Chapter 5

Linear topological spaces

In this chapter, we recall basic background on algebraic completions. See [1, Section 10] and [22, Chapter 2] for related background. This material is used to define our module categories in Chapter 7.

Definition 5.1. A *linear topological vector space* consists of a vector space $\mathcal{X}$ equipped with a topology which admits a basis of open sets centered at 0 consisting of vector subspaces, such that furthermore the addition map $\mathcal{X} \times \mathcal{X} \to \mathcal{X}$ is continuous.

If R is a ring, an *R-module with linear topology* is defined similarly, except that there is a basis of 0 consisting of R-submodules.

Vector spaces with linear topologies are usually specified by picking a decreasing filtration consisting of subspaces $X_i \subseteq \mathcal{X}$ indexed by $i \in I$, where I is some directed and partially ordered set. Such a filtration determines a topology, where we declare a basis to be the sets of the form $x + X_i$, where $i \in I$ and $x \in \mathcal{X}$.

Given a filtration $(X_i)_{i \in I}$ on a vector space $\mathcal{X}$, the *completion* is defined as the inverse limit

$$\boldsymbol{\mathcal{X}} := \varprojlim_{i \in I} \mathcal{X}/X_i.$$

Rather than working with inverse limits, it is sometimes easier to work with Cauchy sequences and nets. Since some of the spaces we are working with are not first countable, we work with Cauchy *nets* instead of the perhaps more familiar Cauchy *sequences*.

We recall that a *net* in a topological space $\mathcal{X}$ is a map $\mathbf{x}\colon I \to \mathcal{X}$, where I is a directed set. We usually write $\mathbf{x} = (x_i)_{i \in I}$ for a net. When $\mathcal{X}$ is a linear topological vector space, we say that a net $(x_i)_{i \in I}$ is *Cauchy* if for each open subspace $E \subseteq \mathcal{X}$, there is an $i_0 \in I$ such that if $i, j \geq i_0$, then $x_i - x_j \in E$.

If $(x_i)_{i \in I}$ and $(y_j)_{j \in J}$ are two nets in $\mathcal{X}$, their sum is the net $(x_i + y_j)_{(i,j) \in I \times J}$. Two Cauchy nets are *equivalent* if their difference converges to $0 \in \mathcal{X}$ as a net. We note that the collection of nets in $\mathcal{X}$ forms a proper class, though the set of Cauchy equivalence classes is a set. The following is well known and is elementary to prove.

Lemma 5.2. *If $\mathcal{X}$ is a linear topological space, then the completion $\boldsymbol{\mathcal{X}}$ is canonically isomorphic to the vector space of equivalence classes of Cauchy nets in $\mathcal{X}$.*

5.1 Linear maps

If $\mathcal{X}$ and $\mathcal{Y}$ are linear topological vector spaces, then a continuous map $f\colon \mathcal{X} \to \mathcal{Y}$ naturally induces a map on completions. We recall the following basic principle, whose proof is elementary.

Lemma 5.3. *Suppose that $\mathcal{X}$ and $\mathcal{Y}$ are linear topological spaces. A linear map $f\colon \mathcal{X} \to \mathcal{Y}$ is continuous if and only if it is continuous at* 0.

Definition 5.4. We define the category of *linear topological vector spaces* over $\mathbb{F}$, denoted $\mathrm{LTS}_{\mathbb{F}}$, as follows. The objects are linear topological vector spaces over $\mathbb{F}$. We define the set of morphisms from $\mathcal{X}$ to $\mathcal{Y}$ in this category to be the set of continuous linear maps from $\boldsymbol{\mathcal{X}}$ to $\boldsymbol{\mathcal{Y}}$ (i.e., continuous linear maps between the completed spaces).

Remark 5.5. By definition, $\mathcal{X}$ and $\boldsymbol{\mathcal{X}}$ are isomorphic in the category $\mathrm{LTS}_{\mathbb{F}}$ for any linear topological $\mathbb{F}$-vector space $\mathcal{X}$.

5.2 Direct products

We now describe some helpful perspectives on the direct sum and product in the context of linear topological spaces. These are important for our work because of the role that direct products play in the surgery formula from [32].

Definition 5.6. If $(X_i)_{i\in I}$ is a family of linear topological spaces, the *direct product* of topological vector spaces $\mathcal{X} = \prod_{i\in I} X_i$ coincides with the direct product in the category of topological spaces. More explicitly, a subspace $E \subseteq \prod_{i\in I} X_i$ is open if and only if there is some finite set $S \subseteq I$ and a set of open subspaces $U_s \subseteq X_s$ ranging over $s \in S$, such that

$$\Big(\prod_{i\in I\setminus S} X_i\Big) \oplus \Big(\prod_{s\in S} U_s\Big) \subseteq E.$$

We declare the product topology on $\bigoplus_{i\in I} X_i$ to be the subspace topology induced by the inclusion $\bigoplus_{i\in I} X_i \subseteq \prod_{i\in I} X_i$.

Note that if $\mathcal{X} = \bigoplus_{i\in I} X_i$ is equipped with the product topology, then $\boldsymbol{\mathcal{X}} \cong \prod_{i\in I} \mathbf{X}_i$.

In our work, we consider frequently the case where each X_i is equipped with the discrete topology. In this case, if $S \subseteq I$ is a finite set, we write

$$\mathcal{X}_{\mathrm{co}(S)} = \bigoplus_{i\notin S} X_i$$

for the fundamental basis of opens.

Example 5.7. In this example, we illustrate the case that $I = \mathbb{Z}$ and $X_i = \mathbb{F}$. We consider $\mathcal{X} = \bigoplus_{i \in I} X_i$. Write x_i for the generator of X_i; the basis of open subspaces as $\mathcal{X}_{\mathrm{co}(\{-n,\dots,n\})} = \mathrm{Span}(x_i : |i| > n)$. We can identify $\mathcal{X}/\mathcal{X}_{\mathrm{co}(\{-n,\dots,n\})}$ with $\mathrm{Span}(x_{-n}, \dots, x_n)$. The inverse limit of $\mathcal{X}/\mathcal{X}_{\mathrm{co}(\{-n,\dots,n\})}$ consists of all tuples $(a_i)_{i \in \mathbb{Z}}$ where $a_i \in X_i$. This is the direct product $\prod_{i \in \mathbb{Z}} X_i$.

The following example illustrates the meaning of a continuous map between direct products equipped with the product topologies.

Example 5.8. Suppose that $F \colon \prod_{i \in I} X_i \to \prod_{j \in J} Y_j$ is a linear map and each X_i and Y_j is equipped with the discrete topology. Write $F_{j,i}$ for $\Pi_j \circ F \circ I_i$, where Π_j and I_i are projections and inclusions. Then F is continuous with respect to the product topologies if and only if for each j, there are only finitely many i such that $F_{j,i} \neq 0$, and furthermore $F = \sum_{j,i} F_{j,i}$.

5.3 Linear compactness

We now recall the notion of linear compactness. This definition is important for the module categories in Chapter 7 since some of the operations we define (such as various versions of the external tensor product) require the underlying spaces of our modules to be linearly compact. All of the link surgery modules we construct are linearly compact.

Many basic statements about compact sets from point-set topology have analogs for linearly compact vector spaces, though linear compactness and point-set compactness are not equivalent in general. There are several equivalent definitions of linear compactness, though the following is the most useful for our purposes.

Definition 5.9. A linear topological space $\mathcal{X}$ is *linearly compact* if each open subspace $U \subseteq \mathcal{X}$ has finite codimension, i.e., $\dim(\mathcal{X}/U) < \infty$.

Remark 5.10. (1) The notion of linear compactness may be found in the classical work of Lefschetz [22, Chapter 2]. Note that Lefschetz formulates linear compactness in terms of the finite intersection property for closed affine subspaces, though this turns out to be equivalent. Since we will not revisit the work of Lefschetz, we will not describe the equivalence.

(2) The direct product of linearly compact spaces is linearly compact. In particular, $\prod_{\mathbb{N}} \mathbb{F}$ is linearly compact.

5.4 Tensor products

Given linear topological vector spaces $\mathcal{X}$ and $\mathcal{Y}$, there are many different ways to topologize the tensor product $\mathcal{X} \otimes \mathcal{Y}$. This phenomenon goes back to work of Grothendieck [8] in the setting of functional analysis and Banach spaces. Despite fundamental differences between the Banach spaces and linear topological vector spaces, similar phenomena occur in both settings.

In this memoir, we focus on three natural topologies on tensor products:

(1) The *standard topology* $\mathcal{X} \otimes^! \mathcal{Y}$.

(2) The $*$-*topology* $\mathcal{X} \otimes^* \mathcal{Y}$.

(3) The *chiral-topology* $\mathcal{X} \vec{\otimes} \mathcal{Y}$.

The following inclusions are continuous:

$$\mathcal{X} \otimes^* \mathcal{Y} \subseteq \mathcal{X} \vec{\otimes} \mathcal{Y} \subseteq \mathcal{X} \otimes^! \mathcal{Y}.$$

The topologies $\otimes^*$ and $\vec{\otimes}$ are due to Beilinson [2]. Additional exposition and analysis of these constructions is given by Positelski [46, Sections 12 and 13], wherein many basic properties, subtleties and counterexamples to natural conjectures are described. These topologies are defined as follows.

Definition 5.11. Suppose that $\mathcal{X}$ and $\mathcal{Y}$ are linear topological spaces.

(1) A subspace $E \subseteq \mathcal{X} \otimes^! \mathcal{Y}$ is open if and only if the following holds: There are open subspaces $U \subseteq \mathcal{X}$ and $V \subseteq \mathcal{Y}$ such that $U \otimes \mathcal{Y} \subseteq E$ and $\mathcal{X} \otimes V \subseteq E$.

(2) A subspace $E \subseteq \mathcal{X} \otimes^* \mathcal{Y}$ is open if and only if the following holds: There are open subspaces $U \subseteq \mathcal{X}$ and $V \subseteq \mathcal{Y}$ such that $U \otimes V \subseteq E$; and for each $x \in \mathcal{X}$ and $y \in \mathcal{Y}$, there are open subspaces $U_y \subseteq \mathcal{X}$ and $V_x \subseteq \mathcal{Y}$ such that $x \otimes V_x \subseteq E$ and $U_y \otimes y \subseteq E$.

(3) A subspace $E \subseteq \mathcal{X} \vec{\otimes} \mathcal{Y}$ is open if and only if the following holds: There is an open subspace $U \subseteq \mathcal{X}$ so that $U \otimes \mathcal{Y} \subseteq E$; and for each $x \in \mathcal{X}$, there is an open $V_x \subseteq \mathcal{Y}$ such that $x \otimes V_x \subseteq E$.

The topologies $\otimes^*$ and $\otimes^!$ are symmetric between $\mathcal{X}$ and $\mathcal{Y}$, though $\vec{\otimes}$ is asymmetric. The motivation for $\vec{\otimes}$ is that it is a natural topology for an algebra whose topology is given by one-sided ideals. This perspective is of particular importance because our algebra $\mathcal{K}$ is naturally topologized using a family of right ideals. See Lemma 7.7.

Remark 5.12. If $\mathcal{X}$ and $\mathcal{Y}$ are equipped with filtrations $(X_i)_{i\in\mathbb{N}}$ and $(Y_i)_{i\in\mathbb{N}}$, such that $X_0 = \mathcal{X}$ and $Y_0 = \mathcal{Y}$, then the topology on $\mathcal{X} \otimes^! \mathcal{Y}$ is equivalent to the one induced by the filtration $(\mathcal{X} \otimes \mathcal{Y})_n = \sum_{i+j=n} X_i \otimes Y_j$. This is because

$$X_{2n} \otimes \mathcal{Y} \oplus \mathcal{X} \otimes Y_{2n} \subseteq \sum_{i+j=2n} X_i \otimes Y_j \subseteq X_n \otimes \mathcal{Y} \oplus \mathcal{X} \otimes Y_n.$$

This is not in general true if we work with filtrations over $\mathbb{Z}$ instead of $\mathbb{N}$, or if we relax the assumption that $X_0 = \mathcal{X}$ and $Y_0 = \mathcal{Y}$.

5.5 Associativity and commutativity relations

In this section, we recall some basic properties about the tensor products $\otimes^!$, $\vec{\otimes}$ and $\otimes^*$. We begin with associativity.

Lemma 5.13. *If $\circ$ is one of $\{*, !, \rightarrow\}$, then the canonical map*

$$\mathcal{X} \otimes^\circ (\mathcal{Y} \otimes^\circ \mathcal{Z}) \cong (\mathcal{X} \otimes^\circ \mathcal{Y}) \otimes^\circ \mathcal{Z}$$

is a homeomorphism.

The reader may consult [46, Proposition 13.4] for details. In contrast, the different tensor product operations are not mutually associative. Nonetheless, there are some relations between different parenthesizations of the tensor products. One important and well-known example is the following (we give a proof for the benefit of the reader).

Lemma 5.14. *Suppose that $\mathcal{X}_1$, $\mathcal{X}_2$, $\mathcal{Y}_1$, and $\mathcal{Y}_2$ are linear topological vector spaces. The natural map*

$$(\mathcal{X}_1 \otimes^! \mathcal{Y}_1) \vec{\otimes} (\mathcal{X}_2 \otimes^! \mathcal{Y}_2) \rightarrow (\mathcal{X}_1 \vec{\otimes} \mathcal{X}_2) \otimes^! (\mathcal{Y}_1 \vec{\otimes} \mathcal{Y}_2)$$

is continuous.

Proof. A subspace $E \subseteq (\mathcal{X}_1 \vec{\otimes} \mathcal{X}_2) \otimes^! (\mathcal{Y}_1 \vec{\otimes} \mathcal{Y}_2)$ is open if

(s) There are open $U_1 \subseteq \mathcal{X}_1$ and $V_1 \subseteq \mathcal{Y}_1$, and for each $x_1 \in \mathcal{X}_1$ and $y_1 \in \mathcal{Y}_1$ there are open $U_2^{x_1} \subseteq \mathcal{X}_2$ and $V_2^{y_1} \subseteq \mathcal{Y}_2$ so that $\mathcal{X}_1 \otimes \mathcal{X}_2 \otimes V_1 \otimes \mathcal{Y}_2$, $\mathcal{X}_1 \otimes \mathcal{X}_2 \otimes y_1 \otimes V_2^{y_1}$, $U_1 \otimes \mathcal{X}_2 \otimes \mathcal{Y}_1 \otimes \mathcal{Y}_2$, and $x_1 \otimes U_2^{x_1} \otimes \mathcal{Y}_1 \otimes \mathcal{Y}_2$ are contained in E.

A subspace $E \subseteq (\mathcal{X}_1 \otimes^! \mathcal{Y}_1) \vec{\otimes} (\mathcal{X}_2 \otimes^! \mathcal{Y}_2)$ is open if

(t-1) There are open $U_1 \subseteq \mathcal{X}_1$ and $V_1 \otimes \mathcal{Y}_1$ so that $U_1 \otimes \mathcal{Y}_1 \otimes \mathcal{X}_2 \otimes \mathcal{Y}_2$, and $\mathcal{X}_1 \otimes V_1 \otimes \mathcal{X}_2 \otimes \mathcal{Y}_2$ are contained in E.

(t-2) For each $x_1 \in \mathcal{X}_1$ and $y_1 \in \mathcal{Y}_1$, there is an open $U_2^{x_1,y_1} \subseteq \mathcal{X}_2$ so that $x_1 \otimes y_1 \otimes U_2^{x_1,y_1} \otimes \mathcal{Y}_2 \subseteq E$.

(t-3) For each $x_1 \in \mathcal{X}_1$, $y_1 \in \mathcal{Y}_1$, $x_2 \in \mathcal{X}_2$, there is an open $V_2^{x_1,y_1,x_2} \subseteq \mathcal{Y}_2$ so that $x_1 \otimes y_1 \otimes x_2 \otimes V_2^{x_1,y_1,x_2} \subseteq E$.

Any subspace E satisfying (s) satisfies (t-1), (t-2) and (t-3), so the stated map is continuous. ∎

Applying Lemma 5.14 repeatedly gives the following corollary.

Corollary 5.15. *Suppose that* $\mathcal{X}_1, \dots, \mathcal{X}_n$ *and* $\mathcal{Y}_1, \dots, \mathcal{Y}_n$ *are linear topological vector spaces.*

(1) *The map*

$$(\mathcal{X}_1 \otimes^! \mathcal{Y}_1) \vec{\otimes} \cdots \vec{\otimes} (\mathcal{X}_n \otimes^! \mathcal{Y}_n) \to (\mathcal{X}_1 \vec{\otimes} \cdots \vec{\otimes} \mathcal{X}_n) \otimes^! (\mathcal{Y}_1 \vec{\otimes} \cdots \vec{\otimes} \mathcal{Y}_n)$$

is continuous.

(2) *The map*

$$(\mathcal{X}_1 \otimes^! \cdots \otimes^! \mathcal{X}_n) \vec{\otimes} (\mathcal{Y}_1 \otimes^! \cdots \otimes^! \mathcal{Y}_n) \to (\mathcal{X}_1 \vec{\otimes} \mathcal{Y}_1) \otimes^! \cdots \otimes^! (\mathcal{X}_n \vec{\otimes} \mathcal{Y}_n)$$

is continuous.

The following is well known.

Lemma 5.16. *Suppose that* $\mathcal{X}$ *and* $\mathcal{Y}$ *are filtered spaces. If either*

(1) $\mathcal{X}$ *is linearly compact, or*

(2) $\mathcal{Y}$ *is discrete,*

then $\mathcal{X} \vec{\otimes} \mathcal{Y} \cong \mathcal{X} \otimes^! \mathcal{Y}$.

Proof. Consider the case that $\mathcal{X}$ is linearly compact and suppose that $E \subseteq \mathcal{X} \vec{\otimes} \mathcal{Y}$ is an open subspace. By assumption, there is some open subspace $U \subseteq \mathcal{X}$ so that $U \otimes \mathcal{Y} \subseteq E$. Since $\mathcal{X}$ is linearly compact, we may pick $x_1, \dots, x_n \in \mathcal{X}$ spanning a complementary vector space to U. By assumption, there are open subspaces $V_1, \dots, V_n \subseteq \mathcal{Y}$ so that $x_i \otimes V_i \subseteq E$. Therefore

$$U \otimes \mathcal{Y} + \mathcal{X} \otimes (V_1 \cap \cdots \cap V_n) \subseteq U \otimes \mathcal{Y} + x_1 \otimes V_1 + \cdots + x_n \otimes V_n \subseteq E$$

so the map $\mathcal{X} \otimes^! \mathcal{Y} \to \mathcal{X} \vec{\otimes} \mathcal{Y}$ is continuous. The map in the opposite direction is obviously continuous, so they are isomorphic.

The claim when $\mathcal{Y}$ is discrete is similar. ■

Remark 5.17. If both $\mathcal{X}$ and $\mathcal{Y}$ are linearly compact, or both are discrete, then all three tensor products $\mathcal{X} \otimes^! \mathcal{Y}$, $\mathcal{X} \vec{\otimes} \mathcal{Y}$ and $\mathcal{X} \otimes^* \mathcal{Y}$ coincide. Similarly, if $\mathcal{X}$ is linearly compact and discrete (i.e., finite-dimensional), then all three tensor products coincide.

Remark 5.18. Even if $\mathcal{X}$ and $\mathcal{Y}$ are first countable, it may not be the case that $\mathcal{X} \vec{\otimes} \mathcal{Y}$ is first countable. For example, if $\mathcal{X} = \bigoplus_{i \in \mathbb{Z}} \mathbb{F}$ is given the discrete topology and $\mathcal{Y} = \mathbb{F}[\![U]\!]$ (equipped with the U-adic topology) then it is not hard to see that $\mathcal{X} \vec{\otimes} \mathcal{Y}$ is not first countable. See [46, Example 13.1 and Lemma 7.4].

Morphisms may also be tensored. We state the following well-known result.

Lemma 5.19. *All three tensor product operations* $\otimes^!$, $\otimes^*$ *and* $\vec{\otimes}$ *are functorial with respect to morphisms, i.e., if* f *and* g *are continuous morphisms, then* $f \otimes^\circ g$ *is also continuous for* $\circ \in \{!, *, \to\}$.

See [46, Lemma 12.3] for a detailed exposition.

5.6 The tree topology

In this section, we introduce a topology on a tensor product of linear topological spaces indexed by vertices of certain directed trees. We use this notion when considering split Alexander modules in Section 7.4.

We say that a tree Γ is *strongly directed* if each edge is given an orientation, and each vertex has at most one outgoing edge. We write $v > v'$ if there is an oriented sequence of edges from v to v'. A strongly directed tree has a unique minimal vertex, which we refer to as the *root*.

If $(\mathcal{X}_v)_{v \in V(\Gamma)}$ is a collection of linear topological spaces, we will describe a linear topology on the tensor product

$$\mathcal{X}_\Gamma := \bigotimes_{v \in V(\Gamma)} \mathcal{X}_v.$$

The construction works more generally when the spaces $\mathcal{X}_v$ are equipped with commuting actions of rings R_e for each edge e adjacent to v. (For our purposes, R_e will be an idempotent ring of a link algebra.)

We describe the topology after introducing some notation. We consider a set of symbols $\{p, \mathcal{O}, X\}$. We call a function

$$s \colon V(\Gamma) \to \{p, \mathcal{O}, X\}$$

an *admissible labeling* if it satisfies the following:

(s-1) s is not constantly p.

(s-2) If $s(v) = X$, then $s(v') = X$ for all $v' < v$.

(s-3) If $s(v) = p$, then $s(v') = p$ for all $v' > v$.

(s-4) If $s(v) = \mathcal{O}$, then $s(v') = p$ for all $v' > v$, and $s(v') = X$ for all $v' < v$.

(s-5) If $s(v) = X$, then there is a vertex v' with an edge pointing to v such that $s(v') \in \{\mathcal{O}, X\}$.

See Figure 5.1 for an example of an admissible labeling. Note that the last axiom prohibits any maximal leaf from being labeled by X. Furthermore, an admissible labeling must label at least one vertex with $\mathcal{O}$.

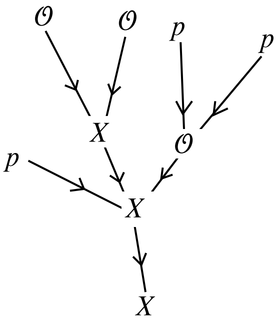


Figure 5.1. An admissible labeling of a strongly directed tree.

Suppose that s is an admissible labeling of Γ. We define a *point-enhancement* $\mathbf{p}$ of s to be a choice of an element $\mathbf{p}(v) \in \mathcal{X}_v$ for each $v \in s^{-1}(p)$. We say an *open-enhancement* $\mathcal{U}$ of s is a choice of open subspaces $\mathcal{U}(v) \subseteq \mathcal{X}_v$ for each $v \in s^{-1}(\mathcal{O})$. Given point- and open-enhancements, $\mathbf{p}$ and $\mathcal{U}$, of s, there is a subspace $\mathcal{W}(\mathcal{U}, \mathbf{p}) \subseteq \mathcal{X}_\Gamma$, obtained by tensoring the spans of $\mathbf{p}(v)$ from $\mathbf{p}$, with the open subspaces $\mathcal{U}(v)$, as well as $\mathcal{X}_v$ ranging over $v \in s^{-1}(X)$.

We now define the *tree topology* on $\mathcal{X}_\Gamma$, when Γ is a strongly directed tree.

Definition 5.20. Suppose that Γ is a strongly directed tree. We say a subspace $E \subseteq \mathcal{X}_\Gamma$ is open if and only if for each admissible labeling s of Γ and each point-enhancement $\mathbf{p}$ of s, there is an open-enhancement $\mathcal{U}_{\mathbf{p}}$ of s so that

$$\mathcal{W}(\mathcal{U}_{\mathbf{p}}, \mathbf{p}) \subseteq E.$$

Example 5.21. If Γ is a linearly ordered graph with vertices $v_n > v_{n-1} > \cdots > v_1$, and $\mathcal{X}_{v_i}$ are spaces, then

$$\mathcal{X}_\Gamma = \mathcal{X}_{v_n} \vec{\otimes} \cdots \vec{\otimes} \mathcal{X}_{v_1}.$$

Remark 5.22. We warn the reader that there are some related (but subtly non-isomorphic) topologies that one can define on these tensor products. As an example, consider a graph Γ with a central root vertex v_0, and two rays pointing into v_0, labeled by vertices $w_1, \ldots, w_n$ and $u_1, \ldots, u_m$. We designate v_0 is minimal, and w_n and u_m are maximal. See the following:

$$\begin{tikzcd}
w_n \arrow[r] & \cdots \arrow[r] & w_1 \arrow[dr] & \\
 & & & v_0. \\
u_m \arrow[r] & \cdots \arrow[r] & u_1 \arrow[ur] &
\end{tikzcd}$$

Note that the topology

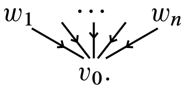
$$\mathcal{Y} := ((\mathcal{X}_{w_1} \vec{\otimes} \cdots \vec{\otimes} \mathcal{X}_{w_n}) \otimes^* (\mathcal{X}_{u_1} \vec{\otimes} \cdots \vec{\otimes} \mathcal{X}_{u_m})) \vec{\otimes} \mathcal{X}_{v_0}$$

is similar to the tree topology $\mathcal{X}_\Gamma$, but is not isomorphic in general. The open subspaces of $\mathcal{Y}$ may also be described in terms of admissible labelings of Γ, however, the description has some subtle differences. For example, given an admissible labeling s with point-enhancements $\mathbf{p}$, the open subspaces in an open-enhancement $\mathcal{U}_\mathbf{p}$ for $\mathcal{X}_\Gamma$ may vary as we change $\mathbf{p}$. For $\mathcal{Y}$, the open set at a vertex labeled $\mathcal{O}$ can only depend on the values of $\mathbf{p}$ for vertices above that vertex. On the other hand, the map

$$\mathcal{X}_\Gamma \to \mathcal{Y}$$

is continuous. Compare Proposition 5.24 and Remark 5.26, below.

Remark 5.23. Suppose that Γ is a star with a minimal root v_0, and n vertices $w_1, \dots, w_n$, each connected to v_0 (and no additional vertices):



If $\mathcal{X}_{v_0}, \mathcal{X}_{w_1}, \dots, \mathcal{X}_{w_n}$ are spaces indexed by the vertices, then

$$\mathcal{X}_\Gamma = (\mathcal{X}_{w_1} \otimes^* \cdots \otimes^* \mathcal{X}_{w_n}) \vec{\otimes} \mathcal{X}_{v_0}.$$

Proposition 5.24. *Suppose that Γ is a strongly directed tree and $\Gamma_0 \subseteq \Gamma$ is a connected subtree. Let $(\mathcal{X}_v)_{v \in V(\Gamma)}$ be a collection of spaces, indexed by vertices of Γ and let $\mathcal{Y}$ be another linear topological space. Suppose that*

$$f \colon \mathcal{X}_{\Gamma_0} \to \mathcal{Y}$$

is a continuous map. Let $\Gamma' = \Gamma / \Gamma_0$ be the strongly directed tree obtained by contracting all vertices of Γ_0 to a single vertex, denoted $[\Gamma_0]$. For $v \in V(\Gamma')$, write $\mathcal{Y}_v$ for $\mathcal{X}_v$ if $v \in V(\Gamma) \setminus V(\Gamma_0)$, and $\mathcal{Y}$ if $v = [\Gamma_0]$. Then the map

$$f \otimes \mathrm{id} \colon \mathcal{X}_\Gamma \to \mathcal{Y}_{\Gamma'}$$

is continuous.

See Figure 5.2 for a schematic.

Proof. We let $\mathcal{V}' \subseteq \mathcal{Y}_{\Gamma'}$ be open. Let s be an admissible labeling of Γ and $\mathbf{p}$ a point-enhancement of s. Our goal is to construct an open-enhancement $\mathcal{U}_\mathbf{p}$ of s so that

$$(f \otimes \mathrm{id})(\mathcal{W}(\mathcal{U}_\mathbf{p}, \mathbf{p})) \subseteq \mathcal{V}'.$$

Our construction of $\mathcal{U}_\mathbf{p}$ varies depending on s.

We partition the admissible labelings s of Γ as follows:

(1) (p-type) $s|_{\Gamma_0} \equiv p$.

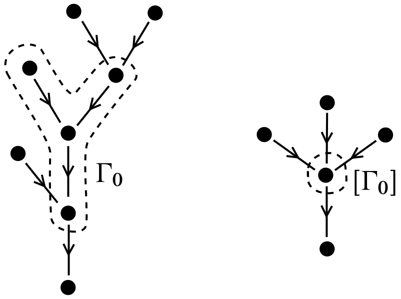


Figure 5.2. The operation considered in Proposition 5.24. On the left is Γ and on the right Γ'. The regions enclosed by dashes indicate Γ_0 and $[\Gamma_0]$.

(2) ($\mathcal{O}$-type) $s|_{\Gamma_0}$ is not constantly p, and all edges pointing into Γ_0 take value p.

(3) (X-type) There is at least one edge pointing into Γ_0 which takes value $\mathcal{O}$ or X.

If s is an admissible labeling, then we define a labeling s' of $V(\Gamma')$ as follows. For $v \neq [\Gamma_0]$, we can view v as a vertex of both Γ and Γ'. We set $s'(v) = s(v)$. If $v = [\Gamma_0]$, we define $s'(v)$ to be p if s is p-type, $\mathcal{O}$ if s is $\mathcal{O}$-type, and X if s is X-type.

We make the following claims:

(1) If s is an admissible labeling of Γ, then s' is an admissible labeling of Γ'.

(2) If s is $\mathcal{O}$-type, then $s|_{\Gamma_0}$ is an admissible labeling.

We leave it to the reader to verify both of the above two claims, as they are straightforward.

We now describe $\mathcal{U}_{\mathbf{p}}$. The construction depends on whether s is p-, $\mathcal{O}$- or X-type. Suppose first that s is p-type. We let s' be the admissible labeling of Γ' constructed above, and we define a point-enhancement $\mathbf{p}'$ of s' by using the points from $\mathbf{p}$ for vertices in $V(\Gamma) \setminus V(\Gamma_0)$, and using $f(\mathbf{p}(v_1) \otimes \cdots \otimes \mathbf{p}(v_n))$ for the vertex $[\Gamma_0]$, where $V(\Gamma_0) = \{v_1, \dots, v_n\}$. Since $\mathcal{V}'$ is open, there is an open-enhancement $\mathcal{U}_{\mathbf{p}'}$ of s' so that

$$\mathcal{W}(\mathcal{U}_{\mathbf{p}'}, \mathbf{p}') \subseteq \mathcal{V}'. \tag{5.1}$$

We can use the same open subspaces from $\mathcal{U}_{\mathbf{p}'}$ to construct an open-enhancement $\mathcal{U}_{\mathbf{p}}$ of s. Equation (5.1) is equivalent to the statement that

$$(f \otimes \mathrm{id})(\mathcal{W}(\mathcal{U}_{\mathbf{p}}, \mathbf{p})) \subseteq \mathcal{V}'.$$

The construction of $\mathcal{U}_{\mathbf{p}}$ when s is X-type is similar. In this case, given a point-enhancement $\mathbf{p}$ of s, we define a point-enhancement $\mathbf{p}'$ of s' by using the points from $\mathbf{p}$ for the vertices in $V(\Gamma') \setminus \{[\Gamma_0]\} = V(\Gamma) \setminus V(\Gamma_0)$. Since $\mathcal{V}'$ is open, there is an open-enhancement $\mathcal{U}_{\mathbf{p}'}$ of s' so that

$$\mathcal{W}(\mathcal{U}_{\mathbf{p}'}, \mathbf{p}') \subseteq \mathcal{V}'.$$

We construct an open-enhancement of s by using the subspaces from $\mathcal{U}_{\mathbf{p}'}$ for $v \notin V(\Gamma_0)$, and using the entire subspaces $\mathcal{X}_v$ for $v \in V(\Gamma_0)$ with $s(v) = \mathcal{O}$. Since $s'([\Gamma_0]) = X$, we have that

$$(f \otimes \mathrm{id})(\mathcal{W}(\mathcal{U}_{\mathbf{p}}, \mathbf{p})) \subseteq \mathcal{W}(\mathcal{U}_{\mathbf{p}'}, \mathbf{p}')$$

which is contained in $\mathcal{V}'$ by construction.

Finally, we consider the case that s is $\mathcal{O}$-type (or equivalently $s'([\Gamma_0]) = \mathcal{O}$). We let $\mathbf{p}$ be a point-enhancement of s. We let $\mathbf{p}'$ be the point-enhancement of s' obtained by restricting $\mathbf{p}$ to the vertices of $V(\Gamma') \setminus \{[\Gamma_0]\}$. Since $\mathcal{V}'$ is open, there is an open-enhancement $\mathcal{U}_{\mathbf{p}'}$ of s' so that

$$\mathcal{W}(\mathcal{U}_{\mathbf{p}'}, \mathbf{p}') \subseteq \mathcal{V}'.$$

Let $U_0' \subseteq \mathcal{Y}$ denote the open subspace $\mathcal{U}_{\mathbf{p}'}([\Gamma_0])$. Since f is continuous, there is an open subspace $U_0 \subseteq \mathcal{X}_{\Gamma_0}$ such that $f(U_0) \subseteq U_0'$.

Write s_0 for $s|_{V(\Gamma_0)}$. As described earlier, s_0 is an admissible labeling. Let $\mathbf{p}_0$ denote the restriction of $\mathbf{p}$ to Γ_0. Since U_0 is open, there is an open-enhancement $\mathcal{U}_0$ of s_0 so that

$$\mathcal{W}(\mathcal{U}_0, \mathbf{p}_0) \subseteq U_0.$$

We now define the open-enhancement $\mathcal{U}_{\mathbf{p}}$ of s as follows. For vertices $w \in V(\Gamma) \setminus V(\Gamma_0)$ with $s(w) = \mathcal{O}$, we set $\mathcal{U}_{\mathbf{p}}(w) = \mathcal{U}_{\mathbf{p}'}(w)$. For the vertices $w \in V(\Gamma_0)$ with $s(w) = \mathcal{O}$, we set $\mathcal{U}_{\mathbf{p}}(w) = \mathcal{U}_0(w)$. By construction

$$(f \otimes \mathrm{id})(\mathcal{W}(\mathcal{U}_{\mathbf{p}}, \mathbf{p})) \subseteq \mathcal{W}(\mathcal{U}_{\mathbf{p}'}, \mathbf{p}') \subseteq \mathcal{V}'.$$

It follows that $(f \otimes \mathrm{id})^{-1}(\mathcal{V}')$ is an open subspace, so the proof is complete. ∎

Remark 5.25. Note that we can extend the above result in several ways.

(1) If $(\mathcal{X}_v)_{v \in V(\Gamma)}$ is a family of linear topological spaces with $\Gamma_0 \subseteq \Gamma$, and if $f\colon \boldsymbol{\mathcal{X}}_{\Gamma_0} \to \boldsymbol{\mathcal{Y}}$ is a continuous map between completions, then there is a well-defined, continuous map

$$f \otimes \mathrm{id}\colon \boldsymbol{\mathcal{X}}_{\Gamma} \to \boldsymbol{\mathcal{Y}}_{\Gamma'}.$$

To see this, note that it suffices to define a map from $\mathcal{X}_\Gamma$ to $\boldsymbol{\mathcal{Y}}_{\Gamma'}$. In this case, observe that if $x \in \mathcal{X}_{\Gamma_0}$, then by definition $f(x) \in \boldsymbol{\mathcal{Y}}$ is the limit of a Cauchy net $\{y_\beta\}_{\beta \in B}$ where $y_\beta \in \mathcal{Y}$. If $\mathbf{x}$ is in the tensor product of the $\mathcal{X}_v$ for $v \notin V(\Gamma_0)$, then we claim that $\{y_\beta \otimes \mathbf{x}\}_{\beta \in B}$ is a Cauchy net in $\mathcal{Y}_{\Gamma'}$. It suffices to prove this when $\mathbf{x}$ is an elementary tensor. Let s' be the admissible labeling on Γ / Γ_0 which assigns $\mathcal{O}$ to $[\Gamma_0]$, assigns

X to each vertex below $[\Gamma_0]$, and assigns p to all other vertices. Let $\mathbf{p}'$ be the point-enhancement of s' which assigns to each vertex w with $s(w') = p$ the corresponding factor from $\mathbf{x}$. If $\mathcal{V}' \subseteq \mathcal{Y}_{\Gamma'}$ is an open subspace, then by definition there is an open-enhancement $\mathcal{U}_{\mathbf{p}'}$ so that

$$\mathcal{W}(\mathcal{U}_{\mathbf{p}'}, \mathbf{p}') \subseteq \mathcal{V}'.$$

The open-enhancement $\mathcal{U}_{\mathbf{p}'}$ consists of a single open subspace $U_0 \subseteq \mathcal{Y}$, since $[\Gamma_0]$ is the only $\mathcal{O}$-labeled vertex. For all sufficiently large β, we have $y_{\beta'} - y_{\beta''} \in U_0$ whenever $\beta', \beta'' \geq \beta$. It follows that $y_{\beta'} \otimes \mathbf{x} - y_{\beta''} \otimes \mathbf{x} \in \mathcal{W}(\mathcal{U}_{\mathbf{p}'}, \mathbf{p}')$ so $\{y_\beta \otimes \mathbf{x}\}_{\beta \in B}$ is a Cauchy sequence in $\mathcal{Y}$. We therefore define $(f \otimes \mathrm{id})(x \otimes \mathbf{x})$ to be the limit of the Cauchy net $\{y_\beta \otimes \mathbf{x}\}_{\beta \in B}$. A similar argument to the previous paragraph shows that this definition of $(f \otimes \mathrm{id})$ is well defined (independent of the choice of net $\{y_\beta\}$). A straightforward extension of Proposition 5.24 shows that $(f \otimes \mathrm{id})\colon \boldsymbol{\mathcal{X}}_\Gamma \to \boldsymbol{\mathcal{Y}}_{\Gamma'}$ is continuous.

(2) It follows from the previous argument that the completion $\mathcal{X}_\Gamma$ is unchanged if we replace one $\mathcal{X}_v$ in our family of spaces by its completion $\boldsymbol{\mathcal{X}}_v$. To see this, let $\mathcal{X}'_\Gamma$ denote the tree complex where one $\mathcal{X}_v$ is replaced by $\boldsymbol{\mathcal{X}}_v$. There is a natural map $\mathcal{X}_\Gamma \to \mathcal{X}'_\Gamma$. Applying the previous remark, there is a natural map in the opposite direction, because the identity map gives a continuous map from the completion of $\boldsymbol{\mathcal{X}}_v$ to the completion of $\mathcal{X}_v$. These maps are easily seen to be inverses.

We highlight an important application of Proposition 5.24.

Remark 5.26. If Γ is a strongly directed graph and $(\mathcal{X}_v)_{v \in V(\Gamma)}$ is a family of spaces, we can form another collection of spaces by “adding parentheses”, in the following sense. Let $\Gamma_0 \subseteq \Gamma$ be a tree, and consider the tree Γ/Γ_0 obtained by collapsing to vertices and edged of Γ_0 to a single vertex, denoted $[\Gamma_0]$. We can define a collection of spaces indexed by Γ/Γ_0, by setting $\mathcal{Y}_v = \mathcal{X}_v$ if $v \notin V(\Gamma_0)$ and $\mathcal{Y}_{[\Gamma_0]} = \mathcal{X}_{\Gamma_0}$. Proposition 5.24 shows that the natural map

$$\mathbb{I}\colon \mathcal{X}_\Gamma \to \mathcal{Y}_{\Gamma/\Gamma_0}$$

is continuous.

In some cases, there is an isomorphism $\mathcal{X}_\Gamma \cong \mathcal{Y}_{\Gamma/\Gamma_0}$. We focus on the following simple case.

Lemma 5.27. *Let Γ be a strongly directed tree and $(\mathcal{X}_v)_{v \in V(\Gamma)}$ a family of spaces and let v_0 and v_1 be edges of Γ which are connected by an edge. Assume that $v_1 > v_0$ and that v_0 has no other incoming edges. See Figure 5.3. Let Γ_0 consist of a subgraph with vertices v_0 and v_1 and the edge connecting them. Let $\mathcal{Y}_{\Gamma/\Gamma_0}$ be the space above. If $\mathcal{X}_{v_1}$ is linearly compact, then there is an isomorphism*

$$\mathcal{X}_\Gamma \cong \mathcal{Y}_{\Gamma/\Gamma_0}.$$

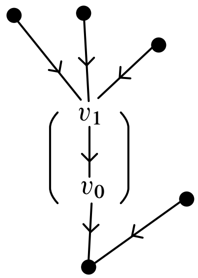


Figure 5.3. A tree from Lemma 5.27. Here, Γ_0 denotes the parenthesized subtree.

Proof. Remark 5.26 shows that the natural map

$$\mathbb{I}\colon \mathcal{X}_\Gamma \to \mathcal{Y}_{\Gamma/\Gamma_0}$$

is continuous. Therefore, it suffices to show that the natural map

$$\mathbb{I}\colon \mathcal{Y}_{\Gamma/\Gamma_0} \to \mathcal{X}_\Gamma$$

is continuous. Equivalently, it suffices to show that each open subspace of $\mathcal{X}_\Gamma$ contains an open subspace of $\mathcal{Y}_{\Gamma/\Gamma_0}$. Let $\mathcal{V} \subseteq \mathcal{X}_\Gamma$ be an open subspace.

We consider admissible labelings s on Γ/Γ_0. We wish to show that for each admissible labeling s' of Γ/Γ_0 and each point-enhancement $\mathbf{p}'$ of s', there is an open-enhancement $\mathcal{U}_{\mathbf{p}'}$ of s' so that

$$\mathcal{W}(\mathcal{U}_{\mathbf{p}'}, \mathbf{p}') \subseteq \mathcal{V}.$$

We break the proof into three cases, depending on whether $s([\Gamma_0])$ is p, $\mathcal{O}$ or X.

If $s([\Gamma_0]) = X$, then we define a labeling s on Γ by setting $s(v_0) = s(v_1) = X$ and setting $s(w) = s'(w)$ for other vertices. If $\mathbf{p}'$ is a point-enhancement of s', then we can view $\mathbf{p}'$ also as a point-enhancement of s, for which we write $\mathbf{p}$. Since $\mathcal{V}$ is open, there is an open-enhancement $\mathcal{U}_{\mathbf{p}}$ so that

$$\mathcal{W}(\mathcal{U}_{\mathbf{p}}, \mathbf{p}) \subseteq \mathcal{V}.$$

Since v_0 and v_1 are labeled X by s', we use the same open subspaces as in $\mathcal{U}_{\mathbf{p}}$ to form an open-enhancement $\mathcal{U}_{\mathbf{p}'}$ of s'. We observe that

$$\mathcal{W}(\mathcal{U}_{\mathbf{p}'}, \mathbf{p}') = \mathcal{W}(\mathcal{U}_{\mathbf{p}}, \mathbf{p}) \subseteq \mathcal{V}.$$

Next, we consider the case that $s'([\Gamma_0]) = p$. In this case, we form an admissible labeling s on Γ by setting $s(v_0) = s(v_1) = p$, and defining s and s' to coincide at other vertices. We let $\mathbf{p}'$ be a point-enhancement of s'. There is a point assigned to $[\Gamma_0]$, which consists of an element of the tensor product $\mathcal{X}_{v_0} \otimes \mathcal{X}_{v_1}$. We can write this as a finite sum of elementary tensors $\sum_{i=1}^{n} p_i \otimes q_i \in \mathcal{X}_{v_0} \otimes \mathcal{X}_{v_1}$. For each elementary tensor $p_i \otimes q_i$, we obtain a point-enhancement $\mathbf{p}_i$ of Γ by using the elements of $\mathbf{p}'$ for p-labeled vertices other than v_0 and v_1, and by using p_i for v_0 and q_i for v_1. Since $\mathcal{V}$ is open, there is an open-enhancement $\mathcal{U}_{\mathbf{p}_i}$ of s so that

$$\mathcal{W}(\mathcal{U}_{\mathbf{p}_i}, \mathbf{p}_i) \subseteq \mathcal{V}.$$

We construct an open-enhancement $\mathcal{U}_{\mathbf{p}'}$ of s' as follows. For a vertex $v \in V(\Gamma) \setminus \{v_0, v_1\}$ we set $\mathcal{U}_{\mathbf{p}'}(v) = \mathcal{U}_{\mathbf{p}_1}(v) \cap \cdots \cap \mathcal{U}_{\mathbf{p}_n}(v)$. We observe that

$$\mathcal{W}(\mathcal{U}_{\mathbf{p}'}, \mathbf{p}') \subseteq \mathcal{W}(\mathcal{U}_{\mathbf{p}_1}, \mathbf{p}_1) + \cdots + \mathcal{W}(\mathcal{U}_{\mathbf{p}_n}, \mathbf{p}_n) \subseteq \mathcal{V}.$$

Finally, we consider the case that $s'([\Gamma_0]) = \mathcal{O}$. If $s'([\Gamma_0]) = \mathcal{O}$, then there are two labelings to consider on Γ. We denote these by $s_{p,\mathcal{O}}$ and $s_{\mathcal{O},X}$. The labeling $s_{p,\mathcal{O}}$ has $s_{p,\mathcal{O}}(v_1) = p$ and $s_{p,\mathcal{O}}(v_0) = \mathcal{O}$. The labeling $s_{\mathcal{O},X}$ has $s_{\mathcal{O},X}(v_1) = \mathcal{O}$ and $s_{\mathcal{O},X}(v_0) = X$. Both $s_{p,\mathcal{O}}$ and $s_{\mathcal{O},X}$ agree with s' for other vertices. It is straightforward to see that $s_{p,\mathcal{O}}$ and $s_{\mathcal{O},X}$ are admissible.

Since the s' and $s_{\mathcal{O},X}$ have the same p-labeled vertices, the collection $\mathbf{p}'$ induces a point-enhancement $\mathbf{p}$ (using the same points) of $s_{\mathcal{O},X}$. Since $\mathcal{V}$ is open, there is an open-enhancement $\mathcal{U}_{\mathbf{p}}$ of $s_{\mathcal{O},X}$ so that

$$\mathcal{W}(\mathcal{U}_{\mathbf{p}}, \mathbf{p}) \subseteq \mathcal{V}.$$

Let U_{v_1} be the open set assigned to v_1.

Since $\mathcal{X}_{v_1}$ is linearly compact, by definition there are elements $p_1, \dots, p_n \in \mathcal{X}_{v_1}$ so that

$$\mathrm{Span}_{\mathbb{F}}(p_1, \dots, p_n) + U_{v_1} = \mathcal{X}_{v_1}. \tag{5.2}$$

We now construct point-enhancements $\mathbf{p}_1, \dots, \mathbf{p}_n$ of $s_{p,\mathcal{O}}$. We use the points from $\mathbf{p}'$ for the p-labeled vertices other than v_0, and we use p_i for v_0. By definition, there are open-enhancements $\mathcal{U}_{\mathbf{p}_1}, \dots, \mathcal{U}_{\mathbf{p}_n}$ so that

$$\mathcal{W}(\mathcal{U}_{\mathbf{p}_i}, \mathbf{p}_i) \subseteq \mathcal{V}.$$

Write $U_{v_0,i} = \mathcal{U}_{\mathbf{p}_i}(v_0)$.

We now define an open-enhancement $\mathcal{U}_{\mathbf{p}'}$ of s'. For a vertex $v \neq [\Gamma_0]$ assigned $\mathcal{O}$, we use the open set

$$\mathcal{U}_{\mathbf{p}}(v) \cap \mathcal{U}_{\mathbf{p}_1}(v) \cap \cdots \cap \mathcal{U}_{\mathbf{p}_n}(v).$$

For the vertex $[\Gamma_0]$, we use the open set

$$U_{v_1} \otimes \mathcal{X}_{v_2} + \mathcal{X}_{v_1} \otimes (U_{v_0,1} \cap \cdots \cap U_{v_0,n}),$$

which is open in $\mathcal{X}_{\Gamma_0} \cong \mathcal{X}_{v_1} \vec{\otimes} \mathcal{X}_{v_0}$ by Lemma 5.16 because $\mathcal{X}_{v_1}$ is linearly compact.

We observe that

$$\mathcal{W}(\mathcal{U}_{\mathbf{p}'}, \mathbf{p}') \subseteq \mathcal{W}(\mathcal{U}_{\mathbf{p}}, \mathbf{p}) + \mathcal{W}(\mathcal{U}_{\mathbf{p}_1}, \mathbf{p}_1) + \cdots + \mathcal{W}(\mathcal{U}_{\mathbf{p}_n}, \mathbf{p}_n) \subseteq \mathcal{V}. \tag{5.3}$$

The above equation is proven as follows. For vertices $V(\Gamma/\Gamma_0)$ which are labeled p by s', we use the same points in all of $\mathbf{p}'$, $\mathbf{p}$ and $\mathbf{p}_i$. For the vertices of $V(\Gamma/\Gamma_0)$ which are labeled $\mathcal{O}$, other than $[\Gamma_0]$, we use the intersection of the opens from $\mathcal{U}_{\mathbf{p}}$, $\mathcal{U}_{\mathbf{p}_1}, \ldots, \mathcal{U}_{\mathbf{p}_n}$, so these factors are contained in the corresponding factors of each summand in the middle of equation (5.3). Finally, for v_0 and v_1, we observe that

$$\begin{aligned} &U_{v_1} \otimes \mathcal{X}_{v_2} + \mathcal{X}_{v_1} \otimes (U_{v_0,1} \cap \cdots \cap U_{v_0,n}) \\ &\qquad \subseteq U_{v_1} \otimes \mathcal{X}_{v_2} + p_1 \otimes U_{v_0,1} + \cdots p_n \otimes U_{v_0,n} \end{aligned}$$

by equation (5.2). Together, the above implies equation (5.3). We conclude that $\mathcal{V}$ is open in $\mathcal{X}_{\Gamma/\Gamma_0}$, completing the proof. ■