Chapter 6

The link surgery formula

In this chapter, we give some background on the link surgery formula of Manolescu and Ozsváth [32], and prove some important preliminary results.

6.1 Statement of the link surgery formula

In this section, we state Manolescu and Ozsváth's link surgery formula. We will describe more detail about the construction in Chapter 9.

If $L \subseteq S^3$ is a link, write $K_1, \dots, K_\ell$ for the components of L. Manolescu and Ozsváth define the affine lattice over $\mathbb{Z}^\ell$

$$\mathbb{H}(L) := \prod_{i=1}^{\ell} \left(\frac{\operatorname{lk}(K_i, L \setminus K_i)}{2} + \mathbb{Z} \right).$$

Whenever $M \subseteq L$, Manolescu and Ozsváth define a reduction map

$$\psi^M : \mathbb{H}(L) \to \mathbb{H}(L \setminus M)$$

as follows. If $K_{i_1} \cup \dots \cup K_{i_j} = L \setminus M$, then they set

$$\psi^M(\mathbf{s}) = (\psi^M_{i_1}(s_{i_1}), \dots, \psi^M_{i_j}(s_{i_j})),$$

where

$$\psi^M_i(s_i) = s_i - \frac{\operatorname{lk}(K_i, M)}{2}.$$

Manolescu and Ozsváth also define a version of the map ψ^M for oriented sublinks of L. See [32, Section 3.7]. The above map ψ^M corresponds to the case that all components of M are oriented consistently with L.

Definition 6.1. Suppose $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{w}, \mathbf{z}, \mathbf{p})$ is a Heegaard link diagram for (Y, L) with free base points $\mathbf{p}$, and link base points $\mathbf{w} \cup \mathbf{z}$. We say that $\mathcal{H}$ is *link minimal* if each link component of L has exactly one base point from $\mathbf{w}$, and one base point from $\mathbf{z}$.

Let $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{w}, \mathbf{z})$ be a link minimal Heegaard diagram for an oriented link $L \subseteq S^3$ with no free base points. If $M \subseteq L$, we write $\mathcal{H}^{L \setminus M}$ for the diagram obtained by deleting $z_i \in \mathbf{z}$ for each component $K_i \subseteq L$. Note that Manolescu also defines a reduction $\mathcal{H}^{L \setminus \vec{M}}$ when $\vec{M}$ is equipped with an orientation. In this case, we delete z_i for each positively oriented component of M, and we delete w_i for each negatively oriented component of M. The diagram $\mathcal{H}^{L\setminus M}$ is a link minimal diagram with $|M|$ free base points.

Recall that if $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{w}, \mathbf{z})$ is a link minimal Heegaard link diagram for L, then there is an ℓ-component Alexander grading

$$A = (A_1, \dots, A_\ell)\colon \mathbb{T}_\alpha \cap \mathbb{T}_\beta \to \mathbb{H}(L).$$

Suppose $\mathcal{H}$ is a link minimal Heegaard diagram for $L \subseteq S^3$ and $\mathcal{H}$ has no free base points. If $\mathbf{s} \in \mathbb{H}(L)$, then Manolescu and Ozsváth define

$$\mathfrak{A}^-(\mathcal{H}^{L\setminus M}, \mathbf{s})$$

to be the free $\mathbb{F}[\![U_1, \dots, U_\ell]\!]$-module generated by monomials

$$U_1^{i_1} \cdots U_n^{i_\ell} \cdot \mathbf{x},$$

where $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, which satisfy $i_j \geq 0$ for all j, and $A_j(\mathbf{x}) \leq s_j + i_j$ whenever $K_j \subseteq L \setminus M$. Manolescu and Ozsváth equip $\mathfrak{A}^-(\mathcal{H}^{L\setminus M}, \mathbf{s})$ with the differential which counts Maslov index 1 holomorphic disks, weighted by the product of $U_i^{n_{w_i}(\phi)}$.

We will also occasionally write

$$A^-(\mathcal{H}^{L\setminus M}, \mathbf{s})$$

for the free $\mathbb{F}[U_1, \dots, U_n]$-module which has the same generators as $\mathfrak{A}^-(\mathcal{H}^{L\setminus M}, \mathbf{s})$.

Let Λ denote an integral framing on L. The underlying group of the link surgery complex is defined to be

$$\mathcal{C}_\Lambda(L) = \bigoplus_{M \subseteq L} \prod_{\mathbf{s} \in \mathbb{H}(L)} \mathfrak{A}^-(\mathcal{H}^{L\setminus M}, \psi^M(\mathbf{s})).$$

The differential $\mathcal{D}\colon \mathcal{C}_\Lambda(L) \to \mathcal{C}_\Lambda(L)$ decomposes as a sum

$$\mathcal{D} = \sum_{N \subseteq L} \sum_{\vec{N} \in \Omega(N)} \Phi^{\vec{N}},$$

where $\Omega(N)$ denotes the set of orientations on N.

Given an oriented sublink $\vec{N} \subseteq L$ (with orientation potentially different from L), Manolescu and Ozsváth define

$$\Lambda_{L,\vec{N}} \in \mathbb{Z}^\ell \cong H_1(S^3 \setminus L)$$

to be the sum of the longitudes of the negatively oriented components of $\vec{N}$.

Suppose $\vec{N} \subseteq L$ is an oriented sublink, and that $L_0 \subseteq L$ is a sublink which contains N. Write $L_1 = L_0 \setminus N$. In this case, the summand $\Phi^{\vec{N}}$ of $\mathcal{D}$ maps $\mathfrak{A}^-(\mathcal{H}^{L_0}, \psi^{L\setminus L_0}(\mathbf{s}))$ to $\mathfrak{A}^-(\mathcal{H}^{L_1}, \psi^{L\setminus L_1}(\mathbf{s} + \Lambda_{L,\vec{N}}))$.

We write $\Phi^{\vec{N}}_{L_0,L_1}$ for the component of $\Phi^{\vec{N}}$ as follows:

$$\Phi^{\vec{N}}_{L_0,L_1}\colon \prod_{\mathbf{s}\in\mathbb{H}(L)} \mathfrak{A}^-(\mathcal{H}^{L_0},\psi^{L\setminus L_0}(\mathbf{s})) \to \prod_{\mathbf{s}\in\mathbb{H}(L)} \mathfrak{A}^-(\mathcal{H}^{L_1},\psi^{L\setminus L_1}(\mathbf{s}+\Lambda_{L,\vec{N}})).$$

Finally, we note that it is often convenient to view $\mathcal{C}_\Lambda(L,\mathcal{A})$ as a 1-dimensional mapping cone, so we establish some notation for this purpose. If $J\subseteq L$ is a component, and $\nu\in\{0,1\}$, then we write $\mathcal{C}^\nu_\Lambda(L)$ for the subcube consisting of $\mathcal{C}_\varepsilon\subseteq\mathcal{C}_\Lambda(L,\mathcal{A})$ ranging over ε with J component equal to ν. We may decompose

$$\mathcal{C}_\Lambda(L)\cong \operatorname{Cone}\big(\mathcal{C}^0_\Lambda(L)\xrightarrow{F^J+F^{-J}}\mathcal{C}^1_\Lambda(L)\big).$$

In the above, F^J (resp. F^{-J}) denotes the sum of $\Phi^{\vec{N}}$ for every oriented sublink $\vec{N}$ of L such that $J\subseteq\vec{N}$ (resp. $-J\subseteq\vec{N}$).

We will review the construction of the hypercube maps $\Phi^{\vec{N}}$ in more detail in Chapter 9.

6.2 Gradings and algebraic reduction maps

In this section, we state several grading formulas which are useful when working with the link surgery formula.

Definition 6.2. Suppose that $\mathcal{H}=(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta},\mathbf{w},\mathbf{z},\mathbf{p})$ is a link minimal Heegaard diagram, where $\mathbf{w}\cup\mathbf{z}$ are link base points and $\mathbf{p}$ are free base points. We call a subset $\mathcal{W}\subseteq\mathbf{w}\cup\mathbf{z}\cup\mathbf{p}$ a *complete set of base points* on $\mathcal{H}$ if $\mathbf{p}\subseteq\mathcal{W}$ and $\mathcal{W}$ contains exactly one base point from each link component.

Suppose $\mathcal{H}$ is a diagram for a link L in Y, each of whose components is null-homologous, and $\mathfrak{s}\in\operatorname{Spin}^c(Y)$ is a torsion. Given a complete set of base points $\mathcal{W}$ on $\mathcal{H}$, there is a well-defined Maslov grading $\mathrm{gr}_{\mathcal{W}}$ on $\mathcal{CFL}(\mathcal{H},\mathfrak{s})$. The grading $\mathrm{gr}_{\mathcal{W}}$ is induced from the absolute grading of $\mathrm{CF}^-(Y,\mathcal{W},\mathfrak{s})$ by declaring the natural quotient map $\mathcal{CFL}(\mathcal{H},\mathfrak{s})\to\mathrm{CF}^-(Y,\mathcal{W},\mathfrak{s})$ to be grading preserving. There is also an ℓ-component Alexander grading $A^{\mathcal{W}}=(A^{\mathcal{W}}_1,\dots,A^{\mathcal{W}}_\ell)$. Note that a complete set of base points also determines an orientation on L, via the convention that L intersects the Heegaard surface Σ negatively at the link base points in $\mathcal{W}$. In particular, we may equivalently specify the Alexander grading A^L by picking an orientation on L.

The variable $\mathscr{U}_i$ has $\mathrm{gr}_{\mathcal{W}}$-grading -2 if $w_i\in\mathcal{W}$, and 0 if $w_i\notin\mathcal{W}$. Similarly, $\mathscr{V}_i$ has $\mathrm{gr}_{\mathcal{W}}$-grading -2 if $z_i\in\mathcal{W}$ and 0 otherwise. The variable U_i for a free base point $p_i\in\mathbf{p}$ has $\mathrm{gr}_{\mathcal{W}}$-grading -2.

Remark 6.3. There are two natural conventions for defining the absolute grading for multi-pointed 3-manifolds. We recall that adding a base point has the effect on $\widehat{\mathrm{HF}}$

of tensoring with a 2-dimensional vector space $V = \mathbb{F} \oplus \mathbb{F}$. See [41, Section 6]. In this memoir, we use the convention that the top degree element of V has absolute grading 0. This is different from the TQFT convention of [54], where V is supported in gradings $1/2$ and $-1/2$.

The following is implicit in the work of Manolescu and Ozsváth [32] (see, in particular, [32, Section 12]). In particular, the results are not new to our memoir, however the presentation is helpful.

Lemma 6.4. *Let $L = K_1 \cup \cdots \cup K_\ell$ denote an ℓ-component link in a 3-manifold Y, such that each component is null-homologous, and suppose Y is equipped with a torsion* Spin^c *structure $\mathfrak{s}$. Let $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{w}, \mathbf{z}, \mathbf{p})$ be a link minimal Heegaard diagram, where* $\mathbf{w}$ *and* $\mathbf{z}$ *are link base points and* $\mathbf{p}$ *are free base points. Let $\mathcal{C}$ denote the link Floer complex $\mathcal{CFL}(\mathcal{H}, \mathfrak{s})$ over $\mathbb{F}[\mathscr{U}_1, \dots, \mathscr{U}_\ell, \mathscr{V}_1, \dots, \mathscr{V}_\ell, U_1, \dots, U_n]$, where $\ell = |\mathbf{w}| = |\mathbf{z}|$ and $n = |\mathbf{p}|$.*

(1) *If $\mathcal{W}$ is a complete collection of base points on $\mathcal{H}$, and $z_i \in \mathbf{z}$ but $z_i \notin \mathcal{W}$, then the quotient map*
$$\mathcal{Q}_{\mathscr{V}_i} : \mathcal{C} \to \mathcal{C}/(\mathscr{V}_i - 1)$$
is grading preserving with respect to $\operatorname{gr}_{\mathcal{W}}$. *Similarly, the quotient map*
$$\mathcal{Q}_{\mathscr{U}_i} : \mathcal{C} \to \mathcal{C}/(\mathscr{U}_i - 1),$$
is grading preserving if $w_i \in \mathbf{w}$ but $w_i \notin \mathcal{W}$.

(2) *Suppose that K_j is a component of L, and let $\mathcal{W}_0 \cup \{w_j\}$ be a complete collection of base points, and write $\{w_j, z_j\} = K_j \cap (\mathbf{w} \cup \mathbf{z})$. Let L and K_j have orientations induced by $\mathcal{W}_0 \cup \{w_j\}$. Then*
$$\operatorname{gr}_{\mathcal{W}_0 \cup \{w_j\}} - \operatorname{gr}_{\mathcal{W}_0 \cup \{z_j\}} = 2A_j^{\mathcal{W}_0 \cup \{w_j\}} - \operatorname{lk}(L \setminus K_j, K_j).$$

(3) *Suppose that $\mathcal{W}$ and $\mathcal{W}'$ are two complete collections of base points which coincide at index j. Then*
$$A_j^{\mathcal{W}} = A_j^{\mathcal{W}'}.$$

(4) *Let $K_i \subseteq L$, and orient both K_i and L to intersect Σ negatively at* $\mathbf{w}$, *and suppose $i \neq j$. Give $\mathcal{C}$ and $\mathcal{C}/(\mathscr{V}_i - 1)$ the Alexander grading A_j induced by the base points* $\mathbf{w}$ *and* $\mathbf{w} \setminus \{w_i\}$. *For $j \neq i$,*
$$A_j(\mathcal{Q}_{\mathscr{V}_i}) = -\frac{1}{2}\operatorname{lk}(K_i, K_j).$$

(5) *Let $K_i \subseteq L$, and orient K_i and L to intersect Σ negatively at* $\mathbf{w}$, *and suppose $j \neq i$. Give $\mathcal{C}$ and $\mathcal{C}/(\mathscr{U}_i - 1)$ the Alexander grading A_j induced by the base points* $\mathbf{w}$ *and* $\mathbf{w} \setminus \{w_i\}$, *respectively. Then,*
$$A_j(\mathcal{Q}_{\mathscr{U}_i}) = \frac{1}{2}\operatorname{lk}(K_i, K_j).$$

Proof. Following [54], the Alexander and Maslov gradings may be defined by representing $Y \setminus N(L)$ as Dehn surgery on a framed link J in the complement of an ℓ-component unlink in S^3. We pick a Heegaard triple $\mathcal{T} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}_0, \boldsymbol{\beta}, \mathbf{w}, \mathbf{z})$ which represents the 2-handle cobordism $X(J)$. We pick a triangle class $\psi \in \pi_2(\mathbf{x}, \mathbf{y}, \mathbf{z})$ on $\mathcal{T}$. In [54, Section 5.5], the following formulas for the Maslov and Alexander gradings are proven:

$$\begin{aligned} \operatorname{gr}_{\mathcal{W}}(\mathbf{z}) &= \operatorname{gr}_{\mathcal{W}}(\mathbf{x}) + \operatorname{gr}_{\mathcal{W}}(\mathbf{y}) - \frac{1}{2}(g(\Sigma) + |\mathcal{W}| - 1) - \mu(\psi) + 2n_{\mathcal{W}}(\psi) \\ &\quad + \frac{c_1(\mathfrak{s}_{\mathcal{W}}(\psi))^2 - 2\chi(X(J)) - 3\sigma(X(J))}{4}, \\ A_j^{\mathcal{W}}(\mathbf{z}) &= A_j^{\mathcal{W}}(\mathbf{x}) + A_j^{\mathcal{W}}(\mathbf{y}) + (n_{\mathcal{W}} - n_{(\mathbf{w}\cup\mathbf{z}\cup\mathbf{p})\setminus\mathcal{W}})(\psi) \\ &\quad + \frac{\langle c_1(\mathfrak{s}_{\mathcal{W}}(\psi)), \widehat{S}_j\rangle - [\widehat{S}]\cdot[\widehat{S}_j]}{2}. \end{aligned} \tag{6.1}$$

In the above, if W is obtained by attaching 2-handles to $S^3 \times [0,1]$ in the complement of an unlink with components $u_1 \cup \cdots \cup u_\ell$, then S_j is the surface $u_j \times [0,1]$ and $S = S_1 + \cdots + S_\ell$. The classes $\widehat{S}$ and $\widehat{S}_j$ are obtained by capping these surfaces with Seifert surfaces in either end.

With the above in place, claim (1) follows because the formula for $\operatorname{gr}_{\mathcal{W}}$ does not change if we delete a base point which is not in $\mathcal{W}$.

Consider now claim (2). An identical argument to [54, Lemma 3.8] implies that if ψ is a homology class of triangles on the surgery diagram for the above cobordism, then

$$\mathfrak{s}_{\mathcal{W}_0\cup\{w_j\}}(\psi) - \mathfrak{s}_{\mathcal{W}_0\cup\{z_j\}}(\psi) = \mathrm{PD}[S_j]. \tag{6.2}$$

We may use the formula in equation (6.1) to evaluate $\operatorname{gr}_{\mathcal{W}_0\cup\{w_j\}} - \operatorname{gr}_{\mathcal{W}_0\cup\{z_j\}}$. Firstly, we recall that by straightforward computation $\widehat{\mathit{HFL}}(\boldsymbol{\alpha}, \boldsymbol{\beta}_0, \mathbf{w}, \mathbf{z}, \mathbf{p})$ is isomorphic to $(\mathbb{F} \oplus \mathbb{F})^{\otimes(|\mathbf{w}|+|\mathbf{p}|-1)}$. This computation can be obtained by using topological invariance of the Heegaard Floer group, and picking a convenient genus 0 Heegaard diagram, so that each w_i is immediately adjacent to z_i, and so that the attaching curves come in isotopic pairs. Similarly, $\widehat{\mathit{HFL}}(\boldsymbol{\beta}_0, \boldsymbol{\beta}, \mathbf{w}, \mathbf{z}, \mathbf{p})$ is isomorphic to $(\mathbb{F} \oplus \mathbb{F})^{\otimes(g(\Sigma)-|J|+|\mathbf{w}|+|\mathbf{p}|-1)}$. Furthermore, for both groups, the non-zero elements of homology decompose into homogeneously graded summands which have the property that

$$\operatorname{gr}_{\mathcal{W}}(\mathbf{x}) = \operatorname{gr}_{\mathcal{W}'}(\mathbf{x}) \quad \text{and} \quad A^{\mathcal{W}}(\mathbf{x}) = A^{\mathcal{W}'}(\mathbf{x}) = 0 \tag{6.3}$$

for all complete collections $\mathcal{W}, \mathcal{W}' \subseteq \mathbf{w} \cup \mathbf{z} \cup \mathbf{p}$. Therefore, in our grading formulas, we may assume that the intersection points $\mathbf{x}$ and $\mathbf{y}$ satisfy equation (6.3). Using equation (6.2), we therefore obtain

$$\begin{aligned} &\operatorname{gr}_{\mathcal{W}_0\cup\{w_j\}}(\mathbf{z}) - \operatorname{gr}_{\mathcal{W}_0\cup\{z_j\}}(\mathbf{z}) \\ &\quad = \frac{c_1^2(\mathfrak{s}_{\mathcal{W}_0\cup\{w_j\}}(\psi)) - (c_1(\mathfrak{s}_{\mathcal{W}_0\cup\{w_j\}}(\psi)) - 2\mathrm{PD}[S_j])^2}{4} - 2(n_{z_j} - n_{w_j})(\psi). \end{aligned}$$

We may rearrange the above to see that

$$\begin{aligned}
&\mathrm{gr}_{\mathcal{W}_0\cup\{w_j\}}(\mathbf{z})-\mathrm{gr}_{\mathcal{W}_0\cup\{z_j\}}(\mathbf{z})\\
&\quad=\langle c_1(\mathfrak{s}_{\mathcal{W}_0\cup\{w_j\}}(\psi)),\widehat{S}_j\rangle-[\widehat{S}_j]\cdot[\widehat{S}_j]-2(n_{z_j}-n_{w_j})(\psi)\\
&\quad=\langle c_1(\mathfrak{s}_{\mathcal{W}_0\cup\{w_j\}}(\psi)),\widehat{S}_j\rangle-[\widehat{S}]\cdot[\widehat{S}_j]-2(n_{z_j}-n_{w_j})(\psi)+[\widehat{S}\setminus\widehat{S}_j]\cdot[\widehat{S}]\\
&\quad=2A_j^{\mathcal{W}_0\cup\{w_j\}}(\mathbf{z})+[\widehat{S}\setminus\widehat{S}_j]\cdot[\widehat{S}_j]\\
&\quad=2A_j^{\mathcal{W}_0\cup\{w_j\}}(\mathbf{z})-\mathrm{lk}(L\setminus K_j,K_j).
\end{aligned}$$

In the last line, we used the equality $[\widehat{S}\setminus\widehat{S}_j]\cdot[\widehat{S}_j]=-\mathrm{lk}(L\setminus K_j,K_j)$, which follows since we capped with negative Seifert surfaces of L (and since we are using the outward normal first convention for boundary orientations).

We now consider claim (3). We consider the case when $\mathcal{W}$ and $\mathcal{W}'$ differ only at a single index $i\neq j$. Assume that $\mathcal{W}$ contains w_i and that $\mathcal{W}'$ contains z_i. Write L' for L with the orientation of K_i reversed. Computing directly from the definition, shows that

$$2(A_j^{\mathcal{W}}-A_j^{\mathcal{W}'})=\langle c_1(\mathfrak{s}_{\mathcal{W}}(\psi))-c_1(\mathfrak{s}_{\mathcal{W}'}(\psi)),\widehat{S}_j\rangle-\widehat{S}\cdot\widehat{S}_j+\widehat{S}'\cdot\widehat{S}_j.$$

Here, $\widehat{S}'=\widehat{S}-2\widehat{S}_i$. As before, $\mathfrak{s}_{\mathcal{W}}-\mathfrak{s}_{\mathcal{W}'}=\mathrm{PD}[S_i]$. Hence, the above equation gives

$$A_j^{\mathcal{W}}-A_j^{\mathcal{W}'}=0.$$

Consider claim (4) now. For this claim, we apply claim (2), with $\mathcal{W}_0=\mathcal{W}\setminus\{w_j\}$. Since $z_i\notin\mathcal{W}_0\cup\{w_j\}$ and $z_i\notin\mathcal{W}_0\cup\{z_j\}$, by part (1) we know that $\mathcal{Q}_{\mathcal{V}_i}$ is homogeneously graded with respect to $\mathrm{gr}_{\mathcal{W}_0\cup\{w_j\}}$ and $\mathrm{gr}_{\mathcal{W}_0\cup\{z_j\}}$. Using claim (2), we obtain

$$2\Delta(A_j)=\mathrm{lk}(L\setminus(K_i\cup K_j),K_j)-\mathrm{lk}(L\setminus K_j,K_j)=-\mathrm{lk}(K_i,K_j).$$

We now consider claim (5). By claim (3), it is sufficient to compute the grading change of $A_j^{\mathcal{W}'}$ where $\mathcal{W}'=(\mathbf{w}\setminus\{w_i\})\cup\mathbf{p}\cup\{z_i\}$. We use claim (4), but note that with respect to $\mathcal{W}'$, the orientation of K_i is reversed, while the orientation of K_j is unchanged. ∎

6.3 On the algebra of the link surgery formula

In this section, we reformulate the algebra of the link surgery formula slightly. We begin with a convenient description of the underlying group.

Lemma 6.5. *Suppose that $\mathcal{H}$ is a link minimal Heegaard diagram for a link L in S^3, which has no free base points. Let M be a sublink of L. Write $S_M \subseteq \mathbb{F}[\mathscr{U}_1, \dots, \mathscr{U}_\ell, \mathscr{V}_1, \dots, \mathscr{V}_\ell]$ for the multiplicatively closed subset generated by $\mathscr{V}_i$ for i such that $K_i \subseteq M$. Then there is an $\mathbb{F}[U_1, \dots, U_\ell]$-equivariant chain isomorphism*

$$\bigoplus_{\mathbf{s} \in \mathbb{H}(L)} A^-(\mathcal{H}^{L \setminus M}, \psi^M(\mathbf{s})) \cong S_M^{-1} \cdot \mathcal{CFL}(\mathcal{H}),$$

where we view U_i as acting by $\mathscr{U}_i \mathscr{V}_i$ on the right-hand side. Furthermore, if $\mathbf{s} \in \mathbb{H}(L)$, this isomorphism intertwines the summand $A^-(\mathcal{H}^{L \setminus M}, \psi^M(\mathbf{s}))$ with the subspace of $S_M^{-1} \cdot \mathcal{CFL}(\mathcal{H})$ in an Alexander multi-grading $\mathbf{s}$.

Proof. The module $S_M^{-1} \cdot \mathcal{CFL}(\mathcal{H})$ decomposes over Alexander gradings $\mathbf{s} \in \mathbb{H}(L)$. Hence, it is sufficient to identify the subspace in an Alexander grading $\mathbf{s}$ with

$$A^-(\mathcal{H}^{L \setminus M}, \psi^M(\mathbf{s}))$$

in a way which commutes with the action of $\mathbb{F}[U_1, \dots, U_\ell]$. We recall that $A^-(\mathcal{H}^{L \setminus M}, \psi^M(\mathbf{s}))$ is generated by monomials $U_1^{i_1} \cdots U_\ell^{i_\ell} \cdot \mathbf{x}$ such that $i_j \geq 0$ for all j, and

$$A_j^{L \setminus M}(\mathbf{x}) - i_j \leq \psi_j^M(s_j)$$

for each j such that $K_j \nsubseteq M$. Here, we are writing $\mathbf{s} = (s_1, \dots, s_\ell)$. Note that by definition of $\psi_j^M(s_j)$ (see [32, Section 3.7]), we have

$$A_j^L(\mathbf{x}) - A_j^{L \setminus M}(\mathbf{x}) = s_j - \psi_j^M(s_j).$$

Compare Lemma 6.4. Hence we may think of $A^-(\mathcal{H}^{L \setminus M}, \psi^M(\mathbf{s}))$ as being generated by monomials $U_1^{i_1} \cdots U_\ell^{i_\ell} \cdot \mathbf{x}$ such that $i_j \geq 0$ for all j and such that

$$A_j^L(\mathbf{x}) - i_j \leq s_j$$

whenever $K_j \nsubseteq M$. We may define a map

$$A^-(\mathcal{H}^{L \setminus M}, \psi^M(\mathbf{s})) \to S_M^{-1} \cdot \mathcal{CFL}(\mathcal{H})$$

via the formula

$$U_1^{i_1} \cdots U_\ell^{i_\ell} \cdot \mathbf{x} \mapsto \mathscr{U}_1^{i_1} \cdots \mathscr{U}_\ell^{i_\ell} \mathscr{V}_1^{s_1 - A_1^L(\mathbf{x}) + i_1} \cdots \mathscr{V}_\ell^{s_\ell - A_\ell^L(\mathbf{x}) + i_\ell} \cdot \mathbf{x}. \qquad \blacksquare$$

Remark 6.6. In their construction of the link surgery formula, Manolescu and Ozsváth also define inclusion maps $\mathcal{I}_{\mathbf{s}}^{\vec{N}}$ between the complexes $\mathfrak{A}^-(\mathcal{H}, \mathbf{s})$ and some generalizations of these complexes. See [32, Section 3.8]. We note that the map $\mathcal{I}_{\mathbf{s}}^{\vec{N}}$ can be identified with the map for localizing at the variables $\mathscr{U}_i$, for i such that $+K_i \subseteq \vec{N}$, and localizing at variables $\mathscr{V}_i$, for i such that $-K_i \subseteq \vec{N}$.

We now discuss some algebraic properties of the maps in the link surgery hypercube. Manolescu and Ozsváth prove that the hypercube maps in their surgery formula $\mathcal{C}_\Lambda(L)$ are $\mathbb{F}[U_1,\dots,U_\ell]$-equivariant. By the previous lemma, there is a more refined action of $\mathbb{F}[\mathcal{U}_1,\dots,\mathcal{U}_\ell,\mathcal{V}_1,\dots,\mathcal{V}_\ell]$ on the complex $\mathcal{C}_\Lambda(L)$, though the hypercube differential does not commute with this action. We now describe how the hypercube maps interact with the action of this larger ring.

Firstly, we define a ring homomorphism

$$\phi^\tau\colon \mathbb{F}[\mathcal{U},\mathcal{V}]\to\mathbb{F}[\mathcal{U},\mathcal{V},\mathcal{V}^{-1}]$$

via the formulas

$$\phi^\tau(\mathcal{U})=\mathcal{V}^{-1}\quad\text{and}\quad\phi^\tau(\mathcal{V})=\mathcal{U}\mathcal{V}^2.$$

We write $\phi^\sigma\colon\mathbb{F}[\mathcal{U},\mathcal{V}]\to\mathbb{F}[\mathcal{U},\mathcal{V},\mathcal{V}^{-1}]$ for the canonical inclusion.

We view $S_M^{-1}\cdot\mathbb{F}[\mathcal{U}_1,\dots,\mathcal{U}_\ell,\mathcal{V}_1,\dots,\mathcal{V}_\ell]$ as the tensor product over $\mathbb{F}$ of ℓ different rings, which are each isomorphic to $\mathbb{F}[\mathcal{U},\mathcal{V}]$ or $\mathbb{F}[\mathcal{U},\mathcal{V},\mathcal{V}^{-1}]$.

If $N\subseteq L\setminus M$, then we define the ring homomorphism

$$\phi^{\vec{N}}\colon S_M^{-1}\cdot\mathbb{F}[\mathcal{U}_1,\dots,\mathcal{U}_\ell,\mathcal{V}_1,\dots,\mathcal{V}_\ell]\to S_{M\cup N}^{-1}\cdot\mathbb{F}[\mathcal{U}_1,\dots,\mathcal{U}_\ell,\mathcal{V}_1,\dots,\mathcal{V}_\ell]$$

by tensoring the map ϕ^τ for each component $K_i\subseteq\vec{N}$ which is oriented oppositely to L, tensoring ϕ^σ for each component $K_i\subseteq\vec{N}$ which is oriented consistently with L, and tensoring the identity map for the remaining components of L.

Lemma 6.7. *Suppose that $M\subseteq L$ and $N\subseteq L\setminus M$. Let $\vec{N}$ denote N with a choice of orientation. If $a\in S_M^{-1}\cdot\mathbb{F}[\mathcal{U}_1,\dots,\mathcal{U}_\ell,\mathcal{V}_1,\dots,\mathcal{V}_\ell]$, and $\mathbf{x}\in S_M^{-1}\cdot\mathcal{CFL}(\mathcal{H})$, then*

$$\Phi^{\vec{N}}(a\cdot\mathbf{x})=\phi^{\vec{N}}(a)\cdot\Phi^{\vec{N}}(\mathbf{x}).$$

Proof. We focus on the length 1 maps in the surgery hypercube. The maps of higher length are analyzed using the same argument. In this case, write $N=K_i$, where K_i is a component of L. To simplify the notation further, we focus on the case that $M=\emptyset$. Let K_i, (resp. $-K_i$), denote K_i equipped with the orientation which coincides with (resp. is opposite to) the orientation from L. The map $\Phi^{K_i}_{L,L\setminus K_i}$ is the canonical inclusion map

$$\mathcal{CFL}(\mathcal{H})\hookrightarrow\mathcal{V}_i^{-1}\cdot\mathcal{CFL}(\mathcal{H}),$$

which is clearly $\mathbb{F}[\mathcal{U}_1,\dots,\mathcal{U}_\ell,\mathcal{V}_1,\dots,\mathcal{V}_\ell]$-equivariant. Note that ϕ^{σ_i} is just the canonical inclusion of $\mathbb{F}[\mathcal{U}_1,\dots,\mathcal{U}_\ell,\mathcal{V}_1,\dots,\mathcal{V}_\ell]$ into $\mathcal{V}_i^{-1}\cdot\mathbb{F}[\mathcal{U}_1,\dots,\mathcal{U}_\ell,\mathcal{V}_1,\dots,\mathcal{V}_\ell]$.

The map $\Phi^{-K_i}_{L,L\setminus K_i}$ is more complicated, as we now describe. Similarly to Lemma 6.5, there are canonical isomorphisms

$$\theta_i\colon\mathcal{U}_i^{-1}\cdot\mathcal{CFL}(\mathcal{H})\to\mathcal{CFL}(\mathcal{H})/(\mathcal{U}_i-1)\otimes\mathbb{F}[\mathbb{H}_i(L)],$$
$$\rho_i\colon\mathcal{V}_i^{-1}\cdot\mathcal{CFL}(\mathcal{H})\to\mathcal{CFL}(\mathcal{H})/(\mathcal{V}_i-1)\otimes\mathbb{F}[\mathbb{H}_i(L)].$$

Here, $\mathbb{F}[\mathbb{H}_i(L)]$ denotes the $\mathbb{F}[T_i, T_i^{-1}]$-module generated by T_i^{α} where $\alpha \in \mathbb{H}_i(L)$. Here, T_i is a formal variable. We also view $\mathcal{CFL}(\mathcal{H})/(\mathcal{U}_i - 1)$ and $\mathcal{CFL}(\mathcal{H})/(\mathcal{V}_i - 1)$ as $\mathbb{F}[U_i, T_i, T_i^{-1}]$-modules, where U_i acts by $\mathcal{V}_i$ on $\mathcal{CFL}(\mathcal{H})/(\mathcal{U}_i - 1)$, and by $\mathcal{U}_i$ on $\mathcal{CFL}(\mathcal{H})/(\mathcal{V}_i - 1)$.

The map θ_i is given as follows. Write

$$\mathcal{Q}_{\mathcal{U}_i}: \mathcal{U}_i^{-1}\mathcal{CFL}(\mathcal{H}) \to \mathcal{CFL}(\mathcal{H})/(\mathcal{U}_i - 1)$$

for the quotient map, which sends $\mathcal{U}_i$ to 1 and which sends $\mathcal{V}_i$ to U_i. Then θ_i is given by the formula

$$\theta_i(\mathbf{x}) = \mathcal{Q}_{\mathcal{U}_i}(\mathbf{x}) \otimes T_i^{A_i^L(\mathbf{x})}.$$

The map ρ_i is defined similarly.

We write $\mathcal{I}_i$ for the inclusion of $\mathcal{CFL}(\mathcal{H})$ into $\mathcal{U}_i^{-1} \cdot \mathcal{CFL}(\mathcal{H})$.

Moving the base point z_i to w_i along a subarc of K_i determines a homotopy equivalence

$$\Psi_{z_i \to w_i}: \mathcal{CFL}(\mathcal{H})/(\mathcal{U}_i - 1) \to \mathcal{CFL}(\mathcal{H})/(\mathcal{V}_i - 1)$$

which preserves $A_j^{L \setminus K_i}$ for $j \neq i$. The map $\Psi_{z_i \to w_i}$ is $\mathbb{F}[U_i, T_i, T_i^{-1}]$-equivariant, and is equivariant with respect to the variables for other link components.

The map $\Phi^{-K_i}_{L, L\setminus K_i}$ is defined as the composition

$$\Phi^{-K_i}_{L, L\setminus K_i} := \rho_i^{-1} \circ (\Psi_{z_i \to w_i} \otimes T_i^{\lambda_i}) \circ \theta_i \circ \mathcal{I}_i.$$

We now consider the interaction of $\Phi^{-K_i}_{L, L\setminus K_i}$ with $\mathcal{U}_i$ and $\mathcal{V}_i$. The map $\mathcal{I}_i$ commutes with $\mathcal{U}_i$ and $\mathcal{V}_i$. The map θ_i has the property that

$$\theta_i(\mathcal{U}_i^n \cdot \mathbf{x}) = T_i^{-n} \cdot \theta_i(\mathbf{x}) \quad \text{and} \quad \theta_i(\mathcal{V}_i^n \cdot \mathbf{x}) = U_i^n T_i^n \cdot \theta_i(\mathbf{x}).$$

Similarly,

$$\rho_i(\mathcal{U}_i^n \cdot \mathbf{x}) = U_i^n T_i^{-n} \cdot \rho_i(\mathbf{x}) \quad \text{and} \quad \rho_i(\mathcal{V}_i^n \cdot \mathbf{x}) = T_i^n \cdot \rho_i(\mathbf{x}),$$

which implies that

$$\rho_i^{-1}(T_i^n \cdot \mathbf{x}) = \mathcal{V}_i^n \cdot \rho_i^{-1}(\mathbf{x}) \quad \text{and} \quad \rho_i^{-1}(U_i^n \cdot \mathbf{x}) = \mathcal{U}_i^n \mathcal{V}_i^n \cdot \rho_i^{-1}(\mathbf{x}).$$

From these relations, it follows that

$$\Phi^{-K_i}_{L, L\setminus K_i}(\mathcal{U}_i \cdot \mathbf{x}) = \mathcal{V}_i^{-1} \cdot \Phi^{-K_i}_{L, L\setminus K_i}(\mathbf{x}) \quad \text{and} \quad \Phi^{-K_i}_{L, L\setminus K_i}(\mathcal{V}_i \cdot \mathbf{x}) = \mathcal{U}_i \mathcal{V}_i^2 \cdot \Phi^{-K_i}_{L, L\setminus K_i}(\mathbf{x}),$$

completing the proof. ■

Example 6.8. We now describe the surgery complex for the unknot $\mathbb{O}$ in our present notation, and compare it to the traditional notation of [43]. The complex $\mathcal{CFK}(\mathbb{O})$ has one generator, 1, over $\mathbb{F}[\mathscr{U},\mathscr{V}]$. In our notation, v sends $\mathscr{U}^i\mathscr{V}^j \in \mathcal{CFK}(\mathbb{O})$ to $\mathscr{U}^i\mathscr{V}^j \in \mathscr{V}^{-1}\mathcal{CFK}(\mathbb{O})$. Similarly, h_λ sends $\mathscr{U}^i\mathscr{V}^j$ to $\mathscr{V}^\lambda\phi^\tau(\mathscr{U}^i\mathscr{V}^j) = \mathscr{U}^j\mathscr{V}^{2j-i+\lambda}$.

Traditionally, one works with the mapping cone formula over $\mathbb{F}[U]$. We identify both $A(\mathbb{O},s) \subseteq \mathcal{CFK}(\mathbb{O})$ and $B(\mathbb{O},s) \subseteq \mathscr{V}^{-1}\mathcal{CFK}(\mathbb{O})$ with $\mathbb{F}[U]$ (where U acts by $\mathscr{U}\mathscr{V}$). We identify $\mathscr{U}^i\mathscr{V}^j \in \mathcal{CFK}(\mathbb{O})$ with $U^{\min(i,j)} \in A(\mathbb{O},s)$, and we identify $\mathscr{U}^i\mathscr{V}^j \in \mathscr{V}^{-1}\mathcal{CFK}(\mathbb{O})$ with $U^i \in B(\mathbb{O},s)$. We can view the previous identification as giving two isomorphisms

$$\mathcal{CFK}(\mathbb{O}) \cong \bigoplus_{s\in\mathbb{Z}} \mathbb{F}[U] \quad \text{and} \quad \mathscr{V}^{-1}\mathcal{CFK}(\mathbb{O}) \cong \bigoplus_{s\in\mathbb{Z}} \mathbb{F}[U].$$

In the original notation of Ozsváth and Szabó,

$$v(U^i) = \begin{cases} U^{i-s} & \text{if } s \le 0, \\ U^i & \text{if } s \ge 0, \end{cases} \qquad h_\lambda(U^i) = \begin{cases} U^i & \text{if } s \le 0, \\ U^{i+s} & \text{if } s \ge 0. \end{cases}$$

The map h_λ sends $A(\mathbb{O},s)$ to $B(\mathbb{O},s+\lambda)$. It is straightforward to see that the above maps form a commutative diagram

$$\begin{array}{ccc} \mathcal{CFK}(\mathbb{O}) & \xrightarrow{\ f\ } & \mathscr{V}^{-1}\mathcal{CFK}(\mathbb{O}) \\ \downarrow \cong & & \downarrow \cong \\ \bigoplus_{s\in\mathbb{Z}} A(\mathbb{O},s) & \xrightarrow{\ f\ } & \bigoplus_{s\in\mathbb{Z}} B(\mathbb{O},s) \end{array}$$

whenever f is one of v or h_λ.