Chapter 7

The knot surgery algebra

In this chapter, we introduce our algebras $\mathcal{K}$ and $\mathcal{L}$. We also describe natural filtrations on $\mathcal{K}$ and $\mathcal{L}$. Using these filtrations, we describe our module categories.

7.1 The algebras $\mathcal{K}$ and $\mathcal{L}$

We recall the *knot surgery algebra* $\mathcal{K}$ from the introduction. We define $\mathcal{K}$ to be an algebra over the idempotent ring

$$\mathbf{I} = \mathbf{I}_0 \oplus \mathbf{I}_1,$$

where each of $\mathbf{I}_i$ is rank 1 over $\mathbb{F} = \mathbb{Z}/2$. We write i_0 and i_1 for the generators of $\mathbf{I}_0$ and $\mathbf{I}_1$, respectively. We set

$$\mathbf{I}_0 \cdot \mathcal{K} \cdot \mathbf{I}_0 = \mathbb{F}[\mathcal{U}, \mathcal{V}] \quad \text{and} \quad \mathbf{I}_1 \cdot \mathcal{K} \cdot \mathbf{I}_1 \cong \mathbb{F}[\mathcal{U}, \mathcal{V}, \mathcal{V}^{-1}].$$

Also,

$$\mathbf{I}_0 \cdot \mathcal{K} \cdot \mathbf{I}_1 = 0.$$

We define

$$\mathbf{I}_1 \cdot \mathcal{K} \cdot \mathbf{I}_0 = \mathbb{F}[\mathcal{U}, \mathcal{V}, \mathcal{V}^{-1}] \otimes \langle \sigma, \tau \rangle.$$

That is, elements of $\mathbf{I}_1 \cdot \mathcal{K} \cdot \mathbf{I}_0$ can be written as sums of monomials of the form $\mathcal{U}^i \mathcal{V}^j \sigma$ and $\mathcal{U}^i \mathcal{V}^j \tau$ where $i \geq 0$ and $j \in \mathbb{Z}$. These algebra elements satisfy the relations

$$\sigma \cdot a = \phi^\sigma(a) \cdot \sigma \quad \text{and} \quad \tau \cdot a = \phi^\tau(a) \cdot \tau,$$

for $a \in \mathbb{F}[\mathcal{U}, \mathcal{V}] = \mathbf{I}_0 \cdot \mathcal{K} \cdot \mathbf{I}_0$. Here, $\phi^\tau \colon \mathbb{F}[\mathcal{U}, \mathcal{V}] \to \mathbb{F}[\mathcal{U}, \mathcal{V}, \mathcal{V}^{-1}]$ is the algebra homomorphism satisfying $\phi^\tau(\mathcal{U}) = \mathcal{V}^{-1}$ and $\phi^\tau(\mathcal{V}) = \mathcal{U}\mathcal{V}^2$. The map $\phi^\sigma \colon \mathbb{F}[\mathcal{U}, \mathcal{V}] \to \mathbb{F}[\mathcal{U}, \mathcal{V}, \mathcal{V}^{-1}]$ is the canonical inclusion.

In particular, $\mathbf{I}_1 \cdot \mathcal{K} \cdot \mathbf{I}_0$ is generated by two special algebra elements σ and τ, together with the left action of $\mathbb{F}[\mathcal{U}, \mathcal{V}, \mathcal{V}^{-1}]$, which satisfy the relations

$$\begin{aligned} \sigma\mathcal{U} &= \mathcal{U}\sigma \quad && \text{and} \quad \sigma\mathcal{V} = \mathcal{V}\sigma, \\ \tau\mathcal{U} &= \mathcal{V}^{-1}\tau \quad && \text{and} \quad \tau\mathcal{V} = \mathcal{U}\mathcal{V}^2\tau. \end{aligned}$$

The link algebra $\mathcal{L}_\ell$ is defined as

$$\mathcal{L}_\ell := \mathcal{K} \otimes_{\mathbb{F}} \cdots \otimes_{\mathbb{F}} \mathcal{K}.$$

We often write just $\mathcal{L}$, when ℓ is determined by context. We view $\mathcal{L}_\ell$ as being an algebra over the idempotent ring

$$\mathbf{E}_\ell := \mathbf{I} \otimes_{\mathbb{F}} \cdots \otimes_{\mathbb{F}} \mathbf{I}.$$

Remark 7.1. The algebra $\mathcal{K}$ admits a more symmetric basis, as follows. We may view $\mathbf{I}_1 \cdot \mathcal{K} \cdot \mathbf{I}_1$ as being $\mathbb{F}[U, T, T^{-1}]$, where $U = \mathscr{U}\mathscr{V}$ and $T = \mathscr{V}$. Then the above relations become

$$\begin{aligned} \sigma\mathscr{U} &= UT^{-1}\sigma \quad \text{and} \quad \sigma\mathscr{V} = T\sigma, \\ \tau\mathscr{U} &= T^{-1}\tau \qquad \text{and} \quad \tau\mathscr{V} = UT\tau. \end{aligned}$$

Remark 7.2. The above symmetry may be packaged into an algebra homomorphism

$$\mathcal{E} \colon \mathcal{K} \to \mathcal{K}$$

as follows. On $\mathbf{I}_0 \cdot \mathcal{K} \cdot \mathbf{I}_0$, we set

$$\mathcal{E}(\mathscr{U}) = \mathscr{V} \quad \text{and} \quad \mathcal{E}(\mathscr{V}) = \mathscr{U}.$$

On $\mathbf{I}_1 \cdot \mathcal{K} \cdot \mathbf{I}_1$, we set

$$\mathcal{E}(U) = U \quad \text{and} \quad \mathcal{E}(\mathscr{V}^i) = \mathscr{V}^{-i}.$$

On $\mathbf{I}_1 \cdot \mathcal{K} \cdot \mathbf{I}_0$, we set

$$\mathcal{E}(\sigma) = \tau \quad \text{and} \quad \mathcal{E}(\tau) = \sigma.$$

7.2 Topologies on $\mathcal{K}$ and $\mathcal{L}$

In this section, we describe the topologies on $\mathcal{K}$ and $\mathcal{L}$ which we use throughout the memoir.

Definition 7.3. Suppose that $n \in \mathbb{N}$ is fixed. We define $J_n \subseteq \mathcal{K}$ to be the $\mathbb{F}$ span of the following set of generators:

1. In $\mathbf{I}_0 \cdot \mathcal{K} \cdot \mathbf{I}_0$, the generators $\mathscr{U}^i\mathscr{V}^j$, for $i \ge n$ or $j \ge n$ (i.e., $\max(i, j) \ge n$).
2. In $\mathbf{I}_1 \cdot \mathcal{K} \cdot \mathbf{I}_0$, the generators $\mathscr{U}^i\mathscr{V}^j\sigma$ for $i \ge n$ or $j \ge n$.
3. In $\mathbf{I}_1 \cdot \mathcal{K} \cdot \mathbf{I}_0$, the generators $\mathscr{U}^i\mathscr{V}^j\tau$ for $j \le 2i - n$ or $i \ge n$.
4. In $\mathbf{I}_1 \cdot \mathcal{K} \cdot \mathbf{I}_1$, the generators $\mathscr{U}^i\mathscr{V}^j$ where $i \ge n$.

The subspace J_n is illustrated in Figure 7.1.

We observe that

$$J_0 = \mathcal{K}, \quad J_{n+1} \subseteq J_n, \quad \text{and} \quad \bigcap_{n \in \mathbb{N}} J_n = \{0\}.$$

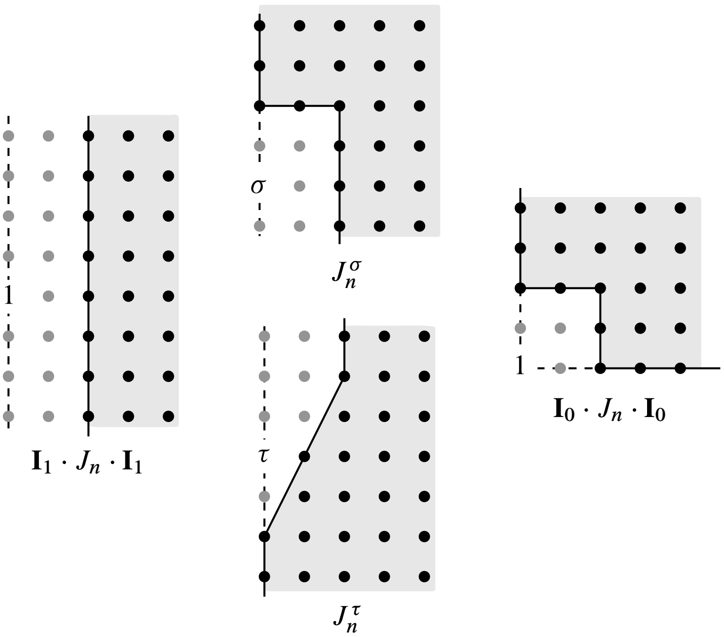

Figure 7.1. The subspace $J_n \subseteq \mathcal{K}$, indicated by the shaded regions. Here, J_n^σ and J_n^τ denote the subspaces of J_n which are in the $\mathcal{K}$ span of σ and τ. The dots indicate monomials in the algebra. The horizontal direction indicates the power of $\mathcal{U}$, and the vertical direction indicates the power of $\mathcal{V}$.

We will use the subspaces J_n as a fundamental basis of 0 in our topology on $\mathcal{K}$. The condition that $\bigcap_{n \in \mathbb{N}} J_n = \{0\}$ is equivalent to the inclusion $\mathcal{K} \to \boldsymbol{\mathcal{K}}$ being injective.

Remark 7.4. The subspaces J_n^σ and J_n^τ can also be described as follows:

$$J_n^\sigma = (U^n) \cdot \sigma + \sigma \cdot (\mathcal{U}^n, \mathcal{V}^n),$$
$$J_n^\tau = (U^n) \cdot \tau + \tau \cdot (\mathcal{U}^n, \mathcal{V}^n).$$

Here, $(U^n) \subseteq \mathbf{I}_1 \cdot \mathcal{K} \cdot \mathbf{I}_1$ denotes the ideal generated by $U = \mathcal{U}\mathcal{V}$, and $(\mathcal{U}^n, \mathcal{V}^n) \subseteq \mathbf{I}_0 \cdot \mathcal{K} \cdot \mathbf{I}_0$ denotes the ideal spanned by the elements $\mathcal{U}^n$ and $\mathcal{V}^n$.

We now consider the link algebra. For our purposes, the most natural topology on the link algebra is the !-topology,

$$\mathcal{L}_\ell = \mathcal{K} \otimes_{\mathbb{F}}^! \cdots \otimes_{\mathbb{F}}^! \mathcal{K}.$$

This topology is first countable, with a basis of opens given by

$$\mathscr{J}_{\ell,n} := \sum_{i=0}^{\ell-1} \mathcal{K}^{\otimes i} \otimes_{\mathbb{F}} J_n \otimes_{\mathbb{F}} \mathcal{K}^{\otimes \ell-i-1} \subseteq \mathcal{L}_\ell.$$

A fundamental result to our memoir is the following.

Proposition 7.5. *The map* $\mu_2\colon \mathcal{K} \vec{\otimes}_{\mathrm{I}} \mathcal{K} \to \mathcal{K}$ *is continuous.*

Remark 7.6. Multiplication is not continuous on $\mathcal{K} \otimes^!_{\mathrm{I}} \mathcal{K}$. As an example, $\mathcal{V}^{-i} \otimes \mathcal{V}^i\sigma \to 0$ in $\mathcal{K} \otimes_{\mathrm{I}} \mathcal{K}$ as $i \to \infty$, whereas $\mu_2(\mathcal{V}^{-i} \otimes \mathcal{V}^i\sigma) = \sigma \not\to 0$.

Proposition 7.5 follows immediately from the following lemma and the definition of $\vec{\otimes}$ in Section 5.4.

Lemma 7.7. *Let* $J_n \subseteq \mathcal{K}$ *be the subspace defined in Definition* 7.3.

(1) *For all* n, $\mu_2(J_n \otimes \mathcal{K}) \subseteq J_n$ *(i.e.,* J_n *is a right ideal of* $\mathcal{K}$*).*

(2) *Suppose* $x \in \mathcal{K}$ *and* $n \in \mathbb{N}$ *are fixed. Then for sufficiently large* m,

$$\mu_2(x \otimes J_m) \subseteq J_n.$$

Proof. We consider the first claim. We wish to show that $\mu_2(x \otimes a) \in J_n$ whenever $x \in J_n$ and $a \in \mathcal{K}$. We break the argument into cases, depending on the idempotents of x and a. If $x, a \in \mathbf{I}_1 \cdot \mathcal{K} \cdot \mathbf{I}_1$, the claim follows because $w_U(x \cdot a) \geq w_U(x) + w_U(a)$, where w_U is the U-adic weight (i.e., $w_U(a) = \max\{i : \exists a', a = U^i \cdot a'\}$). The same argument applies if $x \in \mathbf{I}_1 \cdot \mathcal{K} \cdot \mathbf{I}_1$ and $a \in \mathbf{I}_1 \cdot \mathcal{K} \cdot \mathbf{I}_0$. The situation that $x, a \in \mathbf{I}_0 \cdot \mathcal{K} \cdot \mathbf{I}_0$ is also clear. We now consider the case that $x \in \mathbf{I}_1 \cdot \mathcal{K} \cdot \mathbf{I}_0$ and $a \in \mathbf{I}_0 \cdot \mathcal{K} \cdot \mathbf{I}_0$. Suppose first that $x = \mathcal{U}^i\mathcal{V}^j\sigma$ and $a = \mathcal{U}^s\mathcal{V}^t$ (for $s, t \geq 0$). Then $x \cdot a = \mathcal{U}^{i+s}\mathcal{V}^{j+t}\sigma$, which is clearly also in J_n. Similarly, if $x = \mathcal{U}^i\mathcal{V}^j\tau$ and $a = \mathcal{U}^s\mathcal{V}^t$, then

$$x \cdot a = \mathcal{U}^{i+t}\mathcal{V}^{j-s+2t}\tau.$$

By assumption, either $i \geq n$ or $j \leq 2i - n$. If $i \geq n$, then $i + t \geq n$ so $x \cdot a \in J_n$. The second case is equivalent to $2i - j - n \geq 0$. We note that

$$2(i + t) - (j - s + 2t) - n = s + 2i - j - n \geq 0$$

so $x \cdot a \in J_n$. This completes the first part of the claim.

We leave the second claim to the reader, as it may be handled similarly. ∎

Combining Corollary 5.15 and Proposition 7.5, we also obtain the following.

Corollary 7.8. *Suppose* $\mathcal{L}$ *is topologized as* $\mathcal{K} \otimes^! \cdots \otimes^! \mathcal{K}$. *Then the map*

$$\mu_2\colon \mathcal{L} \vec{\otimes}_{\mathrm{E}} \mathcal{L} \to \mathcal{L}$$

is continuous.

7.3 Alexander modules

In this section, we describe our categories of type-D, A and DA modules. We will refer to these categories as the categories of *Alexander modules*.

Definition 7.9. A *type*-D *Alexander module* consists of a pair $(\mathcal{X}, \delta^1)$ where $\mathcal{X}$ is a linear topological right $\mathbf{E}$-module and

$$\delta^1\colon \mathcal{X} \to \mathcal{X} \vec{\otimes}_{\mathrm{E}} \mathcal{L}$$

is a linear topological morphism which satisfies

$$(\mathrm{id} \otimes \mu_2) \circ (\delta^1 \otimes \mathrm{id}_{\mathcal{L}}) \circ \delta^1 = 0.$$

Recall that a linear topological morphism is the same as a continuous linear map on completions. If $\mathcal{X}^{\mathcal{L}}$ and $\mathcal{Y}^{\mathcal{L}}$, we define $\mathrm{Mor}(\mathcal{X}^{\mathcal{L}}, \mathcal{Y}^{\mathcal{L}})$ to be the space of linear topological morphisms

$$f^1\colon \mathcal{X} \to \mathcal{Y} \vec{\otimes}_{\mathrm{E}} \mathcal{L}.$$

We write $\mathrm{Mod}^{\mathcal{K}}_{\mathfrak{a}}$ for the category of Alexander type-D modules.

Type-A Alexander modules are similar, as we see in the following definition.

Definition 7.10. A *type*-A *Alexander module* $(\mathcal{X}, m_j)$ over $\mathcal{L}$ consists of a linear topological left $\mathbf{E}$-module $\mathcal{X}$ and a collection of linear topological morphisms

$$m_{j+1}\colon \mathcal{L} \vec{\otimes}_{\mathrm{E}} \cdots \vec{\otimes}_{\mathrm{E}} \mathcal{L} \vec{\otimes}_{\mathrm{E}} \mathcal{X} \to \mathcal{X},$$

satisfying the A_∞-relations.

Type-DA Alexander modules are defined by the obvious amalgamation of the above notions.

7.4 Split Alexander modules

We now define the notion of a *split* Alexander type-A or DA module. Consider the link algebra $\mathcal{L}_\ell$ and suppose that $\mathbb{P}$ is a partition of $\{1, \dots, \ell\}$. Suppose that $\mathcal{X}$ is a linear topological left $\mathbf{E}_\ell$-module. A $\mathbb{P}$-*split continuous* linear topological morphism

$$f_{j+1}\colon \mathcal{L}_\ell \otimes_{\mathrm{E}} \cdots \otimes_{\mathrm{E}} \mathcal{L}_\ell \otimes_{\mathrm{E}} \mathcal{X} \to \mathcal{X}$$

is a continuous map on completions, where we give $cL_\ell^{\otimes j} \otimes \mathcal{X}$ the tree topology from Section 5.6, where $\mathcal{X}$ is the central vertex and we have $|\mathbb{P}|$-rays of length j extending from this vertex. If $p_1, \dots, p_{|\mathbb{P}|}$ are the sizes of the elements of $\mathbb{P}$, then each non-root vertex of the i-th ray is labeled with $\mathcal{L}_{p_i}$ (topologized using the !-topology).

Note that $\mathbb{P}$-split continuity is preserved by A_∞-composition of such maps by Proposition 5.24.

Definition 7.11. Suppose that $\mathbb{P}$ is a partition of $\{1,\dots,\ell\}$. A $\mathbb{P}$-*split type*-A *Alexander module* consists of a linear topological $\mathbf{E} = \mathbf{E}_\ell$-module $\mathcal{X}$, equipped with a $\mathbb{P}$-split linear topological morphism

$$m_{j+1}\colon \mathcal{L}_\ell \otimes_{\mathrm{E}} \cdots \otimes_{\mathrm{E}} \mathcal{L}_\ell \otimes_{\mathrm{E}} \mathcal{X} \to \mathcal{X},$$

which satisfies the A_∞-module relations.

If $\mathbb{P}$ is a partition of $\{1,\dots,\ell\}$ and $i_1,\dots,i_n$ are the sizes of the partition elements (so $i_1+\cdots+i_n=\ell$), we will write

$${}_{\mathcal{L}_{i_1}|\cdots|\mathcal{L}_{i_n}}\mathcal{X}$$

for a $\mathbb{P}$-split Alexander module.

Once we introduce box tensor products of split Alexander modules, we will show in Lemma 7.16 that the split-Alexander condition is weaker than the Alexander condition. In particular, any type-A Alexander module ${}_{\mathcal{L}}\mathcal{X}$ is also a split type-A Alexander module for any partition.

$\mathbb{P}$-split Alexander type-DA modules can be defined similarly to the above, though we typically need to assume that the underlying space $\mathcal{X}$ of the module is linearly compact for the type-DA structure relations to be meaningful, and for the A_∞-composition of morphisms to be defined. (Note that all of the link surgery modules we consider in this memoir are linearly compact.) We explain this in the case of a $\{\{1\},\{2\}\}$-split Alexander module ${}_{\mathcal{K}|\mathcal{K}}\mathcal{X}^{\mathcal{K}}$. Consider the spaces involved in the composition of δ_{i+1}^1, δ_{j+1}^1 and $\mathbb{I}_{\mathcal{X}}\otimes\mu_2$, applied to

$$\underbrace{(\mathcal{K}|\mathcal{K}) \otimes_{\mathrm{E}_2} \cdots \otimes_{\mathrm{E}_2} (\mathcal{K}|\mathcal{K})}_{i+j} \otimes_{\mathrm{E}_2} \mathcal{X}.$$

We topologize the above space using the tree topology for the tree where there are 2 length $i+j$ rays attached to a lowest vertex. The lowest vertex is assigned the space $\mathcal{X}$. Proposition 5.24 shows that $\delta_{i+1}^1 \otimes \mathrm{id}$ gives a map from the above space to

$$\underbrace{(\mathcal{K}|\mathcal{K}) \otimes_{\mathrm{E}_2} \cdots \otimes_{\mathrm{E}_2} (\mathcal{K}|\mathcal{K})}_{j} \otimes_{\mathrm{E}_2} (\mathcal{X} \vec{\otimes} \mathcal{K}).$$

The above space is topologized using a tree with a single lowest vertex, and 2 length i rays extending from this ray. The lowest vertex is assigned the space $\mathcal{X} \vec{\otimes} \mathcal{K}$. Lemma 5.27 implies that if $\mathcal{X}$ is linearly compact, then the above topological vector space is isomorphic to the same tensor product, equipped with the tree topology where the tree Γ has 2 length i rays connected to a vertex v_1, which is above a single vertex v_0. We assign v_1 the vector space $\mathcal{X}$, and v_0 the vector space $\mathcal{K}$. See Figure 7.2 for a schematic when $i=j=1$.

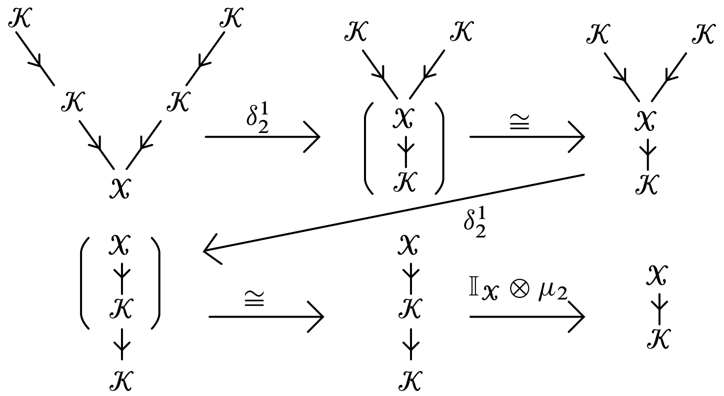

Figure 7.2. Maps appearing in the DA-structure relation for a split Alexander DA bimodule. Here, the arrows labeled $\cong$ correspond to the isomorphism from Lemma 5.27, which requires $\mathcal{X}$ to be linearly compact.

7.5 External tensor products

We now discuss external tensor products, focusing on subtleties involving completions.

We first discuss type-D modules. Suppose that $\mathcal{X}^{\mathcal{L}_m}$ and $\mathcal{Y}^{\mathcal{L}_n}$ are type-D Alexander modules. To form the external tensor products, we assume that $\mathcal{X}$ and $\mathcal{Y}$ are linearly compact, so that $\mathcal{X} \vec{\otimes} \mathcal{L}_m \cong \mathcal{X} \otimes^! \mathcal{L}_m$ by Lemma 5.16, and similarly for $\mathcal{Y} \vec{\otimes} \mathcal{L}_n$. Under this assumption, we may define the structure map on the tensor product as $\delta^1_{\mathcal{X}} \otimes \mathrm{id}_{\mathcal{Y}} \otimes 1_{\mathcal{L}_n} + \mathrm{id}_{\mathcal{X}} \otimes 1_{\mathcal{L}_m} \otimes \delta^1_{\mathcal{Y}}$, with tensor factors reordered, as indicated below:

$$\begin{aligned} \mathcal{X} \otimes^! \mathcal{Y} \to (\mathcal{X} \vec{\otimes} \mathcal{L}_m) \otimes^! (\mathcal{Y} \vec{\otimes} \mathcal{L}_n) &\cong \mathcal{X} \otimes^! \mathcal{L}_m \otimes^! \mathcal{Y} \otimes^! \mathcal{L}_n \\ &\cong (\mathcal{X} \otimes^! \mathcal{Y}) \vec{\otimes} (\mathcal{L}_m \otimes^! \mathcal{L}_n). \end{aligned}$$

Next, we consider tensoring (split) type-A modules. Suppose that $\mathbb{P}$ is a partition of $\{1, \dots, s\}$ and $\mathbb{P}'$ is a partition of $\{1, \dots, r\}$. Let us write $i_1, \dots, i_m$ for the sizes of the elements of $\mathbb{P}$, and $j_1, \dots, j_n$ for those of $\mathbb{P}'$. Suppose that we have $\mathbb{P}$ and $\mathbb{P}'$ split modules

$$ {}_{\mathcal{L}_{i_1}|\cdots|\mathcal{L}_{i_m}}\mathcal{X} \quad \text{and} \quad {}_{\mathcal{L}_{j_1}|\cdots|\mathcal{L}_{j_n}}\mathcal{Y}. $$

We will describe two versions of the external tensor product, namely

$$ {}_{\mathcal{L}_{i_1}|\cdots|\mathcal{L}_{i_m+j_1}|\cdots|\mathcal{L}_{j_n}}(\mathcal{X} \otimes^! \mathcal{Y}) \quad \text{and} \quad {}_{\mathcal{L}_{i_1}|\cdots|\mathcal{L}_{i_m}|\mathcal{L}_{j_1}|\cdots|\mathcal{L}_{j_n}}(\mathcal{X} \otimes^! \mathcal{Y}). $$

We topologize $\mathcal{L}_{r+s}^{\otimes j} \otimes \mathcal{X} \otimes \mathcal{Y}$ using the tree topology for a star with $m + n - 1$ (resp. $m + n$ rays). We label the root $\mathcal{X} \otimes^! \mathcal{Y}$, and we label each other vertex with a $\otimes^!$-tensor product of copies of $\mathcal{K}$.

To construct the external tensor product, one must also pick a cellular diagonal of the associahedron. This construction is recalled in Section 3.5. The fact that the external tensor product behaves well with completions is implied by the following result.

Proposition 7.12. *Suppose that $\mathbb{P}$ and $\mathbb{P}'$ are partitions of $\{1,\dots,r\}$ and $\{1,\dots,s\}$, respectively, and that*

$$f_{j+1}\colon \mathcal{L}_r^{\otimes j}\otimes \mathcal{X}\to \mathcal{X} \quad \textit{and} \quad g_{j+1}\colon \mathcal{L}_s^{\otimes j}\otimes \mathcal{Y}\to \mathcal{Y}$$

are $\mathbb{P}$ and $\mathbb{P}'$ split continuous, then their tensor product $f_{j+1}\otimes g_{j+1}$ is both $\mathbb{P}\sqcup\mathbb{P}'$ and $(\mathbb{P}\sqcup\mathbb{P}')_{p\sim p'}$ split continuous, for any $p\in\mathbb{P}$ and $p'\in\mathbb{P}'$. (Here, we write $(\mathbb{P}\sqcup\mathbb{P}')_{p\sim p'}$ for the partition obtained by merging p and p' in $\mathbb{P}\sqcup\mathbb{P}'$.)

We will prove Proposition 7.12 for the module ${}_{\mathcal{L}_{i_1}|\cdots|\mathcal{L}_{i_m+j_1}|\cdots|\mathcal{L}_{j_n}}(\mathcal{X}\otimes^! \mathcal{Y})$. We will later see in Lemma 7.16 that the module ${}_{\mathcal{L}_{i_1}|\cdots|\mathcal{L}_{i_m}|\mathcal{L}_{j_1}|\cdots|\mathcal{L}_{j_n}}(\mathcal{X}\otimes^! \mathcal{Y})$ is a box tensor product of this module with a split Alexander bimodule ${}_{\mathcal{L}_{i_m}|\mathcal{L}_{j_1}}[\mathbb{I}]^{\mathcal{L}_{i_m+j_1}}$, and hence is also a split Alexander module.

The proof Proposition 7.12 is an immediate consequence of the following general result about tree topologies.

Lemma 7.13. *Let Γ_n^k denote the star which has n rays of length k, which all point towards a root vertex v_0. Let $(\mathcal{X}_v)_{v\in V(\Gamma_n^k)}$ and $(\mathcal{Y}_v)_{v\in V(\Gamma_m^k)}$ denote two families of spaces. Pick a distinguished ray from each of Γ_n^k and Γ_m^k. Form a new collection of spaces $(\mathcal{Z}_v)_{v\in\Gamma_{m+n-1}^k}$ as follows. View Γ_{m+n-1}^k as being obtained by merging vertices of the same height along the distinguished rays. We define $\mathcal{Z}_v$ to be either $\mathcal{X}_v$ or $\mathcal{Y}_v$ for v not along the merged ray. For v along the merged ray, we define $\mathcal{Z}_v$ to be $\mathcal{X}_v\otimes^!\mathcal{Y}_v$. Then the natural map*

$$\mathcal{Z}_{\Gamma_{m+n-1}^k}\to \mathcal{X}_{\Gamma_n^k}\otimes^! \mathcal{Y}_{\Gamma_m^k}$$

is continuous. See Figure 7.3 for a schematic where $m=n=2$.

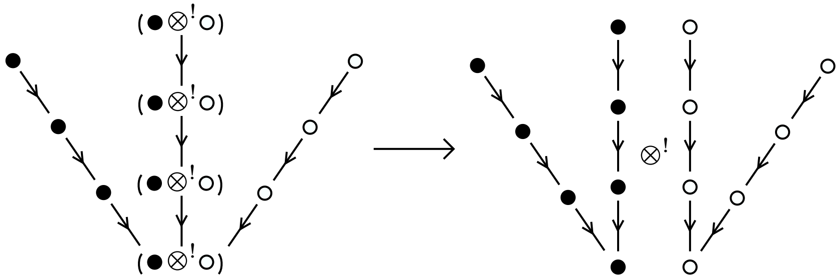

Figure 7.3. A schematic of Lemma 7.13. Solid dots denote $\mathcal{X}_v$ and open dots denote $\mathcal{Y}_v$.

Proof. We abbreviate $\mathcal{X}_{\Gamma_n^k}$, $\mathcal{Y}_{\Gamma_m^k}$ and $\mathcal{Z}_{\Gamma_{m+n-1}^k}$ by $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$, respectively.

Let

$$U_{\mathcal{X}} \otimes \mathcal{Y} + \mathcal{X} \otimes V_{\mathcal{Y}} \subseteq \mathcal{X} \otimes^! \mathcal{Y}$$

be an open subspace, where $U_{\mathcal{X}} \subseteq \mathcal{X}$ and $V_{\mathcal{Y}} \subseteq \mathcal{Y}$ are open subspaces. Let s be an admissible labeling of Γ_{n+m-1}^k. Our goal is to show that if $\mathbf{p}$ is a point-enhancement of s, then there is an open-enhancement $\mathcal{U}_{\mathbf{p}}$ so that

$$\mathcal{W}(\mathcal{U}_{\mathbf{p}}, \mathbf{p}) \subseteq U_{\mathcal{X}} \otimes \mathcal{Y} + \mathcal{X} \otimes V_{\mathcal{Y}}.$$

There are two induced labelings $s_{\mathcal{X}}$ and $s_{\mathcal{Y}}$ of Γ_n^k and Γ_m^k, obtained by restricting s. We make the following claims:

(1) At least one of $s_{\mathcal{X}}$ and $s_{\mathcal{Y}}$ is admissible.

(2) If there is an $\mathcal{O}$-labeled vertex along the merged ray, then both $s_{\mathcal{X}}$ and $s_{\mathcal{Y}}$ are admissible.

The above claims are straightforward to prove, so we leave the argument to the reader.

Consider first the case that the merged ray has no vertices labeled $\mathcal{O}$. In this case, the root is labeled X, but all other vertices along the merged ray are labeled p. Also, in this case, at least one of $s_{\mathcal{X}}$ and $s_{\mathcal{Y}}$ is admissible. Suppose, without loss of generality, that $s_{\mathcal{X}}$ is admissible. Write $v_1, \dots, v_{k-1}$ for the p-labeled vertices along the merged ray. If v is a p-labeled vertex along the merged ray, write $\mathbf{p}(v)$ as a sum of finitely many elementary tensors. For each $v \in \{v_1, \dots, v_{k-1}\}$, write $x_{v,1}, \dots, x_{v,i_v}$ for these elementary tensors. We consider the $N := i_{v_1} \cdots i_{v_{k-1}}$ point-enhancements $\mathbf{p}_{\mathcal{X},j}$, where $\mathbf{p}_{\mathcal{X},j}(w) = \mathbf{p}(w)$ if w is not along the merged ray, and such that $\mathbf{p}_{\mathcal{X},j}$ enumerate all combinations of the $x_{v_i,r}$ for $i \in \{1, \dots, k-1\}$ and $r \in \{1, \dots, i_{v_i}\}$.

Since $U_{\mathcal{X}}$ is open, for each $\mathbf{p}_{\mathcal{X},i}$, there is an open-enhancement $\mathcal{U}_{\mathcal{X},i} \subseteq \mathcal{X}$ so that

$$\mathcal{W}(\mathcal{U}_{\mathcal{X},i}, \mathbf{p}_{\mathcal{X},i}) \subseteq U_{\mathcal{X}}.$$

We define an open-enhancement of s as follows. If v is in Γ_n^k (the tree for $\mathcal{X}$), we define

$$\mathcal{U}_{\mathbf{p}}(v) = \bigcap_{i=1}^{N} \mathcal{U}_{\mathcal{X},i}(v).$$

If v is in Γ_n^k (the tree for $\mathcal{Y}$), set $\mathcal{U}_{\mathbf{p}}(v) = \mathcal{Y}_v$. Then

$$\begin{aligned}\mathcal{W}(\mathcal{U}_{\mathbf{p}}, \mathbf{p}) &\subseteq \mathcal{W}(\mathcal{U}_{\mathcal{X},1}, \mathbf{p}_{\mathcal{X},1}) \otimes \mathcal{Y} + \cdots + \mathcal{W}(\mathcal{U}_{\mathcal{X},N}, \mathbf{p}_{\mathcal{X},N}) \otimes \mathcal{Y} \\ &\subseteq U_{\mathcal{X}} \otimes \mathcal{Y} + \mathcal{X} \otimes V_{\mathcal{Y}},\end{aligned}$$

as we wanted to show.

Next, we consider the case that s labels a vertex v_0 along the merged ray by $\mathcal{O}$. In this case, both $s_{\mathcal{X}}$ and $s_{\mathcal{Y}}$ are admissible. Consider a point-enhancement $\mathbf{p}$ of s. Write $v_1, \dots, v_l$ for the vertices above v_0 (which are labeled p by s). Similar to before, by separating the elements of $\mathbf{p}$ into elementary tensors, we obtain point-enhancements

$\mathbf{p}_{\mathcal{X},i}$ and $\mathbf{p}_{\mathcal{Y},j}$ for $i = 1, \dots, N$ and $j = 1, \dots, M$. For each i and j, there are open-enhancements $\mathcal{U}_{\mathcal{X},i}$ and $\mathcal{U}_{\mathcal{Y},j}$ of $s_{\mathcal{X}}$ and $s_{\mathcal{Y}}$ (respectively) so that

$$\mathcal{W}(\mathcal{U}_{\mathcal{X},i}, \mathbf{p}_{\mathcal{X},i}) \subseteq U_{\mathcal{X}} \quad \text{and} \quad \mathcal{W}(\mathcal{U}_{\mathcal{Y},j}, \mathbf{p}_{\mathcal{X},j}) \subseteq V_{\mathcal{Y}}.$$

We then define an open-enhancement $\mathcal{U}_{\mathbf{p}}$ of s as follows. Suppose $s(v) = \mathcal{O}$. If v is in Γ_n^k not in the merged ray, we set

$$\mathcal{U}_{\mathbf{p}}(v) = \bigcap_{i=1}^{N} \mathcal{U}_{\mathcal{X},i}(v).$$

If v is in Γ_m^k but not in the merged ray, we set

$$\mathcal{U}_{\mathbf{p}}(v) = \bigcap_{j=1}^{M} \mathcal{U}_{\mathcal{Y},j}(v).$$

If v is in the merged ray, we set

$$\mathcal{U}_{\mathbf{p}}(v) = \left(\bigcap_{i=1}^{N} \mathcal{U}_{\mathcal{X},i}(v) \right) \otimes \mathcal{Y} + \mathcal{X} \otimes \left(\bigcap_{j=1}^{M} \mathcal{U}_{\mathcal{Y},j}(v) \right).$$

We observe that

$$\begin{aligned} \mathcal{W}(\mathcal{U}_{\mathbf{p}}, \mathbf{p}) \subseteq{} & (\mathcal{W}(\mathcal{U}_{\mathcal{X},1}, \mathbf{p}_{\mathcal{X},1}) + \cdots + \mathcal{W}(\mathcal{U}_{\mathcal{X},N}, \mathbf{p}_{\mathcal{X},N})) \otimes \mathcal{Y} \\ & + \mathcal{X} \otimes (\mathcal{W}(\mathcal{U}_{\mathcal{Y},1}, \mathbf{p}_{\mathcal{Y},1}) + \cdots + \mathcal{W}(\mathcal{U}_{\mathcal{Y},M}, \mathbf{p}_{\mathcal{Y},M})) \\ \subseteq{} & U_{\mathcal{X}} \otimes \mathcal{Y} + \mathcal{X} \otimes V_{\mathcal{Y}}. \end{aligned}$$

This completes the proof. ■

Taking the external tensor product of two type-DA Alexander modules, when the underlying spaces of the modules are linearly compact, is a straightforward extension.

Remark 7.14. We can see more directly that if ${}_{\mathcal{L}_i}\mathcal{X}$ and ${}_{\mathcal{L}_j}\mathcal{Y}$ are type-A Alexander modules, then ${}_{\mathcal{L}_{i+j}}(\mathcal{X} \otimes^! \mathcal{Y})$ is also an Alexander module. Consider two continuous maps

$$f_{j+1}: \vec{\mathcal{T}}^j \mathcal{L}_n \mathbin{\vec{\otimes}} \mathcal{X} \to \mathcal{X} \quad \text{and} \quad g_{j+1}: \vec{\mathcal{T}}^j \mathcal{L}_m \mathbin{\vec{\otimes}} \mathcal{X} \to \mathcal{X},$$

where

$$\vec{\mathcal{T}}^j \mathcal{L}_n = \underbrace{\mathcal{L}_n \mathbin{\vec{\otimes}} \cdots \mathbin{\vec{\otimes}} \mathcal{L}_n}_{j}.$$

The external tensor product of f_{j+1} and g_{j+1} is continuous since it may be written as the following sequence of continuous maps:

$$\begin{aligned} & \vec{\mathcal{T}}^j (\mathcal{L}_n \otimes^! \mathcal{L}_m) \mathbin{\vec{\otimes}} (\mathcal{X} \otimes^! \mathcal{Y}) \\ & \quad \to (\vec{\mathcal{T}}^j \mathcal{L}_n \mathbin{\vec{\otimes}} \mathcal{X}) \otimes^! (\vec{\mathcal{T}}^j \mathcal{L}_m \mathbin{\vec{\otimes}} \mathcal{Y}) \to \mathcal{X} \otimes^! \mathcal{Y}. \end{aligned}$$

The first map is from Corollary 5.15 and the second map is $f_{j+1} \otimes g_{j+1}$.

7.6 Box tensor products

In this section, we describe the interaction of Lipshitz, Ozsváth and Thurston's box tensor product operation with Alexander modules. The simplest case is boxing an Alexander type-D module $\mathcal{X}^{\mathcal{L}}$ with an Alexander type-A module ${}_{\mathcal{L}}\mathcal{Y}$. In this case,

$$\delta^1\colon \mathcal{X} \to \mathcal{X} \mathbin{\vec{\otimes}_{\mathrm{E}}} \mathcal{L} \quad \text{and} \quad m_{j+1}\colon \mathcal{L} \mathbin{\vec{\otimes}_{\mathrm{E}}} \cdots \mathbin{\vec{\otimes}_{\mathrm{E}}} \mathcal{L} \mathbin{\vec{\otimes}_{\mathrm{E}}} \mathcal{Y} \to \mathcal{Y}$$

are continuous linear maps on the completions. Hence, assuming an appropriate boundedness condition so that only finitely many terms contribute to the box tensor product (e.g., $m_j = 0$ for $j \gg 0$), the differential $\partial^{\boxtimes}$ will be a sum of finitely many continuous maps. Type-DA modules present no additional complications.

Box tensor products involving split Alexander type-A and DA modules are more subtle. If ${}_{\mathcal{L}_{i_1}|\cdots|\mathcal{L}_{i_n}}\mathcal{Y}$ is a split Alexander type-A module, it seems only possible to tensor with an Alexander type-D module which is itself an external tensor product of n modules

$$\mathcal{X}_1^{\mathcal{L}_{i_1}}, \dots, \mathcal{X}_n^{\mathcal{L}_{i_n}}.$$

In this case, we write

$$(\mathcal{X}_1^{\mathcal{L}_{i_1}}, \dots, \mathcal{X}_n^{\mathcal{L}_{i_n}}) \mathbin{\hat{\boxtimes}} {}_{\mathcal{L}_{i_1}|\cdots|\mathcal{L}_{i_n}}\mathcal{Y}$$

for the resulting tensor product. Ignoring completions, the tensor product is formed by taking the external tensor product of the type-D modules $\mathcal{X}_j^{\mathcal{L}_{i_j}}$ to obtain a type-D module over $\mathcal{L}_{i_1+\cdots+i_n}$. This type-D module is then tensored with the type-A module using the standard definition (see equation (3.1)). We topologize the underlying space of the tensor product

$$(\mathcal{X}_1 \otimes \cdots \otimes \mathcal{X}_n) \otimes \mathcal{Y}$$

using the graph topology from Section 5.6 for a tree with one root and n rays. Here, $\mathcal{Y}$ is labeled as the root, and $\mathcal{X}_1, \dots, \mathcal{X}_n$ each form a ray, with an edge pointing to $\mathcal{Y}$. Proposition 5.24 shows that the structure operations are continuous.

We may also form the tensor product when $\mathcal{Y}$ or some of the $\mathcal{X}_i$ are split DA Alexander bimodules. The output will be a split Alexander DA bimodule, whose tensor splitting of the incoming algebras is the concatenation of the tensor-splittings for each of the $\mathcal{X}_i$. To illustrate, if ${}_{\mathcal{L}_{i_1}|\mathcal{L}_{i_2}}\mathcal{Y}^{\mathcal{L}}$, ${}_{\mathcal{L}_{j_1}|\cdots|\mathcal{L}_{j_m}}\mathcal{X}_1^{\mathcal{L}_{i_1}}$ and ${}_{\mathcal{L}_{k_1}|\cdots|\mathcal{L}_{k_t}}\mathcal{X}_2^{\mathcal{L}_{i_2}}$ are split Alexander DA bimodules, then their tensor product, N, will be a split module as indicated in the following equation:

$$({}_{\mathcal{L}_{j_1}|\cdots|\mathcal{L}_{j_m}}\mathcal{X}_1^{\mathcal{L}_{i_1}}, {}_{\mathcal{L}_{k_1}|\cdots|\mathcal{L}_{k_t}}\mathcal{X}_2^{\mathcal{L}_{i_2}}) \mathbin{\hat{\boxtimes}} {}_{\mathcal{L}_{i_1}|\mathcal{L}_{i_2}}\mathcal{Y}^{\mathcal{L}} = {}_{\mathcal{L}_{j_1}|\cdots|\mathcal{L}_{j_m}|\mathcal{L}_{k_1}|\cdots|\mathcal{L}_{k_t}}N^{\mathcal{L}}.$$

Remark 7.15. The box tensor product operation of split Alexander modules seems definable using our techniques only when the underlying vector spaces of our modules are linearly compact. In this case, the underlying vector space of the tensor product $(\mathcal{X}^{\mathcal{K}}, \mathcal{Y}^{\mathcal{K}}) \mathbin{\hat{\boxtimes}} {}_{\mathcal{K}|\mathcal{K}}\mathcal{Z}$ is the standard tensor product $\mathcal{X} \otimes^! \mathcal{Y} \otimes^! \mathcal{Z}$ by Lemma 5.16. Furthermore, assuming linear compactness, the completed box tensor product operation satisfies the following associativity property:

$$\begin{aligned}(\mathcal{X}^{\mathcal{K}}, \mathcal{Y}^{\mathcal{K}}) \mathbin{\hat{\boxtimes}} {}_{\mathcal{K}|\mathcal{K}}\mathcal{Z} &\cong \mathcal{X}^{\mathcal{K}} \mathbin{\hat{\boxtimes}} {}_{\mathcal{K}}(({}_{\mathcal{K}}\mathbb{I}^{\mathcal{K}}, \mathcal{Y}^{\mathcal{K}}) \mathbin{\hat{\boxtimes}} {}_{\mathcal{K}|\mathcal{K}}\mathcal{Z})\\ &\cong \mathcal{Y}^{\mathcal{K}} \mathbin{\hat{\boxtimes}} {}_{\mathcal{K}}((\mathcal{X}^{\mathcal{K}}, {}_{\mathcal{K}}[\mathbb{I}]^{\mathcal{K}}) \mathbin{\hat{\boxtimes}} {}_{\mathcal{K}|\mathcal{K}}\mathcal{Z}).\end{aligned}$$

Note that all of the bordered link surgery modules considered in this memoir are countable direct products of $\mathbb{F}$, so are linearly compact.

We now describe a split type-DA Alexander bimodule ${}_{\mathcal{K}|\mathcal{K}}[\mathbb{I}]^{\mathcal{L}_2}$ which transforms any Alexander module ${}_{\mathcal{L}_2}\mathcal{X}$ into a split module ${}_{\mathcal{K}|\mathcal{K}}\mathcal{X}$. The structure maps are the same as for the identity module.

Lemma 7.16. *The bimodule* ${}_{\mathcal{K}|\mathcal{K}}[\mathbb{I}]^{\mathcal{L}_2}$ *is a split Alexander bimodule. More generally, if* $\mathbb{P}$ *is a partition of* $\{1, \dots, \ell\}$*, and* $\ell_1, \dots, \ell_j$ *are the sizes of the partition elements, then* ${}_{\mathcal{L}_{\ell_1}|\cdots|\mathcal{L}_{\ell_j}}[\mathbb{I}]^{\mathcal{L}_\ell}$ *is a split Alexander bimodule.*

Proof. Consider first ${}_{\mathcal{K}|\mathcal{K}}[\mathbb{I}]^{\mathcal{L}_2}$. In this case, the claim follows from Remark 5.23 and the fact that the natural map

$$\mathcal{K} \otimes^*_{\mathbb{F}} \mathcal{K} \to \mathcal{K} \otimes^!_{\mathbb{F}} \mathcal{K}$$

is continuous. The case of general ℓ and $\mathbb{P}$ is proven by the same argument. ■

Remark 7.17. Lemma 7.16 implies that the ordinary Alexander module condition is stronger than the split Alexander module condition. In particular, given a type-A Alexander module ${}_{\mathcal{L}_\ell}\mathcal{X}$, we may always view $\mathcal{X}$ as a split Alexander module for any partition of $\{1, \dots, \ell\}$.

7.7 Finitely generated $\boldsymbol{\mathcal{K}}$-modules

In this section, we discuss the category of finitely generated type-D modules over $\mathcal{L}$.

We denote by $\boldsymbol{\mathcal{K}}$ the completion of the algebra $\mathcal{K}$ with respect to the topology described above. As a vector space, we have the following isomorphisms:

$$\begin{aligned}\mathbf{I}_0 \cdot \boldsymbol{\mathcal{K}} \cdot \mathbf{I}_0 &\cong \mathbb{F}\llbracket \mathcal{U}, \mathcal{V} \rrbracket,\\ \mathbf{I}_1 \cdot \boldsymbol{\mathcal{K}} \cdot \mathbf{I}_0 &\cong \mathbb{F}[\mathcal{V}, \mathcal{V}^{-1}\rrbracket\llbracket \mathcal{U} \rrbracket\langle \tau \rangle \oplus \mathbb{F}\llbracket \mathcal{V}, \mathcal{V}^{-1}]\llbracket \mathcal{U} \rrbracket\langle \sigma \rangle,\\ \mathbf{I}_1 \cdot \boldsymbol{\mathcal{K}} \cdot \mathbf{I}_1 &\cong \mathbb{F}[\mathcal{V}, \mathcal{V}^{-1}]\llbracket \mathcal{U} \rrbracket.\end{aligned}$$

We may consider the completion of the link algebra $\boldsymbol{\mathcal{L}}$ as well.

Since multiplication is continuous by Proposition 7.5 and Corollary 7.8, we obtain a well-defined map on the completed algebras

$$\mu_2 \colon \boldsymbol{\mathcal{L}} \mathbin{\vec{\otimes}_{\mathrm{E}}} \boldsymbol{\mathcal{L}} \to \boldsymbol{\mathcal{L}}.$$

Since $\boldsymbol{\mathcal{L}}$ is an algebra, we may also consider the ordinary category of type-D modules over this algebra (i.e., ordinary modules and algebras, with no topologies). Of particular interest to us is the category of finitely generated type-D modules. We write $\mathrm{Mod}^{\boldsymbol{\mathcal{L}}}_{\mathrm{fg}}$ for this category.

There is a related category, $\mathrm{Mod}^{\mathcal{L}}_{\mathrm{fg},\mathfrak{a}}$, consisting of Alexander type-D modules over $\mathcal{L}$ which are finitely generated over $\mathbb{F}$.

Proposition 7.18. *The categories* $\mathrm{Mod}^{\mathcal{L}}_{\mathrm{fg},\mathfrak{a}}$ *and* $\mathrm{Mod}^{\boldsymbol{\mathcal{L}}}_{\mathrm{fg}}$ *are equivalent.*

Proof. We define functors in both directions. The functor

$$F \colon \mathrm{Mod}^{\mathcal{L}}_{\mathrm{fg},\mathfrak{a}} \to \mathrm{Mod}^{\boldsymbol{\mathcal{L}}}_{\mathrm{fg}}$$

is obtained by taking completions, noting that the completion of $\mathcal{X} \mathbin{\vec{\otimes}} \mathcal{L}$ coincides with the ordinary tensor product $\boldsymbol{\mathcal{X}} \otimes \boldsymbol{\mathcal{L}}$ when $\mathcal{X}$ is finitely generated. (Note that when $\mathcal{X}$ is finite-dimensional, $\mathcal{X} \cong \boldsymbol{\mathcal{X}}$ if and only if $\mathcal{X}$ is Hausdorff.)

The functor

$$G \colon \mathrm{Mod}^{\boldsymbol{\mathcal{L}}}_{\mathrm{fg}} \to \mathrm{Mod}^{\mathcal{L}}_{\mathrm{fg},\mathfrak{a}}$$

sends a type-D module $(\mathcal{X}, \delta^1)$ to itself, where $\mathcal{X}$ is equipped with the discrete topology. This is well defined due to the fact that if $\mathcal{X}$ and $\mathcal{Y}$ are equipped with the discrete topology, then a map

$$f^1 \colon \mathcal{X} \to \mathcal{Y} \mathbin{\vec{\otimes}} \boldsymbol{\mathcal{L}}$$

is automatically continuous, since $\{0\} \subseteq \mathcal{X}$ is open. We observe that $F \circ G$ is the identity functor.

On the other hand, $G \circ F$ maps $(\mathcal{X}, \delta^1)$ to $(\boldsymbol{\mathcal{X}}, \delta^1)$. Note that in $\mathrm{mod}^{\mathcal{L}}_{\mathrm{fg},\mathfrak{a}}$, the objects $(\mathcal{X}, \delta^1)$ to $(\boldsymbol{\mathcal{X}}, \delta^1)$ are canonically isomorphic, since morphisms are defined to be continuous linear maps on completions. See Definition 5.4 and Remark 5.5. In particular, there is a natural transformation from $G \circ F$ to the identity. The proof is complete. ■