

Chapter 2

Hypercubes and Heegaard Floer homology

In this chapter, we describe some background on Heegaard Floer homology and homological algebra.

2.1 Background on Heegaard Floer homology

We will assume that the reader is familiar with the basics of Heegaard Floer homology [39], as well as its refinements for knots [37, 47] and links [41]. We provide a brief review to establish notation.

Definition 2.1 ([41, Definition 3.1]). We consider the following notions of Heegaard diagram in this memoir:

- (1) A *multi-pointed Heegaard diagram* consists of a tuple

$$(\Sigma, \alpha, \beta, \mathbf{w})$$

as follows. Here, Σ is a surface of genus g and $\mathbf{w} = \{w_1, \dots, w_n\}$ is a non-empty collection of base points. Also, $\alpha = \{\alpha_1, \dots, \alpha_{g+n-1}\}$, is a collection of pairwise disjoint embedded curves on Σ which are linearly independent in $H_1(\Sigma \setminus \mathbf{w})$. We assume $\beta = \{\beta_1, \dots, \beta_{g+n-1}\}$ satisfies the same condition. In particular, $\Sigma \setminus \alpha$ consists of n components, each of which contains a single base point $w_i \in \mathbf{w}$, and similarly for $\Sigma \setminus \beta$.

- (2) A *Heegaard link diagram with free base points* consists of a tuple

$$(\Sigma, \alpha, \beta, \mathbf{w}, \mathbf{z}, \mathbf{p})$$

satisfying the following. We assume that $\mathbf{w} = \{w_1, \dots, w_\ell\}$, $\mathbf{z} = \{z_1, \dots, z_m\}$ and $\mathbf{p} = \{p_1, \dots, p_m\}$ are collections of base points. Let $n = \ell + m$. We assume that $\alpha = \{\alpha_1, \dots, \alpha_{g+n-1}\}$ and $\beta = \{\beta_1, \dots, \beta_{g+n-1}\}$ are collections of pairwise disjoint, embedded curves which each span a rank g summand of $H_1(\Sigma)$. We assume that each component of $\Sigma \setminus \alpha$ (resp. $\Sigma \setminus \beta$) contains either a single base point of \mathbf{p} , or contains one base point from \mathbf{w} and one base point from \mathbf{z} .

Remark 2.2. Occasionally, it is helpful to allow the collections \mathbf{w} and \mathbf{z} to contain free base points. In this case, we assume that if $p \in \mathbf{w} \cup \mathbf{z}$ is the sole base point in a component of $\Sigma \setminus \alpha$, then it is the sole base point in a component of $\Sigma \setminus \beta$.

The curves $\alpha, \beta \subseteq \Sigma$ determine two handlebodies, U_α and U_β , with boundary Σ . A Heegaard link diagram with free base points determines a 3-manifold Y and an oriented link $L \subseteq Y$. In U_α , the link L is a push-off of a collection of smoothly embedded, pairwise disjoint arcs in $\Sigma \setminus \alpha$ which connect the link base points of \mathbf{w} to the link base points of \mathbf{z} . In U_β , we assume L is a push-off of a collection of smoothly embedded, pairwise disjoint arcs which connect the link base points of \mathbf{z} to the link base points of \mathbf{w} . We follow the orientation convention that L intersects Σ positively at \mathbf{z} and negatively at \mathbf{w} .

Remark 2.3. Typically we assume that if w_i and z_j are in a component of $\Sigma \setminus \alpha$, then they are also in a component of $\Sigma \setminus \beta$. This ensures that $(\Sigma, \alpha, \beta, \mathbf{w}, \mathbf{z}, \mathbf{p})$ represents a link with ℓ components and each component has 2 base points.

If $\mathcal{H} = (\Sigma, \alpha, \beta, \mathbf{w})$ is a multi-pointed Heegaard diagram for Y and $\mathfrak{s} \in \text{Spin}^c(Y)$, we define $\text{CF}^-(\mathcal{H}, \mathfrak{s})$ to be the free $\mathbb{F}[U_1, \dots, U_n]$ -module generated by intersection points \mathbf{x} of the Lagrangian tori

$$\mathbb{T}_\alpha = \alpha_1 \times \cdots \times \alpha_{g+n-1} \quad \text{and} \quad \mathbb{T}_\beta = \beta_1 \times \cdots \times \beta_{g+n-1}$$

in the symmetric product $\text{Sym}^{g+n-1}(\Sigma)$, such that $\mathfrak{s}_\mathbf{w}(\mathbf{x}) = \mathfrak{s}$. See [39, Section 2.6] for the definition of the map $\mathfrak{s}_\mathbf{w}: \mathbb{T}_\alpha \cap \mathbb{T}_\beta \rightarrow \text{Spin}^c(Y)$.

The differential counts pseudo-holomorphic representatives of Maslov index 1 flowlines in $\text{Sym}^{g+n-1}(\Sigma)$ via the formula

$$\partial(\mathbf{x}) = \sum_{\substack{\phi \in \pi_2(\mathbf{x}, \mathbf{y}) \\ \mu(\phi)=1}} \#(\mathcal{M}(\phi)/\mathbb{R}) U_1^{n_{w_1}(\phi)} \cdots U_n^{n_{w_n}(\phi)} \cdot \mathbf{y}, \quad (2.1)$$

extended equivariantly over $\mathbb{F}[U_1, \dots, U_n]$.

We will frequently make use of Lipshitz's *cylindrical reformulation* of Heegaard Floer homology [23]. In this formulation, the differential counts holomorphic curves of potentially higher genus in $\Sigma \times [0, 1] \times \mathbb{R}$. We refer the reader to [23] for additional background.

Definition 2.4 (Cf. [39, Section 4.2]). We make the following definitions regarding admissibility of Heegaard diagrams:

- (1) A *periodic domain* on a Heegaard multi-diagram $\mathcal{H} = (\Sigma, \boldsymbol{\gamma}_1, \dots, \boldsymbol{\gamma}_n, \mathbf{w})$ is a 2-chain P on Σ such that ∂P is an integral linear combination of the curves from $\boldsymbol{\gamma}_1, \dots, \boldsymbol{\gamma}_n$ and $n_\mathbf{w}(P) = 0$.
- (2) A Heegaard diagram for a pointed 3-manifold $\mathcal{H} = (\Sigma, \alpha, \beta, \mathbf{w})$ is *weakly admissible* if each non-trivial periodic domain has both positive and negative multiplicities.
- (3) If $\mathcal{H} = (\Sigma, \alpha, \beta, \mathbf{w}, \mathbf{z}, \mathbf{p})$ is a Heegaard link diagram with free base points, then we say that \mathcal{H} is weakly admissible if $(\Sigma, \alpha, \beta, \mathbf{q})$ is weakly admissible

for each collection $\mathbf{q} \subseteq \mathbf{w} \cup \mathbf{z} \cup \mathbf{p}$ such that \mathbf{q} contains all of the free base points and exactly one link base point from each link component.

- (4) We say that a Heegaard multi-diagram $(\Sigma, \boldsymbol{\gamma}_1, \dots, \boldsymbol{\gamma}_n, \mathbf{q})$ is *weakly admissible* if each periodic domain has positive and negative multiplicities. A Heegaard multi-diagram with link base points $(\Sigma, \boldsymbol{\gamma}_1, \dots, \boldsymbol{\gamma}_n, \mathbf{w}, \mathbf{z}, \mathbf{p})$ is called weakly admissible if it is weakly admissible for each collection $\mathbf{q} \subseteq \mathbf{w} \cup \mathbf{z} \cup \mathbf{p}$ as above.

Given a weakly admissible Heegaard diagram \mathcal{H} , we can define a completed version of the Heegaard Floer homology $\mathbf{CF}^-(\mathcal{H})$, which is freely generated by $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ (with no restriction on Spin^c structures) over the ring $\mathbb{F}[[U_1, \dots, U_n]]$.

We now discuss link Floer homology. In this memoir, we use a slightly different version of link Floer homology than appeared in [41]. If $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{w}, \mathbf{z})$ is a Heegaard link diagram with no free base points and with $|\mathbf{w}| = |\mathbf{z}| = \ell$, we use a version of the link Floer complex which is a free, finitely generated chain complex over the ring $\mathbb{F}[[U_1, \dots, U_\ell, V_1, \dots, V_\ell]]$. This version was considered in [53] and [55]. We define $\mathcal{CF}\mathcal{L}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{w}, \mathbf{z}, \mathfrak{s})$ to be generated over $\mathbb{F}[[U_1, \dots, U_\ell, V_1, \dots, V_\ell]]$ by intersection points $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ such that $\mathfrak{s}_{\mathbf{w}}(\mathbf{x}) = \mathfrak{s}$. The differential is similar to (2.1), except that we weight a holomorphic curve by

$$\mathcal{U}_1^{n_{w_1}(\phi)} \mathcal{V}_1^{n_{z_1}(\phi)} \dots \mathcal{U}_\ell^{n_{w_\ell}(\phi)} \mathcal{V}_\ell^{n_{z_\ell}(\phi)}.$$

In the case of a doubly pointed Heegaard knot diagram, $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, w, z)$, we write $\mathcal{CF}\mathcal{K}(\mathcal{H}, \mathfrak{s})$ for the complex which is freely generated over $\mathbb{F}[[U, V]]$ by intersection points $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$.

2.2 Hypercubes and hyperboxes of chain complexes

We now describe Manolescu and Ozsváth's algebraic formalism of *hypercubes* and *hyperboxes* of chain complexes [32, Section 5]. Define

$$\mathbb{E}_n := \{0, 1\}^n \subseteq \mathbb{R}^n,$$

with the convention that $\mathbb{E}_0 = \{0\}$. If $\mathbf{d} = (d_1, \dots, d_n) \in \mathbb{Z}^{\geq 0}$, we define

$$\mathbb{E}(\mathbf{d}) := \{(i_1, \dots, i_n) \in \mathbb{Z}^n : 0 \leq i_j \leq d_j, \forall j\}.$$

If $\varepsilon, \varepsilon' \in \mathbb{E}(\mathbf{d})$, we write $\varepsilon \leq \varepsilon'$ if the inequality holds for each coordinate of ε and ε' . We write $\varepsilon < \varepsilon'$ if $\varepsilon \leq \varepsilon'$ and strict inequality holds at one or more coordinates.

Definition 2.5. An n -dimensional hypercube of chain complexes $(C_\varepsilon, D_{\varepsilon, \varepsilon'})_{\varepsilon \in \mathbb{E}_n}$ consists of the following:

- (1) A group C_ε for each $\varepsilon \in \mathbb{E}_n$.
- (2) For each pair of indices $\varepsilon, \varepsilon' \in \mathbb{E}_n$ such that $\varepsilon \leq \varepsilon'$, a linear map

$$D_{\varepsilon, \varepsilon'}: C_\varepsilon \rightarrow C_{\varepsilon'}.$$

Furthermore, we assume the following compatibility condition holds whenever $\varepsilon \leq \varepsilon''$:

$$\sum_{\substack{\varepsilon' \in \mathbb{E}_n, \\ \varepsilon \leq \varepsilon' \leq \varepsilon''}} D_{\varepsilon', \varepsilon''} \circ D_{\varepsilon, \varepsilon'} = 0. \quad (2.2)$$

Definition 2.6. If $\mathbf{d} \in (\mathbb{Z}^{>0})^n$, a *hyperbox of chain complexes of size \mathbf{d}* consists of a collection of groups $(C_\varepsilon)_{\varepsilon \in \mathbb{E}(\mathbf{d})}$, together with a linear map $D_{\varepsilon, \varepsilon'}$ whenever $\varepsilon \leq \varepsilon'$ and $|\varepsilon' - \varepsilon|_{L^\infty} \leq 1$. Furthermore, we assume that the analog of (2.2) holds whenever $\varepsilon \leq \varepsilon''$ and $|\varepsilon'' - \varepsilon|_{L^\infty} \leq 1$.

Hypercubes and hyperboxes of the same size form dg-categories. If $\mathcal{C} = (C_\varepsilon, D_{\varepsilon, \varepsilon'})_{\varepsilon \in \mathbb{E}_n}$ and $\mathcal{C}' = (C'_{\varepsilon}, D'_{\varepsilon, \varepsilon'})_{\varepsilon \in \mathbb{E}_n}$ are two hypercubes of the same dimension, then a *morphism of hypercubes* F consists of a collection of linear maps

$$F_{\varepsilon, \varepsilon'}: C_\varepsilon \rightarrow C'_{\varepsilon'},$$

ranging over all $\varepsilon, \varepsilon' \in \mathbb{E}_n$ such that $\varepsilon \leq \varepsilon'$. We write $\text{Hom}(\mathcal{C}, \mathcal{C}')$ for the space of morphisms. There is a natural morphism differential ∂_{Mor} on $\text{Hom}(\mathcal{C}, \mathcal{C}')$, given by

$$\partial_{\text{Mor}}(F)_{\varepsilon, \varepsilon''} = \sum_{\substack{\varepsilon' \in \mathbb{E}_n, \\ \varepsilon \leq \varepsilon' \leq \varepsilon''}} D'_{\varepsilon', \varepsilon''} \circ F_{\varepsilon, \varepsilon'} + F_{\varepsilon', \varepsilon''} \circ D_{\varepsilon, \varepsilon'}. \quad (2.3)$$

Remark 2.7. In Section 3.6, we will give an alternate perspective on hypercubes, and show that the category of hypercubes of chain complexes is equivalent to the category of type-D modules over a certain algebra.

2.3 Compressing hyperboxes

Manolescu and Ozsváth [32, Section 5] describe an operation called *compression* which takes a hyperbox $\mathcal{C} = (C_\varepsilon, D_{\varepsilon, \varepsilon'})_{\varepsilon \in \mathbb{E}(\mathbf{d})}$ of size $\mathbf{d} = (d_1, \dots, d_n)$ and returns an n -dimensional hypercube $\widehat{\mathcal{C}} = (\widehat{C}_\varepsilon, \widehat{D}_{\varepsilon, \varepsilon'})_{\varepsilon \in \mathbb{E}_n}$. The underlying complexes are given by the formula

$$\widehat{C}_{(\varepsilon_1, \dots, \varepsilon_n)} = C_{(d_1 \varepsilon_1, \dots, d_n \varepsilon_n)}.$$

The hypercube structure maps are more complicated. We first illustrate the construction in the 1-dimensional case. Suppose \mathcal{C} is the 1-dimensional hyperbox of chain complexes

$$C_0 \xrightarrow{f_{0,1}} C_1 \xrightarrow{f_{1,2}} \dots \xrightarrow{f_{n-1,n}} C_n.$$

The compression of the above hyperbox is the hypercube

$$C_0 \xrightarrow{f_{n-1,n} \circ \cdots \circ f_{0,1}} C_n.$$

In Section 3.6, we give an alternate description of compression in the 1-dimensional case in terms of box-tensor products of DA bimodules over a certain algebra.

The same description also works for hyperboxes of size $(1, \dots, 1, d)$. That is, we view \mathcal{C} as a 1-dimensional hyperbox where the complex at each point of $\mathbb{E}(d)$ is a hypercube of dimension $n - 1$. Compression in this case is given by composition of hypercube morphisms.

For a hyperbox \mathcal{C} of size (d_1, \dots, d_n) , the compression may be defined by iterating the above construction. Doing so requires a choice of ordering of the axis directions. For concreteness, we pick the standard ordering of $1, \dots, n$. Given a hyperbox \mathcal{C} we define the partial compression of the i -th axis direction, denoted $c_i(\mathcal{C})$, to be the following hyperbox of size $(d_1, \dots, d_{i-1}, 1, d_{i+1}, \dots, d_n)$. We view the \mathcal{C} as a 1-dimensional hyperbox (of hyperboxes) of size (d_i) , where the complex at each point in $\mathbb{E}(d_i)$ is a hyperbox of size $(d_1, \dots, d_{i-1}, d_{i+1}, \dots, d_n)$. The hyperbox at point $j \in \{0, \dots, d_i\}$ has an underlying group consisting of the sum of all complexes at points $\varepsilon \in \mathbb{E}(\mathbf{d})$ with $\varepsilon_i = j$. We apply the composition perspective of compression for 1-dimensional hypercubes to obtain $c_i(\mathcal{C})$. We then define the compression

$$\widehat{\mathcal{C}} = (c_n \circ \cdots \circ c_1)(\mathcal{C}).$$

Compare [30, Section 4.1.2].

Of course, we can compress the hyperbox \mathcal{C} using any ordering of the axis directions. The following is well known, and we do not claim originality for the following, though we include it for completeness.

Lemma 2.8. *If \mathcal{C} is a hypercube of chain complexes and $\widehat{\mathcal{C}}$ is its compression, then $\widehat{\mathcal{C}}$ is independent up to homotopy equivalence of hypercubes from the choice of ordering of the axis directions.*

Proof. Given the inductive nature of the construction, it suffices to prove the case when $n = 2$. Let \mathcal{C} be a hyperbox of size (d_1, d_2) . Write $c_2(c_1(\mathcal{C}))$ and $c_1(c_2(\mathcal{C}))$ for the two compressions of \mathcal{C} . Note that the length 1 maps of $c_2(c_1(\mathcal{C}))$ and $c_1(c_2(\mathcal{C}))$ coincide, so it suffices to consider the diagonal maps. Since we are considering hypercubes of dimension 2, it suffices to show that the two diagonal maps are themselves chain homotopic since such a chain homotopy can be used to build a morphism of hypercubes $F: c_2(c_1(\mathcal{C})) \rightarrow c_1(c_2(\mathcal{C}))$ which is a homotopy equivalence. The components of F which preserve cube points act by the identity. We define F to have no components which increase only one cube point. We will define F to have a component from cube point $(0, 0)$ to cube point $(1, 1)$, which acts by the above chain homotopy between the length 2 maps of $c_2(c_1(\mathcal{C}))$ and $c_1(c_2(\mathcal{C}))$. Assuming we can

construct such a chain homotopy, the map F is clearly a homotopy equivalence. We now describe how to construct this chain homotopy.

Write $C_*(P_2)$ for the following chain complex over $\mathbb{F} = \mathbb{Z}/2$. There are two generators in degree 0, denoted e_1e_2 and e_2e_1 . We define $\partial(e_ie_j) = 0$. There is a single generator in degree 1, denoted $h_{1,2}$, which satisfies

$$\partial h = e_1e_2 + e_2e_1.$$

We view e_1e_2 and e_2e_1 as the paths along the boundary of a 2-dimensional cube from $(0, 0)$ to $(1, 1)$, and we think of $h_{1,2}$ as the diagonal. We think of $C_*(P_2)$ as the cell complex for the 2-dimensional permutohedron.

We now define another chain complex $C_*(P_{d_1, d_2})$. The generators consist of concatenations of e_1 , e_2 and $h_{1,2}$ such that the total number of 1 subscripts appearing is d_1 , and the total number of 2 subscripts appearing is d_2 . The differential sums over all ways of breaking a single $h_{1,2}$ into $(e_1e_2 + e_2e_1)$. We think of the generators of $C_*(P_{d_1, d_2})$ as corresponding to paths in $[0, d_1] \times [0, d_2]$ from $(0, 0)$ to (d_1, d_2) which consist of concatenations of three types of segments (horizontal unit length segments, vertical unit length segments, or diagonal segments of L^1 -length 2).

Note that the hyperbox structure maps of \mathcal{C} give a chain map

$$C_*(P_{d_1, d_2}) \rightarrow \text{Hom}(C_{0,0}, C_{d_1, d_2}).$$

The two compression procedures give chain maps

$$c_{1,2}, c_{2,1}: C_*(P_2) \rightarrow C_*(P_{d_1, d_2}).$$

Note that $c_{i,j}(e_1e_2) = e_1^{d_1}e_2^{d_2}$ and $c_{i,j}(e_2e_1) = e_2^{d_2}e_1^{d_1}$ for both choices of i, j . In order to show that the diagonals of $\widehat{\mathcal{C}}_{1,2}$ and $\widehat{\mathcal{C}}_{2,1}$ are chain homotopic, it suffices to show that the maps $c_{1,2}$ and $c_{2,1}$ are themselves chain homotopic. To do this, it suffices to show that the homology of P_{d_1, d_2} satisfies

$$H_n(P_{d_1, d_2}) \cong \begin{cases} \mathbb{F} & \text{if } n = 0, \\ 0 & \text{if } n > 0. \end{cases}$$

To establish this, we describe a filtration $\mathcal{F}_{\leq n}$ on $C_*(P_{d_1, d_2})$. Given an arrow sequence $\omega \in P_{d_1, d_2}$, view ω as giving an embedded piecewise linear curve in $[0, d_1] \times [0, d_2] \subseteq \mathbb{R}^2$. Consider the region R between ω and the arrow sequence $e_1^{d_1}e_2^{d_2}$. We define the filtration level of ω to be the number of unit squares in $[0, d_1] \times [0, d_2]$ with integral corners and whose interiors intersect R non-trivially. Note that the subspace $\mathcal{F}_{\leq n}$ is preserved by ∂ .

We now define a homotopy $H: C_*(P_{d_1, d_2}) \rightarrow C_{*+1}(P_{d_1, d_2})$. The map H sums over all ways of replacing one subword e_2e_1 with $h_{1,2}$. We observe that $\mathbb{I} + [\partial, H]$ acts by the identity on $e_1^{d_1}e_2^{d_2}$. We leave it to the reader to verify that if ω is a word

with filtration level $n > 0$, then $(\mathbb{I} + [\partial, H])(\omega)$ is a sum of arrow words of filtration level at most $n - 1$. Therefore, $(\mathbb{I} + [\partial, H])^N$ gives a homotopy equivalence between $C_*(P_{d_1, d_2})$ and \mathbb{F} (concentrated in degree 0) for $N \gg 0$. This completes the proof. ■

2.4 A_∞ -categories and twisted complexes

We now recall the notion of a *twisted complex* in an A_∞ -category. We refer to [50, Section 3.1] for more background on this construction.

We recall that an A_∞ -category \mathcal{C} consists of a set of objects $\mathcal{O}(\mathcal{C})$, a collection of morphisms $\text{Hom}(L, L')$ for $L, L' \in \mathcal{O}(\mathcal{C})$, together with a collection of higher compositions $(\mu_n)_{n \geq 1}$, as follows. We assume that if $L_0, L_1 \in \mathcal{O}(\mathcal{C})$, then $\text{Hom}(L_0, L_1)$ is a vector space. We assume that for each sequence $L_0, \dots, L_n \in \mathcal{O}(\mathcal{C})$, the map μ_n takes the form

$$\mu_n: \text{Hom}(L_0, L_1) \otimes \cdots \otimes \text{Hom}(L_{n-1}, L_n) \rightarrow \text{Hom}(L_0, L_1).$$

Furthermore, we assume that for all $\mathbf{x}_1, \dots, \mathbf{x}_n$ (where $\mathbf{x}_i \in \text{Hom}(L_{i-1}, L_i)$) the following associativity condition holds:

$$\sum_{i=1}^n \sum_{j=i}^n \mu_{n-j+i}(\mathbf{x}_1, \dots, \mathbf{x}_{i-1}, \mu_{j-i+1}(\mathbf{x}_i, \dots, \mathbf{x}_j), \mathbf{x}_{j+1}, \dots, \mathbf{x}_n) = 0.$$

If \mathcal{C} is an A_∞ -category, the *additive enlargement* $\Sigma\mathcal{C}$ of \mathcal{C} is defined as follows. The objects of $\Sigma\mathcal{C}$ consist of formal sums of the form $\bigoplus_{i \in I} X_i \otimes V_i$. Here, X_i denote objects of \mathcal{C} , V_i denote graded vector spaces over $\mathbb{F} = \mathbb{Z}/2$, and I denotes a finite index set. The symbols \otimes and \bigoplus are just notation. (For our purposes, we can take $V_i = \mathbb{F}$ for all i .)

A morphism in $\Sigma\mathcal{C}$ from $\bigoplus_{i \in I} X_i \otimes V_i$ to $\bigoplus_{j \in J} Y_j \otimes W_j$ consists of a collection $(\theta_{i,j} \otimes f_{i,j})_{i \in I, j \in J}$ where $\theta_{i,j}: X_i \rightarrow X_j$ is a morphism in \mathcal{C} and $f_{i,j}: V_i \rightarrow W_j$ is a linear map of vector spaces. We will typically write such a morphism as $\sum_{(i,j) \in I \times J} \theta_{i,j} \otimes f_{i,j}$.

If \mathcal{C} is an A_∞ -category, then $\Sigma\mathcal{C}$ is naturally also an A_∞ -category, with compositions

$$\begin{aligned} \mu_n^{\Sigma\mathcal{C}} & \left(\sum_{i_0, i_1} \theta_{i_0, i_1} \otimes f_{i_0, i_1}, \dots, \sum_{i_{n-1}, i_n} \theta_{i_{n-1}, i_n} \otimes f_{i_{n-1}, i_n} \right) \\ & := \sum_{i_0, i_1, \dots, i_n} \mu_n^{\mathcal{C}}(\theta_{i_0, i_1}, \dots, \theta_{i_{n-1}, i_n}) \otimes (f_{i_{n-1}, i_n} \circ \cdots \circ f_{i_0, i_1}). \end{aligned}$$

It is straightforward to verify that $\Sigma\mathcal{C}$ is an A_∞ -category.

Finally, we define the category of twisted complexes $\text{Tw } \mathcal{C}$. A *twisted complex* (X, δ) in \mathcal{C} consists of an object $X = (X_i, V_i)_{i \in I}$ in $\Sigma\mathcal{C}$ where I is a finite partially

ordered set, together with an endomorphism $\delta \in \text{Hom}_{\Sigma\mathcal{C}}(X, X)$. If $\delta = \sum_{i,j} \delta_{i,j} \otimes f_{i,j}$, we assume that $\delta_{i,j} = 0$ unless $i < j$. Furthermore, we assume the following version of the Maurer–Cartan equation is satisfied:

$$\sum_{n \geq 1} \mu_n^{\Sigma\mathcal{C}}(\underbrace{\delta, \dots, \delta}_n) = 0.$$

A morphism in $\text{Tw}\mathcal{C}$ is the same as a morphism in $\Sigma\mathcal{C}$. Finally, we note that $\text{Tw}\mathcal{C}$ is itself an A_∞ -category, with compositions

$$\mu_n^{\text{Tw}}(\theta_1, \dots, \theta_n) := \sum_{i_0, \dots, i_n \geq 0} \mu_{n+i_0+\dots+i_n}^{\Sigma\mathcal{C}}(\underbrace{\delta, \dots, \delta}_{i_0}, \theta_1, \underbrace{\delta, \dots, \delta}_{i_1}, \theta_2, \dots, \theta_n, \underbrace{\delta, \dots, \delta}_{i_n}).$$

2.5 Hypercubes of attaching curves

The notion of a hypercube of chain complexes adapts to other categories. We now describe the appropriate notion of a hypercube in the Fukaya category. See [32, Section 8] and [24, Section 3], where such objects are referred to as *hyperboxes of Heegaard diagrams*, and *chain complexes of attaching curves*, respectively. These may naturally be viewed as twisted complexes in the Fukaya category, using the framework described in Section 2.4.

Definition 2.9. *A hypercube of handleslide equivalent attaching curves*

$$\mathcal{L}_\beta = (\beta_\varepsilon, \Theta_{\varepsilon, \varepsilon'})_{\varepsilon \in \mathbb{E}_n}$$

on (Σ, \mathbf{w}) consists of a collection of beta attaching curves β_ε , which are each pairwise related by a sequence of handleslides and isotopies, together with distinguished morphisms $\Theta_{\varepsilon, \varepsilon'} \in \mathbf{CF}^-(\Sigma, \beta_\varepsilon, \beta_{\varepsilon'}, \mathbf{w})$ whenever $\varepsilon < \varepsilon'$. We assume, furthermore, that the Heegaard multi-diagram containing all 2^n beta curves is weakly admissible (see Definition 2.4). Finally, we assume that for each pair $\varepsilon < \varepsilon' \in \mathbb{E}_n$, the following relation is satisfied:

$$\sum_{\varepsilon = \varepsilon_1 < \dots < \varepsilon_n = \varepsilon'} f_{\beta_{\varepsilon_1}, \dots, \beta_{\varepsilon_n}}(\Theta_{\varepsilon_1, \varepsilon_2}, \dots, \Theta_{\varepsilon_{n-1}, \varepsilon_n}) = 0. \quad (2.4)$$

In the above, $f_{\beta_{\varepsilon_1}, \dots, \beta_{\varepsilon_n}}$ denotes the holomorphic polygon counting map.

Sometimes in Heegaard Floer theory it is helpful to have a notion of a hypercube of *beta* attaching curves, or a hypercube of *alpha* attaching curves. We define a hypercube of beta attaching curves to be the object described in Definition 2.9, however, we define a *hypercube of alpha attaching curves* as the following modification: for each $\varepsilon < \varepsilon'$ we have a chain $\Theta_{\varepsilon', \varepsilon} \in \mathbf{CF}^-(\Sigma, \alpha_{\varepsilon'}, \alpha_\varepsilon, \mathbf{w})$, and the compatibility condition in

equation (2.4) holds, as long as we write the indices in the opposite order. Clearly any hypercube of attaching curves can be viewed as either a hypercube of alpha attaching curves or beta attaching curves (by possibly reversing the order of indices), but the terminology is occasionally helpful.

If \mathcal{L}_α and \mathcal{L}_β are hypercubes of handleslide equivalent attaching curves on (Σ, \mathbf{w}) , of dimension m and n respectively, then there is an $(n + m)$ -dimensional hypercube of chain complexes

$$(C_{(\varepsilon, \nu)}, D_{(\varepsilon, \nu), (\varepsilon', \nu')})_{(\varepsilon, \nu) \in \mathbb{E}_n \times \mathbb{E}_m} = \mathbf{CF}^-(\Sigma, \mathcal{L}_\alpha, \mathcal{L}_\beta, \mathbf{w}),$$

whenever the diagram containing all attaching curves is weakly admissible. For each $(\varepsilon, \nu) \in \mathbb{E}_{n+m}$, we set

$$C_{(\varepsilon, \nu)} = \mathbf{CF}^-(\Sigma, \alpha_\varepsilon, \beta_\nu, \mathbf{w}).$$

The structure maps are given by the following formula:

$$\begin{aligned} & D_{(\varepsilon, \nu), (\varepsilon', \nu')}(\mathbf{x}) \\ &= \sum_{\substack{\varepsilon = \varepsilon_1 < \dots < \varepsilon_i = \varepsilon', \\ \nu = \nu_1 < \dots < \nu_j = \nu'}} f_{\alpha_{\varepsilon_i}, \dots, \alpha_{\varepsilon_1}, \beta_{\nu_1}, \dots, \beta_{\nu_j}} (\Theta_{\varepsilon_i, \varepsilon_{i-1}}, \dots, \Theta_{\varepsilon_2, \varepsilon_1}, \mathbf{x}, \Theta_{\nu_1, \nu_2}, \dots, \Theta_{\nu_{j-1}, \nu_j}). \end{aligned}$$

It is straightforward to see that $\mathbf{CF}^-(\Sigma, \mathcal{L}_\alpha, \mathcal{L}_\beta, \mathbf{w})$ is a hypercube of chain complexes.

The case when there are link base points \mathbf{w}, \mathbf{z} base points is a straightforward modification of the above construction, using variables \mathcal{U}_i and \mathcal{V}_i in the definition of the Floer complexes.

We observe that a hypercube of attaching curves $\mathcal{L} = (\gamma_\varepsilon, \theta_{\varepsilon, \varepsilon'})_{\varepsilon \in \mathbb{E}_n}$ may naturally be interpreted as a twisted complex of Lagrangians in the Fukaya category. The index set I in the definition of a twisted complex is the cube \mathbb{E}_n , and the partial order is the natural ordering of cube points.

Correspondingly, there is a natural notion of a morphism between two hypercubes of attaching curves, and the resulting category is an A_∞ -category. If $\mathcal{L} = (\gamma_\varepsilon, \mathbf{x}_{\varepsilon, \varepsilon'})_{\varepsilon \in \mathbb{E}_n}$ and $\mathcal{L}' = (\gamma'_{\varepsilon'}, \mathbf{y}_{\varepsilon, \varepsilon'})_{\varepsilon \in \mathbb{E}_n}$ are n -dimensional hypercubes of attaching curves, then a morphism Φ from \mathcal{L} to \mathcal{L}' consists of a collection of chains $\Phi_{\varepsilon, \varepsilon'} \in \mathbf{CF}^-(\gamma_\varepsilon, \gamma'_{\varepsilon'}, \mathbf{w})$ ranging over all $\varepsilon, \varepsilon' \in \mathbb{E}_n$ with $\varepsilon \leq \varepsilon'$. Of course, the set of morphisms is none other than the Floer complex $\mathbf{CF}^-(\Sigma, \mathcal{L}, \mathcal{L}', \mathbf{w})$.

Suppose $\mathcal{L}_1, \dots, \mathcal{L}_n$ are n -dimensional hypercubes of attaching curves, where $\mathcal{L}_i = (\gamma_{i, \varepsilon}, (\Theta_i)_{\varepsilon, \varepsilon'})_{\varepsilon \in \mathbb{E}_n}$. The composition operator

$$\mu_n^{\text{Tw}}: \mathbf{CF}^-(\mathcal{L}_1, \mathcal{L}_2) \otimes \dots \otimes \mathbf{CF}^-(\mathcal{L}_{n-1}, \mathcal{L}_n) \rightarrow \mathbf{CF}^-(\mathcal{L}_1, \mathcal{L}_n)$$

is given as follows. If $\Phi_{i,i+1} \in \mathbf{CF}^-(\mathcal{L}_i, \mathcal{L}_{i+1})$, then

$$\begin{aligned} & \mu_n^{\text{Tw}}(\Phi_{1,2}, \dots, \Phi_{n-1,n}) \\ &= \sum_{i_1, \dots, i_n \geq 0} \mu_{i_1 + \dots + i_n + n} \left(\underbrace{\Theta_1, \dots, \Theta_1}_{i_1}, \Phi_{1,2}, \underbrace{\Theta_2, \dots, \Theta_2}_{i_2}, \right. \\ & \quad \left. \Phi_{2,3}, \dots, \Phi_{n-1,n}, \underbrace{\Theta_n, \dots, \Theta_n}_{i_n} \right). \end{aligned}$$

In the above, $\Theta_i = \sum_{\varepsilon < \varepsilon'} (\Theta_i)_{\varepsilon, \varepsilon'}$, viewed as an element in the group $\bigoplus_{\varepsilon < \varepsilon'} \mathbf{CF}^-(\gamma_{i,\varepsilon}, \gamma_{i,\varepsilon'})$, and similarly for $\Phi_{i,i+1}$. Additionally, $\mu_{i_1 + \dots + i_n + n}$ is declared to vanish on non-composable Floer chains, e.g., $\mu_2(\mathbf{x}, \mathbf{y}) = 0$ if $\mathbf{x} \in \mathbf{CF}^-(\gamma_0, \gamma_1)$ and $\mathbf{y} \in \mathbf{CF}^-(\gamma'_1, \gamma_2)$ where $\gamma_1 \neq \gamma'_1$.

2.6 More hypercubes

We now collect several additional constructions and basic properties. Firstly, if $\mathcal{C} = (C_\varepsilon, D_{\varepsilon, \varepsilon'})_{\varepsilon \in \mathbb{E}_n}$ and $\mathcal{C}' = (C'_v, D'_{v, v'})_{v \in \mathbb{E}_m}$ are two hypercubes of chain complexes over a ring R , we may define their *external tensor product*

$$\mathcal{E} := \mathcal{C} \otimes_R \mathcal{C}',$$

as follows. The hypercube $\mathcal{E} = (E_{(\varepsilon, v)}, \delta_{(\varepsilon, v), (\varepsilon', v')})_{(\varepsilon, v) \in \mathbb{E}_n \times \mathbb{E}_m}$ is the tensor product in the ordinary sense: As groups, we set

$$E_{(\varepsilon, v)} = C_\varepsilon \otimes_R C'_v$$

and the structure maps are given by

$$\delta_{(\varepsilon, v), (\varepsilon', v')} = \begin{cases} D_{\varepsilon, \varepsilon'} \otimes \mathbb{I} & \text{if } v = v', \\ \mathbb{I} \otimes D'_{v, v'} & \text{if } \varepsilon = \varepsilon', \\ 0 & \text{if } \varepsilon < \varepsilon' \text{ and } v < v'. \end{cases}$$

Additionally, if \mathcal{C} and \mathcal{C}' are two hyperboxes of size $\mathbf{d}_0 \times \{d\}$ and $\mathbf{d}_0 \times \{d'\}$, such that the restrictions $\mathcal{C}|_{\mathbb{E}(\mathbf{d}_0) \times \{d\}}$ and $\mathcal{C}'|_{\mathbb{E}(\mathbf{d}_0) \times \{0\}}$ coincide (in both underlying groups and structure morphisms), then we may stack \mathcal{C} and \mathcal{C}' to obtain a hyperbox of size $\mathbf{d}_0 \times \{d + d'\}$, which we denote $\text{St}(\mathcal{C}, \mathcal{C}')$. The following lemma records some interactions between stacking and the external tensor product.

Lemma 2.10. *The operations of stacking and compressing satisfy the following:*

- (1) *Stacking hyperboxes of attaching curves commutes with pairing: If \mathcal{L}_α , \mathcal{L}_β and $\mathcal{L}_{\beta'}$ are hyperboxes of attaching curves, and \mathcal{L}_β and $\mathcal{L}_{\beta'}$ may be stacked along a codimension 1 face, then*

$$\text{St}(\mathbf{CF}^-(\mathcal{L}_\alpha, \mathcal{L}_\beta), \mathbf{CF}^-(\mathcal{L}_\alpha, \mathcal{L}_{\beta'})) \cong \mathbf{CF}^-(\mathcal{L}_\alpha, \text{St}(\mathcal{L}_\beta, \mathcal{L}_{\beta'})).$$

- (2) *Stacking hyperboxes of chain complexes commutes with tensoring hyperboxes: If \mathcal{C} and \mathcal{C}' are stackable hyperboxes of chain complexes, and \mathcal{D} is another hyperbox of chain complexes, then*

$$\mathrm{St}(\mathcal{C}, \mathcal{C}') \otimes \mathcal{D} \cong \mathrm{St}(\mathcal{C}, \mathcal{D}) \otimes \mathrm{St}(\mathcal{C}', \mathcal{D}).$$

- (3) *Compressing hyperboxes of chain complexes commutes with tensor products of hyperboxes. If \mathcal{C} and \mathcal{D} are hyperboxes of chain complexes, and $\widehat{\mathcal{C}}$ denotes compression, then there is a choice of ordering of the axis directions used to compute the compression so that*

$$\widehat{\mathcal{C}} \otimes \widehat{\mathcal{D}} \cong \widehat{\mathcal{C} \otimes \mathcal{D}}.$$

Proof. Claim (1) is essentially the definition of the pairing between hyperboxes. Similarly, (2) is clear. For claim (3), we use the inductive perspective on compression from Section 2.3. Suppose that \mathcal{C} is a hyperbox of size (d_1, \dots, d_n) and \mathcal{D} is a hyperbox of size (d'_1, \dots, d'_m) . We use the partial compression operations c described in Section 2.3. To show the main claim, it suffices to show that

$$c_i(\mathcal{C} \otimes \mathcal{D}) \cong c_i(\mathcal{C}) \otimes \mathcal{D} \quad \text{and} \quad c_{j+n}(\mathcal{C} \otimes \mathcal{D}) = \mathcal{C} \otimes c_j(\mathcal{D}) \quad (2.5)$$

if $1 \leq i \leq n$ and $1 \leq j \leq m$. We focus on the equation $c_i(\mathcal{C} \otimes \mathcal{D}) \cong c_i(\mathcal{C}) \otimes \mathcal{D}$ when $1 \leq i \leq n$. Recall that the differential on $\mathcal{C} \otimes \mathcal{D}$ is the tensor product $D_{\mathcal{C}} \otimes \mathbb{I} + \mathbb{I} \otimes D_{\mathcal{D}}$. Therefore, if we view $\mathcal{C} \otimes \mathcal{D}$ as a hyperbox of size (d_i) , then only the components of $D_{\mathcal{C}} \otimes \mathbb{I}$ will increase the $\mathbb{E}(d_i)$ index, while the components of $\mathbb{I} \otimes D_{\mathcal{D}}$ will preserve the $\mathbb{E}(d_i)$ index. From this observation, equation (2.5) is straightforward, and the main claim follows. ■

A more sophisticated extension of Lemma 2.10 concerns commuting a tensor product operation on attaching curves and the pairing operation on hypercubes of attaching curves. See Propositions 10.1 and 11.8.

