Chapter 1

Introduction

In this memoir, we define several algebraic objects associated to compact, oriented 3-manifolds with parametrized torus boundary, and prove a gluing formula. For the purposes of this memoir, we use the following definition (compare [27, Section 1.1]).

Definition 1.1. A *bordered manifold with torus boundary* consists of a compact and oriented manifold M such that $\partial M \cong \mathbb{T}^2$ and M is equipped with a choice of oriented basis (μ, λ) of $H_1(\partial M)$.

If M_1 and M_2 are two bordered manifolds with torus boundaries, a natural gluing operation is to glue μ_1 to μ_2 and glue λ_1 to $-\lambda_2$. This is motivated by the following fact. Suppose that M_1 and M_2 are the complements of two oriented knots K_1 and K_2 in S^3, which are given integral framings λ_1 and λ_2. Let $\phi\colon \partial M_1 \to \partial M_2$ denote the orientation reversing diffeomorphism which sends μ_1 to μ_2 and λ_1 to $-\lambda_2$. Then there is a diffeomorphism

$$M_1 \cup_\phi M_2 \cong S^3_{\lambda_1+\lambda_2}(K_1 \# K_2).$$

This basic topological fact is reviewed in Appendix A.

The algebraic objects we define are reformulations of Manolescu and Ozsváth's link surgery formula [32], recast using the framework of type-D and type-A modules of Lipshitz, Ozsváth and Thurston [27].

Using the above topological observation about surgeries on connected sums, our gluing formula amounts to a formula for the behavior of the link surgery formula under connected sums.

1.1 The knot surgery algebra

We begin by describing our associative algebra $\mathcal{K}$, which is an algebra over the idempotent ring $\mathbf{I} = \mathbf{I}_0 \oplus \mathbf{I}_1$. Here each $\mathbf{I}_i$ is a copy of $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$. We define

$$\mathbf{I}_0 \cdot \mathcal{K} \cdot \mathbf{I}_0 = \mathbb{F}[\mathscr{U}, \mathscr{V}] \quad \text{and} \quad \mathbf{I}_1 \cdot \mathcal{K} \cdot \mathbf{I}_1 = \mathbb{F}[\mathscr{U}, \mathscr{V}, \mathscr{V}^{-1}],$$

where $\mathbb{F}[\mathscr{U}, \mathscr{V}]$ denotes a polynomial ring in two variables and $\mathbb{F}[\mathscr{U}, \mathscr{V}, \mathscr{V}^{-1}]$ is its localization at $\mathscr{V}$. We set $\mathbf{I}_0 \cdot \mathcal{K} \cdot \mathbf{I}_1 = \{0\}$. There are two special algebra elements $\sigma, \tau \in \mathbf{I}_1 \cdot \mathcal{K} \cdot \mathbf{I}_0$, and we define

$$\mathbf{I}_1 \cdot \mathcal{K} \cdot \mathbf{I}_0 = \mathbb{F}[\mathscr{U}, \mathscr{V}, \mathscr{V}^{-1}] \otimes \langle \sigma \rangle \oplus \mathbb{F}[\mathscr{U}, \mathscr{V}, \mathscr{V}^{-1}] \otimes \langle \tau \rangle,$$

where $\langle\sigma\rangle$ and $\langle\tau\rangle$ denote copies of $\mathbb{F}$. The elements σ and τ satisfy the relations

$$\sigma\mathscr{U} = \mathscr{U}\sigma \quad \text{and} \quad \sigma\mathscr{V} = \mathscr{V}\sigma,$$
$$\tau\mathscr{U} = \mathscr{V}^{-1}\tau \quad \text{and} \quad \tau\mathscr{V} = \mathscr{U}\mathscr{V}^2\tau.$$

We typically write U for the product $\mathscr{U}\mathscr{V}$.

The algebra $\mathcal{K}$ is an associative algebra, i.e., we can think of it as an A_∞-algebra which has vanishing actions μ_j for $j \neq 2$. Although seemingly asymmetric between $\mathscr{U}$ and $\mathscr{V}$, as well as between σ and τ, the above algebra admits a more symmetrical description. See Remark 7.1. We work with an asymmetric presentation of the algebra $\mathcal{K}$ because it mirrors an asymmetric construction of the Heegaard Floer surgery formulas which frequently simplifies computations.

1.2 Surgery formulas as $\mathcal{K}$-modules

Ozsváth and Szabó [43] proved that if $K \subseteq S^3$ is a knot, then the Heegaard Floer homology groups $\mathit{HF}^-(S^3_\lambda(K))$ are computable from the knot Floer complex of K [37,47]. If λ is an integral framing on K, their construction produced a mapping cone complex $\mathbb{X}_\lambda(K)$, which satisfied

$$H_*(\mathbb{X}_\lambda(K)) \cong \mathbf{HF}^-(S^3_\lambda(K)).$$

(The bold font denotes coefficients in the power series ring $\mathbb{F}[[U]]$.)

Manolescu and Ozsváth [32] extended this formula to links in S^3. They proved that if $L \subseteq S^3$ is a link with integral framing Λ, then there is a complex $\mathcal{C}_\Lambda(L)$, defined using the link Floer homology of L [41], such that

$$H_*(\mathcal{C}_\Lambda(L)) \cong \mathbf{HF}^-(S^3_\Lambda(L)).$$

In Section 8.5, we describe how to naturally reinterpret the data of Ozsváth and Szabó's mapping cone complex $\mathbb{X}_\lambda(K)$ both in terms of a right type-D module $\mathcal{X}_\lambda(K)^{\mathcal{K}}$ and a left type-A module ${}_{\mathcal{K}}\mathcal{X}_\lambda(K)$.

In Section 8.6, we extend this interpretation to the link surgery formula. If ℓ is a positive integer, we define the ℓ-component *link algebra* to be

$$\mathcal{L}_\ell = \bigotimes_{\mathbb{F}}^{\ell} \mathcal{K} = \mathcal{K} \otimes_{\mathbb{F}} \cdots \otimes_{\mathbb{F}} \mathcal{K}.$$

To an ℓ-component link $L \subseteq S^3$, we define in Section 8.6 a type-D module

$$\mathcal{X}_\Lambda(L)^{\mathcal{L}_\ell}.$$

Remark 1.2. We think of the map v in Ozsváth and Szabó's mapping cone formula as being encoded by the algebra element $\sigma \in \mathcal{K}$. The map h is encoded by the algebra element $\tau \in \mathcal{K}$.

The description of the link surgery formula given by Manolescu and Ozsváth is an infinite direct product of free modules over $\mathbb{F}[\![U_1, \dots, U_\ell]\!]$. In particular, it is very large and sometimes challenging to do direct computations with. On the other hand, the type-D module $\mathcal{X}_\Lambda(L)^{\mathcal{L}_\ell}$ that we describe contains all of the algebraic information of the link surgery formula, but is always finite-dimensional over $\mathbb{F}$. For example, the complement of the 0-framed trefoil has the type-D module shown in Figure 1.1.

$$\begin{tikzcd}
& \mathbf{x}^0 \arrow[dl, "\mathscr{V}"'] \arrow[dr, "\mathscr{U}"] & \\
\mathbf{y}^0 \arrow[dr, "\mathscr{V}^{-2}(U\sigma+\tau)"'] & & \mathbf{z}^0 \arrow[dl, "\sigma+U\tau"] \\
& \mathbf{z}^1 &
\end{tikzcd}$$

Figure 1.1. The type-D module for the 0-framed trefoil complement. The superscript indicates the idempotent.

In our memoir, the most fundamental object is the type-D module $\mathcal{X}_\Lambda(L)^{\mathcal{L}_\ell}$. To change a type-D algebra factor of $\mathcal{K}$ to a type-A algebra factor, we tensor with an algebraically defined "identity bimodule"

$$ {}_{\mathcal{K}|\mathcal{K}}[\mathbb{I}^{\supseteq}]. $$

Ignoring completions, ${}_{\mathcal{K}|\mathcal{K}}[\mathbb{I}^{\supseteq}]$ is a type-A module over $\mathcal{K} \otimes_{\mathbb{F}} \mathcal{K}$. (We write $\mathcal{K}|\mathcal{K}$ instead of $\mathcal{K} \otimes_{\mathbb{F}} \mathcal{K}$ to indicate a specific behavior with respect to completions, which we call the *split Alexander condition*.)

The bimodule ${}_{\mathcal{K}|\mathcal{K}}[\mathbb{I}^{\supseteq}]$ extends to a type-AA bimodule ${}_{\mathcal{K}|\mathcal{K}}[\mathbb{I}^{\supseteq}]_{\mathbb{F}[U]}$, which is also useful to consider. Unlike the type-D modules $\mathcal{X}_\Lambda(L)^{\mathcal{L}_\ell}$, which are all finitely generated over $\mathbb{F}$, the type-A and DA bimodules in our theory are usually not finitely generated.

Manolescu and Ozsváth's link surgery formula $\mathcal{C}_\Lambda(L)$ is recovered by tensoring our type-D module $\mathcal{X}_\Lambda(L)^{\mathcal{L}_\ell}$ with ℓ-copies of the 0-framed solid torus module, which we denote by ${}_{\mathcal{K}}[\mathcal{D}_0]_{\mathbb{F}[U]}$. Here, a "0-framed solid torus" refers to the complement of a 0-framed unknot in S^3.

Algebraic completions (e.g., direct products and power series coefficients) interact with the algebra $\mathcal{K}$ in subtle and interesting ways. The algebra $\mathcal{K}$ has a natural filtration by right ideals

$$ J_0 \supseteq J_1 \supseteq \cdots . $$

This filtration equips $\mathcal{K}$ with a topology where the subspaces J_n form a basis of opens centered at 0. This equips $\mathcal{K}$ with the structure of a *linear topological chiral algebra*, which is important for our theory. In Chapter 7, we define algebraic categories of *Alexander modules*, which consist of linear topological vector spaces with appropriately continuous actions of $\mathcal{K}$. These are the categories where our link surgery modules live.

1.3 Pairing theorems

One of the central results of our memoir is a connected sum formula for the Manolescu–Ozsváth integer surgery formula. Suppose L_1 and L_2 are two links in S^3, which have framings Λ_1 and Λ_2, as well as distinguished components $K_1 \subseteq L_1$ and $K_2 \subseteq L_2$. In Section 8.6, we describe how to compute $\mathcal{C}_{\Lambda_1+\Lambda_2}(L_1 \# L_2)$ in terms of $\mathcal{C}_{\Lambda_1}(L_1)$ and $\mathcal{C}_{\Lambda_2}(L_2)$. Here, $L_1 \# L_2$ denotes the connected sum of L_1 and L_2 along K_1 and K_2, and $\Lambda_1 + \Lambda_2$ denotes the framing where we sum the framings on K_1 and K_2 and leave the others unchanged. We will interpret this connected sum formula as an A_∞ tensor product over the algebra $\mathcal{K}$. We use the algebraic formalism of Lipshitz, Ozsváth and Thurston [25, 27, 28], though our proof of the pairing theorem is independent of their proof in the standard bordered setting [27].

Our formula is simplest to state when we are working with knots.

Theorem 1.3. *Suppose that (Y_1, K_1) and (Y_2, K_2) are two knots in integer homology 3-spheres with integral framings λ_1 and λ_2. Then*

$$\mathbb{X}_{\lambda_1+\lambda_2}(Y_1 \# Y_2, K_1 \# K_2) \simeq \mathcal{X}_{\lambda_1}(Y_1, K_1)^{\mathcal{K}} \mathbin{\hat{\boxtimes}}{}_{\mathcal{K}} \mathcal{X}_{\lambda_2}(Y_2, K_2).$$

Here, $\hat{\boxtimes}$ denotes a completed version of Lipshitz, Ozsváth and Thurston's box tensor product [27]. There is also a version of the above theorem for links.

Theorem 1.4. *Suppose that L_1 and L_2 are two links in S^3 with integral framings Λ_1 and Λ_2, and that we have two distinguished components $K_1 \subseteq L_1$ and $K_2 \subseteq L_2$. Then*

$$\mathcal{C}_{\Lambda_1+\Lambda_2}(L_1 \# L_2) \simeq \mathcal{X}_{\Lambda_1}(L_1)^{\mathcal{K}} \mathbin{\hat{\boxtimes}}{}_{\mathcal{K}} \mathcal{X}_{\Lambda_2}(L_2).$$

In Section 8.6, we describe several more general versions of the pairing theorem. One version allows us to compute

$$\mathcal{X}_{\Lambda_1+\Lambda_2}(L_1 \# L_2)^{\mathcal{L}_{\ell_1+\ell_2-1}} \mathbin{\hat{\boxtimes}}{}_{\mathcal{K}} [\mathcal{D}_0]_{\mathbb{F}[U]}. \tag{1.1}$$

In the above, $\mathcal{D}_0$ denotes the module for the 0-framed solid torus. In the tensor product, we are tensoring the $\mathcal{K}$-action on $\mathcal{D}_0$ with the $\mathcal{K}$-factor of $\mathcal{L}_{\ell_1+\ell_2-1}$ which corresponds to $K_1 \# K_2$. The reader should think of tensoring $\mathcal{D}_0$ as corresponding to gluing in a solid torus $K_1 \# K_2$. See Remark 3.6 for more details about this tensor product.

We will show in Section 15.3 that the module in equation (1.1) is homotopy equivalent to an appropriate tensor product of the three modules

$$\mathcal{X}_{\Lambda_1}(L_1)^{\mathcal{L}_{\ell_1}}, \quad \mathcal{X}_{\Lambda_2}(L_2)^{\mathcal{L}_{\ell_2}}, \quad \text{and} \quad {}_{\mathcal{K}|\mathcal{K}}[\mathbb{I}^{\supset}]_{\mathbb{F}[U]}.$$

We think of tensoring with ${}_{\mathcal{K}|\mathcal{K}}[\mathbb{I}^{\supset}]_{\mathbb{F}[U]}$ as corresponding to gluing torus boundary components together.

In some sense, the tensor product formulas above use up the two actions of $\mathcal{K}$, and hence are not convenient for taking connected sums of more than two links along a single component. To take the connected sum of more than two components, we define *pair-of-pants* bimodules

$${}_{\mathcal{K}|\mathcal{K}}W^{\mathcal{K}}_{\alpha\beta,\alpha} \quad \text{and} \quad {}_{\mathcal{K}|\mathcal{K}}W^{\mathcal{K}}_{\alpha\beta,\beta}.$$

In Chapter 15, we study these bimodules, and we prove that they can be used to compute the type-D module over $\mathcal{K}$ after taking the connected sum of link components. There are additional modules for other combinations of α and β subscripts. There are multiple modules because of different choices of *arc systems* in the construction of the surgery formula. (See Section 1.7 for more details.)

In Section 15.4, we relate the pair-of-pants bimodules to a surgery description of the 3-manifold $P \times S^1$, where P is a pair-of-pants (i.e., a twice punctured disk).

Remark 1.5. The above pairing theorems give formulas for gluing two bordered manifolds along torus boundary components. Another important operation is to glue two boundary components of a single bordered 3-manifold together (i.e., self-gluing). Our pairing theorems do not cover this. We will investigate this in a future work.

1.4 Splicing knot complements

A basic topological operation is to glue the complements of two knots together using an orientation reversing diffeomorphism of their boundaries. 3-manifolds obtained in this manner are referred to as *splices*. Dehn surgery is a special case of splicing.

Splicing in the context of Heegaard Floer homology has been studied by many authors. See [5, 9, 10, 18] for some examples. Using bordered Heegaard Floer homology [27], it is possible to compute $\widehat{\mathit{HF}}$ of spliced manifolds. Despite the interest in splicing, there is to date no general description of HF^- of splices.

In this memoir, we give a description of HF^- of splices in terms of knot Floer homology. If $\phi\colon \mathbb{T}^2 \to \mathbb{T}^2$ is an orientation preserving diffeomorphism, then we will describe a bimodule ${}_{\mathcal{K}}[\phi]^{\mathcal{K}}$. We think of this bimodule as the bordered invariant for $[0,1] \times \mathbb{T}^2$ with basis $(\mu, -\lambda)$ on $\{0\} \times \mathbb{T}^2$ (where we write μ and λ for a standard basis of $H_1(\mathbb{T}^2)$), and $(\phi(\mu), \phi(\lambda))$ on $\{1\} \times \mathbb{T}^2$.

Theorem 1.6. *Let K_1 and K_2 be knots in S^3 and let M_1 and M_2 be their complements. Let $\phi\colon \partial M_1 \to -\partial M_2$ be an orientation preserving diffeomorphism. Then*

$$\mathbf{CF}^-(M_1 \cup_\phi M_2)_{\mathbb{F}[\![U]\!]} \simeq \mathcal{X}_{\lambda_1}(K_1)^{\mathcal{K}} \hat{\boxtimes} {}_{\mathcal{K}}[\phi]^{\mathcal{K}} \hat{\boxtimes} {}_{\mathcal{K}}\mathcal{X}_{\lambda_2}(K_2)_{\mathbb{F}[\![U]\!]}.$$

The bimodule ${}_{\mathcal{K}}[\phi]^{\mathcal{K}}$ is constructed by factoring the diffeomorphism ϕ into the following generators of $\mathrm{MCG}(\mathbb{T}^2)$:

(1) $\phi(\mu) = \mu$ and $\phi(\lambda) = \lambda \pm \mu$.

(2) $\phi(\mu) = \pm\lambda$ and $\phi(\lambda) = \mp\mu$.

In both of the above cases, the bimodule ${}_{\mathcal{K}}[\phi]^{\mathcal{K}}$ is obtained from the surgery complex for the Hopf link. We remark that the Hopf link has complement $[0,1] \times \mathbb{T}^2$, so taking a connected sum with the Hopf link and then surgering on the original knot has the effect of changing the boundary parametrization.

In general, we do not yet have a closed formula for the bimodule ${}_{\mathcal{K}}[\phi]^{\mathcal{K}}$ for an arbitrary diffeomorphism ϕ, nor do our techniques imply independence of the bimodule ${}_{\mathcal{K}}[\phi]^{\mathcal{K}}$ from the above presentation. We plan to address both of these questions in a future work.

Remark 1.7. When K_1 and K_2 are knots in S^3, the modules $\mathcal{X}_{\lambda_1}(K_1)^{\mathcal{K}}$ and ${}_{\mathcal{K}}\mathcal{X}_{\lambda_2}(K_2)_{\mathbb{F}[\![U]\!]}$ are determined by the full knot Floer complexes $\mathrm{CFK}^\infty(K_1)$ and $\mathrm{CFK}^\infty(K_2)$, up to homotopy equivalence.

1.5 Dual knots

As a special case, our techniques recover the dual knot formula of Eftekhary [4] and Hedden–Levine [11] by taking a connected sum with the Hopf link. We recall that they computed the knot Floer complex of the dual knot $\mu \subseteq S^3_\lambda(K)$ in terms of the knot Floer complex $\mathrm{CFK}^\infty(K)$. Note that the dual knot $\mu \subseteq S^3_\lambda(K)$ may be obtained by taking the connected sum of K with a Hopf link, and then performing λ-framed surgery on K.

In our present memoir case, taking the connected sum of K with a Hopf link and performing λ-framed surgery corresponds to taking the tensor product of $\mathcal{X}_\lambda(K)^{\mathcal{K}}$ with the Hopf link surgery complex, viewed as a DA bimodule ${}_{\mathcal{K}}\mathcal{H}_\Lambda^{\mathcal{K}}$. Using homological perturbation theory, we describe a smaller model, which we denote by ${}_{\mathcal{K}}\mathcal{Z}_\Lambda^{\mathcal{K}}$. Taking the box tensor product with this minimal model and then restricting to idempotent $\mathbf{I}_0$ recovers the model of Eftekhary and Hedden–Levine (modulo differing notational conventions). We sketch the equivalence of our model of the dual knot complex with the Eftekhary and Hedden–Levine model in Section 17.4.

Our analysis of the Hopf link complex builds on joint work by the author with Hendricks, Hom and Stoffregen in [14], where an extension of the dual knot formula is proven for Hendricks and Manolescu's involutive knot Floer homology [15].

1.6 Examples

We compute many basic examples in this memoir. The most fundamental is the type-D module for the Hopf link. We compute this in Chapter 16, and compute a minimal model of the corresponding DA bimodule in Chapter 17. This example is important for applications and computations. Combined with the tensor product formulas, it also gives a computation of the Heegaard Floer homology of all 3-manifolds obtained as the boundary of a plumbing of disk bundles over S^2, plumbed along a tree. This is explored further in [56], wherein the equivalence of Heegaard Floer homology and the lattice homology of Némethi [34, 35] is proven.

In Section 18.2, we describe the type-D modules of rationally framed solid tori $\mathcal{D}^{\mathcal{K}}_{p/q}$ for $p/q \in \mathbb{Q} \cup \{\infty\}$. The type-D modules $\mathcal{D}^{\mathcal{K}}_{p/q}$ can be viewed as naturally recovering the rational surgeries complex $\mathbb{X}_{p/q}(K)$ of Ozsváth and Szabó [44] in the sense that

$$\mathbb{X}_{p/q}(K) \cong \mathcal{D}^{\mathcal{K}}_{p/q} \mathbin{\hat{\boxtimes}}_{\mathcal{K}} \mathcal{X}_0(K)$$

for any knot K in S^3.

Remark 1.8. If $L \subseteq S^3$, then by tensoring type-D modules of the form $\mathcal{D}^{\mathcal{K}}_{p/q}$ with the type-A module ${}_{\mathcal{K}|\cdots|\mathcal{K}}\mathcal{X}(L)$, we obtain a version of the link surgery formula which allows for rational surgeries on L.

As another example, we show that there is a homotopy equivalence of type-D modules

$$\mathcal{D}^{\mathcal{K}}_{\infty} \simeq \operatorname{Cone}(f^1 \colon \mathcal{D}^{\mathcal{K}}_{n} \to \mathcal{D}^{\mathcal{K}}_{n+1}),$$

for some type-D morphism f^1. Tensoring with this homotopy equivalence recovers the surgery exact triangle in Heegaard Floer homology. This example is reminiscent of the original setting of bordered Heegaard Floer homology [27, Section 11.2].

1.7 Arc systems

We also prove several important technical results about the link surgery formula of Manolescu and Ozsváth [32]. We recall that the link surgery formula requires, for each knot component $K_i \subseteq L$, a choice of embedded arc $\mathcal{A}_i \subseteq S^3$ which connects two base points on K_i. We furthermore require the arcs to be pairwise disjoint and to be disjoint from L except at their endpoints. We call any such collection $\mathcal{A}$ a *system of arcs*. To a system of arcs, Manolescu and Ozsváth's construction produces a chain complex $\mathcal{C}_\Lambda(L, \mathcal{A})$.

Given a knot $K \subseteq Y$, there are two natural choices of arc systems on K. These are obtained by connecting the two base points $w, z \in K$ with an arc that runs parallel to K. We say the arc system is *beta-parallel* if the arc runs from z to w and is oriented in the same direction as K. We say that the arc system is *alpha-parallel* if it runs from z to w and is oriented oppositely to K. Given a Heegaard knot splitting Σ for K, the arc system is *alpha-parallel* if the arc is parallel to the subarc of K which lies in the alpha handlebody, and similarly for beta-parallel arcs. To a link $L \subseteq Y$, there are $2^{|L|}$ arc systems which can be constructed via this procedure, where we choose each arc to be either alpha-parallel or beta-parallel. Our theory also allows arc systems where some of the arcs are neither alpha-parallel nor beta-parallel. Although these seem less natural, they appear naturally in our proof of the connected sum formula.

There is an analogous asymmetry in Lipshitz, Ozsváth and Thurston's theory [27]. In their theory, for a bordered 3-manifold Y they consider Heegaard diagrams with boundary where some set of the attaching curves are properly embedded arcs. For each boundary component of Y, they must choose whether the arcs which abut that boundary component are alpha arcs or beta arcs. See Figure 1.2 for an example of a bordered Heegaard diagram with alpha arcs.

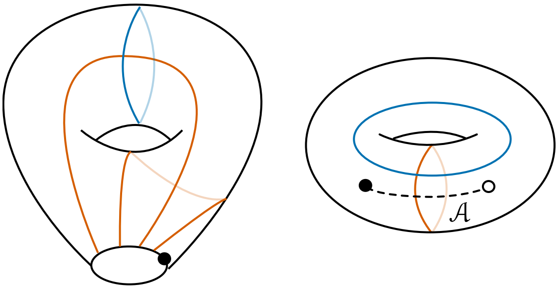

Figure 1.2. Left: A bordered Heegaard diagram for a solid torus in the Lipshitz–Ozsváth–Thurston theory. Right: A Heegaard diagram of an unknot with a beta-parallel arc system $\mathcal{A}$ (passing through the alpha curve). In our theory, the corresponding bordered manifold would be represented by the module for an unknot (whose complement is a solid torus). The choice of alpha arcs on the boundary of the bordered Heegaard diagram on the left is analogous to the choice of a beta-parallel arc system on the right.

For a system of arcs $\mathcal{A}$ we obtain a surgery complex $\mathcal{C}_\Lambda(L, \mathcal{A})$ and a type-D module $\mathcal{X}_\Lambda(L, \mathcal{A})^{\mathcal{L}}$. Manolescu and Ozsváth's result may be interpreted as showing if each arc of $\mathcal{A}$ is alpha-parallel or beta-parallel, then

$$\mathbf{HF}^-(S^3_\Lambda(L)) \cong H_*(\mathcal{C}_\Lambda(L, \mathcal{A}))$$

as modules over $\mathbb{F}[\![U]\!]$. They only consider arc systems which are beta-parallel. In our work, we study the dependence on the arcs system, which turns out to be rather subtle. As a first step, we prove the following theorem.

Theorem 1.9. *If $\mathcal{A}$ is any system of arcs for L, then there is relatively graded isomorphism of vector spaces*

$$\mathbf{HF}^-(S^3_\Lambda(L)) \cong H_*(\mathcal{C}_\Lambda(L,\mathcal{A})).$$

If the arc $a_i \in \mathcal{A}$ is either alpha- or beta-parallel, then the isomorphism is of $\mathbb{F}[\![U]\!]$-modules, where U acts by U_i on $H_*(\mathcal{C}_\Lambda(L,\mathcal{A}))$.

The isomorphism between the link surgery complexes for different systems of arcs is somewhat subtle. For example, we will construct two systems of arcs $\mathcal{A}_1$ and $\mathcal{A}_2$ on the Hopf link H for which

$$\mathcal{X}_\Lambda(H,\mathcal{A}_1)^{\mathcal{L}_2} \not\simeq \mathcal{X}_\Lambda(H,\mathcal{A}_2)^{\mathcal{L}_2} \tag{1.2}$$

as type-D modules. See Section 16.3. On the other hand, for type-D modules associated to knots, alpha-parallel and beta-parallel arc systems give the same modules. This follows from [56, Remark 5.7].

In Chapter 13, we will describe a general formula for the effect of changing the arcs in the link surgery formula. One important special case is when we want to change an arc from alpha-parallel to beta-parallel, or vice versa. This may be achieved by tensoring with a rank 1 DA bimodule ${}_{\mathcal{K}}\mathcal{T}^{\mathcal{K}}$. In Chapter 14 we study this bimodule. It is algebraically related to the base point moving map investigated by Sarkar [49] and later by the author [53].

The arc systems also play an important role in the connected sum theorems and our construction of the pair-of-pants modules in Chapter 15. Note that in the Lipshitz, Ozsváth, Thurston bordered theory, the gluing formula requires one to tensor an "alpha bordered" module with an "alpha bordered" module. From a Heegaard diagrammatic perspective, this is natural because it is most natural to glue an "alpha bordered" Heegaard diagram to an "alpha bordered" Heegaard diagram to get a Heegaard diagram of the glued manifold (since we cannot glue alpha arcs to beta arcs).

It is helpful to extend the Lipshitz, Ozsváth and Thurston convention of gluing alpha-to-alpha and beta-to-beta into our theory. In our theory, gluing is accomplished by way of taking the tensor product with algebraically defined bimodules

$${}_{\mathcal{K}|\mathcal{K}}[\mathbb{I}^{\Supset}], \quad {}_{\mathcal{K}|\mathcal{K}}W^{\mathcal{K}}_{\alpha\beta,\beta}, \quad \text{and} \quad {}_{\mathcal{K}|\mathcal{K}}W^{\mathcal{K}}_{\alpha\beta,\alpha}.$$

(There are further modules for all combinations of α and β subscripts.) For connected sums involving ${}_{\mathcal{K}|\mathcal{K}}[\mathbb{I}^{\Supset}]$, the most basic theorem we prove requires one module to be alpha bordered, and one to be beta bordered. Therefore, it is most natural to think of the module ${}_{\mathcal{K}|\mathcal{K}}[\mathbb{I}]^{\Supset}$ as alpha bordered on one component, and beta bordered on the other. The labeling of the modules ${}_{\mathcal{K}|\mathcal{K}}W^{\mathcal{K}}_{\alpha\beta,\beta}$, ${}_{\mathcal{K}|\mathcal{K}}W^{\mathcal{K}}_{\alpha\beta,\alpha}$ indicates the borderedness. For example, tensoring two beta bordered type-D modules for knots into ${}_{\mathcal{K}|\mathcal{K}}W^{\mathcal{K}}_{\alpha\beta,\alpha}$ gives the alpha bordered module for their connected sum.

1.8 General bordered manifolds with torus boundary

The connected sum formula of Theorem 1.4 and the topological interpretation of surgeries on connected sums in terms of gluing torus boundary components motivate a definition of a potential invariant of a bordered 3-manifold with torus boundary components.

We suppose that M is a bordered 3-manifold with n toroidal boundary components. We may describe M as surgery on a link L, in the complement of an unlink $\mathbb{U}_n$ in S^3, in a way which is compatible with the parametrization of ∂M.

We may define a type-D module $\mathcal{X}(M)^{\mathcal{L}_n}$ as follows. We begin with the link surgery formula for $L \cup \mathbb{U}_n$, interpreted as a type-D module over our algebra $\mathcal{L}_{n+\ell}$, where $\ell = |L|$. We write $\mathcal{X}_\Lambda(L \cup \mathbb{U}_n)^{\mathcal{L}_{n+\ell}}$ for this module. We define $\mathcal{X}(M)^{\mathcal{L}_n}$ by tensoring the module $\mathcal{X}_\Lambda(L \cup \mathbb{U}_n)^{\mathcal{L}_{n+\ell}}$ with ℓ type-A modules for 0-framed solid tori. This corresponds topologically to performing Dehn surgery on the components L, while identifying the components of $\partial \nu(\mathbb{U}_n)$ with ∂M. Theorem 1.4 may be interpreted as a pairing theorem for gluing two bordered 3-manifolds with torus boundary components together. We prove in Section 18.4 that the modules $\mathcal{X}(M)^{\mathcal{L}_n}$, as defined above, are always homotopy equivalent to finitely generated type-D modules, i.e., the underlying vector space $\mathcal{X}(M)$ may be taken to be finite-dimensional over $\mathbb{F}$.

In a future work, we hope to understand in what sense $\mathcal{X}(M)^{\mathcal{L}_n}$, defined as above, is an invariant of the bordered manifold M. In light of equation (1.2), the modules will in general depend on the choice of system of arcs $\mathcal{A}$ in S^3. We make the following conjecture.

Conjecture 1.10. *If M is a bordered manifold with toroidal boundary components which is equipped with a system of arcs $\mathcal{A} \subseteq M$, then the type-*D *module $\mathcal{X}(M, \mathcal{A})^{\mathcal{L}}$ is an invariant of $(M, \mathcal{A})$ up to homotopy equivalence. (In a subsequent work* [57]*, we prove the above conjecture for arc systems where each arc is either alpha-parallel or beta-parallel.)*

Remark 1.11. Our bordered modules $\mathcal{X}(M)^{\mathcal{K}}$ give useful surgery formulas for homologically essential knots in 3-manifolds with $b_1 > 0$. This is a setting where the techniques of Ozsváth and Szabó [43] do not seem to have a simple adaptation. As a concrete example, the ∞-framed solid torus may be viewed as the knot complement of a fiber in $S^1 \times \{\mathrm{pt}\} \subseteq S^1 \times S^2$. We compute in Section 18.2 the associated type-D module, which is concentrated in idempotent 1 and has two generators, and has δ^1 encoded by the diagram

$$\begin{tikzcd} \mathbf{x}^1 \arrow[r, "1+\mathcal{V}"] & \mathbf{y}^1 \end{tikzcd}.$$