Chapter 4

Homological perturbation theory

In this chapter, we describe several versions of the homological perturbation lemma. In this chapter and throughout the rest of the memoir, we assume that every ring and module has characteristic 2.

4.1 A_∞-modules

Kadeishvili proved that A_∞-algebra structures could be transferred along homotopy equivalences of the underlying chain complex, when coefficients were in a field [17]. A formula for the perturbed A_∞-module structure in terms of trees may be found in work of Kontsevich and Soibelman [21, Section 6.4]. See [19, Section 3.3] or [50, Remark 1.15] for more historical context and development of ideas. The following is a statement of the homological perturbation lemma for A_∞-modules.

Lemma 4.1. *Suppose that $\mathcal{A}$ is an associative algebra over a ring* $\mathbf{k}$*, ${}_{\mathcal{A}}M$ is an A_∞-module, (Z, m_1^Z) is a chain complex over* $\mathbf{k}$*, and that we have three maps of left* $\mathbf{k}$*-modules*

$$i\colon Z \to M, \quad \pi\colon M \to Z, \quad \text{and} \quad h\colon M \to M$$

satisfying the following:

(1) *i and π are chain maps.*

(2) $\pi \circ i = \mathrm{id}_Z$.

(3) $i \circ \pi = \mathrm{id}_M + \partial_{\mathrm{Mor}}(h)$.

(4) $h \circ i = 0$.

(5) $\pi \circ h = 0$.

(6) $h \circ h = 0$.

In the above, $\partial_{\mathrm{Mor}}(h)$ denotes the morphism differential for chain complexes, i.e., $\partial_{\mathrm{Mor}}(h) = h \circ m_1^M + m_1^M \circ h$. Then there is an A_∞-module structure on Z, extending m_1^Z, such that the maps π, i and h extend to A_∞-module morphisms which satisfy all of the above relations as A_∞-module maps.

One extremely useful aspect of the homological perturbation lemma is that all of the maps have a concrete description. The structure maps on Z are given by the diagrams shown in Figure 4.1. Therein, $m_{>1}$ denotes the A_∞ structure maps of M, with m_1 excluded.

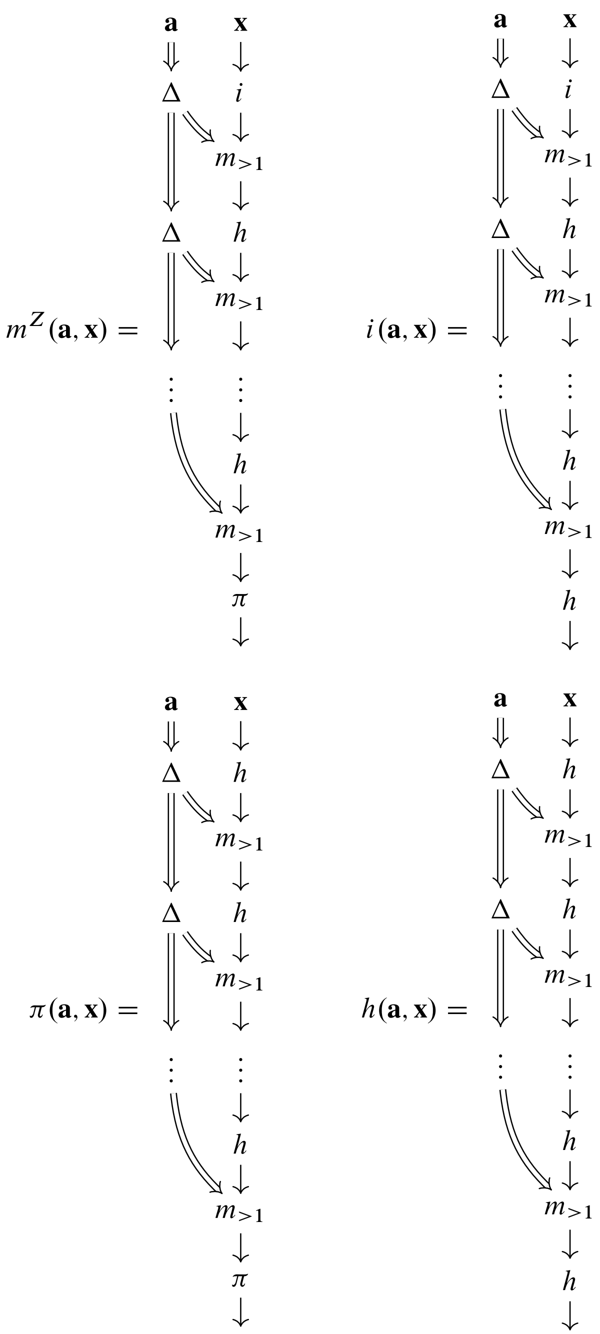

Figure 4.1. The maps appearing in the homological perturbation lemma for A_∞-modules.

4.2 DA bimodules

There is a homological perturbation lemma for DA bimodules, which we will use extensively. The statement and proof are similar to the statement for A_∞-modules. If $({}_{\mathcal{A}}M^{\mathcal{B}}, \delta^1_j)$ is a DA bimodule, we will write $M^{\mathcal{B}}$ for the associated type-D module which has the same generators as ${}_{\mathcal{A}}M^{\mathcal{B}}$, and whose structure map is given by the map δ^1_1 from ${}_{\mathcal{A}}M^{\mathcal{B}}$.

Lemma 4.2. *Suppose that $\mathcal{A}$ and $\mathcal{B}$ are associative algebras, and that ${}_{\mathcal{A}}M^{\mathcal{B}}$ is a type-*DA *bimodule, $Z^{\mathcal{B}}$ is a type-*D *module over $\mathcal{B}$, and that we have three morphisms of type-*D *modules*

$$i^1 \colon Z^{\mathcal{B}} \to M^{\mathcal{B}}, \quad \pi^1 \colon M^{\mathcal{B}} \to Z^{\mathcal{B}}, \quad \textit{and} \quad h^1 \colon M^{\mathcal{B}} \to M^{\mathcal{B}}$$

satisfying the following:

(1) *i^1 and π^1 are homomorphisms of type-*D *modules (i.e., $\partial_{\mathrm{Mor}}(i^1)$ and $\partial_{\mathrm{Mor}}(\pi^1)$ vanish).*

(2) $\pi^1 \circ i^1 = \mathrm{id}_Z$.

(3) $i^1 \circ \pi^1 = \mathrm{id}_M + \partial_{\mathrm{Mor}}(h^1)$.

(4) $h^1 \circ i^1 = 0$.

(5) $\pi^1 \circ h^1 = 0$.

(6) $h^1 \circ h^1 = 0$.

*In the above, $\partial_{\mathrm{Mor}}(h^1)$ denotes the morphism differential for type-*D *modules over $\mathcal{B}$; see equation (3.2). Then there is an induced* DA *bimodule structure on ${}_{\mathcal{A}}Z^{\mathcal{B}}$, extending $Z^{\mathcal{B}}$, such that the maps π^1, i^1 and h^1 extend to* DA *bimodule morphisms which satisfy all of the above relations as* DA *bimodule morphisms.*

We refer the reader to [42, Lemma 2.12] for a proof. Note that the maps are similar to those appearing in Figure 4.1. The structure map δ^1_{j+1} is indicated in Figure 4.2. The other maps i^1_j, π^1_j and h^1_j are constructed by a similar modification of the other maps in Figure 4.1. Here, $\delta^1_{>1}$ denotes δ^1_j for some $j > 1$. The final Π map means to multiply all of the inputs using several applications of μ_2.

4.3 Hypercubes

In this section, we review a version of the homological perturbation lemma for hypercubes. See [14, Section 2.7] for more details. Lemma 4.3 is similar to other formulations of the homological perturbation lemma in the context of filtered chain complexes. See, e.g., [16]. A similar, though slightly less explicit, construction for transferring hypercube structures along homotopy equivalences is described by Liu [30, Section 5.6].

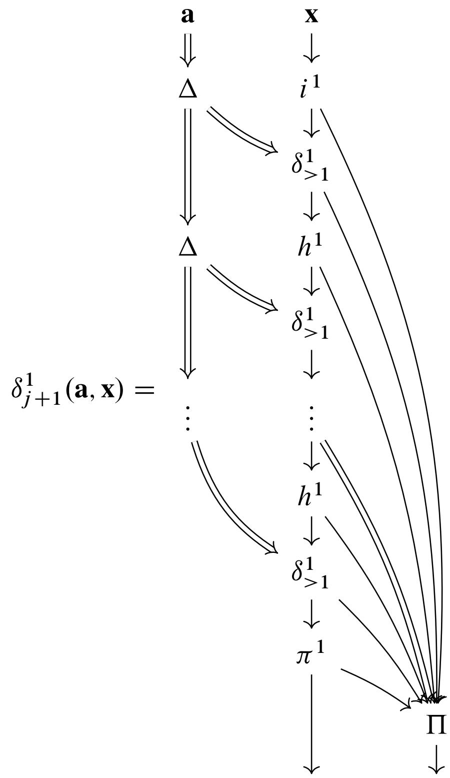

Figure 4.2. The structure relation δ^1_{j+1} from the homological perturbation lemma for DA bimodules.

Lemma 4.3. *Suppose that $\mathcal{C} = (C_\varepsilon, D_{\varepsilon,\varepsilon'})_{\varepsilon\in\mathbb{E}_n}$ is a hypercube of chain complexes, and $(Z_\varepsilon)_{\varepsilon\in\mathbb{E}_n}$ is a collection of chain complexes, indexed by $\varepsilon \in \mathbb{E}_n$. Furthermore, suppose that for each $\varepsilon \in \mathbb{E}_n$, we have chosen maps*

$$\pi_\varepsilon\colon C_\varepsilon \to Z_\varepsilon, \quad i_\varepsilon\colon Z_\varepsilon \to C_\varepsilon, \quad h_\varepsilon\colon C_\varepsilon \to C_\varepsilon,$$

satisfying

$$\pi_\varepsilon \circ i_\varepsilon = \mathrm{id}, \quad i_\varepsilon \circ \pi_\varepsilon = \mathrm{id} + \partial_{\mathrm{Mor}}(h_\varepsilon), \quad h_\varepsilon \circ h_\varepsilon = 0,$$
$$\pi_\varepsilon \circ h_\varepsilon = 0, \quad \textit{and} \quad h_\varepsilon \circ i_\varepsilon = 0,$$

and such that π_ε and i_ε are chain maps. With the above data chosen, there are canonical hypercube structure maps $\delta_{\varepsilon,\varepsilon'}\colon Z_\varepsilon \to Z_{\varepsilon'}$ so that $\mathcal{Z} = (Z_\varepsilon, \delta_{\varepsilon,\varepsilon'})$ is a hypercube *of chain complexes, and also there are morphisms of hypercubes*

$$\Pi\colon \mathcal{C} \to \mathcal{Z}, \quad I\colon \mathcal{Z} \to \mathcal{C}, \quad \textit{and} \quad H\colon \mathcal{C} \to \mathcal{C}$$

such that I and Π are chain maps, and satisfy

$$\Pi \circ I = \mathrm{id}, \quad I \circ \Pi = \mathrm{id} + \partial_{\mathrm{Mor}}(H), \quad H \circ H = 0, \quad \Pi \circ H = 0, \quad \textit{and} \quad H \circ I = 0.$$

It is important to understand that the structure maps $\delta_{\varepsilon,\varepsilon'}$ and the morphisms Π and I are determined completely by an explicit formula. We now describe $\delta_{\varepsilon,\varepsilon'}$. Suppose that $\varepsilon < \varepsilon'$ are points in $\mathbb{E}_n$. The hypercube structure maps $\delta_{\varepsilon,\varepsilon'}$ are given by the following sum:

$$\delta_{\varepsilon,\varepsilon'} := \sum_{\varepsilon=\varepsilon_1<\cdots<\varepsilon_j=\varepsilon'} \pi_{\varepsilon'} \circ D_{\varepsilon_{j-1},\varepsilon_j} \circ h_{\varepsilon_{j-1}} \circ D_{\varepsilon_{j-2},\varepsilon_{j-1}} \circ \cdots \circ h_{\varepsilon_2} \circ D_{\varepsilon_1,\varepsilon_2} \circ i_{\varepsilon}.$$

The component of the map I sending coordinate ε to coordinate ε' is given via the formula

$$I_{\varepsilon,\varepsilon'} := \sum_{\varepsilon=\varepsilon_1<\cdots<\varepsilon_j=\varepsilon'} h_{\varepsilon_j} \circ D_{\varepsilon_{j-1},\varepsilon_j} \circ \cdots \circ h_{\varepsilon_2} \circ D_{\varepsilon_1,\varepsilon_2} \circ i_{\varepsilon}.$$

Similarly, Π is given by the formula

$$\Pi_{\varepsilon,\varepsilon'} := \sum_{\varepsilon=\varepsilon_1<\cdots<\varepsilon_j=\varepsilon'} \pi_{\varepsilon'} \circ D_{\varepsilon_{j-1},\varepsilon_j} \circ h_{\varepsilon_{j-1}} \circ D_{\varepsilon_{j-2},\varepsilon_{j-1}} \circ \cdots \circ D_{\varepsilon_1,\varepsilon_2} \circ h_{\varepsilon_1}.$$

4.4 Hypercubes of DA bimodules

There is a version of the homological perturbation lemma which naturally generalizes both the homological perturbation lemmas for DA bimodules and hypercubes of chain complexes.

Definition 4.4. A *hypercube of* DA *bimodules* consists of a DA bimodule (C, δ^1_*), such that C decomposes as $C = \bigoplus_{\varepsilon\in\mathbb{E}_n} C_\varepsilon$, where each C_ε is itself a $(\mathbf{j}, \mathbf{k})$-module. Furthermore, the structure map δ^1_{j+1} decomposes as a sum of maps

$$\delta^1_{j+1,\varepsilon,\varepsilon'}\colon \mathcal{A}^{\otimes j} \otimes C_\varepsilon \to C_{\varepsilon'} \otimes \mathcal{B},$$

where $j \geq 0$, and $\varepsilon \leq \varepsilon'$.

The above definition is a special case of Lipshitz, Ozsváth and Thurston's notion of a *filtered* DA *bimodule* [24, Section 2.2].

Lemma 4.5. *Suppose that $\mathcal{A}$ and $\mathcal{B}$ are* dg-*algebras. Suppose also that ${}_{\mathcal{A}}\mathcal{C}^{\mathcal{B}} = ({}_{\mathcal{A}}C_\varepsilon^{\mathcal{B}}, D^1_{*,\varepsilon,\varepsilon'})_{\varepsilon\in\mathbb{E}_n}$ is a hypercube of* DA *bimodules, and suppose that for each $\varepsilon \in \mathbb{E}_n$ we have chosen type-*D *modules $(Z_\varepsilon^{\mathcal{B}})_{\varepsilon\in\mathbb{E}_n}$, as well as morphisms of type-*D *modules*

$$\pi_\varepsilon^1 \colon C_\varepsilon^{\mathcal{B}} \to Z_\varepsilon^{\mathcal{B}}, \quad i_\varepsilon^1 \colon Z_\varepsilon^{\mathcal{B}} \to C_\varepsilon^{\mathcal{B}}, \quad \textit{and} \quad h_\varepsilon^1 \colon C_\varepsilon^{\mathcal{B}} \to C_\varepsilon^{\mathcal{B}}$$

satisfying

$$\pi_\varepsilon^1 \circ i_\varepsilon^1 = \mathrm{id}_{Z_\varepsilon}, \quad i_\varepsilon^1 \circ \pi_\varepsilon^1 + \mathrm{id}_{C_\varepsilon} = \partial_{\mathrm{Mor}}(h_\varepsilon^1), \quad h_\varepsilon^1 \circ h_\varepsilon^1 = 0,$$
$$h_\varepsilon^1 \circ i_\varepsilon^1 = 0, \quad \textit{and} \quad \pi_\varepsilon^1 \circ h_\varepsilon^1 = 0,$$

*and such that π_ε^1 and i_ε^1 are homomorphisms of type-*D *modules (i.e., cycles). Such data determines structure maps $\delta^1_{j+1,\varepsilon,\varepsilon'} \colon \mathcal{A}^{\otimes j} \otimes Z_\varepsilon \to Z_{\varepsilon'} \otimes \mathcal{B}$, which make ${}_{\mathcal{A}}\mathcal{Z}^{\mathcal{B}} = ({}_{\mathcal{A}}Z_\varepsilon^{\mathcal{B}}, \delta^1_{*,\varepsilon,\varepsilon'})$ a hypercube of* DA *bimodules. Furthermore, there are morphisms of hypercubes of* DA *bimodules*

$$\Pi^1_* \colon {}_{\mathcal{A}}\mathcal{C}^{\mathcal{B}} \to {}_{\mathcal{A}}\mathcal{Z}^{\mathcal{B}}, \quad I^1_* \colon {}_{\mathcal{A}}\mathcal{Z}^{\mathcal{B}} \to {}_{\mathcal{A}}\mathcal{C}^{\mathcal{B}}, \quad \textit{and} \quad H^1_* \colon {}_{\mathcal{A}}\mathcal{C}^{\mathcal{B}} \to {}_{\mathcal{A}}\mathcal{C}^{\mathcal{B}},$$

which satisfy analogous relations to i_ε^1, π_ε^1 and h_ε^1.

We now describe the maps δ^1_*, Π^1_*, I^1_* and H^1_*. We focus on the structure map $\delta^1_{j+1,\varepsilon,\varepsilon'}$. This map is defined by modifying Figure 4.2 as follows. In place of each $\delta^1_{>1}$ labeled therein, we are allowed one of two maps:

(1) A hypercube map $D^1_{j+1,\varepsilon,\varepsilon'}$ for $\mathcal{C}$, such that $\varepsilon < \varepsilon'$. Furthermore, we assume only that $j \geq 0$ (so that we allow the case that there are no algebra inputs here).

(2) An internal structure map $D^1_{>1,\varepsilon,\varepsilon}$ of some ${}_{\mathcal{A}}C_\varepsilon^{\mathcal{B}}$ (so in this case we do not allow instances of the internal differential $D^1_{1,\varepsilon,\varepsilon}$).

The maps Π^1_*, I^1_* and H^1_* are defined by small modifications, similar to Figure 4.1. We leave the proof of the above lemma to the reader, as it is a straightforward variation of Lemma 4.2.