

## Chapter 3

### $A_\infty$ -modules

In this chapter, we recall some background about  $A_\infty$ -modules. We follow the notation of [25, 27]. See [19, 20] for further background and historical context.

#### 3.1 Type-A and type-D modules

In this section and throughout the memoir, we work over characteristic 2.

**Definition 3.1.** Suppose  $\mathcal{A}$  is an associative algebra. A *left  $A_\infty$ -module*  ${}_{\mathcal{A}}M$  over  $\mathcal{A}$  is a left  $\mathbf{k}$ -module  $M$  equipped with a  $\mathbf{k}$ -module map

$$m_{j+1}: \mathcal{A}^{\otimes j} \otimes_{\mathbf{k}} M \rightarrow M$$

for each  $j \geq 0$ , such that if  $a_n, \dots, a_1 \in \mathcal{A}$  and  $\mathbf{x} \in M$ , then

$$\begin{aligned} & \sum_{j=0}^n m_{n-j+1}(a_n, \dots, a_{j+1}, m_{j+1}(a_j, \dots, a_1, \mathbf{x})) \\ & + \sum_{k=1}^{n-1} m_n(a_n, \dots, a_{k+1}a_k, \dots, a_1, \mathbf{x}) = 0. \end{aligned}$$

An  $A_\infty$ -module  ${}_{\mathcal{A}}M$  is *strictly unital* if  $m_2(1, \mathbf{x}) = \mathbf{x}$  for all  $\mathbf{x} \in M$ , and such that for each  $j > 1$ ,  $m_{j+1}(a_j, \dots, a_1, \mathbf{x}) = 0$  if  $a_i \in \mathbf{k}$  for some  $i$ .

**Definition 3.2.** If  $\mathcal{A}$  is an associative algebra over  $\mathbf{k}$ , a *right type-D module*  $N^{\mathcal{A}}$  is right  $\mathbf{k}$ -module  $N$ , equipped with a  $\mathbf{k}$ -linear structure map

$$\delta^1: N \rightarrow N \otimes_{\mathbf{k}} \mathcal{A},$$

which satisfies

$$(\text{id}_N \otimes \mu_2) \circ (\delta^1 \otimes \text{id}_{\mathcal{A}}) \circ \delta^1 = 0.$$

Note that in the above, we are only considering associative algebras, instead of dg-algebras or  $A_\infty$ -algebras. See [25, Definition 2.2.23] for the general definition.

#### 3.2 Bimodules

We also need to consider  $A_\infty$ -bimodules. Our exposition and notation follows [25], though we restrict to the case of associative algebras (as opposed to dg-algebras or  $A_\infty$ -algebras).

**Definition 3.3.** Suppose that  $\mathcal{A}$  and  $\mathcal{B}$  are algebras over  $\mathbf{j}$  and  $\mathbf{k}$ , respectively. A type-DA bimodule, denoted  ${}_{\mathcal{A}}M^{\mathcal{B}}$ , consists of a  $(\mathbf{j}, \mathbf{k})$ -module  $M$ , equipped with  $(\mathbf{j}, \mathbf{k})$ -linear structure morphisms

$$\delta_{j+1}^1: \mathcal{A}^{\otimes j} \otimes_{\mathbf{j}} M \rightarrow M \otimes_{\mathbf{k}} \mathcal{B},$$

for  $j \geq 0$ . These are required to satisfy the following structure relation:

$$\begin{aligned} & \sum_{j=0}^n ((\text{id}_M \otimes \mu_2) \circ (\delta_{n-j+1}^1 \otimes \text{id}_{\mathcal{B}}))(a_n, \dots, a_{j+1}, \delta_{j+1}^1(a_j, \dots, a_1, \mathbf{x})) \\ & + \sum_{k=1}^{n-1} \delta_n^1(a_n, \dots, a_{k+1}a_k, \dots, a_1, \mathbf{x}) = 0. \end{aligned}$$

### 3.3 The Lipshitz–Ozsváth–Thurston box tensor product

In this section, we recall the *box tensor product* of Lipshitz, Ozsváth and Thurston [27, Section 2.4].

Suppose that  $\mathcal{A}$  is an associative algebra over  $\mathbf{k}$ . If  $N^{\mathcal{A}}$  is a type-D module and  ${}_{\mathcal{A}}M$  is a type-A module, then the *box tensor product*  $N^{\mathcal{A}} \boxtimes {}_{\mathcal{A}}M$  is the chain complex  $N \otimes_{\mathbf{k}} M$ , with differential  $\partial^{\boxtimes}$  as follows. Following [27], it is convenient to make the inductive notation that

$$\delta^j: N \rightarrow N \otimes \mathcal{A}^{\otimes j}$$

is the map given inductively by  $\delta^j = (\delta^1 \otimes \text{id}_{\mathcal{A}^{\otimes j-1}}) \circ \delta^{j-1}$  and  $\delta^0 = \text{id}$ . Then the differential has the formula

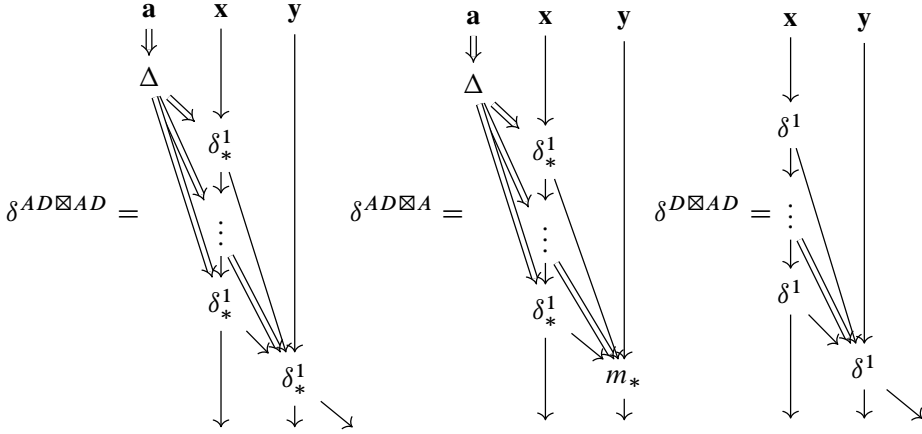
$$\partial^{\boxtimes}(\mathbf{x} \otimes \mathbf{y}) = \sum_{j=0}^{\infty} (\text{id}_N \otimes m_{j+1})(\delta^j(\mathbf{x}), \mathbf{y}).$$

The differential is usually depicted via the following diagram:

$$\begin{array}{ccc} & \mathbf{x} & \mathbf{y} \\ & \downarrow & \downarrow \\ & \delta^1 & \\ & \downarrow & \downarrow \\ \partial^{\boxtimes}(\mathbf{x} \otimes \mathbf{y}) = \sum & \vdots & \\ & \downarrow & \downarrow \\ & \delta^1 & \\ & \downarrow & \downarrow \\ & m_* & \\ & \downarrow & \downarrow \end{array} \tag{3.1}$$

According to [27, Lemma 2.30], the map  $\partial^\boxtimes$  is a differential whenever one of  $M$  or  $N$  satisfies a *boundedness* assumption; see [25, Definitions 2.2.18 and 2.2.29].

There are also tensor products between various types of bimodules. The following diagram indicates schematically the box tensor products of the form  $AD \boxtimes AD$ ,  $AD \boxtimes A$  and  $D \boxtimes AD$ :



### 3.4 Morphisms of modules and bimodules

We now recall the definitions of morphisms of type-D, type-A and type-DA modules and bimodules from [25]. We assume that all algebras are associative algebras with vanishing differential. We refer the reader to [25] for the case of dg or  $A_\infty$ -algebras.

Suppose  $\mathcal{A}$  is an algebra over  $\mathbf{k}$  and  $N^{\mathcal{A}}$  and  $M^{\mathcal{A}}$  are two type-D modules. A morphism  $f^1: N^{\mathcal{A}} \rightarrow M^{\mathcal{A}}$  is a  $\mathbf{k}$ -linear map

$$f^1: N \rightarrow M \otimes_{\mathbf{k}} \mathcal{A}.$$

There is a morphism differential, given by

$$\partial_{\text{Mor}}(f^1) = (\text{id}_M \otimes \mu_2) \circ (f^1 \otimes \text{id}_{\mathcal{A}}) \circ \delta^1 + (\text{id}_M \otimes \mu_2) \circ (\delta^1 \otimes \text{id}_{\mathcal{A}}) \circ f^1. \quad (3.2)$$

The map  $\partial_{\text{Mor}}$  squares to zero. A type-D *homomorphism* (or *cycle*) is a morphism which satisfies  $\partial_{\text{Mor}}(f^1) = 0$ .

A morphism  $f_*$  between two type-A modules  ${}_{\mathcal{A}}M$  and  ${}_{\mathcal{A}}N$  is a collection of  $\mathbf{k}$ -linear maps

$$f_{j+1}: \mathcal{A}^{\otimes j} \otimes_{\mathbf{k}} M \rightarrow N.$$

There is a morphism differential  $\partial_{\text{Mor}}(f_*)$  given by

$$\begin{aligned} \partial_{\text{Mor}}(f_*)_{n+1}(a_n, \dots, a_1, \mathbf{x}) &= \sum_{j=0}^n m_{n-j+1}(a_n, \dots, a_{j+1}, f_{j+1}(a_j, \dots, a_1, \mathbf{x})) \\ &\quad + \sum_{j=0}^n f_{n-j+1}(a_n, \dots, a_{j+1}, m_{j+1}(a_j, \dots, a_1, \mathbf{x})) \\ &\quad + \sum_{i=1}^{n-1} f_n(a_n, \dots, a_{i+1}a_i, \dots, a_1, \mathbf{x}). \end{aligned}$$

Finally, if  ${}_{\mathcal{A}}M^{\mathcal{B}}$  and  ${}_{\mathcal{A}}N^{\mathcal{B}}$  are DA bimodules, then a morphism  $f_*^1$  from  ${}_{\mathcal{A}}M^{\mathcal{B}}$  to  ${}_{\mathcal{A}}N^{\mathcal{B}}$  is a collection of maps

$$f_{j+1}^1: \mathcal{A}^{\otimes j} \otimes M \rightarrow N \otimes \mathcal{B}$$

for  $j \geq 0$ . The morphism differential is given by

$$\begin{aligned} \partial_{\text{Mor}}(f_*^1)_{n+1}(a_n, \dots, a_1, \mathbf{x}) &= \sum_{j=0}^n (\text{id}_N \otimes \mu_2)(\delta_{n-j+1}^1 \otimes \text{id}_{\mathcal{B}})(a_n, \dots, a_{j+1}, f_{j+1}^1(a_j, \dots, a_1, \mathbf{x})) \\ &\quad + \sum_{j=0}^n (\text{id}_N \otimes \mu_2)(f_{n-j+1}^1 \otimes \text{id}_{\mathcal{B}})(a_n, \dots, a_{j+1}, \delta_{j+1}^1(a_j, \dots, a_1, \mathbf{x})) \\ &\quad + \sum_{i=1}^{n-1} f_n^1(a_n, \dots, a_{i+1}a_i, \dots, a_1, \mathbf{x}). \end{aligned}$$

Of course, the morphism differential  $\partial_{\text{Mor}}$  also squares to zero for morphisms of type-A and DA morphisms. See [25, Definition 2.2.43].

If  $f_*^1: {}_{\mathcal{A}}M^{\mathcal{B}} \rightarrow {}_{\mathcal{A}}N^{\mathcal{B}}$  and  $g_*^1: {}_{\mathcal{A}}N^{\mathcal{B}} \rightarrow {}_{\mathcal{A}}Q^{\mathcal{B}}$  are DA bimodule morphisms, Lipshitz–Ozsváth–Thurston define the composition  $f_*^1 \circ g_*^1$  via the following diagram:

$$g_*^1 \circ f_*^1 = \begin{array}{ccc} \mathbf{a} & & \mathbf{x} \\ \Downarrow & & \downarrow \\ \Delta & \searrow & f_*^1 \\ & \searrow & \downarrow \\ & & g_*^1 \\ & & \downarrow \\ & & \mu_2 \end{array} \tag{3.3}$$

In equation (3.3),  $\Delta$  denotes the co-multiplication

$$\Delta: \mathcal{A}^{\otimes n} \rightarrow \bigoplus_{i=0}^n \mathcal{A}^{\otimes i} \otimes \mathcal{A}^{\otimes (n-i)}$$

given by

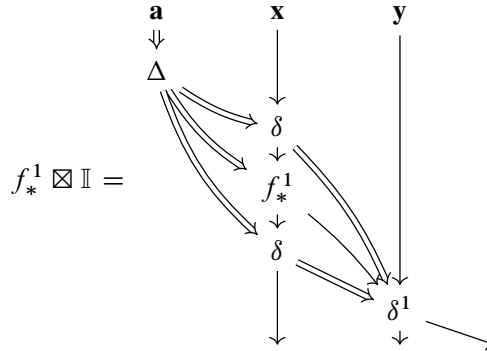
$$\Delta(a_1 | \dots | a_n) = \sum_{i=1}^n (a_1 | \dots | a_i) \otimes (a_{i+1} | \dots | a_n).$$

In the above,  $|$  denotes tensor product.

We now discuss pairing morphisms of modules and bimodules. We focus on the case of DA bimodule morphisms. The other types of pairings which are relevant to our memoir are defined by forgetting about the input algebra or output algebra in the diagrams below. The natural algebraic operation is to pair a bimodule morphism with the identity (as opposed to pairing two morphisms together). Suppose that  ${}_{\mathcal{A}}N^{\mathcal{B}}$  and  ${}_{\mathcal{A}}M^{\mathcal{B}}$  and  ${}_{\mathcal{B}}P^{\mathcal{C}}$  are DA bimodules, and

$$f_*^1: {}_{\mathcal{A}}N^{\mathcal{B}} \rightarrow {}_{\mathcal{A}}M^{\mathcal{B}}$$

is a morphism of DA bimodules. There is a morphism of DA bimodules  $f_*^1 \boxtimes \mathbb{I}: N \boxtimes P \rightarrow M \boxtimes P$ , defined by the following diagram:

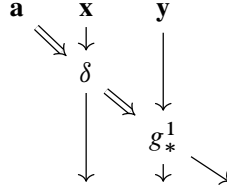


In the above, the map  $\delta$  denotes the sum over  $n \geq 0$  of the maps

$$\delta_{j+1}^n: \underbrace{\mathcal{A} \otimes \dots \otimes \mathcal{A}}_j \otimes M \rightarrow M \otimes \underbrace{\mathcal{B} \otimes \dots \otimes \mathcal{B}}_n$$

obtained by composing  $\delta_*^1$  repeatedly, but not multiplying the  $\mathcal{B}$ -outputs. (Note that by convention  $\delta_*^0 = \mathbb{I}$ .)

If instead  $g_*^1: \mathcal{B}P^{\mathcal{C}} \rightarrow \mathcal{B}Q^{\mathcal{C}}$  is a morphism of type-DA modules, then  $\mathbb{I} \boxtimes g_*^1$  is given by the following diagram:



### 3.5 External tensor products

In this section, we recall a basic algebraic operation called the *external tensor product*. We consider these for type-D and type-DA modules over associative algebras, and we prove some basic properties.

We begin with the setting of type-D modules. Let  $\mathcal{A}$  and  $\mathcal{B}$  algebras over rings  $\mathbf{i}$  and  $\mathbf{j}$  of characteristic 2, and let  $N^{\mathcal{A}} = (N, \delta^1)$  and  $M^{\mathcal{B}} = (M, d^1)$  be type-D modules. Then the *external tensor product*, denoted  $(N \otimes_{\mathbb{F}} M)^{\mathcal{A} \otimes_{\mathbb{F}} \mathcal{B}}$ , is the group  $M \otimes_{\mathbb{F}} N$  equipped with the structure map  $\text{id}_N \otimes 1_{\mathcal{A}} \otimes d^1 + \delta^1 \otimes \text{id}_M \otimes 1_{\mathcal{B}}$  (with tensor factors reordered).

The external tensor product of two type-DA bimodules is more complicated. In general, one must choose a cellular diagonal of the associahedron. We follow the presentation of Lipshitz, Ozsváth and Thurston [27, 28]. The main goal of this section is to give a concrete formula (not involving diagonals of the associahedron) of the external tensor product of two DA bimodules, which we use throughout the memoir. This formula appears in the statement of Proposition 3.7.

Our description is in terms of a more basic operation, which we call *extension of scalars* (and is also inspired by [25]). Suppose that  $\mathcal{A}$ ,  $\mathcal{B}$  and  $\mathcal{C}$  are algebras over rings  $\mathbf{i}$ ,  $\mathbf{j}$  and  $\mathbf{k}$ , respectively. If  ${}_{\mathcal{A}}N^{\mathcal{B}}$  is a bimodule, then we will describe a bimodule  ${}_{\mathcal{A} \otimes \mathcal{C}}\mathcal{E}(N; \mathcal{C})^{\mathcal{B} \otimes \mathcal{C}}$  which we say is formed by extension of scalars. We will relate this operation with an external tensor product with the identity bimodule  ${}_{\mathcal{C}}[\mathbb{I}]^{\mathcal{C}}$  in Proposition 3.18.

As an  $(\mathbf{i} \otimes \mathbf{k}, \mathbf{j} \otimes \mathbf{k})$ -module, we declare  $\mathcal{E}(N; \mathcal{C}) := N \otimes_{\mathbb{F}} \mathbf{k}$ . If  $k \in \mathbf{k}$ , then we define

$$\delta_{j+1}^1(a_j | c_j, \dots, a_1 | c_1, \mathbf{x} | k) = \delta_{j+1}^1(a_j, \dots, a_1, \mathbf{x}) \otimes k \otimes (c_j \cdots c_1). \tag{3.4}$$

Note that  $\delta_1^1(\mathbf{x} | k) = \delta_1^1(\mathbf{x}) \otimes k \otimes 1_{\mathcal{C}}$ . (Note we are implicitly reordering the  $k$  factor and the  $\mathcal{B}$  factor above.)

Given a morphism of DA bimodules  $f_*^1: {}_{\mathcal{A}}N^{\mathcal{B}} \rightarrow {}_{\mathcal{A}}M^{\mathcal{B}}$ , there is a morphism

$$\mathcal{E}(f)_*^1: {}_{\mathcal{A} \otimes \mathcal{C}}\mathcal{E}(N; \mathcal{C})^{\mathcal{B} \otimes \mathcal{C}} \rightarrow {}_{\mathcal{A} \otimes \mathcal{C}}\mathcal{E}(M; \mathcal{C})^{\mathcal{B} \otimes \mathcal{C}}$$

given by replacing  $\delta_{j+1}^1$  with  $f_{j+1}^1$  in equation (3.4).

**Remark 3.4.** If  ${}_{\mathcal{A}}M^{\mathcal{B}}$  is operationally bounded (i.e.,  $\delta_j^1 = 0$  for large  $j$ ), then the same is true for  $\mathcal{E}(M, \mathcal{C})$ . A similar statement holds for morphisms.

**Lemma 3.5.** *Suppose that  $\mathcal{A}$ ,  $\mathcal{B}$ , and  $\mathcal{C}$  are algebras and suppose that  ${}_{\mathcal{A}}N^{\mathcal{B}}$  is a DA bimodule.*

- (1)  ${}_{\mathcal{A} \otimes \mathcal{C}}\mathcal{E}(N; \mathcal{C})^{\mathcal{B} \otimes \mathcal{C}}$  satisfies the DA bimodule structure relations.
- (2) If  $f_*^1$  is a morphism of DA bimodules, then  $\partial_{\text{Mor}}(\mathcal{E}(f_*^1)) = \mathcal{E}(\partial_{\text{Mor}}(f_*^1))$ .
- (3) If  ${}_{\mathcal{A}}M^{\mathcal{B}}$  and  ${}_{\mathcal{B}}N^{\mathcal{D}}$  are bimodules and  ${}_{\mathcal{B}}N^{\mathcal{D}}$  is operationally bounded, then there is a canonical isomorphism of DA bimodules

$${}_{\mathcal{A} \otimes \mathcal{C}}\mathcal{E}({}_{\mathcal{A}}M^{\mathcal{B}} \boxtimes {}_{\mathcal{B}}N^{\mathcal{D}}; \mathcal{C})^{\mathcal{D} \otimes \mathcal{C}} \cong {}_{\mathcal{A} \otimes \mathcal{C}}\mathcal{E}(M; \mathcal{C})^{\mathcal{B} \otimes \mathcal{C}} \boxtimes {}_{\mathcal{B} \otimes \mathcal{C}}\mathcal{E}(N; \mathcal{C})^{\mathcal{D} \otimes \mathcal{C}}.$$

- (4) With respect to the above isomorphism, we also have

$$\mathcal{E}(f_*^1 \boxtimes \mathbb{I}) = \mathcal{E}(f_*^1) \boxtimes \mathbb{I} \quad \text{and} \quad \mathcal{E}(\mathbb{I} \boxtimes g_*^1) = \mathbb{I} \boxtimes \mathcal{E}(g_*^1),$$

for any DA bimodule morphisms  $f_*^1$  and  $g_*^1$ .

*Proof.* The first claim follows from the bimodule relations for  $N$ , together with the fact that multiplication on  $\mathcal{C}$  is associative.

For the second claim, we note that each term of  $\partial_{\text{Mor}}(\mathcal{E}(f_*^1))$ , when evaluated on  $(b_n|c_n), \dots, (b_1|c_1), \mathbf{x}|1_{\mathbf{k}}$ , is equal to a corresponding term of  $\partial_{\text{Mor}}(f_*^1)$ . It is easy to see that all of the terms cancel.

For the third claim, the map is  $f_1^1(\mathbf{x} \otimes \mathbf{y}) = \mathbf{x} \otimes \mathbf{y} \otimes 1_{\mathcal{A} \otimes \mathcal{C}}$  and  $f_j^1 = 0$  if  $j > 1$ . It is easy to see that this is an isomorphism of DA bimodules.

The final claim is verified in a similar manner. ■

**Remark 3.6.** The external tensor product gives a way to tensor a type-D module  $\mathcal{X}^{\mathcal{A} \otimes \mathcal{B}}$  with a type-A module  ${}_{\mathcal{A}}\mathcal{Y}$ . Namely, we take the ordinary box tensor product of  $\mathcal{X}^{\mathcal{A} \otimes \mathcal{B}}$  with  ${}_{\mathcal{A} \otimes \mathcal{B}}\mathcal{E}(\mathcal{Y}, \mathcal{B})^{\mathcal{B}}$ . The reader may recognize that this procedure is identical to Lipshitz, Ozsváth and Thurston's box tensor product between a type-DD module and a type-A module [25, Figure 4].

We now consider an additional commutativity property of extension of scalars.

**Proposition 3.7.** *Suppose that  ${}_{\mathcal{A}}M^{\mathcal{B}}$  and  ${}_{\mathcal{C}}N^{\mathcal{D}}$  are DA bimodules over associative algebras. Then*

$${}_{\mathcal{A} \otimes \mathcal{C}}\mathcal{E}(M, \mathcal{C})^{\mathcal{B} \otimes \mathcal{C}} \boxtimes {}_{\mathcal{B} \otimes \mathcal{C}}\mathcal{E}(N, \mathcal{B})^{\mathcal{B} \otimes \mathcal{D}}$$

and

$${}_{\mathcal{A} \otimes \mathcal{C}}\mathcal{E}(N, \mathcal{A})^{\mathcal{A} \otimes \mathcal{D}} \boxtimes {}_{\mathcal{A} \otimes \mathcal{D}}\mathcal{E}(M, \mathcal{D})^{\mathcal{B} \otimes \mathcal{D}}$$

are homotopy equivalent as DA bimodules. In fact, both are homotopy equivalent to the external tensor product of  ${}_{\mathcal{A}}M^{\mathcal{B}}$  and  ${}_{\mathcal{C}}N^{\mathcal{D}}$  along a cellular diagonal of the associahedron (defined below).

**Remark 3.8.** If one of  $M$  and  $N$  has the property that  $\delta_j^1 = 0$  for  $j > 2$ , then the above proposition may be easily proven directly since the structure maps on the two tensor products are identical.

As alluded to in its statement, Proposition 3.7 is most naturally proven by considering a more general tensor product operation on bimodules, called the *external tensor product*. The construction requires a choice of a cellular diagonal of Stasheff’s associahedron [51, 52]. We recall some background about diagonals presently. Our exposition will follow the work of Lipshitz, Ozsváth and Thurston [28], though we refer the reader to earlier work of Saneblidze–Umble [48] on diagonals of the associahedron. See also [33] and [31].

We recall that the  $(n - 2)$ -dimensional associahedron, denoted  $K_n$ , has a cell decomposition which admits a simple combinatorial description, as follows. We consider a chain complex  $C_*^{\text{cell}}(K_n)$  over  $\mathbb{F}$  whose generators are identified with planar trees  $T$ , which have  $n + 1$  valence 1 vertices, which are partitioned into  $n$  “input” vertices and one “output” vertex. Furthermore, trees  $T$  are assumed to have no vertices of valence 2. The dimension of a generator  $[T]$  is given by

$$\text{deg}(T) = \sum_{v \in V_{\text{int}}(T)} (\text{val}(v) - 3),$$

where  $V_{\text{int}}(T)$  denotes the interior vertices of  $T$ . The differential on  $C_*^{\text{cell}}(K_n)$  is defined by setting  $\partial(T)$  to be the sum of all trees obtained by splitting an interior vertex into two interior vertices by adding a single edge.

The external tensor product operation has as input a pair of DA bimodules  ${}_{\mathcal{A}}M^{\mathcal{B}}$  and  ${}_{\mathcal{C}}N^{\mathcal{D}}$ , and has as output a DA bimodule  ${}_{\mathcal{A} \otimes \mathcal{C}}(M \otimes_{\Gamma} N)^{\mathcal{B} \otimes \mathcal{D}}$ . The underlying vector space of  $M \otimes_{\Gamma} N$  is the ordinary tensor product  $M \otimes_{\mathbb{F}} N$ . The DA bimodule structure map  $\delta_{j+1}^1$  depends on an extra piece of data, called a *cellular diagonal of the associahedron*  $\Gamma$ .

**Definition 3.9** ([28, Definition 2.13]). A *cellular diagonal of the associahedron*  $\Gamma$  consists of a collection of degree preserving chain maps

$$\Gamma_n: C_*^{\text{cell}}(K_n) \rightarrow C_*^{\text{cell}}(K_n) \otimes_{\mathbb{F}} C_*^{\text{cell}}(K_n),$$

for  $n \geq 2$ , which are compatible with concatenation of trees (in the sense described below), and such that  $\Gamma_2$  is the canonical isomorphism between  $C_*^{\text{cell}}(K_2) \cong \mathbb{F}$  and  $C_*^{\text{cell}}(K_2) \otimes_{\mathbb{F}} C_*^{\text{cell}}(K_2) \cong \mathbb{F}$ .

The compatibility condition is as follows. Firstly, there is a natural splicing operation

$$\circ_i: C_*^{\text{cell}}(K_n) \otimes C_*^{\text{cell}}(K_m) \rightarrow C_*^{\text{cell}}(K_{n+m-1}),$$

where  $T \circ_i S$  joins the output of  $S$  into the  $i$ -th input of  $T$ . We define  $\circ_i$  also for tensors of trees by the formula

$$(T_1 \otimes T_2) \circ_i (S_1 \otimes S_2) = (T_1 \otimes_i S_1) \otimes (T_2 \otimes_i S_2).$$

The compatibility condition of Definition 3.9 is that if  $T$  and  $S$  have  $n$  and  $m$  inputs, respectively, then

$$\Gamma_{n+m-1}(T \circ_i S) = \Gamma_n(T) \circ_i \Gamma_m(S).$$

We write  $\Psi_n \in C_*^{\text{cell}}(K_n)$  for the tree with  $n$  inputs and exactly one interior vertex. The generator  $\Psi_n$  is called the *corolla* with  $n$  inputs and has homological degree  $n - 2$ . This is the maximal non-trivial grading of  $C_*^{\text{cell}}(K_n)$ .

If  ${}_{\mathcal{A}}M^{\mathcal{B}} = (M, \delta_*^1)$  is a DA bimodule, then to each planar tree  $T$  with  $(n + 1)$ -inputs, we may form an evaluation map  $\delta_T^1: \mathcal{A}^{\otimes n} \otimes M \rightarrow M \otimes \mathcal{B}$  as follows. We first compose the maps  $\delta_j^1$  according to the tree  $T$ . This naturally gives a map from  $\mathcal{A}^{\otimes n} \otimes M$  to  $M \otimes \mathcal{B}^{\otimes k}$ , where  $k$  is the number of vertices in the right-most path of  $T$ . We then apply  $\mu_2^{\mathcal{B}}$  repeatedly to obtain a map from  $\mathcal{A}^{\otimes n} \otimes M$  to  $M \otimes \mathcal{B}$ .

This gives a map

$$\text{ev}: C_*^{\text{cell}}(K_{n+1}) \rightarrow \text{Hom}_{(\mathbf{k}, \mathbf{j})}(\mathcal{A}^{\otimes n} \otimes M, M \otimes \mathcal{B}).$$

We equip  $\text{Hom}_{(\mathbf{k}, \mathbf{j})}(\mathcal{A}^{\otimes n} \otimes M, M \otimes \mathcal{B})$  with a differential  $\partial$  given by

$$\partial(f_{j+1}^1) = \delta_1^1 \circ f_{j+1}^1 + f_{j+1}^1 \circ \delta_1^1.$$

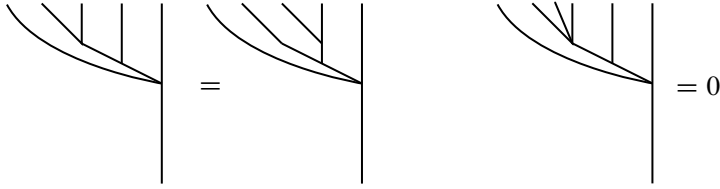
The statement that the structure maps  $\delta_*^1$  of  ${}_{\mathcal{A}}M^{\mathcal{B}}$  satisfy the DA bimodule relations is equivalent to the statement that  $M^{\mathcal{B}} \boxtimes_{\mathcal{B}} \mathcal{B}$  is a chain complex, and that  $\text{ev}$  is a chain map.

We define  $\delta_1^1$  on  $M \otimes_{\mathbb{F}} N$  via the Leibniz rule, and for  $n > 0$  we define the structure map  $\delta_{n+1}^1$  as  $(\text{ev} \otimes \text{ev})(\Gamma_{n+1}(\Psi_{n+1}))$ . The fact that the DA bimodule structure relations hold is immediate.

Since we are focused only on associative algebras, it is helpful to instead focus our attention on a slightly more coarse definition of a diagonal.

**Definition 3.10.** For  $n \geq 2$ , we define a chain complex  $M_{n,*}^{\text{red}}$  as the following modification of  $C_*^{\text{cell}}(K_n)$ . We call  $M_{n,*}^{\text{red}}$  the complex of *left-reduced module trees*. The vector space  $M_{n,*}^{\text{red}}$  is generated by planar trees  $T$  with  $n$ -inputs, such that all vertices of valence 4 or more occur along the right-most path in  $T$  from an input to the output. Additionally, we quotient by the relation that  $T \circ_i S = T \circ_i S'$  whenever  $i < n$  and  $S$  and  $S'$  have degree 0 (i.e., only valence 3 internal vertices) and the same number of inputs. See Figure 3.1 for examples.

We define the differential  $\partial$  on  $M_{n,*}^{\text{red}}$  the same as in  $C_*^{\text{cell}}(K_n)$ . It is not hard to see that  $\partial^2 = 0$ .



**Figure 3.1.** Equalities of trees in  $M_{n,*}^{\text{red}}$ .

For the purposes of defining the diagonal tensor product of two DA bimodules over associative algebras, we will use the following notion.

**Definition 3.11.** A *left-reduced module diagonal* is a collection of grading preserving chain maps

$$\Gamma_n : M_{n,*}^{\text{red}} \rightarrow M_{n,*}^{\text{red}} \otimes M_{n,*}^{\text{red}}$$

which are compatible with splicing of trees.

**Lemma 3.12.** *The complex  $M_{n,*}^{\text{red}}$  has homology  $\mathbb{F}$  in degree 0, and 0 in other degrees.*

*Proof.* We claim that the homology is generated by a tree

$$T_{\text{gen}} := \Psi_2 \circ_1 T_0,$$

where  $T_0$  is a degree 0 tree with  $n - 1$  inputs (all such trees are equal in  $M_{n,*}^{\text{red}}$ ). To see this, we argue as follows. We define a map

$$H : M_{n,*}^{\text{red}} \rightarrow M_{n,*+1}^{\text{red}}$$

as follows. If  $T$  is a tree, we say the *right-most path* of  $T$  is the path from the right-most input of  $T$  to its root. We write  $\text{RP}(T)$  for this path. Given a tree  $T$ , if  $T$  has more than one interior vertex along  $\text{RP}(T)$ , then we contract the top two interior vertices along this path to define a tree  $T'$ . We set  $H(T) = T'$ . Otherwise we define  $H(T) = 0$ .

We claim that if  $N \gg 0$ , the map  $(\mathbb{I} + [\partial, H])^N$  is a projection onto the span of  $T_{\text{gen}}$ . Note that this clearly proves the claim. To see this, we argue as follows. Given a tree, we define the *right-path weight* to be

$$w_R(T) := \sum_{v \in \text{RP}(T)} (\text{val}(v) - 2).$$

We define the subspace  $\mathcal{F}_j \subseteq M_{n,*}^{\text{red}}$  to be the span of trees  $T$  with  $w_R(T) \leq j$ .

It is straightforward to check that if  $j > 1$ , then  $\mathbb{I} + [\partial, H]$  maps  $\mathcal{F}_j$  into  $\mathcal{F}_{j-1}$ . Furthermore,  $\mathbb{I} + [\partial, H]$  is the identity on  $\mathcal{F}_1$ . Therefore, for  $N$  large enough,

$(\mathbb{I} + [\partial, H])^N$  gives a homotopy equivalence onto  $\mathcal{F}_1$  (viewing  $\mathcal{F}_1$  as a subcomplex with vanishing differential). However,  $\mathcal{F}_1$  is exactly the span of  $T_{\text{gen}}$ , so the proof is complete.  $\blacksquare$

For our theory, we need to show that if  ${}_{\mathcal{A}}M$  and  ${}_{\mathcal{B}}N$  are  $A_\infty$ -modules over associative algebras  $\mathcal{A}$  and  $\mathcal{B}$ , then  ${}_{\mathcal{A} \otimes \mathcal{B}}(M \otimes_\Gamma N)$  is independent of the choice of left-reduced module diagonal  $\Gamma$ . The argument is identical to the work in [28, Section 3], though we include it for completeness. Firstly, we introduce the following notion.

**Definition 3.13.** A *left-reduced module morphism tree* consists of a planar input tree  $T$  which has exactly one distinguished vertex along its right-most path, and whose vertices of valence 4 or more are all along the right-most path. We call the distinguished vertex the *morphism vertex*. This vertex has valence 2 or greater, whereas the other interior vertices have valence 3 or greater. We quotient this set of trees by the relation that  $T \circ_i S = T \circ_i S'$  whenever  $T$  has  $n$  inputs,  $i < n$ , and  $S$  and  $S'$  have degree 0 and the same number of inputs.

The degree of a left-reduced module tree  $T$  is

$$\deg(T) = 1 + \sum_{v \in V_0(T)} (\text{val}(v) - 3).$$

We write  $X_{n,*}^{\text{red}}$  for the vector space of left-reduced module morphism trees. We can naturally equip  $X_{n,*}^{\text{red}}$  with a differential which sums over all ways of splitting one interior vertex into two interior vertices while leaving a valid left-reduced module morphism tree.

**Lemma 3.14.** *The chain complex  $X_{n,*}^{\text{red}}$  has homology equal to  $\mathbb{F}$  in degree 0, and 0 in other degrees.*

The proof of Lemma 3.14 is very similar to the proof of Lemma 3.12, and we leave the details to the reader.

**Definition 3.15.** If  $\Gamma$  and  $\Gamma'$  are two left-reduced module diagonals, then a left-reduced module morphism diagonal from  $\Gamma$  to  $\Gamma'$  consists of a collection of chain maps

$$\hat{\Gamma}_n: X_{n,*}^{\text{red}} \rightarrow X_{n,*}^{\text{red}} \otimes X_{n,*}^{\text{red}},$$

such that  $\hat{\Gamma}_1$  is an isomorphism, and  $\hat{\Gamma}_n$  is compatible with splicing in the following sense. If  $S \in X_{n,*}^{\text{red}}$  and  $T \in M_{m,*}^{\text{red}}$ , then

$$\hat{\Gamma}_{n+m-1}(S \circ_n T) = \hat{\Gamma}_n(S) \circ_n \Gamma_m(T) \quad \text{and} \quad \hat{\Gamma}_{n+m-1}(T \circ_m S) = \Gamma'_m(T) \circ_m \hat{\Gamma}_n(S).$$

Finally, if  $T_0 \in M_{m,0}^{\text{red}}$  is a degree 0 tree and  $S \in X_{n,*}^{\text{red}}$ , then

$$\hat{\Gamma}_{n+m-1}(S \circ_i T_0) = \hat{\Gamma}_n(S) \circ_i (T_0 \otimes T_0).$$

Using Lemma 3.14, left-reduced module morphism diagonals may be constructed by a straightforward inductive argument.

**Remark 3.16.** If  $A$  and  $B$  are associative algebras and  $f_*: {}_A M_1 \rightarrow {}_A M_2$  and  $g_*: {}_B N_1 \rightarrow {}_B N_2$  are two  $A_\infty$ -module morphisms and  $\hat{\Gamma}$  is a left-reduced module morphism diagonal, then we can define an  $A_\infty$ -module morphism

$$(f_* \otimes_{\hat{\Gamma}} g_*): {}_{A \otimes B}(M_1 \otimes_{\Gamma} N_1) \rightarrow {}_{A \otimes B}(M_2 \otimes_{\Gamma'} N_2).$$

Applying the above construction to the case that  $M_1 = M_2 = M$  and  $N_1 = N_2 = N$ , we obtain a chain map  $I_*$  from  $(M \otimes_{\Gamma} N)$  to  $(M \otimes_{\Gamma'} N)$  which is a homotopy equivalence because  $I_1 = \mathbb{I}_{M \otimes N}$  is an isomorphism. We may similarly use  $\hat{\Gamma}$  to tensor DA bimodule morphisms together when the underlying algebras are associative algebras.

The motivation for left-reduced module diagonals is that it is straightforward to describe explicitly a diagonal over them. We now define two left-reduced module diagonals

$$L\Gamma_n, R\Gamma_n: M_n^{\text{red}} \rightarrow M_n^{\text{red}} \otimes M_n^{\text{red}}.$$

We only focus on  $L\Gamma_n$ , since  $R\Gamma_n$  is obtained by switching the coordinates of the codomain.

To describe  $L\Gamma_n$ , we define the *right-join* of two input trees  $T_1$  and  $T_2$ , denoted  $\text{RiJ}(T_1, T_2)$ , to be the tree obtained by stacking the output of  $T_2$  into the right-most input vertex of  $T_1$ . (This is the mirror of Lipshitz, Ozsváth and Thurston’s *left-join*, as adapted to our handedness conventions on modules.) If  $T_1, \dots, T_n$  is a collection of trees, we write

$$\text{RiJ}(T_1, \dots, T_n) = \text{RiJ}(T_1, \dots, \text{RiJ}(T_{n-1}, T_n)),$$

If  $T_1, \dots, T_n$  are  $n$  trees, the *root-join* of  $T_1, \dots, T_n$ , denoted  $\text{RoJ}(T_1, \dots, T_n)$ , is obtained by adjoining the outputs of  $T_1, \dots, T_n$  into a single corolla  $\Psi_n$ . See [28, Definition 2.28].

For  $n > 1$ , we write  $S_n \in H_0^{\text{cell}}(K_n)$  for the generator (thought of as a tree with  $n$  inputs and only valence 3 interior vertices).

The tree pairs appearing as summands of  $L\Gamma_n(\Psi_n)$  consist of the sum of all pairs  $T \otimes S$  where

$$T = \text{RiJ}(\Psi_{n_k+1}, \dots, \Psi_{n_1+1})$$

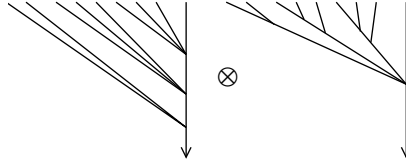
for some numbers  $n_1, \dots, n_k > 0$  (where  $k > 0$ ) satisfying

$$n_1 + \dots + n_k + 1 = n,$$

and

$$S = \text{RoJ}(S_{n_1}, \dots, S_{n_k}, \downarrow).$$

This root join is taken in such a way that  $\downarrow$  corresponds to the module input of the tree. See Figure 3.2 for an example.



**Figure 3.2.** An example of a tree contributing to  $L\Gamma_9(\Psi_9)$ .

**Lemma 3.17.** *The maps  $R\Gamma_n$  and  $L\Gamma_n$ , defined above, are left-reduced module diagonals.*

The proof is straightforward and left to the reader.

*Proof of Proposition 3.7.* We observe that, immediately from the definitions, we have isomorphisms

$$\begin{aligned} \mathcal{A} \otimes_{\mathcal{C}} \mathcal{E}(M, \mathcal{C})^{\mathcal{B} \otimes \mathcal{C}} \boxtimes_{\mathcal{B} \otimes \mathcal{C}} \mathcal{E}(N, \mathcal{B})^{\mathcal{B} \otimes \mathcal{D}} &\cong \mathcal{A} \otimes_{\mathcal{C}} (N \otimes_{R\Gamma} M)^{\mathcal{B} \otimes \mathcal{D}}, \\ \mathcal{A} \otimes_{\mathcal{C}} \mathcal{E}(N, \mathcal{A})^{\mathcal{A} \otimes \mathcal{D}} \boxtimes_{\mathcal{A} \otimes \mathcal{D}} \mathcal{E}(M, \mathcal{D})^{\mathcal{B} \otimes \mathcal{D}} &\cong \mathcal{A} \otimes_{\mathcal{C}} (N \otimes_{L\Gamma} M)^{\mathcal{B} \otimes \mathcal{D}}, \end{aligned}$$

where  $R\Gamma$  and  $L\Gamma$  denote the left-reduced module diagonals from Lemma 3.17. This implies the main claim, because any two choices of module diagonals give homotopy equivalent bimodules by the argument described in Remark 3.16. ■

We now prove a result which relates our extension of scalars operation with the external tensor product.

**Proposition 3.18.** *Let  $\mathcal{A}$ ,  $\mathcal{B}$  and  $\mathcal{C}$  be associative algebras. If  ${}_{\mathcal{A}}N^{\mathcal{B}}$  is a DA bimodule, then  ${}_{\mathcal{A} \otimes_{\mathcal{C}}} \mathcal{E}(N, \mathcal{C})^{\mathcal{B} \otimes \mathcal{C}}$  is isomorphic to the external tensor product of  ${}_{\mathcal{A}}N^{\mathcal{B}}$  and  $\mathcal{C}[\mathbb{I}]^{\mathcal{C}}$ , for any choice of cellular diagonal of the associahedron (or more generally, module diagonal or left-reduced module diagonal).*

The key to the above result is the following lemma.

**Lemma 3.19.** *Suppose that  $\Gamma$  is a cellular diagonal of the associahedron (or a module diagonal, or a left-reduced module diagonal). For each  $n$ , write  $\Gamma_n(\Psi_n) = \Psi_n \otimes X + Y$ , where  $X$  has degree 0 and  $Y \in C_*^{\text{cell}}(K_n) \otimes C_{>0}^{\text{cell}}(K_n)$ . Write  $X = T_1 + \cdots + T_j$ , where each  $T_i$  is a tree of degree 0. Then  $j \equiv 1 \pmod{2}$ .*

*Proof.* The argument is the same, regardless of what kind of diagonal  $\Gamma$  is, so we assume that  $\Gamma$  is an ordinary cellular diagonal of the associahedron. There is a quasi-isomorphism  $\Pi: C_*^{\text{cell}}(K_n) \rightarrow \mathbb{F}$ , which is 0 on  $C_{>0}^{\text{cell}}(K_n)$ . The map  $\Pi$  sends a tree in grading 0 to 1 in  $\mathbb{F}$ , and is 0 on all other trees. It is straightforward to check that  $\Pi$  is a chain map. By tensoring, we obtain a quasi-isomorphism  $\mathbb{I} \otimes \Pi: C_*^{\text{cell}}(K_n) \otimes C_*^{\text{cell}}(K_n) \rightarrow C_*^{\text{cell}}(K_n)$ .

The claim in the statement is equivalent to the claim that  $(\mathbb{I} \otimes \Pi)(\Gamma_n(\Psi_n)) = \Psi_n$ . This claim is equivalent to the claim that  $(\mathbb{I} \otimes \Pi)(\Gamma_n(\Psi_n))$  is non-zero, since  $\Psi_n$  is the only tree in its grading. We argue by induction. The case that  $n = 2$  is clear.

By induction and the fact that  $\Gamma$  respects splicing, we observe that

$$(\mathbb{I} \otimes \Pi)(\Gamma_n(\partial\Psi_n)) = \partial\Psi_n.$$

On the other hand,

$$(\mathbb{I} \otimes \Pi)(\Gamma_n(\partial\Psi_n)) = \partial((\mathbb{I} \otimes \Pi)(\Gamma_n(\Psi_n))).$$

For this to be  $\partial\Psi_n$ , we must have that  $(\mathbb{I} \otimes \Pi)(\Gamma_n(\Psi_n))$  is non-zero, so the proof is complete. ■

*Proof of Proposition 3.18.* The proof is the same, regardless of what kind of diagonal  $\Gamma$  is, so we assume that  $\Gamma$  is an ordinary cellular diagonal of the associahedron. The map  $\delta_{n+1}^1$  on the external tensor product is determined by the structure map on  $N$  as well as the chain  $\Gamma_{n+1}(\Psi_{n+1})$ . If  $T_i \otimes S_j$  is any summand of  $\Gamma_{n+1}(\Psi_{n+1})$ , where  $T_i$  and  $S_j$  are trees with  $n + 1$  inputs of degree  $i$  and  $j$ , respectively, then the corresponding summand of  $\delta_{n+1}^1$  on  $\mathcal{E}(N, \mathcal{C})$  will vanish unless  $j = 0$  and  $i = n - 1$  since  $\mathcal{C}$  is an associative algebra. In particular, the only contributions to  $\delta_{n+1}^1$  will be from tree pairs of the form  $\Psi_{n+1} \otimes T_0$ , where  $T_0$  is a degree 0 tree. By Lemma 3.19 there are an odd number of such summands of  $\Gamma_{n+1}(\Psi_{n+1})$ . It follows that the external tensor product is exactly the extension of scalars construction we gave earlier. ■

### 3.6 Hypercubes as type-D modules

In this section, we present a helpful perspective of hypercubes and hyperboxes in terms of type-D modules over certain algebras which we call the *cube* and *box* algebras. Many natural hyperbox operations, such as compression, have natural interpretations as  $A_\infty$ -module operations with respect to these algebras.

We will not use the cube and box algebras explicitly in this memoir, but they are related to our reinterpretation of the Manolescu and Ozsváth’s link surgery formula in terms of type-D modules over the algebra  $\mathcal{L}$ .

We begin with the 1-dimensional cube algebra  $\mathcal{C}_1$ . The algebra  $\mathcal{C}_1$  is an algebra over the idempotent ring  $\mathbf{I}_0 \oplus \mathbf{I}_1$ , where  $\mathbf{I}_\varepsilon \cong \mathbb{F}$ . If  $\varepsilon \in \{0, 1\}$ , we write  $i_\varepsilon$  for the generator of  $\mathbf{I}_\varepsilon$ . We set  $\mathbf{I}_\varepsilon \cdot \mathcal{C}_1 \cdot \mathbf{I}_\varepsilon = \mathbf{I}_\varepsilon$ , and we set

$$\mathbf{I}_1 \cdot \mathcal{C}_1 \cdot \mathbf{I}_0 = \langle \theta_{0,1} \rangle,$$

i.e., we set  $\mathbf{I}_1 \cdot \mathcal{C}_1 \cdot \mathbf{I}_0 \cong \mathbb{F}$ , generated by an element  $\theta_{0,1}$ .

Next, we introduce the  $n$ -dimensional cube algebra  $\mathcal{C}_n$ . The definition is

$$\mathcal{C}_n = \mathcal{C}_1 \otimes_{\mathbb{F}} \cdots \otimes_{\mathbb{F}} \mathcal{C}_1 = \bigotimes_{\mathbb{F}}^n \mathcal{C}_1.$$

Note that this is an algebra over the idempotent ring  $\mathbf{E}_n = \bigotimes_{\mathbb{F}}^n \mathbf{I}$ . If  $\varepsilon \in \{0, 1\}^n$ , we will write  $\mathbf{E}_\varepsilon$  for  $\mathbf{I}_{\varepsilon_1} \otimes \cdots \otimes \mathbf{I}_{\varepsilon_n}$ . We will write  $i_\varepsilon$  for the generator of  $\mathbf{E}_\varepsilon \cong \mathbb{F}$ .

The importance of the cube algebra is the following.

**Lemma 3.20.** *The category of  $n$ -dimensional hypercubes of chain complexes is equivalent to the category of type-D modules over  $\mathcal{C}_n$ .*

*Proof.* We describe how to obtain a type-D module over  $\mathcal{C}_n$  from a hypercube of chain complexes. Suppose that  $X = (X_\varepsilon, D_{\varepsilon, \varepsilon'})_{\varepsilon \in \mathbb{E}_n}$  is a hypercube of chain complexes. We define a type-D module  $\mathcal{X}^{\mathcal{C}_n}$  as follows. The underlying group of  $\mathcal{X}$  is the total space of  $X$ , i.e.,  $\mathcal{X} = \bigoplus_{\varepsilon \in \mathbb{E}_n} X_\varepsilon$ . This is naturally a module over the idempotent ring  $\mathbf{E}_n$ .

We define  $\delta^1: \mathcal{X} \rightarrow \mathcal{X} \otimes_{\mathbf{E}_n} \mathcal{C}_n$  as follows. If  $\varepsilon \leq \varepsilon'$ , there is an algebra element  $\theta_{\varepsilon, \varepsilon'}$  which is the tensor product of  $\theta_{0,1}$  over all  $i$  such that  $\varepsilon_i < \varepsilon'_i$ . If  $\varepsilon = \varepsilon'$ , we set  $\theta_{\varepsilon, \varepsilon} = i_\varepsilon$ . With this notation, we set

$$\delta^1(\mathbf{x}) = \sum_{\varepsilon \leq \varepsilon'} D_{\varepsilon, \varepsilon'}(\mathbf{x}) \otimes \theta_{\varepsilon, \varepsilon'}.$$

The type-D structure relations for  $(\mathcal{X}, \delta^1)$  are straightforward to verify.

If  $X = (X_\varepsilon, D_{\varepsilon, \varepsilon'})$  and  $Y = (Y_\varepsilon, D'_{\varepsilon, \varepsilon'})$  are two  $n$ -dimensional hypercubes of chain complexes, it is straightforward to extend the above construction to give an isomorphism between morphism sets

$$F: \text{Mor}_{\mathbb{E}_n}(X, Y) \rightarrow \text{Mor}^{\mathcal{C}_n}(\mathcal{X}^{\mathcal{C}_n}, \mathcal{Y}^{\mathcal{C}_n}),$$

where  $\text{Mor}_{\mathbb{E}_n}$  denotes the group of hypercube morphisms and  $\text{Mor}^{\mathcal{C}_n}$  denotes the group of type-D morphisms. Clearly,  $F$  is an isomorphism and  $F(f \circ g) = F(f) \circ F(g)$ . Hence, we have constructed a functor from the category of  $n$ -dimensional hypercubes to the category of type-D modules over  $\mathcal{C}_n$ . An inverse functor is straightforward to describe.  $\blacksquare$

The above construction extends naturally to the setting of hyperboxes, as we now describe. We begin with the 1-dimensional box algebra  $\mathcal{B}_{(d)}$  of size  $d > 0$ . This is an algebra over the idempotent ring

$$\mathbf{I}_{(d)} := \mathbf{I}_0 \oplus \mathbf{I}_1 \oplus \cdots \oplus \mathbf{I}_d,$$

where each  $\mathbf{I}_i \cong \mathbb{F}$ . We set

$$\mathbf{I}_i \cdot \mathcal{B}_{(d)} \cdot \mathbf{I}_i = \mathbb{F}.$$

We set

$$\mathbf{I}_j \cdot \mathcal{B}_{(d)} \cdot \mathbf{I}_i = \begin{cases} \langle \theta_{i,i+1} \rangle & \text{if } j = i + 1, \\ \{0\} & \text{otherwise.} \end{cases}$$

Multiplication is given by setting  $\theta_{j,j+1} \cdot \theta_{i,i+1} = 0$  for all  $i$  and  $j$ .

If  $\mathbf{d} = (d_1, \dots, d_n)$  is a tuple of positive integers, we define the *hyperbox algebra* of size  $\mathbf{d}$ , denoted  $\mathcal{B}_{\mathbf{d}}$  to be the tensor product

$$\mathcal{B}_{\mathbf{d}} := \mathcal{B}_{(d_1)} \otimes_{\mathbb{F}} \cdots \otimes_{\mathbb{F}} \mathcal{B}_{(d_n)}.$$

We have the following straightforward extension of Lemma 3.20.

**Lemma 3.21.** *The category of hyperboxes of size  $\mathbf{d}$  is equivalent to the category of type-D modules over the algebra  $\mathcal{B}_{\mathbf{d}}$ .*

Manolescu and Ozsváth’s compression operation also has a conceptually simple description in terms of the cube and box algebras. With the present perspective, compression takes the form of a DA bimodule

$$\mathcal{B}_{(d)}[\mathbb{C}_{(d)}]^{\mathcal{E}_1},$$

defined as follows.

If  $d = 1$ , we define  $\mathbb{C}_{(d)}$  to be the identity bimodule. Henceforth, assume  $d > 1$ . As a group,  $\mathbb{C}_{(d)}$  is a copy of  $\mathbf{I} = \mathbf{I}_0 \oplus \mathbf{I}_1$ . The right action of  $\mathbf{I}$  is the obvious one. The left action of the box idempotent ring  $\mathbf{I}_{(d)}$  is given by having  $\mathbf{I}_{(0)}$  and  $\mathbf{I}_{(d)}$  act by  $\mathbf{I}_0$  and  $\mathbf{I}_1$ , respectively, and having  $\mathbf{I}_j$  act trivially for  $j = 1, \dots, d - 1$ .

The bimodule  $[\mathbb{C}_{(d)}]$  has  $\delta_j^1 = 0$  for all  $j$  except  $\delta_2^1$  and  $\delta_{d+1}^1$ . We set  $\delta_2^1$  to be the unique map satisfying  $\delta_2^1(i_\varepsilon, i_\nu) = (i_\varepsilon \cdot i_\nu) \otimes 1$ , where  $\varepsilon \in \{0, \dots, d\}$  and  $\nu \in \{0, 1\}$ , and such that  $\delta_2^1(\theta_{i,i+1}, i_\nu) = 0$  for all  $i$ . We define

$$\delta_{d+1}^1(\theta_{d-1,d}, \dots, \theta_{0,1}, i_0) = i_1 \otimes \theta.$$

**Example 3.22.** Consider a hyperbox of dimension 1,

$$X = (C_0 \xrightarrow{f_{1,0}} C_1 \xrightarrow{f_{2,1}} \cdots \xrightarrow{f_{d-1,d}} C_d).$$

The associated type-D module over  $\mathcal{B}_{(d)}$  takes the following form:

$$\mathcal{X}^{\mathcal{B}_{(d)}} = (C_0 \xrightarrow{f_{0,1} \otimes \theta_{0,1}} C_1 \xrightarrow{f_{1,2} \otimes \theta_{1,2}} \cdots \xrightarrow{f_{d-1,d} \otimes \theta_{d-1,d}} C_d).$$

The underlying group of the box tensor product  $\mathcal{X}^{\mathcal{B}_{(d)}} \boxtimes_{\mathcal{B}_{(d)}} [\mathbb{C}_{(d)}]^{\mathcal{E}_1}$  is  $C_0 \oplus C_d$ . One easily computes that the differential on this box tensor product is

$$\mathcal{X}^{\mathcal{B}_{(d)}} \boxtimes_{\mathcal{B}_{(d)}} [\mathbb{C}_{(d)}]^{\mathcal{E}_1} = (C_0 \xrightarrow{(f_{d-1,d} \circ \cdots \circ f_{0,1}) \otimes \theta} C_d).$$

The right-hand side is the compression of  $X$ , viewed as a type-D module over  $\mathcal{C}_1$ , because it coincides with the function composition description of compression that is discussed in Section 2.3.

More generally, if  $\mathbf{d} = (d_1, \dots, d_n)$  is an  $n$ -tuple of positive integers, we can define a bimodule  $\mathcal{B}_{\mathbf{d}}[\mathcal{C}_{\mathbf{d}}]^{\mathcal{C}_n}$  as the external tensor product of the DA bimodules  $\mathcal{B}_{(d_i)}[\mathcal{C}_{(d_i)}]^{\mathcal{C}_1}$  ranging over  $i = 1, \dots, n$ . Note that defining the external tensor product requires a choice of ordering of the axis directions, cf. Section 3.5.

**Lemma 3.23.** *If  $X$  is a hyperbox of size  $\mathbf{d} = (d_1, \dots, d_n)$ , and  $\mathcal{X}^{\mathcal{B}_{\mathbf{d}}}$  is the corresponding type-D module, then the compression of  $X$  coincides with the hypercube corresponding to  $\mathcal{X}^{\mathcal{B}_{\mathbf{d}}} \boxtimes \mathcal{B}_{\mathbf{d}}[\mathcal{C}_{\mathbf{d}}]^{\mathcal{C}_n}$ .*

*Proof.* The bimodule  $\mathcal{B}_{\mathbf{d}}[\mathcal{C}_{\mathbf{d}}]^{\mathcal{C}_n}$  requires a choice of module diagonal to construct. Alternatively, by the proof of Proposition 3.7, we can construct  $\mathcal{B}_{\mathbf{d}}[\mathcal{C}_{\mathbf{d}}]^{\mathcal{C}_n}$  using the extension of scalars construction. Write  $\mathbf{d}_i = (1, \dots, 1, d_i, \dots, d_n)$ . We apply the extension of scalars construction to  $\mathcal{B}_{d_i}[\mathcal{C}_{(d_i)}]^{\mathcal{C}_1}$  to obtain a bimodule

$$\mathcal{B}_{d_i}[\mathcal{C}_{\mathbf{d}_i, i}]^{\mathcal{B}_{d_i+1}}.$$

Using Proposition 3.7, we see that

$$\mathcal{B}_{\mathbf{d}}[\mathcal{C}_{\mathbf{d}}]^{\mathcal{C}_n} \simeq \mathcal{B}_{\mathbf{d}}[\mathcal{C}_{\mathbf{d}_1, 1}]^{\mathcal{B}_{d_1+1}} \boxtimes \dots \boxtimes \mathcal{B}_{\mathbf{d}_n}[\mathcal{C}_{\mathbf{d}_n, n}]^{\mathcal{C}_n}. \quad (3.5)$$

Tensoring with  $\mathcal{B}_{d_i}[\mathcal{C}_{\mathbf{d}_i, i}]^{\mathcal{B}_{d_i+1}}$  has the same effect as applying the partial compression operation  $c_i$  from Section 2.3, so the main claim follows from equation (3.5). ■