheless, there is a simple way to compare some of our pseudometrics and $Z^{\mathcal{F}}$.

Lemma 2.87. *In the setting above we have*

$$Z^{\mathcal{F}}(Y, X) \leq 4 \cdot \widehat{\bar{\delta}^{\mathcal{F}}}(Y, X).$$

Proof. Assume that $\widehat{\bar{\delta}^{\mathcal{F}}}(Y, X) = r$. This means that there exists $k \in \mathbb{R}$ and a sequence of exact triangles η_∞ in $\mathcal{C}_\infty$

$$\begin{tikzcd}
0 \arrow[rr] & & Y_1 \arrow[r] \arrow[dl, dashed] & \cdots \arrow[r] & Y_i \arrow[rr] & & Y_{i+1} \arrow[r] \arrow[dl, dashed] & \cdots \arrow[r] & Y_{n-1} \arrow[rr] & & Y_n \arrow[dl, dashed] \\
& X_1 \arrow[ul] \arrow[u, phantom, "\Delta_1"] & & & & X_{i+1} \arrow[ul] \arrow[u, phantom, "\Delta_{i+1}"] & & & & X_n \arrow[ul] \arrow[u, phantom, "\Delta_n"] &
\end{tikzcd} \tag{2.43}$$

with $X_i \in \mathcal{F}$ for all indices except for X_j, in which case $X_j = \Sigma^l T^{-1} X$ for some $l \in \mathbb{R}$. Moreover, $Y_n = Y$ and $\bar{w}(\Delta_i) = r_i$ with $\sum_i r_i = r + \epsilon$ (for any small ϵ).

The aim is to construct another sequence $\bar{\eta}$ of exact triangles, this time exact in $\mathcal{C}_0$,

$$\begin{tikzcd}
0 \arrow[rr] & & \bar{Y}_1 \arrow[r] \arrow[dl, dashed] & \cdots \arrow[r] & \bar{Y}_i \arrow[rr] & & \bar{Y}_{i+1} \arrow[r] \arrow[dl, dashed] & \cdots \arrow[r] & \bar{Y}_{n-1} \arrow[rr] & & \bar{Y}_n \arrow[dl, dashed] \\
& \bar{X}_1 \arrow[ul] \arrow[u, phantom, "\Delta'_1"] & & & & \bar{X}_{i+1} \arrow[ul] \arrow[u, phantom, "\Delta'_{i+1}"] & & & & \bar{X}_n \arrow[ul] \arrow[u, phantom, "\Delta'_n"] &
\end{tikzcd} \tag{2.44}$$

such that for each i there is a $k_i \in \mathbb{R}$ with $\bar{X}_i = \Sigma^{k_i} X_i$ and there is some $\phi : Y \to \bar{Y}_n$ that is a $2(r + \epsilon)$-isomorphism. Assuming this construction is achieved, we deduce from Lemma 2.85 that Y and $\bar{Y}_n$ are $2(r + \epsilon)$-interleaved and thus $\rho(\phi) \leq 4(r + \epsilon)$, which implies the claim.

The first step is to replace the sequence η_∞ with a sequence η of strict exact triangles in $\mathcal{C}$:

$$\begin{cases}
\Delta_1 : X'_1 \to 0 \to Y'_1 \to \Sigma^{-r_1} T X'_1 \\
\Delta_2 : X'_2 \to Y'_1 \to Y'_2 \to \Sigma^{-r_2} T X'_2 \\
\Delta_3 : X'_3 \to Y''_2 \to Y'_3 \to \Sigma^{-r_3} T X'_3 \\
\qquad\qquad \vdots \\
\Delta_n : X'_n \to Y'_{n-1} \to Y'_n \to \Sigma^{-r_n} T X'_n,
\end{cases}$$

where the $(-)'$ are appropriate shifts of the corresponding nodes of the sequence η_∞. Such triangles exist by the definition of the weight $\overline{w}$. We now proceed by induction: we assume that we have constructed the first k triangles as in $\bar{\eta}$ together with an $s_k = 2(r_1 + r_2 + \cdots + r_k)$-isomorphism $\phi_k : Y'_k \to \overline{Y}_k$. We now consider the strict exact triangle Δ_{k+1}:

$$\begin{tikzcd}
\Delta_{k+1} : & X'_{k+1} \arrow[r, "u_{k+1}"] & Y'_k \arrow[r] & Y''_{k+1} \arrow[r] \arrow[d, "f"] & TX'_{k+1} \\
 & & & Y'_{k+1} \arrow[r] & \Sigma^{-r_{k+1}} TX'_{k+1}
\end{tikzcd}$$

as in Definition 2.42 with f an r_{k+1}-isomorphism and the top row an exact triangle in $\mathcal{C}_0$. We consider the two exact triangles in $\mathcal{C}_0$,

$$\begin{tikzcd}
X'_{k+1} \arrow[r, "u_{k+1}"] \arrow[d] & Y'_k \arrow[r] \arrow[d, "\phi_k"] & Y''_{k+1} \arrow[r] \arrow[d, "h"] & TX'_{k+1} \arrow[d] \\
X'_{k+1} \arrow[r, "u'"] & \overline{Y}_k \arrow[r] & Y'''_{k+1} \arrow[r] & TX'_{k+1}
\end{tikzcd} \tag{2.45}$$

where $u' = \phi_k \circ u_{k+1}$ and h is induced from the first square on the left. In particular, h is an s_k-isomorphism. So now we consider

$$Y'_{k+1} \xrightarrow{g} \Sigma^{-r_{k+1}} Y''_{k+1} \xrightarrow{h} \Sigma^{-r_{k+1}} Y'''_{k+1},$$

where g is a left r_{k+1}-inverse of f, and we notice that $h \circ g$ is an s_{k+1}-isomorphism. We will take the map ϕ_{k+1} to be the composition $\phi_{k+1} = h \circ g$ and we put $\overline{Y}_{k+1} = \Sigma^{-r_{k+1}} Y'''_{k+1}$. We take the triangle

$$\Sigma^{-r_{k+1}} X'_{k+1} \xrightarrow{u'} \Sigma^{-r_{k+1}} \overline{Y}_k \longrightarrow \Sigma^{-r_{k+1}} Y'''_{k+1} \longrightarrow \Sigma^{-r_{k+1}} TX'_{k+1},$$

which is the bottom row in (2.45) shifted by $\Sigma^{-r_{k+1}}$ as the $(k+1)$st exact triangle in the sequence $\bar{\eta}$. Finally, we adjust the first k triangles already constructed by shifting them all down by $\Sigma^{-r_{k+1}}$. This produces a sequence of $k+1$ triangles, each exact in $\mathcal{C}_0$, with the desired properties, together with the map ϕ_{k+1}; this completes the induction step. ■

Possibly more useful than the actual statement of Lemma 2.87 is the method of proof: we produced a sequence $\bar{\eta}$ of exact triangles in $\mathcal{C}_0$, as in (2.44), and a $2r$-isomorphism $\phi : Y_n \to \overline{Y}_n$ out of the sequence of triangles in $\mathcal{C}_\infty$ in (2.43) whose sum of weights is r.

Using a right inverse $\psi : \Sigma^{2r} \overline{Y}_n \to Y_n$ of ϕ we can transform the last $\mathcal{C}_0$ exact triangle into a strict exact triangle of weight $4r$. The interest of this construction –

and this will be used in the applications in Section 3.1 – is that we obtain in this way a method to bound $\widehat{\bar{\delta}^{\mathcal{F}}}(Y, X)$ both from below and from above by a simpler quantity $Q^{\mathcal{F}}(Y, X)$, defined as the infimum of the sum of the weights of the triangles in decompositions as in (2.43), but with the first $n-1$ triangles of weight 0. Thus the weight of such a decomposition equals the weight of Δ_n. To summarize what was discussed above we have:

Corollary 2.88. *For any objects X, Y, we have*

$$\widehat{\bar{\delta}^{\mathcal{F}}}(Y, X) \leq Q^{\mathcal{F}}(Y, X) \leq 4 \cdot \widehat{\bar{\delta}^{\mathcal{F}}}(Y, X).$$

Remark 2.89. If in the inequality above one could avoid the factor 4, then we would have a simpler description of the fragmentation pseudometrics discussed here by replacing sequences of strict exact triangles in $\mathcal{C}$ with corresponding sequences in $\mathcal{C}_0$, followed by an s-isomorphism, where s is the sum of the weights of the initial triangles. However, this coefficient has to do with the fact that left (or right) inverses of k-isomorphisms are, in general, only $2k$-isomorphisms (see also Remark 2.34), and a factor of at least 2 is basically unavoidable.

2.5 Examples

2.5.1 Filtered dg-categories

The key property of dg-categories, introduced in [12] (see also [27]), is that they admit natural, pre-triangulated closures. The 0-cohomological category of this closure is triangulated. We will see here that there is a natural notion of *filtered* dg-categories. Such a category also admits a pre-triangulated closure, defined using filtered twisted complexes, following closely [12]. Its 0-cohomological category is a triangulated persistence category.

2.5.1.1 Basic definitions. Following a standard convention, we work in a co-homological setting and we keep all the sign conventions as in [12]. For our purposes it is convenient to view a filtered cochain complex over the field $\mathbf{k}$ as a triple (X, ∂, ℓ) consisting of a cochain complex (X, ∂) and a filtration function $\ell : X \to \mathbb{R} \cup \{-\infty\}$ such that for any $a, b \in X$ and $\lambda \in \mathbf{k} \setminus \{0\}$, $\ell(\lambda a + b) \leq \max\{\ell(a), \ell(b)\}$, $\ell(a) = -\infty$ if and only if $a = 0$, and $\ell(\partial a) \leq \ell(a)$. We denote by $X^{\leq r} = \{x \in X \mid \ell(x) \leq r\} \subset X$ the filtration induced on X by the filtration function ℓ. Clearly, $X^{\leq r}$ is again a filtered cochain complex. The family $\{X^{\leq r}\}_{r \in \mathbb{R}}$ determines the function ℓ. The cohomology of a filtered cochain complex is a persistence module: $V^r(X) = H(X^{\leq r}; \mathbf{k})$, whose structural maps $i_{r,s}$ are induced by the inclusions $\iota_{r,s} : X^{\leq r} \hookrightarrow X^{\leq s}$, $r \leq s$. We have omitted the grading here, as is customary. If it needs to be indicated, we write, for instance, $[V^r(X)]^i = H^i(X^{\leq r}; \mathbf{k})$. We denote this (graded) persistence module by $\mathbb{V}(X)$:

$$\mathbb{V}(X) := (\{V^r(X)\}_{r\in\mathbb{R}}, \{i_{r,s} : V^r(X) \to V^s(X)\}_{r,s\in\mathbb{R}, r\leq s}). \tag{2.46}$$

Given two filtered cochain complexes $X = (X, \partial^X, \ell_X)$ and $Y = (Y, \partial^Y, \ell_Y)$, their tensor product is a filtered cochain complex $(X \otimes Y, \partial^{\otimes}, \ell_{\otimes})$ given by $(X \otimes Y)_k = \bigoplus_{i+j=k}(X_i \otimes Y_j)$ and

$$\partial^{\otimes}(x \otimes y) = \partial^X(x) \otimes y + (-1)^{|x|} x \otimes \partial^Y(y) \;\; \ell_{\otimes}(a \otimes b) = \ell_X(a) + \ell_Y(b).$$

If (X, ℓ_X) and (Y, ℓ_Y) are filtered vector spaces, we call a linear map $\phi : X \to Y$ *r-filtered* if $\ell_Y(\phi(x)) \leq \ell_X(x) + r$ for all $x \in X$. A 0-filtered map is sometimes called (for brevity) filtered. For more background on this formalism, see [59].

The next definition is an obvious analog of the notion of dg-category in [12, Section 1].

Definition 2.90. A *filtered dg-category* is a preadditive category $\mathcal{A}$ where

(i) for any $A, B \in \mathrm{Obj}(\mathcal{A})$, the hom-set $\hom_{\mathcal{A}}(A, B)$ is a filtered cochain complex with filtrations denoted by $\hom^{\leq r}_{\mathcal{A}}(A, B)$ such that, for each identity element, we have $\ell(\mathbb{1}_A) = 0$ and $\mathbb{1}_A$ is closed;

(ii) the composition is a filtered chain map:

$$\hom_{\mathcal{A}}(B, C) \otimes \hom_{\mathcal{A}}(A, B) \xrightarrow{\circ} \hom_{\mathcal{A}}(A, C);$$

(iii) for any inclusions $\iota^{AB}_{r,r'}$ and $\iota^{BC}_{s,s'}$, the composition morphism satisfies the compatibility condition $\iota^{BC}_{s,s'}(g) \circ \iota^{AB}_{r,r'}(f) = \iota^{AC}_{r+s,r'+s'}(g \circ f)$ for any $f \in \hom^{\leq r}_{\mathcal{A}}(A, B)$ and $g \in \hom^{\leq s}_{\mathcal{A}}(B, C)$.

Remark 2.91. A filtered dg-category is trivially a persistence category by forgetting the boundary maps on each $\hom_{\mathcal{A}}(A, B)$. Explicitly, for any $A, B \in \mathrm{Obj}(\mathcal{A})$, define $E_{AB} : (\mathbb{R}, \leq) \to \mathrm{Vect}_{\mathbf{k}}$ by $E_{AB}(r) = \hom^{\leq r}_{\mathcal{A}}(A, B)$ and $E_{AB}(i_{r,s}) = \iota_{r,s} : \hom^{\leq r}_{\mathcal{A}}(A, B) \to \hom^{\leq s}_{\mathcal{A}}(A, B)$.

The (co)homology category of a filtered dg-category $\mathcal{A}$, denoted by $\mathrm{H}(\mathcal{A})$, is a category with

$$\mathrm{Obj}(\mathrm{H}(\mathcal{A})) = \mathrm{Obj}(\mathcal{A})$$

and, for any $A, B \in \mathrm{Obj}(\mathrm{H}(\mathcal{A}))$,

$$\hom_{\mathrm{H}(\mathcal{A})}(A, B) := \mathbb{V}(\hom_{\mathcal{A}}(A, B)) = \left(\left\{H^*\left(\hom^{\leq r}_{\mathcal{A}}(A, B)\right)\right\}_{r\in\mathbb{R}}, \{i_{r,s}\}_{r\leq s}\right)$$

is the persistence module as described in (2.46). It is immediate to see that, for any filtered dg-category $\mathcal{A}$, its (co)homology category $\mathrm{H}(\mathcal{A})$ is a (graded) persistence category.

2.5.1.2 Twisted complexes. It is easy to construct a formal shift completion of a dg-category.

Definition 2.92. Let $\mathcal{A}$ be a filtered dg-category. The *shift completion* $\Sigma\mathcal{A}$ of $\mathcal{A}$ is a filtered dg-category such that

(i) the objects of $\Sigma\mathcal{A}$ are

$$\mathrm{Obj}(\Sigma\mathcal{A}) = \left\{\Sigma^r A[d] \mid A \in \mathrm{Obj}(\mathcal{A}),\, r \in \mathbb{R}, \text{ and } d \in \mathbb{Z}\right\},$$

where

$$\Sigma^0 A = A, \quad \Sigma^s(\Sigma^r A) = \Sigma^{r+s} A, \quad A[0] = A,$$
$$(A[d_1])[d_2] = A[d_1 + d_2], \quad (\Sigma^r A)[d] = \Sigma^r(A[d])$$

for any $r, s \in \mathbb{R}$ and $d_1, d_2, d \in \mathbb{Z}$;

(ii) for any $\Sigma^r A[d_A], \Sigma^s B[d_B] \in \mathrm{Obj}(\mathcal{A})$, the hom-set $\hom(\Sigma^r A[d_A], \Sigma^s B[d_B])$ is a filtered cochain complex with the same underlying cochain complex of $\hom(A, B)$ but with degree shifted by $d_B - d_A$ and filtration function $\ell_{\Sigma^r A[d_A]\, \Sigma^s B[d_B]} = \ell_{AB} + s - r$.

Remark 2.93. It is immediate to check that $\Sigma\mathcal{A}$ as given in Definition 2.92 is still a filtered dg-category.

The category $\Sigma\mathcal{A}$ carries a natural functor $\Sigma : (\mathbb{R}, +) \to \mathcal{P}\mathrm{End}(\Sigma\mathcal{A})$ defined on objects by $\Sigma^r(A) = \Sigma^r A$ and with an obvious definition on morphisms such that Σ^r is filtration preserving. For any $r, s \in \mathbb{R}$, the natural transformations $\eta_{r,s} : \Sigma^r \to \Sigma^s$ are such that $(\eta_{r,s})_A : \Sigma^r A \to \Sigma^s A$ is induced by the identity map $\mathbb{1}_A$ for each $A \in \mathrm{Obj}(\mathcal{A})$. In this context, we have a natural definition of (one-sided) twisted complexes obtained by adjusting [12, Section 4, Definition 1] to the filtered case.

Definition 2.94. Let $\mathcal{A}$ be a filtered dg-category. A *filtered (one-sided) twisted complex* of $\Sigma\mathcal{A}$ is a pair $A = \left(\bigoplus_{i=1}^n \Sigma^{r_i} A_i[d_i], q = (q_{ij})_{1 \leq i,j \leq n}\right)$ such that the following conditions hold:

(i) $\Sigma^{r_i} A_i[d_i] \in \mathrm{Obj}(\Sigma\mathcal{A})$, where $r_i \in \mathbb{R}$ and $d_i \in \mathbb{Z}$.

(ii) $q_{ij} \in \hom_{\Sigma\mathcal{A}}(\Sigma^{r_j} A_j[d_j], \Sigma^{r_i} A_i[d_i])$ is of degree 1, and $q_{ij} = 0$ for $i \geq j$.

(iii) $d_{\hom} q_{ij} + \sum_{k=1}^n q_{ik} \circ q_{kj} = 0$.

(iv) For any q_{ij}, $\ell_{\Sigma^{r_j} A_j[d_j] \Sigma^{r_i} A_i[d_i]}(q_{ij}) \leq 0$.

Remark 2.95. We will mostly work with *filtered* one-sided twisted complexes as defined above, but, more generally, the pair $A = \left(\bigoplus_{i=1}^n \Sigma^{r_i} A_i[d_i], q = (q_{ij})_{1 \leq i,j \leq n}\right)$, subject only to (i), (ii), and (iii), is called a one-sided twisted complex.

It is easy to see that there are at least as many filtered one-sided twisted complexes as one-sided twisted complexes; this follows from the statement below, whose proof we leave to the reader.

Lemma 2.96. *Given a twisted complex* $\big(\bigoplus_{i=1}^n A_i[d_i], q = (q_{ij})\big)$*, there exist* $(r_i)_{1\le i\le n}$ *such that condition (iv) in Definition 2.94 is satisfied for the filtration-shifted twisted complex* $\big(\bigoplus_{i=1}^n \Sigma^{r_i} A_i[d_i], q = (q_{ij})\big)$.

2.5.1.3 Pre-triangulated completion. We will see next that the filtered twisted complexes over $\mathcal{A}$ form a category that provides a (pre-)triangulated closure of $\mathcal{A}$. The 0-cohomology category of this completion is a triangulated persistence category.

Definition 2.97. Given a filtered dg-category $\mathcal{A}$, define its *filtered pre-triangulated completion*, denoted by $\mathrm{Tw}(\mathcal{A})$, to be the category with the following properties:

(i) Its objects are

$$\mathrm{Obj}(\mathrm{Tw}(\mathcal{A})) := \{\text{filtered one-sided twisted complexes of } \Sigma\mathcal{A}\}.$$

(ii) For $A = \big(\bigoplus \Sigma^{r_j} A_j[d_j], q\big)$ and $A' = \big(\bigoplus \Sigma^{r'_i} A'_i[d'_i], q'\big)$ in $\mathrm{Obj}(\mathrm{Tw}(\mathcal{A}))$, a morphism $f \in \hom_{\mathrm{Tw}(\mathcal{A})}(A, A')$ is a matrix of morphisms in $\mathcal{A}$, denoted by $f = (f_{ij}) : A \to A'$, where

$$f_{ij} \in \hom_{\Sigma\mathcal{A}}\big(\Sigma^{r_j} A_j[d_j], \Sigma^{r'_i} A'_i[d'_i]\big).$$

(iii) The hom-differential is defined as follows. For any $f \in \hom_{\mathrm{Tw}(\mathcal{A})}(A, A')$ as in (ii) above, define

$$d_{\mathrm{Tw}(\mathcal{A})}(f) := (d_{\mathrm{hom}} f_{ij}) + q' f - (-1)^l f q,$$

where $\deg(f_{ij}) = l$, and the right-hand side is written in matrix form. The composition $f' \circ f$ is given by matrix multiplication.

Lemma 2.98. *Given a filtered dg-category* $\mathcal{A}$*, its filtered pre-triangulated completion* $\mathrm{Tw}(\mathcal{A})$ *is a filtered dg-category.*

Proof. The key step is to note that there exists a filtration function on $\hom_{\mathrm{Tw}(\mathcal{A})}(A, A')$ for any $A, A' \in \mathrm{Obj}(\mathrm{Tw}(\mathcal{A}))$. For any $f = (f_{ij}) \in \hom_{\mathrm{Tw}(\mathcal{A})}(A, A')$, set

$$\ell_{AA'}(f) = \max_{i,j}\Big\{\ell_{\Sigma^{r_j} A_j[d_j]\, \Sigma^{r'_i} A'_i[d'_i]}(f_{ij})\Big\}.$$

It is easily checked that $\ell_{AA'}$ is a filtration function and that it satisfies the other required properties. ∎

The first step towards triangulation is to define an appropriate cone of a morphism.

Definition 2.99. Let $\mathcal{A}$ be a filtered dg-category and $\mathrm{Tw}(\mathcal{A})$ be its pre-triangulated completion. Let

$$A = \Big(\bigoplus \Sigma^{r_j} A_j[d_j], q = (q_{ij})_{1\le i,j\le n}\Big),$$
$$A' = \Big(\bigoplus \Sigma^{r_i'} A_i'[d_i'], q = (q_{ij}')_{1\le i,j\le m}\Big)$$

be two objects of $\mathrm{Tw}(\mathcal{A})$, and let $f : A \to A'$ be a closed, degree-preserving morphism. Define the *λ-filtered mapping cone* of f, where $\lambda \ge \ell_{AA'}(f)$, by

$$\mathrm{Cone}^{\lambda}(f) := \Big(\bigoplus_i \Sigma^{r_i'} A_i'[d_i'] \oplus \bigoplus_j \Sigma^{r_j+\lambda} A_j[d_j+1], q_{\mathrm{co}}\Big),$$

where $q_{\mathrm{co}} = \begin{pmatrix} q' & f \\ 0 & -q \end{pmatrix}$, with q', q, f all block matrices.

Remark 2.100. (a) The condition $\lambda \ge \ell_{AA'}(f)$ guarantees that $\mathrm{Cone}^{\lambda}(f)$ is indeed a filtered one-sided twisted complex over $\Sigma\mathcal{A}$. Therefore, $\mathrm{Tw}(\mathcal{A})$ is closed under taking degree shifts, filtration shifts, and filtered mapping cones of (degree-preserving) closed morphisms.

(b) Notice that a λ-filtered cone can also be written as a 0-filtered cone but for a different map.

(c) Given a filtered dg-category $\mathcal{A}$, it is easy to see that every object in $\mathrm{Tw}(\mathcal{A})$ can be obtained from objects in $\Sigma\mathcal{A}$ by taking iterated 0-filtered mapping cones.

The 0-cohomological category associated to a dg-category is a triangulated category. The next result is the analog in the filtered case.

Proposition 2.101. *If $\mathcal{A}$ is a filtered dg-category and $\mathrm{Tw}(\mathcal{A})$ is its filtered pre-triangulated completion, then the degree-0 cohomology category $H^0(\mathrm{Tw}(\mathcal{A}))$ is a triangulated persistence category.*

In view of this result, it is natural to call a filtered dg-category $\mathcal{A}$ *pre-triangulated* if the inclusion $\mathcal{A} \hookrightarrow \mathrm{Tw}(\mathcal{A})$ is an equivalence of filtered dg-categories.

Corollary 2.102. *Let $\mathcal{A}$ be a filtered pre-triangulated dg-category. Then its degree-0 cohomology category $\mathrm{H}^0(\mathcal{A})$ is a triangulated persistence category.*

Proof of Proposition 2.101. It is trivial to notice that the category $H^0(\mathrm{Tw}(\mathcal{A}))$ is a persistence category. It is endowed with an obvious shift functor as defined in Section 2.5.1.2. The first thing to check is that the 0-level category $[H^0(\mathrm{Tw}(\mathcal{A}))]_0$, with the same objects as $H^0(\mathrm{Tw}(\mathcal{A}))$ and only the shift 0-morphisms, is triangulated; see Definition 2.21. The family of triangles that will provide the exact ones are the triangles of the form

$$A \xrightarrow{f} B \xrightarrow{i} \mathrm{Cone}^0(f) \xrightarrow{\pi} A[1]$$

associated to the 0-cones, as given in Definition 2.99. From this point on, checking that $[H^0(\mathrm{Tw}(\mathcal{A}))]_0$ is triangulated comes down to the usual verifications showing that the H^0 of a dg-category is triangulated, with a bit of care to ensure that the relevant homotopies preserve filtration. We leave this verification to the reader. It is then automatic that Σ^r is triangulated when restricted to $[H^0(\mathrm{Tw}(\mathcal{A}))]_0$. The last step is to show that the morphism $\eta_r^A : \Sigma^r A \to A$ has an r-acyclic cone in $\mathrm{Tw}(\mathcal{A})$. In this context of filtered dg-categories, an object K is r-acyclic if the identity $\mathbb{1}_K \in \hom_{\mathrm{Tw}(\mathcal{A})}(K, K)$ is a boundary of some element $\eta \in \hom^{\leq r}_{\mathrm{Tw}(\mathcal{A})}(K, K)$.

The map $\eta_r^A \in \mathrm{Mor}^0_{\mathrm{Tw}(\mathcal{A})}(\Sigma^r A, A)$ is induced by the identity. By definition, we have $\mathrm{Cone}^0(\eta_r^A) = A \oplus \Sigma^r A[1]$ and

$$q_{\mathrm{co}} = \begin{pmatrix} q & \eta_r^A \\ 0 & -q' \end{pmatrix},$$

where q is the structural map of the twisted complex A and $q' = \Sigma^r q$. Consider a homotopy

$$K = \begin{pmatrix} 0 & 0 \\ (\eta_{0,r})_A & 0 \end{pmatrix} : \mathrm{Cone}^0(\eta_r^A) \to \mathrm{Cone}^0(\eta_r^A)[1].$$

Note that $\ell(K) = r$. We have

$$\begin{aligned} dK &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} q & \eta_r^A \\ 0 & -q' \end{pmatrix} \begin{pmatrix} 0 & 0 \\ (\eta_{0,r})_A & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ (\eta_{0,r})_A & 0 \end{pmatrix} \begin{pmatrix} q & \eta_r^A \\ 0 & -q' \end{pmatrix} \\ &= \begin{pmatrix} \mathbb{1}_A & 0 \\ -q' \circ (\eta_{0,r})_A + (\eta_{0,r})_A \circ q & \mathbb{1}_{\Sigma^r A[1]} \end{pmatrix} = \mathbb{1}_{\mathrm{Cone}^0(\eta_r^A)}, \end{aligned}$$

because $\Sigma^r q \circ (\eta_{0,r})_A = (\eta_{0,r})_A \circ q$ and this concludes the proof. ■

Remark 2.103. (a) In the filtered dg-category $\mathrm{Tw}(\mathcal{A})$, we can replicate all the constructions in Section 2.3 at the chain level, similarly to the definition of r-acyclic objects mentioned in the proof above. For instance, r-isomorphisms are replaced by r-quasi-isomorphisms (meaning filtration-preserving morphisms that induce an r-isomorphism in homology), and all the functorial-type constructions of that section can be pursued at the chain level, by replacing commutativity at the chain level by commutativity up to homotopy.

(b) One advantage of working at the chain level instead of in the general setting of triangulated persistence categories is that the maps c induced on cones through diagrams of the form

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "a"] & B \arrow[r] \arrow[d, "b"] & \mathrm{Cone}(f) \arrow[d, "c"] \\ A' \arrow[r, "f'"] & B' \arrow[r] & \mathrm{Cone}(f') \end{tikzcd}$$

are defined explicitly in terms of the homotopy making the square on the left commutative. An example relevant for this memoir is that, in the homological category of a filtered dg-category, the measurement $Z^{\mathcal{F}}(-,-)$ from Section 2.4.3.5 satisfies the triangle inequality. The proof follows closely the arguments in [10, Lemma 6.11], with all weakly filtered maps there being replaced with filtered ones here.

2.5.2 Filtered cochain complexes

In this section we discuss the main example of a filtered dg-category, the category of filtered co-chain complexes. As we shall see, this is pre-triangulated and thus, in view of Corollary 2.102, its homotopy category is a triangulated persistence category.

We will work over a field **k** and will denote the resulting category by $\mathcal{FK}_{\mathbf{k}}$. The objects of this category are filtered cochain complexes (X, ∂, ℓ), where (X, ∂) is a cochain complex and ℓ is a filtration function, as in Section 2.5.1.1. Given two filtered cochain complexes (X, ∂_X, ℓ_X) and (Y, ∂_Y, ℓ_Y), the morphisms $\hom_{\mathcal{FK}_{\mathbf{k}}}(X, Y)$ are linear graded maps $f : X \to Y$ such that the quantity

$$\ell(f) = \inf\{r \in \mathbb{R} \mid \ell_Y(f(x)) \leq \ell_X(x) + r \ \forall x \in X\} \tag{2.47}$$

is finite. The filtration function on $\hom_{\mathcal{FK}_k}(X, Y)$ is then defined through (2.47). The differential on $\hom_{\mathcal{FK}_k}(X, Y)$ is given, as usual, by $\partial(f) = \partial_Y \circ f - (-1)^{|f|} f \circ \partial_X$, and it obviously preserves filtrations. The composition of morphisms is also obviously compatible with the filtration and therefore $\mathcal{FK}_{\mathbf{k}}$ is a filtered dg-category.

There is a natural shift functor on $\mathcal{FK}_{\mathbf{k}}$, $\Sigma : (\mathbb{R}, +) \to \mathcal{P}\mathrm{End}(\mathcal{FK}_k)$, defined by

$$\Sigma^r(X, \partial, \ell_X) = (X, \partial, \ell_X + r)$$

and

$$\Sigma^r(f) = f \quad \text{for any } f \in \mathrm{Mor}_{\mathcal{FK}_k}(X, Y).$$

Moreover, for $r, s \in \mathbb{R}$, there is a natural transformation from Σ^r to Σ^s induced by the identity.

Assume that $f : (X, \partial_X, \ell_X) \to (Y, \partial_Y, \ell_Y)$ is a cochain morphism such that $\ell(f) \leq 0$. In this case, the usual cone construction $\mathrm{Cone}(f) = (Y \oplus X[1], \partial_{\mathrm{co}})$, with

$$\partial_{\mathrm{co}} = \begin{pmatrix} \partial_Y & f \\ 0 & -\partial_X \end{pmatrix},$$

produces a filtered complex and fits into a triangle of maps with $\ell \leq 0$:

$$X \xrightarrow{f} Y \xrightarrow{i} \mathrm{Cone}(f) \xrightarrow{\pi} X[1].$$

The standard properties of this construction immediately imply that the dg-category $\mathcal{FK}_{\mathbf{k}}$ is pre-triangulated and thus the 0-cohomological category, $H^0\mathcal{FK}_{\mathbf{k}}$, is a triangulated persistence category.

It is useful to make explicit some of the properties of this category:

(i) The objects of $H^0\mathcal{FK}_{\mathbf{k}}$ are filtered cochain complexes (X, ∂_X, ℓ_X).

(ii) The morphisms in $\hom^r_{H^0\mathcal{FK}_{\mathbf{k}}}(X, Y)$ are cochain maps $f : (X, \partial_X, \ell_X) \to (Y, \partial_Y, \ell_Y)$ such that $\ell(f) \leq r$ up to chain homotopy $h : f \simeq f'$ with $\ell(h) \leq r$.

(iii) A filtered complex (K, ∂_K, ℓ_K) is r-acyclic if the identity $\mathbb{1}_K$ is chain homotopic to 0 through a chain homotopy $h : \mathbb{1}_K \simeq 0$ with $\ell(h) \leq r$.

(iv) The construction of weighted exact triangles, as well as their properties, can be pursued in this context by following closely the scheme in Section 2.3.3.

(v) The limit category $[H^0\mathcal{FK}_{\mathbf{k}}]_\infty$ has as morphisms chain homotopy classes of cochain maps (where both the cochain maps and the homotopies are assumed to be of bounded shifts). Its objects are still filtered cochain complexes. It is triangulated, with translation functor $TX = X[1]$, as expected.

Remark 2.104. The example of the dg-category $\mathcal{FK}_{\mathbf{k}}$ can be extended in a number of ways, and we mention a couple of them here.

(a) Assume that we fix a filtered dg-category $\mathcal{A}$. There is a natural notion of a filtered (left/right) module $\mathcal{M}$ over $\mathcal{A}$. Such modules, together with filtered maps relating them, form a new filtered dg-category, denoted by $\mathrm{Mod}_{\mathcal{A}}$. The 0-cohomology category associated to this filtered dg-category, $H^0\mathrm{Mod}_{\mathcal{A}}$, is pre-triangulated because the category $\mathrm{Mod}_{\mathcal{A}}$ is naturally endowed with a shift functor, just like $\mathcal{FK}_{\mathbf{k}}$, as well as an appropriate cone construction over filtered, closed, degree-preserving morphisms.

(b) Similarly to (a), we may take $\mathcal{A}$ to be a filtered A_∞-category and consider the category $\mathrm{Mod}_{\mathcal{A}}$ of filtered modules over $\mathcal{A}$. Again this is a filtered dg-category and it is pre-triangulated (the formalism required to establish this fact appears in [10], in a version dealing with weakly filtered structures).

As mentioned at the beginning of Section 1.1, there exists a quantitative comparison between two filtered cochain complexes X, Y, called the bottleneck distance and denoted by $d_{\mathrm{bot}}(X, Y)$. This is best expressed in the barcode language from [3] or [59].

For completeness, we specify the version of barcodes used here. A *barcode* $\mathcal{B} = \{(I_j, m_j)\}_{j \in \mathcal{J}}$ is a collection of pairs consisting of intervals $I_j \subset \mathbb{R}$ and positive integers $m_j \in \mathbb{Z}_{>0}$, indexed by a set $\mathcal{J}$, and satisfying the following *admissibility* conditions:

- $\mathcal{J}$ is assumed to be either finite or equal to $\mathbb{Z}_{\geq 0}$.
- Each interval I_j is of the form $I_j = [a_j, b_j)$, with $-\infty < a_j < b_j \leq \infty$.
- In the case $\mathcal{J} = \mathbb{Z}_{\geq 0}$, we assume that $a_j \to \infty$ as $j \to \infty$.

The intervals I_j are called bars and, for each j, m_j is called the multiplicity of the bar I_j. To such a barcode one can associate a persistence module $V(\mathscr{B})$ that satisfies the following conditions:

- (Lower semi-continuity) For any $s \in \mathbb{R}$ and any $t \geq s$ sufficiently close to s, the map $i_{s,t} : M^s \to M^t$ is an isomorphism.
- (Lower boundedness) For s sufficiently small, $M^s = 0$.
- (Tameness) For every $s \in \mathbb{R}$, $\dim_{\mathbf{k}}(M^s) < \infty$.

The module $V(\mathscr{B})$ is defined as the direct sum of the elementary persistence modules $V(I)$ for each bar I in the barcode $\mathscr{B}$. Here $V([a,b))^s = \mathbf{k}$ if $s \in [a,b)$ and $V([a,b))^s = 0$ if $s \notin [a,b)$. Conversely, the Normal Form Theorem in [47, Section 2.1], or the main result in [25], says that any persistence module M with the three properties above can be decomposed as a direct sum of persistence modules of the form $V([a,b))$ and $V([c,\infty))$ in a unique way, up to permutation. Thus we can associate to it a barcode $\mathscr{B}(M)$ consisting of the intervals $[a,b)$ and $[c,\infty)$ appearing in the decomposition.

The homology $H(X)$ of a filtered cochain complex X is a persistence module whose barcode can be read out of the normal form of X. More precisely, by [59, Proposition 7.4] (see also [3]) there is a filtered isomorphism (in the category $H^0\mathcal{F}\mathcal{K}_{\mathbf{k}}$) as follows:

$$X \simeq \bigoplus_{[a,+\infty)\in\mathscr{B}(X)} E_1(a) \oplus \bigoplus_{[c,d)\in\mathscr{B}(X)} E_2(c,d), \tag{2.48}$$

where $E_1(a), E_2(c,d) \in \mathrm{Obj}(\mathcal{F}\mathcal{K}_{\mathbf{k}})$ are filtered cochain complexes defined by

$$E_1(a) = ((\cdots \to 0 \to \mathbf{k}\langle x\rangle \to 0 \to \cdots), \ell(x) = a)$$

and

$$E_2(c,d) = ((\cdots \to 0 \to \mathbf{k}\langle y\rangle \xrightarrow{\partial} \mathbf{k}\langle x\rangle \to 0 \to \cdots),\ \ell(y) = c, \ell(x) = d),$$

where $c \geq d$ and $\partial(y) = \kappa x$ for some $0 \neq \kappa \in \mathbf{k}$. The notation $\mathscr{B}(X)$ in (2.48) stands for a collection of intervals of two types: finite or semi-infinite intervals in $\mathbb{R}$ of the form $[c,d)$, with $c < d$ and possibly with $d = +\infty$; and intervals of length 0, $[c,d]$, with $c = d$.

In what follows, sometimes for brevity, we write $E_*(I)$ for $E_1(a)$ when $I = [a,+\infty)$, and for $E_2(c,d)$ when $I = [c,d)$, with $I \in \mathscr{B}(X)$. Then $d_{\mathrm{bot}}(X,Y)$ is defined as the infimum τ satisfying the following conditions: there exist some subsets consisting of certain "short intervals" $\mathscr{B}(X)_{\mathrm{short}} \subset \mathscr{B}(X)$ and $\mathscr{B}(Y)_{\mathrm{short}} \subset \mathscr{B}(Y)$ such that

(i) each short interval $[c,d)$ satisfies $2(d-c) \leq \tau$;

(ii) there is a bijection $\sigma : \mathscr{b}(x) \setminus \mathscr{b}(x)_{\mathrm{short}} \to \mathscr{b}(y) \setminus \mathscr{b}(y)_{\mathrm{short}}$;

(iii) if $\sigma([a,b)) = [c,d)$, then $\max\{|a-c|, |b-d|\} \leq \tau$;

(iv) if $\sigma([a,\infty)) = [c,\infty)$, then $|a-c| \leq \tau$.

In what follows, we assume that the cardinalities of the barcodes, $\#|\mathcal{B}(X)|$ and $\#|\mathcal{B}(Y)|$, are both finite. The following result compares the fragmentation pseudometric $d^{\mathcal{F}}$ defined in Definition 2.58 with the bottleneck distance d_{bot} defined above.

Proposition 2.105. *Let $\mathcal{C} = H^0\mathcal{FK}_{\mathbf{k}}$ and let $\mathcal{F} \subset \mathrm{Obj}(\mathcal{C})$ be a subset containing* 0. *Then*

$$d^{\mathcal{F}}(X,Y) \leq C_{X,Y}\, d_{\mathrm{bot}}(X,Y),$$

where $C_{X,Y} = 4\min\{\#|\mathcal{B}(X)|, \#|\mathcal{B}(Y)|\} + 1$.

Proof. It is immediate to see that we may assume that both $\mathcal{B}'(X)$ and $\mathcal{B}'(Y)$ do not contain any 0-length bars, and thus $\mathcal{B}'(X) = \mathcal{B}(X)$, and similarly for Y. It suffices to prove the conclusion when $\mathcal{B}(X)$ and $\mathcal{B}(Y)$ have the same cardinality of the infinite-length bars (otherwise by definition $d_{\mathrm{bot}}(X,Y) = +\infty$ and the conclusion holds trivially). Let $\tau := d_{\mathrm{bot}}(X,Y) + \epsilon$ for an arbitrarily small $\epsilon > 0$. Since $d^{\mathcal{F}}(\cdot,\cdot)$ is invariant under filtered isomorphisms (applied to either of its two inputs), by (2.48) and by reordering summands we obtain

$$\begin{aligned} d^{\mathcal{F}}(X,Y) \leq d^{\mathcal{F}}\Bigg(&\bigoplus_{I\in\mathcal{B}(X)\setminus\mathcal{B}(X)_{\mathrm{short}}} E_*(I), \bigoplus_{\sigma(I)\in\mathcal{B}(Y)\setminus\mathcal{B}(Y)_{\mathrm{short}}} E_*(\sigma(I))\Bigg) \\ &+ d^{\mathcal{F}}\Bigg(\bigoplus_{J\in\mathcal{B}(X)_{\mathrm{short}}} E_2(J), \bigoplus_{J'\in\mathcal{B}(Y)_{\mathrm{short}}} E_2(J')\Bigg), \end{aligned}$$

where the inequality is given by the triangle inequality of $d^{\mathcal{F}}$ with respect to the direct sum; see Proposition 2.60. For $d^{\mathcal{F}}$ with short intervals, both $\bigoplus_{J\in\mathcal{B}(X)_{\mathrm{short}}} E_2(J)$ and $\bigoplus_{J'\in\mathcal{B}(Y)_{\mathrm{short}}} E_2(J')$ are acyclic objects in $\mathcal{C}$; therefore by (i) in the definition of d_{bot} above, triangles

$$0 \to 0 \to \bigoplus_{J\in\mathcal{B}(X)_{\mathrm{short}}} E_2(J) \to 0 \quad \text{and} \quad 0 \to 0 \to \bigoplus_{J'\in\mathcal{B}(Y)_{\mathrm{short}}} E_2(J') \to 0$$

are weight-$\frac{\tau}{2}$ exact triangles (here we identify $\Sigma^{\lambda}0$ with 0 for any shift $\lambda \in \mathbb{R}$). Thus,

$$\begin{aligned} d^{\mathcal{F}}\Bigg(\bigoplus_{J\in\mathcal{B}(X)_{\mathrm{short}}} E_2(J), \bigoplus_{J'\in\mathcal{B}(X)_{\mathrm{short}}} E_2(J')\Bigg) &\leq d^{\mathcal{F}}\Bigg(\bigoplus_{J\in\mathcal{B}(X)_{\mathrm{short}}} E_2(J), 0\Bigg) \\ &+ d^{\mathcal{F}}\Bigg(0, \bigoplus_{J'\in\mathcal{B}(X)_{\mathrm{short}}} E_2(J')\Bigg) \leq \tau. \end{aligned}$$

On the other hand, by Proposition 2.60 again, for $d^{\mathcal{F}}$ with non-short intervals, we have

$$\begin{aligned} & d^{\mathcal{F}}\left(\bigoplus_{I\in\mathcal{B}(X)\setminus\mathcal{B}(X)_{\text{short}}} E_*(I), \bigoplus_{\sigma(I)\in\mathcal{B}(Y)} E_*(\sigma(I))\right) \\ & \quad \leq \sum_{I\in\mathcal{B}(X)\setminus\mathcal{B}(X)_{\text{short}}} d^{\mathcal{F}}\big(E_*(I), E_*(\sigma(I))\big). \end{aligned}$$

Since $d_{\text{bot}}(X,Y) < +\infty$, the bijection σ will always map a finite interval to a finite interval and a semi-infinite interval to a semi-infinite interval, so it suffices to consider the following two cases.

Case I. Estimate $d^{\mathcal{F}}(E_1(a), E_1(c))$. We need to build the desired cone decomposition. Without loss of generality, assume $a \geq c$. Then the identity map $\langle x\rangle_{E_1(a)} \to \langle x\rangle_{E_1(c)}$ (with *negative* filtration shift) implies that the triangle $E_1(a) \to E_1(c) \to K \to E_1(a)[1]$ is a weight-0 exact triangle (in fact in $\mathcal{C}_0$), where K is the filtered mapping cone. Then in the following cone decomposition (with linearization $(0, E_1(c))$):

$$\begin{cases} 0 \to 0 \to K \to 0 \\ E_1(c) \to K \to E_1(a) \to E_1(c)[1] \end{cases}$$

the first triangle is a weight-$(c-a)$ exact triangle, since it is readily verified that K is $(c-a)$-acyclic. Then $\delta^{\mathcal{F}}(E_1(a), E_1(c)) \leq (c-a) + 0 \leq \tau$ by (iv) in the definition of d_{bot} above. On the other hand, consider the following cone decomposition with linearization $(0, E_1(a), 0)$ (note that, by definition, $\Sigma^{-(a-c)}E_1(a) = E_1(c)$):

$$\begin{cases} 0 \to 0 \to 0 \to 0 \\ E_1(a) \to 0 \to \Sigma^{-(a-c)}E_1(a) \to \Sigma^{-(a-c)}E_1(a)[1] \\ 0 \to E_1(c) \to E_1(c) \to 0, \end{cases} \tag{2.49}$$

where the second triangle has weight $a - c > 0$ by Remark 2.50 (b). Therefore, $\delta^{\mathcal{F}}(E_1(c), E_1(a)) \leq 0 + (a-c) + 0 \leq \tau$, which implies that

$$d^{\mathcal{F}}(E_1(a), E_1(c)) \leq \tau. \tag{2.50}$$

Case II. Estimate $d^{\mathcal{F}}(E_2(a,b), E_2(c,d))$. We will carry out the estimation as follows:

$$d^{\mathcal{F}}(E_2(a,b), E_2(c,d)) \leq d^{\mathcal{F}}(E_2(a,b), E_2(c,b)) + d^{\mathcal{F}}(E_2(c,b), E_2(d,b)).$$

Moreover, we will only estimate $d^{\mathcal{F}}(E_2(a,b), E_2(c,b))$ with $a \geq c$; other situations can be handled in a similar and symmetric way. Similarly to Case I above, consider the cone decomposition

$$\begin{cases} 0 \to 0 \to K \to 0 \\ E_2(c,b) \to K \to E_2(a,b) \to E_2(c,b)[1], \end{cases}$$

where $E_2(a,b) \to E_2(c,b)$ is the identity map $\langle x\rangle_{E_2(a,b)} \to \langle x\rangle_{E_2(c,b)}$ (and similarly for the generator y), with a negative filtration shift, and K is the cone. Since K is $(a-c)$-acyclic, we have $\delta^{\mathcal{F}}(E_2(a,b), E_2(c,b)) \leq 0 + (a-c) \leq \tau$. On the other hand,

$$\begin{aligned} &\delta^{\mathcal{F}}(E_2(c,b), E_2(a,b)) \\ &\qquad \leq \delta^{\mathcal{F}}(E_2(c,b), \Sigma^{a-c}E_2(c,b)) + \delta^{\mathcal{F}}(E_2(a,b+a-c), E_2(a,b)), \end{aligned}$$

where $\delta^{\mathcal{F}}(E_2(c,b), \Sigma^{a-c}E_2(c,b)) \leq a-c$ by a similar cone decomposition to that in (2.49). Meanwhile, since $b+a-c \geq b$, the identity map from $E_2(a,b+a-c)$ to $E_2(a,b)$ (with negative filtration shift) yields $\delta^{\mathcal{F}}(E_2(a,b+a-c), E_2(a,b)) \leq a-c$. Therefore, together we have, by (iii) in the definition of d_{bot} above,

$$d^{\mathcal{F}}(E_2(a,b), E_2(c,b)) \leq 2(a-c) \leq 2\tau,$$

which implies

$$d^{\mathcal{F}}(E_2(a,b), E_2(c,d)) \leq 4\tau. \tag{2.51}$$

Therefore, by (2.50) and (2.51) together, we have

$$\begin{aligned} d^{\mathcal{F}}(X,Y) &\leq \#|\mathcal{B}(X) \setminus \mathcal{B}(X)_{\mathrm{short}}| \cdot 4\tau + \tau \\ &\leq (4\,\#|\mathcal{B}(X) \setminus \mathcal{B}(X)_{\mathrm{short}}| + 1)(d_{\mathrm{bot}}(X,Y) + \epsilon) \\ &\leq (4\min\{\#|\mathcal{B}(X)|, \#|\mathcal{B}(Y)|\} + 1)(d_{\mathrm{bot}}(X,Y) + \epsilon), \end{aligned}$$

where the last inequality holds since σ is a bijection by (ii) in the definition of d_{bot} above. Let $\epsilon \to 0$, and we complete the proof. ■

2.5.3 Topological spaces +

There are many topological categories consisting of topological spaces endowed with additional structures (indicated by the + in the title of the subsection) that can be analyzed with the tools discussed before. We will discuss here two elementary examples. They both fit the following scheme: we will have a triple consisting of a (small) category $\mathcal{K}$, an endofunctor $T_{\mathcal{K}} : \mathcal{K} \to \mathcal{K}$, and a class of triangles $\Delta_{\mathcal{K}}$ in $\mathcal{K}$ of the form

$$A \to B \to C \to T_{\mathcal{K}}A.$$

In these cases, the objects of $\mathcal{K}$ have an underlying structure as topological spaces and, similarly, the morphisms in $\mathcal{K}$ are continuous maps, while the functor $T_{\mathcal{K}}$ corresponds to the suspension of spaces.

The aim is to define fragmentation pseudometrics on the objects of $\mathcal{K}$ by first associating to the triangles in $\Delta_{\mathcal{K}}$ a weight $\bar{w}_{\mathcal{K}} : \Delta_{\mathcal{K}} \to \mathbb{R}$ with some reasonable properties, and then defining the quantities $\delta^{\mathcal{F}}(X, Y)$ and $\underline{\delta}^{\mathcal{F}}(X, Y)$ as in, respectively, (2.2) and (2.5), taking into account only decompositions appealing to triangles $\Delta_i \in \Delta_{\mathcal{K}}$. Notice that $\delta^{\mathcal{F}}$ is not generally defined in this setting as its definition requires to desuspend spaces. On the other hand, as soon as $\bar{w}_{\mathcal{K}}$ is given, $\underline{\delta}^{\mathcal{F}}$ can be defined by formula (2.39) with $w_{\mathcal{K}}$ replacing w_{∞} there, and with each triangle in the sequence (2.38) being replaced with a triangle in $\Delta_{\mathcal{K}}$. We assume that the family $\mathcal{F}$ is such that $0 \in \mathcal{F}$, and in most cases we assume implicitly that $\mathcal{F}$ consists of all the objects F such that there are triangles in $\Delta_{\mathcal{K}}$ of the form $F \to A \to B \to T_{\mathcal{K}} F$. The resulting $\underline{\delta}^{\mathcal{F}}$ trivially satisfies the triangle inequality. The pseudometric $\underline{d}^{\mathcal{F}}$ obtained by the symmetrization of $\underline{\delta}^{\mathcal{F}}$ exists in this case too (see Remark 2.5 (c)). Based on the various constructions discussed earlier in the memoir, there are two approaches to define a weight $\bar{w}_{\mathcal{K}}$ (that is not flat), and they both require some more structure:

(A) The additional structure in this case is a functor $\Phi : \mathcal{K} \to \mathcal{C}_{\infty}$, where $\mathcal{C}$ is a TPC; in the examples below, $\mathcal{C} = H^0\mathcal{F}\mathcal{K}_{\mathbf{k}}$ – the triangulated persistence homotopy category of filtered cochain complexes. We also require that Φ commutes with T (at least up to some natural equivalence), and that for each $\Delta \in \Delta_{\mathcal{K}}$ the image $\Phi(\Delta)$ of Δ is exact in $\mathcal{C}_{\infty}$ (and thus $\bar{w}(\Phi(\Delta)) < \infty$, where $\bar{w}$ is the persistence weight introduced in Definition 2.63). In this case, for each $\Delta \in \Delta_{\mathcal{K}}$ we put

$$\bar{w}_{\mathcal{K}}(\Delta) = \bar{w}(\Phi(\Delta)).$$

(B) This second approach requires first that the morphisms $\hom_{\mathcal{K}}(A, B)$ are endowed with a natural increasing filtration compatible with the composition. Secondly, there should be a shift functor $\Sigma_{\mathcal{K}} : (\mathbb{R}, +) \to \mathrm{End}(\mathcal{K})$ compatible with the filtration on morphisms and that commutes with $T_{\mathcal{K}}$. Moreover, the triangles in $\Delta_{\mathcal{K}}$ have to be part of a richer structure, such as a model category or a Waldhausen category (compatible with the functor $\Sigma_{\mathcal{K}}$). In this case, the definition of weighted triangles can be carried out by following the steps in Section 2.3.3, but at the space level, without moving to an algebraic category. This approach goes beyond the scope of this memoir and will not be pursued here.

Remark 2.106. Of course, it is also possible to mix in some sense the two approaches mentioned above. For instance, in the two examples below the category $\mathcal{K}$ carries a shift functor $\Sigma_{\mathcal{K}}$ as in (B), but also a functor Φ as in (A), such that Φ commutes with the shift functors in the domain and target. In that case we can use Φ to pull back to $\mathcal{K}$ more of the structure and weights in $\mathcal{C}$ (of course, this remains less precise than constructing weights at the space level).

2.5.3.1 Topological spaces with action functionals. We will discuss here a category denoted by $\mathcal{ATop}_*$. The objects of this category are pairs (A, f_A) where $A = (A, *_A)$ is a pointed topological space and $f_A : A \to \mathbb{R}$ is a continuous function bounded from below by the value $f_A(*_A)$ of f_A at the base point $*_A$ of A. We will refer to f_A as the *action functional* associated to A. The morphisms in this category are pointed continuous maps $u : A \to B$ such that there exists $r \in \mathbb{R}$ with the property that $f_B(u(x)) \leq f_A(x) + r$ for all $x \in A$.

We will see that there is a natural *contravariant* functor

$$\Phi : \mathcal{ATop}_* \to [H^0\mathcal{FK}_{\mathbf{k}}]_\infty \tag{2.52}$$

inducing a weight $\overline{w}_{\mathcal{ATop}_*}$ and the associated pseudometrics $\underline{d}^{\mathcal{F}}$ on $\mathrm{Obj}(\mathcal{ATop}_*)$ along the lines of point (A) above.

Remark 2.107. The condition on f_A being bounded from below is one possible choice in this construction. Its role is to allow the constant map $u : (A, f_A) \to (B, f_B)$ to be part of the morphisms of $\mathcal{ATop}_*$.

Before proceeding with the construction of the functor Φ, we discuss some features of $\mathcal{ATop}_*$. Notice first that the morphisms are filtered, with the rth stage being

$$\begin{aligned}
&\hom^{\leq r}_{\mathcal{ATop}_*}(A, B)\\
&\quad = \{u : A \to B \mid u \text{ continuous}, u(*_A) = *_B, f_B(u(x)) \leq f_A(x) + r \ \forall x \in A\}.
\end{aligned}$$

There is an obvious family of functors $\Sigma_{\mathcal{ATop}_*} : (\mathbb{R}, +) \to \mathcal{ATop}_*$ defined by

$$\Sigma^s_{\mathcal{ATop}_*}(A, f_A) = (A, f_A + s)$$

and being the identity on morphisms. The next step is to define the translation functor $T_{\mathcal{ATop}_*}$. At the underlying topological level this is just the topological suspension, but we need to be more precise about the action functional. Given an object (A, f_A), we first define the cone (CA, f_{CA}). We take CA to be the reduced cone; in other words, the quotient topological space $CA = A \times [0, 1]/(A \times \{1\} \cup *_A \times [0, 1])$. To define f_{CA} we first consider the homotopy $h_A : A \times [0, 1] \to \mathbb{R}$,

$$h_A(x, t) = \begin{cases} f_A(x) & \text{if } 0 \leq t \leq \frac{1}{2}, \\ (2 - 2t)(f_A(x) - f_A(*_A)) + f_A(*_A) & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

The map $f_{CA} : CA \to \mathbb{R}$ is induced by h_A. We now define the reduced suspension, $SA = CA/A \times \{0\}$, and take f_{SA} to be the map induced to the quotient by the homotopy $h'_A : A \times [0,1] \to \mathbb{R}$,

$$h'_A(x,t) = \begin{cases} 2t(f_A(x) - f_A(*_A)) + f_A(*_A) & \text{if } 0 \le t \le \frac{1}{2}, \\ (2-2t)(f_A(x) - f_A(*_A)) + f_A(*_A) & \text{if } \frac{1}{2} \le t \le 1. \end{cases}$$

We put $T(A, f_A) = (SA, f_{SA})$. It is immediate to see that T extends to a functor on $\mathcal{ATop}_*$ and that it commutes with Σ. Moreover, both Σ and T so defined commute and are compatible with the filtration of the morphisms, in the sense that they take $\hom^{\le r}$ to $\hom^{\le r}$ for each r. Moreover, composition of morphisms is also compatible with the filtrations, in the sense that it takes $\hom^{\le r_1}(B,C) \times \hom^{\le r_2}(A,B)$ to $\hom^{\le r_1+r_2}(A,C)$.

We now define the class of exact triangles $\Delta_{\mathcal{ATop}_*}$. For this we consider a morphism $u : (A, f_A) \to (B, f_B)$ and we first define its cone $\mathrm{Cone}(u)$. As a topological space this is, as expected, the quotient topological space $(B \cup CA)/\sim$, where the equivalence relation $\sim$ is generated by $f(x) \sim x \times \{0\}$. The base point of $\mathrm{Cone}(u)$ is the same as that of B. The action functional $f_{\mathrm{Cone}(u)}$ is induced to the respective quotient by

$$G(x) = \begin{cases} f_B(x) & \text{if } x \in B, \\ (1-2t)\big(f_B(u(y)) - f_B(*_B)\big) & \\ \quad + 2t\big(f_A(y) - f_A(*_A)\big) + f_B(*_B) & \text{if } x = (y,t) \in A \times [0, \frac{1}{2}], \\ (2-2t)\big(f_A(y) - f_A(*_A)\big) + f_B(*_B) & \text{if } x = (y,t) \in A \times [\frac{1}{2}, 1]. \end{cases}$$

There is an obvious inclusion $i : (B, f_B) \to \mathrm{Cone}(u)$, and there is also a projection $p : \mathrm{Cone}(u) \to TA$ (that contracts B to a point). This map belongs to our class of morphisms because the functional f_B is bounded from below. The class $\Delta_{\mathcal{ATop}_*}$ consists of triangles Δ:

$$\Delta : A \xrightarrow{u} B \xrightarrow{i} \mathrm{Cone}(u) \xrightarrow{p} TA. \tag{2.53}$$

We finally construct the functor $\Phi : \mathcal{ATop}_* \to [H^0\mathcal{FK}_{\mathbf{k}}]_\infty$. This functor will be contravariant, since the objects of $\mathcal{FK}_{\mathbf{k}}$ are cochain complexes (rather than chain complexes).

First we fix some notation. For a pointed topological space X, we denote by $\widetilde{C}_*(X)$ the reduced singular chain complex of X with coefficients in $\mathbf{k}$, and by $\widetilde{C}^*(X)$ the reduced singular cochain complex. We denote by $C_*(X)$ and $C^*(X)$ the corresponding non-reduced complexes. If $Y \subset X$ is a pointed subspace, then $\widetilde{C}_*(X,Y)$ and $\widetilde{C}^*(X,Y)$ are the relative (co)chains. Consider an object of $\mathcal{ATop}_*$, (A, f_A), and let $A^{\le r} = (f_A)^{-1}(-\infty, r]$. Notice that the spaces $A^{\le r}$ are pointed (if non-void). There is a filtration of $C^*(A)$ defined by

$$\widetilde{C}^*(A)^{\le -r} = \mathrm{im}\big\{\widetilde{C}^*(A, A^{\le r}) \to \widetilde{C}^*(A)\big\}.$$

Thus the filtration up to $s \in \mathbb{R}$ of $\widetilde{C}^*(A)$ consists of the cochains in A that vanish over the singular chains of $A^{\leq -s} \subset A$. It is clear that the cochain differential preserves this filtration. Moreover, the filtration is increasing, and if $f \in \hom^{\leq r}_{\mathcal{ATop}_*}(A, B)$, then f pulls back the cochains in B that vanish over $B^{\leq a}$ to cochains in A that vanish over $A^{\leq a-r}$. As a result, $C^*(f) : \widetilde{C}^*(B) \to \widetilde{C}^*(A)$ shifts filtration by r. Finally, we define the functor Φ. For each object (A, f_A) of $\mathcal{ATop}_*$ we take $\Phi(A, f_A)$ to consist of the cochain complex $\widetilde{C}^*(A)$ together with the filtration $\{\widetilde{C}^*(A)^{\leq r}\}$ defined above. For a morphism $u : (A, f_A) \to (B, f_B)$ we take $\Phi(u) = [\widetilde{C}^*(u)]$, where $[-]$ represents the cochain-homotopy class of the respective cochain morphism.

The definition of the morphisms in $\mathcal{ATop}_*$ implies that $\Phi(u)$ is indeed a morphism in $[H^0\mathcal{FK}_{\mathbf{k}}]_\infty$. Moreover, because we are using everywhere reduced cochain complexes (and we work in the pointed category), we have that $\Phi(\Delta)$ is exact in $[H^0\mathcal{FK}_{\mathbf{k}}]_\infty$ for each of the triangles in $\Delta_{\mathcal{ATop}_*}$. Further, the functor Φ also interchanges the shift functors in the domain and the target.

In all cases, the weight $\bar{w}_{\mathcal{ATop}_*}$ is well defined, as are the associated fragmentation pseudometrics $\underline{d}^{\mathcal{F}}(-,-)$ on the objects of $\mathcal{ATop}_*$. Roughly speaking, these fragmentation pseudometrics measure how much “weight” is needed to obtain a given topological space via successive cone attachments of spaces in $\mathcal{F}$.

Remark 2.108. (a) The choice of the class $\Delta_{\mathcal{ATop}_*}$ given above is quite restrictive, with the consequence that the resulting pseudometrics are often infinite. One alternative is to enlarge this class to all triangles in $\mathcal{ATop}_*$ that are homotopy equivalent to those in the initial class through maps (and homotopies) of filtration 0.

(b) From some points of view, working in the *pointed* category of spaces endowed with an action functional is not natural. Other choices are possible, in particular ones for which the translation functor T more closely imitates dynamical stabilization.

(c) The restriction of Φ to compact topological spaces admits an obvious lift to $H^0\mathcal{FK}_{\mathbf{k}}$. However, without this restriction, such a lift does not seem to be available in full generality.

2.5.3.2 Metric spaces. The category $\mathcal{M}etr_0$ that we will consider here has as objects path-connected metric spaces (X, d_X) of finite diameter. The morphisms are Lipschitz maps. Recall that $\phi : X \to Y$ is a Lipschitz map if there exists a constant $c \in [0, \infty)$, called the Lipschitz constant of ϕ, with the property that $d_Y(\phi(x), \phi(y)) \leq c\, d_X(x, y)$ for all $x, y \in X$.

Remark 2.109. The finite-diameter condition imposed here – indicated by the subscript 0 – is necessary for some of the constructions below. The connectivity assumption is more a matter of convenience.

We will construct a functor as in (2.52) with one main modification. For convenience, we prefer to define a covariant functor and thus our target category will not be a category of cochain complexes but rather one of filtered *chain complexes* (the passage from one to the other is formal, replacing C^* by C_{-*} and vice versa). We will denote the category of filtered chain complexes over $\mathbf{k}$ by $\mathcal{FK}'_{\mathbf{k}}$. This behaves just as a usual dg-category, except that the differential on the space of morphisms is of degree -1. With this change, we will construct

$$\Phi' : \mathcal{M}etr_0 \to [H_0\mathcal{FK}'_{\mathbf{k}}]_\infty \tag{2.54}$$

as well as related structures on $\mathcal{M}etr_0$, as in point (A) at the beginning of the section (see also Remark 2.106).

We start by noting that there is an obvious increasing filtration of the morphisms in $\mathcal{M}etr_0$, with

$$\hom^{\leq r}_{\mathcal{M}etr_0}(X, Y) = \{u : X \to Y \mid \text{the Lipschitz constant of } u \text{ is at most } e^r\}.$$

It is immediate to see that this filtration is compatible with composition. There is also a family of functors $\Sigma_{\mathcal{M}etr_0} : (\mathbb{R}, +) \to \mathcal{M}etr_0$ defined by rescaling the metric, $\Sigma^s(A, d_A) = (A, e^s d_A)$, and by acting as the identity on morphisms. As in the example in the previous section, we will next define the translation functor $T_{\mathcal{M}etr_0}$ and the class of triangles $\Delta_{\mathcal{M}etr_0}$. The first step is to construct the metric cone $C'A$ for an object (A, d_A) in our class. Topologically, the cone $C'A$ will be this time the *unreduced* cone over A. Thus it is defined by $C'A = A \times [0, 1]/A \times \{0\}$. To define the metrics $d_{C'A}$, first let D_A be the diameter of A. We then put

$$d_{C'A}((x,t),(y,t')) = \frac{D_A}{2}|t - t'| + \min\{t, t'\}\, d_A(x, y).$$

It is immediate to see that this does indeed define a metric on $C'A$. A similar construction yields $T(A, d_A)$. Topologically, we will first define the (*non-reduced*) suspension $S'A$ as the topological quotient of $A \times [-\frac{1}{2}, \frac{1}{2}]$, with $A \times \{-\frac{1}{2}\}$ identified to a point S and $A \times \{+\frac{1}{2}\}$ identified to a different point N. We now define $d_{S'A}$ by

$$d_{S'A}((x,t),(y,t')) = \frac{D_A}{2}|t - t'| + \min\{\tfrac{1}{2} - |t|, \tfrac{1}{2} - |t'|\} d_A(x, y),$$

and again it is immediate to see that this defines a metric on $S'A$. We now put $T(A, d_A) = (S'A, d_{S'A})$. The next step is to define the triangles in $\Delta_{\mathcal{M}etr_0}$. For this we assume that $u : (A, d_A) \to (B, d_B)$ is a morphism in our category, and we want to define the (non-reduced) cone of u, $\mathrm{Cone}'(u)$. Topologically, this is, as usual, $B \cup C'A/[\{x\} \times \{0\} \sim u(x) \mid x \in A]$. To define a metric on $\mathrm{Cone}'(u)$, we first notice that, given a map $g : X \to Y$ and a pseudometric d_Y on Y, there is a pullback pseudometric on X given by $g^* d_Y(a, b) = d_Y(g(a), g(b))$. We now let $A' = u(A) \subset B$ and we denote by $\bar{u} : A' \hookrightarrow B$ the inclusion. Notice that $\mathrm{Cone}'(\bar{u}) \subset C'B$. Thus $\mathrm{Cone}'(\bar{u})$ is endowed with a metric given by the restriction of the metric $d_{C'B}$ on $C'B$. There are obvious projections $\pi : \mathrm{Cone}'(u) \to \mathrm{Cone}'(\bar{u})$ and $p : \mathrm{Cone}'(u) \to S'A$. Here p collapses B to the point S in the suspension and sends each point $(x, t) \in C'A$ to $(x, t - \frac{1}{2})$. We now define

$$d_{\mathrm{Cone}'(u)} := \pi^* d_{C'B} + p^* d_{S'A}.$$

Notice that if u is not injective and B is not a single point, then both pseudometrics in the right-hand side of the equality are degenerate. Nonetheless, $d_{\mathrm{Cone}'(u)}$ is non-degenerate. Finally, the class of triangles $\Delta_{\mathcal{M}etr_0}$ consists of triangles

$$A \xrightarrow{u} B \xrightarrow{i} \mathrm{Cone}'(u) \xrightarrow{p} S'A,$$

where i is the inclusion and p is the projection above.

With this preparation, we can now define the functor Φ' from (2.54). Consider an object (A, d_A) in our category and the associated singular complex $C_*(A)$. This chain complex is filtered as follows:

$$C_k^{\leq r}(A) = \left\{ \sum_i a_i \sigma_i \mid a_i \in \mathbf{k},\ \sigma_i \text{ a singular simplex of diameter at most } e^r \right\}.$$

In other words, in the expression above, $\sigma_i : \Delta^k \to A$ is a continuous map with the standard k-simplex as domain and such that $d_A(\sigma_i(x), \sigma_i(y)) \leq e^r$ for any $x, y \in \Delta^k$. Consider the constant map $c : A \to *$. This induces an obvious surjection $C_* \to C_*(*)$, and we denote by $\overline{C}_*(A)$ the kernel of this map (this is quasi-isomorphic to the reduced singular chain complex of A – because A is connected – but is independent of the choice of base-point). There is an induced filtration $\overline{C}_*^{\leq r}(A)$. We now put

$$\Phi'(A, d_A) = \overline{C}_*(A) \quad \text{with the filtration } \{\overline{C}_*^{\leq r}(A)\}_r.$$

Further, for a morphism $u : (A, d_A) \to (B, d_B)$ we take $\Phi'(u) = [C_*(u)]$, the chain homotopy class of the singular chain map $C_*(u)$ (restricted to the $\overline{C}(-)$ complexes).

It is easy to see that this Φ' is indeed a functor as desired and that $\Phi'(\Delta)$ is exact for each triangle Δ as defined above and, again, Φ' interchanges the shift functors in the domain and target. In summary, the weight $\overline{w}_{\mathcal{M}etr_0}$ is well defined, as are the quantities $\underline{\delta}^{\mathcal{F}}$ and the pseudometrics associated to them.

Remark 2.110. (a) Similarly to Remark 2.108, the definition of the triangles in $\Delta_{\mathcal{M}etr_0}$ is highly restrictive and, in this case, even the objects in our category are subject to a constraint – finiteness of the diameter – that might be a hindrance in applications. One way to apply the methods above to the study of spaces of infinite diameter is to consider triangles of the form $\Delta : A \to B \to C \to S'A$, where A is of finite diameter and such that $S'A$ admits a metric as above, and to analyze when $\Phi'(\Delta)$ is of finite persistence weight in $[H_0 \mathcal{F}\mathcal{K}'_{\mathbf{k}}]_\infty$.

(b) In studying metric spaces of infinite diameter by these methods, it is likely that the most appropriate structure fitting with the cone construction is that of a *length structure*, in the sense of Gromov, as in [35, Chapter 1, Section A]. We will not pursue this theme further here.

2.5.3.3 Further remarks on topological examples. (A) In the topological examples above – for instance in $\mathcal{AT}op_*$ – it is natural to see what the quantities $\underline{\delta}^{\mathcal{F}}(-)$ mean even for the flat weight w_0, which associates to each exact triangle the value 1. Of course, in this case $\underline{\delta}^{\mathcal{F}}(A, B)$ simply counts the minimal number of cone attachments in the category $\mathcal{AT}op_*$ needed to obtain A out of the space B by attaching cones over spaces in the family $\mathcal{F}$ using the family of triangles $\Delta_{\mathcal{AT}op_*}$. Given that the weight is flat, the question is independent of filtrations and shift functors, and it reduces to the identical question in the category of pointed spaces $\mathcal{T}op_*$. In the examples below we will focus on this category and on $\underline{\delta}^{\mathcal{F}}(A, *)$, which is one of the most basic quantities involved.

It is useful to keep in mind that there are two more choices that are essential in defining $\underline{\delta}^{\mathcal{F}}(A, *)$: the choice of family $\mathcal{F}$ and the choice of the class of exact triangles $\Delta_{\mathcal{T}op_*}$; see also Remark 2.108 (a).

(i) $\mathcal{F} = \{S^0, S^1, \ldots, S^k, \ldots\}$; $\Delta_{\mathcal{T}op_*}$ are the triangles $A \xrightarrow{u} B \to \mathrm{Cone}(u) \to SA$ as in (2.53) (but omitting the action functionals). In this case, $\underline{\delta}^{\mathcal{F}}(A, *) = k < \infty$ means that A has the structure of a finite CW-complex with k cells.

(ii) $\mathcal{F} = \{S^0, S^1, \ldots, S^k, \ldots\}$; we now take $\Delta_{\mathcal{T}op_*}$ to be the triangles that are homotopy equivalent to the triangles $A \xrightarrow{u} B \to \mathrm{Cone}(u) \to SA$ from (2.53). In this case, $\underline{\delta}^{\mathcal{F}}(A, *) = k$ means that A is homotopy equivalent to a CW-complex with k cells. This number is obviously a homotopy invariant. It is clearly bounded from below by the sum of the Betti numbers of A.

(iii) $\mathcal{F}$ consists of all pointed spaces with the homotopy type of CW-complexes; $\Delta_{\mathcal{T}op_*}$ are as at (ii). In this case, the definition of $\underline{\delta}^{\mathcal{F}}(A, *)$ coincides with that of the cone-length, $\mathrm{Cl}(A)$, of A (for a space A with the homotopy type of a CW-complex). Cone-length is a homotopical invariant of interest because it is at most one greater than the Lusternik–Schnirelmann category [19], which, in turn, provides a lower bound for the minimal number of critical points of smooth functions on manifolds. Incidentally, as noted by Smale [54], a version of the Lusternik–Schnirelmann category provides also a measure of the complexity of algorithms; see [22] for more on this subject.

(iv) At this point, we will change the underlying category and place ourselves in the pointed category of finite-type, simply-connected *rational* spaces $\mathcal{T}op_1^{\mathbb{Q}}$ (see [31]). We take $\mathcal{F}$ to consist of finite wedges of rational spheres of dimension at least 2. The triangles $\Delta_{\mathcal{T}op_*^{\mathbb{Q}}}$ are as in (ii) (in the category of rational spaces), but we will also allow in $\Delta_{\mathcal{T}op_*^{\mathbb{Q}}}$ "formal" triangles of the form $S^{-1}F \to * \to F$, where $F \in \mathcal{F}$ (desuspending is not possible in our category, but we still want to have, for a rational 2-sphere $S^2_{\mathbb{Q}}$, $\underline{\delta}^{\mathcal{F}}(S^2_{\mathbb{Q}}, *) = 1$). In this setting, it turns out that

$$\underline{\delta}^{\mathcal{F}}(A, *) = \mathrm{Cl}(A) = \mathrm{nil}(A)$$

(see [20]). Both equalities here are non-trivial: the first because in the definition of Cl we use cones over arbitrary (rational) spaces, while in this example $\mathcal{F}$ consists of only wedges of spheres. For the second equality, $\mathrm{nil}(A)$ is the minimal order of nilpotence of the augmentation ideal $\bar{\mathcal{A}}$ of a rational differential graded commutative algebra $\mathcal{A}$ representing A (recall that, by a celebrated result of Sullivan [55], the homotopy category of rational simply-connected spaces is equivalent to the homotopy category of rational differential graded commutative algebras, the representative of a given space being given by the so-called PL-de Rham complex of A).

(B) One of the difficulties of extracting a triangulated persistence category from a topological category such as those considered in this section is very basic and has to do with the difference between stable and unstable homotopy. In essence, recall that if $\mathcal{C}$ is a TPC, then the 0-level category $\mathcal{C}_0$ is required to be triangulated. However, in unstable settings, homotopy categories of spaces are not triangulated.

(i) An instructive example is a variant of our discussion concerning the category $\mathcal{M}etr_0$. In this case the morphisms $\hom_{\mathcal{M}etr_0}(A, B)$ carry an obvious topology as well as a filtration, as described in Section 2.5.3.2. We can now consider a new category, $\widetilde{\mathcal{M}etr}_0$, with the same objects as $\mathcal{M}etr_0$ but with morphisms $\hom_{\widetilde{\mathcal{M}etr}_0}(A, B) = S_*(\hom_{\mathcal{M}etr_0}(A, B))$, where $S_*(-)$ stands for cubical chain complexes. These morphisms carry an obvious filtration obtained by applying the cubical chains to the filtration of $\hom_{\mathcal{M}etr_0}(A, B)$. The composition in this category is given by applying cubical chains to the composition $\hom_{\mathcal{M}etr_0}(B, C) \times \hom_{\mathcal{M}etr_0}(A, B) \to \hom_{\mathcal{M}etr_0}(A, C)$ and composing with the map

$$\begin{aligned} &S_*(\hom_{\mathcal{M}etr_0}(B, C)) \otimes S_*(\hom_{\mathcal{M}etr_0}(A, B)) \\ &\qquad \to S_*(\hom_{\mathcal{M}etr_0}(B, C) \times \hom_{\mathcal{M}etr_0}(A, B)) \end{aligned}$$

induced by taking products of cubes. It follows that $\widetilde{\mathcal{M}etr}_0$ is a filtered dg-category (in homological formalism). Thus all the machinery in Section 2.5.1 is applicable in this case. Moreover, this category carries an obvious shift functor. However, $[H_0\widetilde{\mathcal{M}etr}_0]_\infty$ is not triangulated and thus $\widetilde{\mathcal{M}etr}_0$

is not pre-triangulated (quite far from it). Indeed, $\hom_{[H_0\widetilde{\mathcal{M}etr}_0]_\infty}(A, B)$ is the free abelian group generated by the homotopy classes of Lipschitz maps from A to B. As a result, the translation functor (which is in our case the topological suspension) is certainly not an isomorphism.

(ii) As mentioned before, in point (B) at the beginning of Section 2.5.3, one way to bypass these issues is to introduce a sort of filtered Waldhausen category or a similar formalism and to develop a machinery parallel to that of TPCs in this unstable context. The structure present in $\mathcal{ATop}_*$ and $\mathcal{M}etr_0$ suggests that such a construction is possible and will be relevant in these cases.

(iii) There is yet another, more geometric, approach to associating to each of $\mathcal{ATop}_*$ and $\mathcal{M}etr_0$ a triangulated persistence category. This is based on moving from these categories to stable categories, where the underlying objects are the spectra obtained by stabilizing the objects of the original categories and the morphisms come with an appropriate filtration induced from the respective structures (action functionals or, respectively, metrics) on the initial objects. This seems likely to work and to directly produce a TPC, but we will not pursue the details at this time.

2.5.4 Filtrations in Tamarkin's category

This section is devoted to an example of a triangulated persistence category that comes from the filtration structure present in Tamarkin's category. This category was originally defined in [56], based on singular supports of sheaves, and was used to prove some non-displaceability results in symplectic geometry, as well as other more recent results related to Hamiltonian dynamics (see [36]).

2.5.4.1 Background on Tamarkin's category. Let X be a manifold, and let $\mathcal{D}(\mathbf{k}_X)$ be the derived category of sheaves of $\mathbf{k}$-modules over X. In particular, this is a triangulated category. For any $A \in \mathrm{Obj}(\mathcal{D}(\mathbf{k}_X))$, due to microlocal sheaf theory as established in [40], one can define the singular support of A, denoted by $\mathrm{SS}(A)$, as a conical (singular) subset of T^*X. We refer to [40, Chapter V] for the precise definition of $\mathrm{SS}(A)$ and a detailed study of its properties. Now, let $X = M \times \mathbb{R}$, where M is a closed manifold, and denote by τ the co-vector coordinate of $T^*\mathbb{R}$ in $T^*(M \times \mathbb{R})$. Consider the full subcategory $\mathcal{D}_{\{\tau\leq 0\}}(\mathbf{k}_{M\times\mathbb{R}})$ of $\mathcal{D}(\mathbf{k}_{M\times\mathbb{R}})$ defined by

$$\mathrm{Obj}(\mathcal{D}_{\{\tau\leq 0\}}(\mathbf{k}_{M\times\mathbb{R}})) = \{A \in \mathrm{Obj}(\mathcal{D}(\mathbf{k}_{M\times\mathbb{R}})) \mid \mathrm{SS}(A) \subset \{\tau \leq 0\}\}.$$

If $A \to B \to C \to A[1]$ is an exact triangle in $\mathcal{D}(\mathbf{k}_{M\times\mathbb{R}})$, then $\mathrm{SS}(C) \subset \mathrm{SS}(A) \cup \mathrm{SS}(B)$. This implies that $\mathcal{D}_{\{\tau\leq 0\}}(\mathbf{k}_{M\times\mathbb{R}})$ is a triangulated subcategory of $\mathcal{D}_{\mathbf{k}_{M\times\mathbb{R}}}$.

Tamarkin's category is defined by

$$\mathcal{T}(M) := \mathcal{D}_{\{\tau \leq 0\}}(\mathbf{k}_{M\times\mathbb{R}})^{\perp,l}, \tag{2.55}$$

where $\perp, l$ denotes the left orthogonal complement of $\mathcal{D}_{\{\tau\leq 0\}}(\mathbf{k}_{M\times\mathbb{R}})$ in $\mathcal{D}(\mathbf{k}_{M\times\mathbb{R}})$. Then $\mathcal{T}(M)$ is also a triangulated subcategory. Note that $\mathcal{T}(M) \subset \mathcal{D}_{\{\tau\geq 0\}}(\mathbf{k}_{M\times\mathbb{R}})$ by definition. When $M = \{\mathrm{pt}\}$, Tamarkin's category $\mathcal{T}(\{\mathrm{pt}\})$, together with a constructibility condition, can be identified with the category of persistence **k**-modules (see [63, Section A.1]).

Remark 2.111. There exists a restricted version of Tamarkin's category, denoted by $\mathcal{T}_V(M)$, where $V \subset T^*M$ is a closed subset (see [63, Section 3.2]). This restricted Tamarkin category is useful to prove the non-displaceability of some subsets in T^*M (see [2]). In this memoir, we will only focus on $\mathcal{T}(M)$.

One way to understand the definition (2.55) is that $\mathcal{T}(M)$ is an admissible subcategory (see [44, Definition 1.8]), in the sense that for any object A in $\mathcal{D}(\mathbf{k}_{M\times\mathbb{R}})$ one can always split A in the form of an exact triangle

$$B \to A \to C \to B[1] \tag{2.56}$$

in $\mathcal{D}(\mathbf{k}_{M\times\mathbb{R}})$, where $B \in \mathcal{T}(M)$ and $C \in \mathcal{D}_{\{\tau\leq 0\}}(\mathbf{k}_{M\times\mathbb{R}})$. In fact, this splitting can be achieved in a rather concrete manner, which involves an important operator called sheaf convolution on objects in $\mathcal{D}(\mathbf{k}_{M\times\mathbb{R}})$. Explicitly, for any two objects A, B in $\mathcal{D}(\mathbf{k}_{M\times\mathbb{R}})$, the sheaf convolution of A and B is defined by

$$A * B := \delta^{-1} R s_!(\pi_1^{-1} A \otimes \pi_2^{-1} B),$$

where $\pi_i : (M\times\mathbb{R})^2 \to M\times\mathbb{R}$ are the projections to each factor of $M\times\mathbb{R}$, s leaves the $(M\times M)$-part unchanged but adds up two inputs on the $\mathbb{R}$-factors, and δ is the diagonal embedding from M to $M\times M$. For instance, $\mathbf{k}_{M\times[0,\infty)} * \mathbf{k}_{M\times[0,\infty)} = \mathbf{k}_{M\times[0,\infty)}$, where $\mathbf{k}_V$ for a closed subset V denotes the constant sheaf supported on V. Moreover, this operator is commutative and associative. An important characterization of an object in $\mathcal{T}(M)$ is that (see [56, Proposition 2.1])

$$A \in \mathrm{Obj}(\mathcal{T}(M)) \quad \text{if and only if} \quad A * \mathbf{k}_{M\times[0,\infty)} = A,$$

which implies that (i) for any object A in $\mathcal{D}(\mathbf{k}_{M\times\mathbb{R}})$, the sheaf convolutions $B := A * \mathbf{k}_{M\times[0,\infty)}$ and $C := A * \mathbf{k}_{M\times(0,\infty)}[1]$ provide the desired exact triangle for a splitting of A in (2.56); (ii) sheaf convolution is a well-defined operator on $\mathcal{T}(M)$.

With the help of sheaf convolution, the $\mathbb{R}$-component generates a filtration structure in $\mathcal{T}(M)$ as follows. For any $r \in \mathbb{R}$, consider the map $T_r : M\times\mathbb{R} \to M\times\mathbb{R}$ defined by $(m, a) \mapsto (m, a + r)$. One can show that for any object A in $\mathcal{T}(M)$, the induced object is $(T_r)_* A = A * \mathbf{k}_{M\times[r,\infty)}$ (see [63, Lemma 3.2]). In fact, $\{(T_r)_*\}_{r\in\mathbb{R}}$

defines an $\mathbb{R}$-family of functors on $\mathcal{T}(M)$. Moreover, if $r \leq s$, then by the restriction map $\mathbf{k}_{M\times[r,\infty)} \to \mathbf{k}_{M\times[s,\infty)}$, we have a canonical morphism $\tau_{r,s}(A) : (T_r)_* A \to (T_s)_* A$. At this point, notice that for $r \leq s$ there does not exist a non-zero morphism from $\mathbf{k}_{M\times[s,\infty)}$ to $\mathbf{k}_{M\times[r,\infty)}$, so the canonical map $\tau_{r,s}$ respects the partial order $\leq$ on $\mathbb{R}$. For any $r \leq s$, $\tau_{r,s}$ is viewed as a natural transformation from $(T_r)_*$ to $(T_s)_*$. Finally, we call an object A in $\mathcal{T}(M)$ a c-torsion element if $\tau_{0,c}(A) : A \to (T_c)_* A$ is zero. For instance, when $M = \{\mathrm{pt}\}$, the constant sheaf $\mathbf{k}_{[a,b)} \in \mathcal{T}(\{\mathrm{pt}\})$ with a finite interval $[a, b)$ is a $(b - a)$-torsion.

We end this subsection with a discussion of the hom-set in $\mathcal{T}(M)$. It is more convenient to consider derived hom, that is, $\mathrm{Rhom}_{\mathcal{T}(M)}(A, B)$ for any two objects A, B in $\mathcal{T}(M)$. Lemma 3.3 in [63] (or Lemma 3.8 (1) in [56]) provides a more explicit way to express such Rhom, that is,

$$\mathrm{Rhom}_{\mathcal{T}(M)}(A, B) = \mathrm{Rhom}_{\mathbb{R}}(\mathbf{k}_{[0,\infty)}, R\pi_* \mathcal{H}om^*(A, B)). \tag{2.57}$$

By taking the cohomology at degree 0, we obtain $\hom_{\mathcal{T}(M)}(A, B)$ as a $\mathbf{k}$-module. Here, $\pi : M \times \mathbb{R} \to \mathbb{R}$ and $\mathcal{H}om^*(\cdot, \cdot)$ is the right adjoint functor to the sheaf convolution (see [2, Definition 3.1]). The right-hand side of (2.57) is relatively computable, since they are all (complexes of) sheaves over $\mathbb{R}$ (cf. [63, Section A.2]). Moreover, by using the adjoint relation between $\mathcal{H}om^*(\cdot, \cdot)$ and the sheaf convolution, one obtains a shifted version of (2.57), that is,

$$\mathrm{Rhom}_{\mathcal{T}(M)}(A, (T_r)_* B) = \mathrm{Rhom}_{\mathbb{R}}(\mathbf{k}_{[0,\infty)}, (T_r)_*(R\pi_* \mathcal{H}om^*(A, B))). \tag{2.58}$$

Therefore, for any $r \leq s$, there exists a well-defined morphism

$$\iota_{r,s}^{A,B} : \hom_{\mathcal{T}(M)}(A, (T_r)_* B) \to \hom_{\mathcal{T}(M)}(A, (T_s)_* B), \tag{2.59}$$

which is induced by the morphism $\tau_{r,s}(R\pi_* \mathcal{H}om^*(A, B))$. Finally, we have a canonical isomorphism,

$$T_r : \hom_{\mathcal{T}(M)}((T_r)_* A, (T_r)_* B) \simeq \hom_{\mathcal{T}(M)}(A, B), \tag{2.60}$$

which is induced by the sheaf convolution with $\mathbf{k}_{M\times[r,\infty)}$. In particular, T_r commutes with the morphism $\iota_{r,s}^{A,B}$ defined in (2.59).

2.5.4.2 Persistence category from Tamarkin's shift functors. We have seen before that Tamarkin's category is endowed with a shift functor. We now discuss the persistence structure induced by this shift functor; see Remark 2.23 (d).

Definition 2.112. Given the category $\mathcal{T}(M)$ as above, define an enriched category $\mathcal{P}(M)$ as follows. The object set of $\mathrm{Obj}(\mathcal{P}(M))$ is the same as $\mathrm{Obj}(\mathcal{T}(M))$, and the hom-set is defined by

$$\hom_{\mathcal{P}(M)}(A, B) = \left\{\{\hom_{\mathcal{T}(M)}(A, (T_r)_* B)\}_{r\in\mathbb{R}}, \{\iota_{r,s}^{A,B}\}_{r,s\in\mathbb{R},\, r\leq s}\right\}$$

for any two objects A, B in $\mathcal{P}(M)$, where $\iota_{r,s}^{A,B}$ is the morphism defined in (2.59).

Remark 2.113. Definition 2.112 can be regarded as a generalization of (2.15) in Section 2.2.4, since when $M = \{\mathrm{pt}\}$, Tamarkin's category $\mathcal{T}(\{\mathrm{pt}\})$ can be identified with the category of persistence $\mathbf{k}$-modules. Also, Definition 2.112 fits with geometric examples. Indeed, recall a concrete computation of $\hom_{\mathcal{T}(M)}(A, B)$ when both A and B are sheaves coming from generating functions on M (see [63, Section 3.9]). In this case, $\hom_{\mathcal{P}(M)}(A, B)$ can be identified to a (Morse) persistence $\mathbf{k}$-module in the classical sense.

Lemma 2.114. *The category $\mathcal{P}(M)$ from Definition 2.112 is a persistence category.*

Proof. Consider the functor $E_{A,B} : (\mathbb{R}, \leq) \to \mathrm{Vect}_{\mathbf{k}}$ defined by

$$E_{A,B}(r)(= \hom^r(A, B)) = \hom_{\mathcal{T}(M)}(A, (T_r)_* B),$$

and, for the morphism $i_{r,s}$ when $r \leq s$, set $E_{A,B}(i_{r,s}) = \iota^{A,B}_{r,s}$. Notice that the composition $E_{A,B}(r) \times E_{B,C}(s) \to E_{A,C}(r + s)$ is well defined due to (2.60). Indeed, for any $f \in E_{A,B}(r)$ and $g \in E_{B,C}(s)$, the composition is defined by

$$(f, g) \mapsto T_r(g) \circ f \in \hom_{\mathcal{T}(M)}(A, (T_{r+s})_* C).$$

Then, for any $r \leq r'$, $s \leq s'$, we have

$$\left(T_{r'}\left(\iota^{B,C}_{s,s'}(g)\right)\right) \circ \iota^{A,B}_{r,r'}(f) = \iota^{A,C}_{r+s,r'+s'}(T_r(g) \circ f),$$

which completes the proof that $\mathcal{P}(M)$ is a persistence category. ■

We now list some properties of the persistence category $\mathcal{P}(M)$.

(a) The 0-level category $\mathcal{P}(M)_0$ has the same objects as $\mathcal{P}(M)$, but

$$\hom_{\mathcal{P}(M)_0}(A, B) = \hom_{\mathcal{T}(M)}(A, B).$$

We use the fact that $(T_0)_* = \mathbb{1}$. Thus, $\mathcal{P}(M)_0 = \mathcal{T}(M)$. This category is triangulated, as we have seen above.

(b) The ∞-level, $\mathcal{P}(M)_\infty$, has the same objects as $\mathcal{P}(M)$, but

$$\hom_{\mathcal{P}(M)_\infty}(A, B) = \varinjlim_{r \to \infty} \hom_{\mathcal{T}(M)}(A, (T_r)_* B),$$

where the direct limit is taken via the map $\iota^{A,B}_{r,s}$. This limit category has been considered in [37, formula (81) and Proposition 6.7], where it is viewed from the perspective of a categorical localization on torsion elements. This can be regarded as a special case of Proposition 2.39 in Section 2.3.2, where the localization is established for a general triangulated persistence category.

(c) On $\mathcal{P}(M)$, each $(T_r)_*$ is a persistence functor for any $r \in \mathbb{R}$, i.e., $(T_r)_* \in \mathcal{P}(\mathrm{End}(\mathcal{P}(M)))$, since T_r commutes with $\tau_{r,s}$.

(d) There exists a natural shift functor on $\mathscr{P}(M)$. Define

$$\Sigma : (\mathbb{R}, +) \to \mathscr{P}(\mathrm{End}(\mathscr{P}(M)))$$

by $\Sigma(r)(= \Sigma^r) = (T_{-r})_*$. For any $r, s \in \mathbb{R}$ and $\eta_{r,s} \in \hom_{\mathbb{R}}(r, s)$, define

$$\Sigma(\eta_{r,s}) := \mathbb{1}_{(T_{-r})_*(-)}.$$

Then, for any object A in $\mathscr{P}(M)$,

$$\begin{aligned}\Sigma(\eta_{r,s})_A = \mathbb{1}_{(T_{-r})_*A} &\in \hom_{\mathcal{T}(M)}((T_{-r})_*A, (T_{-r})_*A)\\ &= \hom_{\mathcal{T}(M)}((T_{-r})_*A, (T_{s-r})_*((T_{-s})_*A))\\ &= E_{(T_{-r})_*A,(T_{-s})_*A}(s-r)\\ &= \hom^{s-r}((T_{-r})_*A, (T_{-s})_*A).\end{aligned}$$

In other words, $\Sigma(\eta_{r,s})_A$ is a natural transformation of shift $s - r$. In particular, the morphism $\eta_r^A = i_{-r,0}((\eta_{r,0})_A) \in \hom^0(\Sigma^r A, A)$ is well defined for any $r \geq 0$. It is easy to check that $\eta_r^A = (\tau_{-r,0})(A)$.

(e) The r-acyclic objects in $\mathscr{P}(M)$ are precisely the r-torsion elements in $\mathcal{T}(M)$. Indeed, by definition, an object A in $\mathscr{P}(M)$ is r-acyclic if and only if $\eta_r^A = \tau_{-r,0}(A) : (T_{-r})_*A \to A$ is the zero morphism, which coincides with the definition of an r-torsion element under the isomorphism (2.60).

(f) Recall that for each r, $\hom^r(-, X) = \hom_{\mathcal{T}(M)}(-, (T_r)_*X)$. This is an exact functor due to (2.58) on $\mathscr{P}(M)_0 = \mathcal{T}(M)$. Similarly, $\hom^r(X, -)$ is also an exact functor on $\mathscr{P}(M)_0 = \mathcal{T}(M)$.

Lemma 2.115. *For any $r \geq 0$ and any object A in $\mathscr{P}(M)$, the morphism*

$$\eta_r^A : (T_{-r})_*A \to A$$

embeds into the exact triangle

$$(T_{-r})_*A \xrightarrow{\eta_r^A} A \to K \to A \tag{2.61}$$

in $\mathcal{T}(M) = \mathscr{P}(M)_0$, where K is r-acyclic.

Proof. Since $\mathcal{T}(M)$ is a triangulated category, the morphism η_r^A embeds into an exact triangle as (2.61). By item (e) above, we need to show that K is an r-torsion element. By [37, Lemma 6.3 (ii)], which provides a criterion for testing whether an object in an exact triangle is a torsion element, it suffices to verify that the following diagram is commutative for some morphism α:

$$\begin{tikzcd}
(T_{-r})_*A \arrow[r, "\eta_r^A"] \arrow[d, "\tau_{0,r}((T_{-r})_*A)"'] & A \arrow[d, "\tau_{0,r}(A)"] \arrow[dl, "\alpha"'] \\
A \arrow[r, "T_r(\eta_r^A)"'] & (T_r)_*A
\end{tikzcd}$$

Indeed, this diagram is commutative if we choose $\alpha = \mathbb{1}_A$ and use the functorial properties of T_{-r}. ■

Remark 2.116. By the definition of an r-isomorphism given earlier, Lemma 2.115 implies that the morphism $\eta_r^A \in \hom^0((T_r)_*A, A)$ is an r-isomorphism. On the other hand, [63, Section 3.10] defines an interleaving relation between two objects in $\mathcal{T}(M)$, which is similar to r-isomorphism in the sense that A and $(T_r)_*A$ are r-interleaved.

Example 2.117. Let $M = \{\mathrm{pt}\}$ and consider $\mathcal{T}(\{\mathrm{pt}\})$. For $A = \mathbf{k}_{[0,\infty)}$, we know that $(T_{-r})_*A = \mathbf{k}_{[-r,\infty)}$ for any $r \geq 0$. Then we have an exact triangle in $\mathcal{T}(\{\mathrm{pt}\})$:

$$\mathbf{k}_{[-r,\infty)} \xrightarrow{\tau_{-r,0}(A)} \mathbf{k}_{[0,\infty)} \to \mathbf{k}_{[-r,0)}[1] \to \mathbf{k}_{[-r,\infty)}[1],$$

where, as we have seen, $\mathbf{k}_{[-r,0)}[1]$ is an r-torsion element (so r-acyclic). Here, by definition, $\tau_{-r,0}(A)$ is the restriction map from $\mathbf{k}_{[-r,\infty)}$ to $\mathbf{k}_{[0,\infty)}$, and the exact triangle is from [40, (2.6.33)].

The properties stated in (a) and (d) above, together with Lemmas 2.114 and 2.115, imply the main consequence of this section.

Corollary 2.118. *The category $\mathcal{P}(M)$, introduced in Definition 2.112, is a triangulated persistence category.*