Chapter 8

Link surgery modules over $\mathcal{K}$ and $\mathcal{L}$

In this chapter, we describe some basic link surgery modules over the algebras $\mathcal{K}$ and $\mathcal{L}$. We begin by defining the type-D and type-A modules for a solid torus and the algebraic merge module. Subsequently, we interpret the knot and link surgery formulas as type-D modules over $\mathcal{K}$ and $\mathcal{L}$.

8.1 The type-D module for a solid torus

If λ is an integer, we refer to the complement of a λ-framed unknot in S^3 as the λ-*framed solid torus*. We will define a type-D module $\mathcal{D}_\lambda^{\mathcal{K}}$ for the λ-framed solid torus as follows. We set

$$\mathcal{D}_\lambda^{\mathcal{K}} \cdot \mathbf{I}_0 = \langle \mathbf{x}^0 \rangle \quad \text{and} \quad \mathcal{D}_\lambda^{\mathcal{K}} \cdot \mathbf{I}_1 = \langle \mathbf{x}^1 \rangle,$$

where $\langle \mathbf{x}^\varepsilon \rangle = \mathbb{F}$, spanned by a generator $\mathbf{x}^\varepsilon$. Here, $\varepsilon \in \{0, 1\}$ denotes the idempotent. We define the structure map via the formula

$$\delta^1(\mathbf{x}^0) = \mathbf{x}^1 \otimes (\sigma + \mathcal{V}^n \tau).$$

8.2 The type-A module for a solid torus

Suppose that λ is an integer. We now define the type-A module ${}_{\mathcal{K}}\mathcal{D}_\lambda$ for the solid torus as follows. We set

$$\mathbf{I}_0 \cdot \mathcal{D}_\lambda = \mathbb{F}[\mathcal{U}, \mathcal{V}] \quad \text{and} \quad \mathbf{I}_1 \cdot \mathcal{D}_\lambda = \mathbb{F}[\mathcal{U}, \mathcal{V}, \mathcal{V}^{-1}].$$

We define the type-A structure map m_j on $\mathcal{D}_\lambda$ to be 0 unless $j = 2$. We define m_2 on $\mathcal{D}_\lambda$ as follows. If $f \in \mathbf{I}_i \cdot \mathcal{K} \cdot \mathbf{I}_i$ and $x \in \mathbf{I}_j \cdot \mathcal{D}_j$, then we define $m_2(f, x)$ to be $f \cdot x$ (ordinary multiplication of polynomials) if $i = j$ and to be 0 otherwise. If $x \in \mathbf{I}_0 \cdot \mathcal{D}_\lambda$, we define

$$m_2(\sigma, x) = \phi^\sigma(x) \in \mathbb{F}[\mathcal{U}, \mathcal{V}, \mathcal{V}^{-1}] = \mathbf{I}_1 \cdot \mathcal{D}_\lambda,$$

where ϕ^σ is the canonical inclusion of localization. Similarly, we define

$$m_2(\tau, x) = \mathcal{V}^\lambda \cdot \phi^\tau(x) \in \mathbb{F}[\mathcal{U}, \mathcal{V}, \mathcal{V}^{-1}],$$

where $\cdot$ denotes ordinary multiplication of polynomials.

We view $\mathcal{D}_\lambda$ as being the direct sum of the 1-dimensional spans of monomials, and give $\mathcal{D}_\lambda$ the product topology with respect to this decomposition. See Definition 5.6. With respect to this topology, the completion is given by

$$\mathbf{I}_0 \cdot \boldsymbol{\mathcal{D}}_\lambda = \mathbb{F}[[\mathcal{U}, \mathcal{V}]], \quad \text{and} \quad \mathbf{I}_1 \cdot \boldsymbol{\mathcal{D}}_\lambda = \mathbb{F}[[\mathcal{U}, \mathcal{V}, \mathcal{V}^{-1}]].$$

(Note that the latter object is defined only as a vector space, and not as a ring, since we are taking completions with respect to $\mathcal{V}$ and $\mathcal{V}^{-1}$.)

Lemma 8.1. *The map*

$$m_2 \colon \mathcal{K} \vec{\otimes}_{\mathrm{I}} \mathcal{D}_\lambda \to \mathcal{D}_\lambda$$

is continuous.

Proof. Write $\mathcal{D}$ for $\mathcal{D}_\lambda$. Continuity amounts to two claims. The first is that if $E \subseteq \mathcal{D}$ is an open subspace, then there is some n so that $m_2(J_n \otimes \mathcal{D}) \subseteq E$. Additionally, we need to show that if E is as above and $a \in \mathcal{K}$, then there is some open $V_a \subseteq \mathcal{D}$ so that $m_2(a \otimes V_a) \subseteq E$. By definition, we may assume that E is $\mathcal{D}_{\mathrm{co}(S)}$ for some finite set of basis elements in $\mathcal{D}$. (Recall that $\mathcal{D}_{\mathrm{co}(S)}$ is the span of the basis elements not in S.)

For the first claim, it suffices to show that if n is sufficiently large, then $J_n \otimes \mathcal{D}$ is mapped into $\mathcal{D}_{\mathrm{co}(S)}$. This may be proven by considering each idempotent of $\mathcal{K}$ separately. To ensure $(\mathbf{I}_0 \cdot J_n \cdot \mathbf{I}_0) \otimes \mathcal{D}$ is sent into $\mathcal{D}_{\mathrm{co}(S)}$, it is sufficient to pick n larger than $\max(p, q)$ ranging over all monomials $\mathcal{U}^p\mathcal{V}^q$ appearing in $\mathbf{I}_0 \cdot S$. For $(\mathbf{I}_1 \cdot J_n \cdot \mathbf{I}_1) \otimes \mathcal{D}$ to be sent into $\mathcal{D}_{\mathrm{co}(S)}$, we pick n larger than p for each monomial $\mathcal{U}^p\mathcal{V}^q$ appearing in $\mathbf{I}_1 \cdot S$. Finally, to ensure that $(\mathbf{I}_1 \cdot J_n \cdot \mathbf{I}_0) \otimes \mathcal{D}$ is sent into $\mathcal{D}_{\mathrm{co}(S)}$, it is sufficient to pick n larger than $\max(p, q, 2p - q)$, ranging over monomials $\mathcal{U}^p\mathcal{V}^q$ appearing in a summand of $\mathbf{I}_1 \cdot S$. To see that this is sufficient, note that if $\mathcal{U}^i\mathcal{V}^j\tau \otimes \mathcal{U}^s\mathcal{V}^t \in J_n \otimes \mathcal{D}$, then $\max(i, 2i - j) \geq n$. We observe that $\mathcal{U}^i\mathcal{V}^j\tau\mathcal{U}^s\mathcal{V}^t = \mathcal{U}^{i+t}\mathcal{V}^{j-s+2t}\tau$ lies in $\mathcal{D}_{\mathrm{co}(S)}$, since

$$\max(i + t, 2(i + t) - (j - s + 2t)) = \max(i + t, 2i - j + s) \geq \max(i, 2i - j) \geq n.$$

The fact that $\mathcal{U}^i\mathcal{V}^j\sigma \otimes \mathcal{U}^s\mathcal{V}^t \in J_n \otimes \mathcal{D}$ is sent into $\mathcal{D}_{\mathrm{co}(s)}$ is similar.

Next, we need to show that if $a \in \mathcal{K}$ is fixed, then there is a finite subset T of generators of $\mathcal{D}$ so that $a \otimes \mathcal{D}_{\mathrm{co}(T)}$ is mapped into $\mathcal{D}_{\mathrm{co}(S)}$. Without loss of generality, we assume that a is a monomial. We observe the following basic fact: if $a \in \mathbf{I}_{\varepsilon'} \cdot \mathcal{K} \cdot \mathbf{I}_\varepsilon$ and $y \in \mathbf{I}_{\varepsilon'} \cdot \mathcal{D}$ are non-zero monomials, then there is at most 1 (in particular, finitely many) $x \in \mathbf{I}_\varepsilon \cdot \mathcal{D}$ such that $m_2(a, x) = y$. In particular, there exists a finite set of generators T such that $a \otimes \mathcal{D}_{\mathrm{co}(T)}$ is sent into $\mathcal{D}_{\mathrm{co}(S)}$, completing the proof. ∎

Remark 8.2. Note that m_2 is *not* continuous as a map from $\mathcal{K} \otimes^! \mathcal{D}$ to $\mathcal{D}$. Compare Remark 7.6. In fact, m_2 does not induce a map from the completion of $\mathcal{K} \otimes^! \mathcal{D}_\lambda$

to $\mathfrak{D}_\lambda$. For example, in the completion of $\mathcal{K} \otimes^! \mathcal{D}_\lambda$ the sum $\sum_{i=0}^{\infty} \mathcal{V}^i \otimes \mathcal{V}^{-i} i_1$ converges since $\mathcal{V}^{-i} i_1 \to 0$ in the topology of $\mathcal{D}_\lambda$, while m_2 cannot be defined on such infinite sums.

The module for a solid torus extends to a type-AA bimodule

$$_{\mathcal{K}}[\mathcal{D}_\lambda]_{\mathbb{F}[U]}.$$

Here, we have U act on the right via polynomial multiplication with $\mathcal{U}\mathcal{V}$ (summed over both idempotents). Since $[\mathcal{U}\mathcal{V}, \tau] = 0$ and $[\mathcal{U}\mathcal{V}, \sigma] = 0$, the construction above defines an AA-bimodule structure on $\mathcal{D}_\lambda$. Note that only $m_{1,1,0}$ and $m_{0,1,1}$ are nontrivial.

Additionally, we can also think of the module for a solid torus as corresponding instead to a DA bimodule $_{\mathcal{K}}[\mathcal{D}_\lambda]^{\mathbb{F}[U]}$, in such a way that

$$_{\mathcal{K}}[\mathcal{D}_\lambda]^{\mathbb{F}[U]} \mathbin{\widehat{\boxtimes}} {}_{\mathbb{F}[U]}\mathbb{F}[U]_{\mathbb{F}[U]} \cong {}_{\mathcal{K}}[\mathcal{D}_\lambda]_{\mathbb{F}[U]}.$$

For this description, the generators of $\mathbf{I}_0 \cdot {}_{\mathcal{K}}[\mathcal{D}_\lambda]^{\mathbb{F}[U]}$ are $\mathcal{U}^i$ and $\mathcal{V}^j$ for $i, j \geq 0$. The generators of $\mathbf{I}_1 \cdot {}_{\mathcal{K}}[\mathcal{D}_\lambda]^{\mathbb{F}[U]}$ are $\mathcal{V}^i$ for $i \in \mathbb{Z}$. This module has $\delta_j^1 = 0$ unless $j = 2$. We illustrate the structure map δ_2^1 of $_{\mathcal{K}}[\mathcal{D}_0]^{\mathbb{F}[U]}$ in Figure 8.1 and leave the structure relations to the reader for other framings.

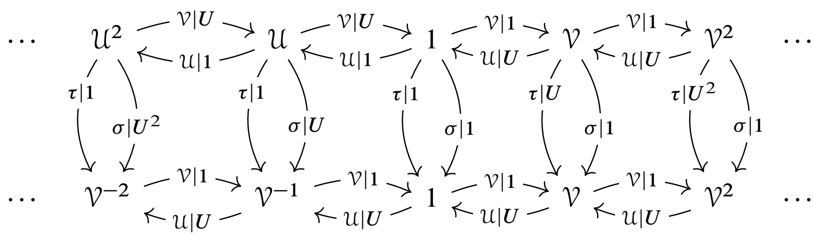


Figure 8.1. The DA bimodule $_{\mathcal{K}}\mathcal{D}_0^{\mathbb{F}[U]}$. An arrow decorated with $a|b$ from $\mathbf{x}$ to $\mathbf{y}$ means that $\delta_2^1(a, \mathbf{x}) = \mathbf{y} \otimes b$.

8.3 The merge bimodule

In this section, we define a bimodule $_{\mathcal{K}|\mathcal{K}}M^{\mathcal{K}}$, which we call the *merge bimodule*.

Whenever R is a commutative algebra over $\mathbb{F}$, there is a morphism of algebras $\phi\colon R \otimes_{\mathbb{F}} R \to R$, given by $\phi(a \otimes b) = ab$. A ring homomorphism $\psi\colon A \to B$ always determines a bimodule $_A[\psi]^B$ whose underlying vector space is the ground ring, and has structure map $\delta_1^1 = 0$ and $\delta_2^1(a \otimes 1) = 1 \otimes \psi(a)$. In particular, there is a natural bimodule $_{R\otimes R}[\phi]^R$ for any commutative ring R.

The algebra $\mathcal{K}$ is not commutative, though the bimodule ${}_{\mathcal{K}|\mathcal{K}}M^{\mathcal{K}}$ is somewhat analogous.

We now describe the bimodule M. To make the notation clear, in this section we will write $|$ for $\otimes_{\mathbb{F}}$ and $\otimes$ for $\otimes_{\mathbf{I}}$.

As a vector space, M is $\mathbf{I} = \mathbf{I}_0 \oplus \mathbf{I}_1$. There is a left action of $\mathbf{I}|\mathbf{I}$, given by

$$(i_1|i_2) \cdot i = i_1 i_2 \cdot i.$$

The right $\mathbf{I}$-action is the standard action.

On M, only δ_2^1 and δ_3^1 are non-trivial. The map δ_2^1 is defined as follows. Suppose that $a_1|a_2$ is an elementary tensor in either

$$(\mathbf{I}_0|\mathbf{I}_0) \cdot (\mathcal{K}|\mathcal{K}) \cdot (\mathbf{I}_0|\mathbf{I}_0) \quad \text{or} \quad (\mathbf{I}_1|\mathbf{I}_1) \cdot (\mathcal{K}|\mathcal{K}) \cdot (\mathbf{I}_1|\mathbf{I}_1).$$

In this case, we set

$$\delta_2^1(a_1|a_2 \otimes i) = i \otimes a_1 a_2.$$

On any other elementary tensor, we set δ_2^1 to vanish.

We define δ_3^1 as follows. We set

$$\delta_3^1(1|\sigma, \sigma|1 \otimes i_0) = i_1 \otimes \sigma \quad \text{and} \quad \delta_3^1(1|\tau, \tau|1 \otimes i_0) = i_1 \otimes \tau.$$

More generally, if a, b, c, d are monomials concentrated in single idempotents, then we set

$$\delta_3^1(a|b\sigma, c\sigma|d, i_0) = i_0 \otimes abc\sigma d$$

and similarly for the τ terms. We set

$$\delta_3^1(\sigma|1, 1|\sigma \otimes i_0) = 0,$$

and similarly if τ replaces σ. The DA bimodule relations are straightforward to verify.

Lemma 8.3. *The merge module ${}_{\mathcal{K}|\mathcal{K}}M^{\mathcal{K}}$ is a split Alexander module (using the discrete partition on the two incoming algebra factors).*

Remark 8.4. The structure maps of the merge module are *not* continuous when we view the bimodule as a non-split module over $\mathcal{K} \otimes^! \mathcal{K}$ or $\mathcal{K} \otimes^* \mathcal{K}$, i.e., as ${}_{\mathcal{K}\otimes^!\mathcal{K}}M^{\mathcal{K}}$ or ${}_{\mathcal{K}\otimes^*\mathcal{K}}M^{\mathcal{K}}$. To see this, observe that discontinuity with respect to the topology on $\mathcal{K} \otimes_{\mathbb{F}}^* \mathcal{K}$ implies discontinuity with respect to $\mathcal{K} \otimes_{\mathbb{F}}^! \mathcal{K}$ since there is a continuous map $\mathcal{K} \otimes_{\mathbb{F}}^* \mathcal{K} \to \mathcal{K} \otimes_{\mathbb{F}}^! \mathcal{K}$. Note that a sequence of tensors

$$\mathbf{x}_n = (x_n|y_n) \otimes (z_n|w_n) \in (\mathcal{K} \otimes_{\mathbb{F}}^* \mathcal{K}) \vec{\otimes}_{\mathrm{E}} (\mathcal{K} \otimes_{\mathbb{F}}^* \mathcal{K})$$

will converge to 0 if x_n is constant in n and $y_n \to 0$. In particular, we need not make any assumptions about z_n or w_n. We note that $\mathscr{V}^n\sigma \to 0$ in $\mathcal{K}$, as each of the ideals

J_m (which form a basis of open sets centered at 0 for our topology on $\mathcal{K}$) contains $\mathcal{V}^n\sigma$ for all sufficiently large n. Therefore,

$$(1|\mathcal{V}^n\sigma) \otimes (\mathcal{V}^{-n}\sigma|1) \to 0.$$

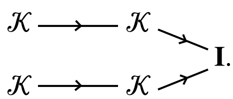
On the other hand,

$$\delta^1_3(1|\mathcal{V}^n\sigma, \mathcal{V}^{-n}\sigma|1, i_0) = i_1 \otimes \sigma \not\to 0,$$

so δ^1_3 is not continuous.

Proof of Lemma 8.3. We consider first δ^1_2. In this case, the only non-trivial actions on i_0 or i_1 are from pairs $a|b$ where a and b are both in $\mathbf{I}_0 \cdot \mathcal{K} \cdot \mathbf{I}_0$ or both in $\mathbf{I}_1 \cdot \mathcal{K} \cdot \mathbf{I}_1$. In this case, the claim is clear.

We now consider δ^1_3. Recall that verifying the split Alexander condition amounts to showing that δ^1_3 is continuous if we use the tree topology from Section 5.6 using the following tree:



We define a function

$$\widetilde{\mu}\colon \mathcal{K} \otimes^*_{\mathbb{F}} \mathcal{K} \to \mathcal{K}$$

given on elementary tensors of monomials by

$$\widetilde{\mu}(a\sigma, b\sigma) = ab\sigma \quad \text{and} \quad \widetilde{\mu}(a\tau, b\tau) = ab\tau,$$

with $\widetilde{\mu}$ vanishing on other elementary tensors of monomials. Additionally, we consider a map

$$\Pi\colon (\mathcal{K} \otimes_{\mathrm{I}} \mathcal{K}) \otimes_{\mathbb{F}} (\mathcal{K} \otimes_{\mathrm{I}} \mathcal{K}) \to (\mathcal{K} \otimes_{\mathrm{I}} \mathcal{K}) \otimes_{\mathbb{F}} (\mathcal{K} \otimes_{\mathrm{I}} \mathcal{K})$$

such that Π is the identity on the elementary tensors of monomials

$$(a\sigma \otimes a')|(b \otimes b'\sigma) \quad \text{and} \quad (a\tau \otimes a')|(b \otimes b'\tau),$$

and such that Π vanishes on other elementary tensors of monomials.

We observe that

$$\delta^1_3(\mathbf{a}, 1) = 1 \otimes (\widetilde{\mu} \circ (\mu_2|\mu_2) \circ \Pi)(\mathbf{a}).$$

The map Π is clearly continuous with respect to the tree topology, and furthermore, $\mu_2|\mu_2$ is continuous by Propositions 7.5 and 5.24. Note that if we topologize $(\mathcal{K}|\mathcal{K}) \otimes_{\mathrm{I}|\mathrm{I}} \mathbf{I}$ with the tree topology, it is isomorphic to $\mathcal{K} \otimes^*_{\mathbb{F}} \mathcal{K}$. (See Remark 5.23.) Hence, it suffices to show that $\widetilde{\mu}$ is continuous. This amounts to the following:

(m-1) For each $n \in \mathbb{N}$, there are $k, m \in \mathbb{N}$ so that

$$\widetilde{\mu}(J_k \otimes J_m) \subseteq J_n.$$

(m-2) For each $n \in \mathbb{N}$ and $x \in \mathcal{K}$, there is an $m \in \mathbb{N}$ so that

$$\widetilde{\mu}(x \otimes J_m) \subseteq J_n \quad \text{and} \quad \widetilde{\mu}(J_m \otimes x) \subseteq J_n.$$

Both of the arguments are elementary, and we illustrate the idea. Recall that by our construction in Definition 7.3, $\mathcal{U}^i\mathcal{V}^j\sigma \in J_n$ if and only if $i \geq 0$ and $\max(i, j) \geq n$. If $\mathcal{U}^i\mathcal{V}^j\sigma$ and $\mathcal{U}^s\mathcal{V}^t\sigma$ are two monomials in J_n, then at least one of two situations occurs: both $j, t \geq n$; or one of i and s is at least n. If $j, t \geq n$, then $\mathcal{U}^{i+s}\mathcal{V}^{j+t}\sigma \in J_n$ since $j + t \geq 2n$. If one of i and s is at least n, then $i + s \geq n$. A symmetric analysis holds for the multiples of τ, and we see that

$$\widetilde{\mu}(J_n \otimes J_n) \subseteq J_n,$$

so (m-1) holds. A similar and elementary argument implies (m-2). ∎

There is an asymmetry in the merge module. Recall that we set

$$\delta_3^1(1|\sigma, \sigma|1, i_0) = i_1 \otimes \sigma \quad \text{and} \quad \delta_3^1(\sigma|1, 1|\sigma, i_0) = 0.$$

We define a different module ${}_{\mathcal{K}|\mathcal{K}}N^{\mathcal{K}}$ by instead setting

$$\delta_3^1(1|\sigma, \sigma|1, i_0) = 0 \quad \text{and} \quad \delta_3^1(\sigma|1, 1|\sigma, i_0) = i_1 \otimes \sigma.$$

We make a similar change to the map δ_3^1 on τ inputs.

This construction turns out to produce homotopy equivalent bimodules.

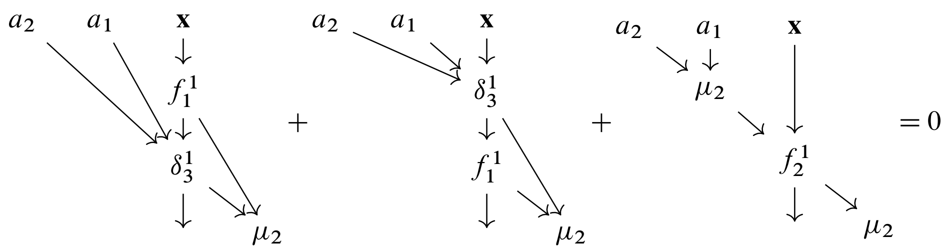
Lemma 8.5. *There is a homotopy equivalence of* DA *bimodules*

$${}_{\mathcal{K}|\mathcal{K}}M^{\mathcal{K}} \simeq {}_{\mathcal{K}|\mathcal{K}}N^{\mathcal{K}}.$$

Proof. We construct a morphism f_{j+1}^1 between the two bimodules. We set $f_1^1(\mathbf{x}) = \mathbf{x} \otimes 1$. The only other non-vanishing term is f_2^1. This map is determined by the relations

$$f_2^1(\sigma|\sigma, i_0) = i_1 \otimes \sigma \quad \text{and} \quad f_2^1(\tau|\tau, i_0) = i_1 \otimes \tau.$$

The DA bimodule morphism structure relations for two algebra inputs which both change idempotent are given by the schematic



which is easily verified. The structure relations for $n \neq 2$ algebra inputs are also easily verified. ∎

8.4 The type-A identity bimodule

It is helpful to introduce another bimodule, which we denote by

$$_{\mathcal{K}|\mathcal{K}}[\mathbb{I}^{\supset}] := {}_{\mathcal{K}|\mathcal{K}}M^{\mathcal{K}} \mathbin{\hat{\boxtimes}} {}_{\mathcal{K}}\mathcal{D}_0.$$

We call $_{\mathcal{K}|\mathcal{K}}[\mathbb{I}^{\supset}]$ the *type*-A *identity module*. We will use $_{\mathcal{K}|\mathcal{K}}[\mathbb{I}^{\supset}]$ to transform type-D actions into type-A actions. The type-A module $_{\mathcal{K}|\mathcal{K}}[\mathbb{I}^{\supset}]$ appears in the pairing theorem, stated in Theorem 1.4.

There is a further refinement which takes into account the right $\mathbb{F}[U]$-action on $\mathcal{D}_0$. We define

$$_{\mathcal{K}|\mathcal{K}}[\mathbb{I}^{\supset}]_{\mathbb{F}[U]} := {}_{\mathcal{K}|\mathcal{K}}M^{\mathcal{K}} \mathbin{\hat{\boxtimes}} {}_{\mathcal{K}}[\mathcal{D}_0]_{\mathbb{F}[U]}.$$

Verifying the pairing theorem with the bimodule $_{\mathcal{K}|\mathcal{K}}[\mathbb{I}^{\supset}]_{\mathbb{F}[U]}$ turns out to be more subtle than for $_{\mathcal{K}|\mathcal{K}}[\mathbb{I}^{\supset}]$. Nonetheless, we prove the pairing theorem with $_{\mathcal{K}|\mathcal{K}}[\mathbb{I}^{\supset}]_{\mathbb{F}[U]}$ in Section 15.3 once we introduce the pair-of-pants bimodules.

8.5 The knot surgery formula over $\mathcal{K}$

We now describe how to view the mapping cone formula of Ozsváth and Szabó in terms of the algebra $\mathcal{K}$.

Recall that Ozsváth and Szabó's mapping cone formula [43, Theorem 1.1] states that if λ is an integral framing on $K \subseteq S^3$, then

$$\mathbf{CF}^-(S^3_\lambda(K)) \cong \mathbb{X}_\lambda(K) = \operatorname{Cone}\big(\mathbb{A}(K) \xrightarrow{v+h_\lambda} \mathbb{B}(K)\big).$$

Recall, as in Lemma 6.5, that $\mathbb{A}(K)$ may be identified with a completion of $\mathcal{CFK}(K)$ and $\mathbb{B}(K)$ may be identified with a completion of $\mathcal{V}^{-1}\mathcal{CFK}(K)$.

We will describe the following algebraic perspectives of the mapping cone formula:

(1) A chain complex $\mathbb{X}_\lambda(K)$ over $\mathbb{F}[\![U]\!]$.

(2) A right type-D module $\mathcal{X}_\lambda(K)^{\mathcal{K}}$.

(3) A left type-A module $_{\mathcal{K}}\mathcal{X}_\lambda(K)$.

These will satisfy the following relations:

$$\mathbb{X}_\lambda(K) \cong \mathcal{X}_\lambda(K)^{\mathcal{K}} \mathbin{\hat{\boxtimes}} {}_{\mathcal{K}}[\mathcal{D}_0] \simeq \mathcal{D}_0^{\mathcal{K}} \mathbin{\hat{\boxtimes}} {}_{\mathcal{K}}\mathcal{X}_\lambda(K).$$

We begin with the description as a type-D module. Let $\mathbf{x}_1, \dots, \mathbf{x}_n$ be a free basis of $\mathcal{CFK}(K)$ over $\mathbb{F}[\mathcal{U}, \mathcal{V}]$. We declare the underlying $\mathbf{I}$-module of $\mathcal{X}_\lambda(K)^{\mathcal{K}}$ to be

$$\mathcal{X}_\lambda(K)^{\mathcal{K}} = \operatorname{Span}_{\mathbb{F}}(\mathbf{x}_1, \dots, \mathbf{x}_n) \otimes_{\mathbb{F}} \mathbf{I}.$$

In particular, over $\mathbb{F}$ each $\mathbf{x}_i$ contributes one generator in each idempotent. We denote these generators by $\mathbf{x}_i^0$ and $\mathbf{x}_i^1$.

The map δ^1 on $\mathcal{X}_\lambda(K)^{\mathcal{K}}$ is as follows. Firstly, there are internal δ^1 summands from the differential of $\mathcal{CFK}(K)$. If $\partial(\mathbf{x})$ contains a summand of $\mathbf{y} \cdot \mathcal{U}^n\mathcal{V}^m$, then $\delta^1(\mathbf{x}^\varepsilon)$ contains a summand of $\mathbf{y}^\varepsilon \otimes \mathcal{U}^n\mathcal{V}^m$, for $\varepsilon \in \{0, 1\}$.

In Ozsváth and Szabó's mapping cone formula, v is the canonical inclusion of $\mathcal{CFK}(K)$ into $\mathcal{V}^{-1}\mathcal{CFK}(K)$. Correspondingly, $\delta^1(\mathbf{x}^0)$ contains a summand of the form $\mathbf{x}^1 \otimes \sigma$.

If $\mathbf{y} \cdot \mathcal{U}^i\mathcal{V}^j$ is a summand of $h_\lambda(\mathbf{x})$ then we define $\delta^1(\mathbf{x}^0)$ to have a summand of the form

$$\mathbf{y}^1 \otimes \mathcal{U}^i\mathcal{V}^j\tau.$$

Lemma 8.6. *$\mathcal{X}_\lambda(K)^{\mathcal{K}}$ is a type-*D *module.*

Proof. The proof is a formal consequence of the fact that $\partial^2 = 0$ on $\mathcal{CFK}(K)$, that v and h_λ are chain maps, and that v and h_λ satisfy the equivariance properties of Lemma 6.7.

In more detail, suppose that $\mathbf{x}_1, \dots, \mathbf{x}_n$ is a free basis of $\mathcal{CFK}(K)$, and consider

$$((\mathrm{id}_{\mathcal{X}} \otimes \mu_2) \circ (\delta^1 \otimes \mathrm{id}_{\mathcal{K}}) \circ \delta^1)(\mathbf{x}_i^\varepsilon) \tag{8.1}$$

For concreteness, consider the case when $\varepsilon = 0$. There are three sources of terms in the above expression: those arising from two applications of ∂, those arising from one v and one ∂, and those arising from one h_λ and one ∂. Consider the terms with two ∂ terms. Write $\partial(\mathbf{x}_i) = \sum_{j=1}^n \mathbf{x}_j \cdot f_{j,i}$. The terms with two ∂ terms are simply

$$\sum_{k=1}^n \mathbf{x}_k^0 \otimes \left(\sum_{j=1}^n f_{k,j} f_{j,i} \right),$$

which is 0 since $\partial^2 = 0$ on $\mathrm{CFK}(K)$. Similarly, the terms corresponding to one ∂ and one v are given by the formula

$$\sum_{j=1}^n \mathbf{x}_j^1 \otimes (\sigma f_{j,i} + f_{j,i}\sigma),$$

which vanishes because $\sigma f_{j,i} = f_{j,i}\sigma$ in $\mathcal{K}$.

Finally, we consider the terms corresponding to one ∂ and one h_λ. Write $h_\lambda(\mathbf{x}_i) = \sum_{j=1}^n \mathbf{x}_j \cdot g_{j,i}$, where $g_{j,i} \in \mathbb{F}[\mathcal{U}, \mathcal{V}, \mathcal{V}^{-1}]$. The corresponding terms of equation (8.1) are

$$\sum_{k=1}^n \mathbf{x}_k^1 \otimes \left(\sum_{j=1}^n g_{k,j}\tau f_{j,i} + f_{k,j} g_{j,i}\tau \right) = \sum_{k=1}^n \mathbf{x}_k^1 \otimes \left(\sum_{j=1}^n g_{k,j}\phi^\tau(f_{j,i}) + f_{k,j} g_{j,i} \right)\tau. \tag{8.2}$$

However, $\sum_{j=1}^{n} g_{k,j}\phi^{\tau}(f_{j,i}) + f_{k,j}g_{j,i}$ is the $\mathbf{x}_k$ coefficient of $[\partial, h_\lambda](\mathbf{x}_i)$, since h_λ satisfies $h_\lambda(\mathbf{x}\cdot a) = h_\lambda(\mathbf{x})\cdot\phi^{\tau}(a)$, by Lemma 6.7. In particular, equation (8.2) vanishes. The case when $\varepsilon = 1$ is similar. ∎

Having defined the type-D module $\mathcal{X}_\lambda(K)^{\mathcal{K}}$, we may now define the type-A module

$$ {}_{\mathcal{K}}\mathcal{X}_\lambda(K) := \mathcal{X}_\lambda(K)^{\mathcal{K}} \mathbin{\hat{\boxtimes}} {}_{\mathcal{K}|\mathcal{K}}[\mathbb{I}^{\supseteq}]. $$

8.6 The link surgery formula over $\mathcal{L}$

In this section, we describe how to view Manolescu and Ozsváth's link surgery formula as a type-D module.

Recall that we define

$$ \mathcal{L}_\ell := \mathcal{K}_1 \otimes_{\mathbb{F}} \cdots \otimes_{\mathbb{F}} \mathcal{K}_\ell, $$

where each $\mathcal{K}_i$ denotes a copy of $\mathcal{K}$.

Let L be an ℓ-component link in S^3 with integral framing Λ. The link surgery complexes require a choice of auxiliary data, which we call a *system of arcs* $\mathcal{A}$. We will discuss these in detail in Section 9.1 below. If $\mathcal{A}$ is chosen, Manolescu and Ozsváth's construction produces a chain complex $\mathcal{C}_\Lambda(L,\mathcal{A})$ over $\mathbb{F}[U_1,\dots,U_\ell]$. In this section, we will describe how to construct a type-D module

$$ \mathcal{X}_\Lambda(L,\mathcal{A})^{\mathcal{L}_\ell}, $$

based on their construction of $\mathcal{C}_\Lambda(L,\mathcal{A})$.

We view the link algebra $\mathcal{L}_\ell$ as being an algebra over the idempotent ring

$$ \mathbf{E}_\ell := \underbrace{\mathbf{I} \otimes_{\mathbb{F}} \cdots \otimes_{\mathbb{F}} \mathbf{I}}_{\ell}. $$

We can view $\mathbf{E}_\ell$ as being a ring with 2^ℓ elementary idempotents, each identified with a point in the cube $\mathbb{E}_\ell$. If $\varepsilon \in \mathbb{E}_\ell$, we write

$$ \mathbf{E}_\ell := \mathbf{I}_{\varepsilon_1} \otimes \cdots \otimes \mathbf{I}_{\varepsilon_\ell} \cong \mathbb{F} $$

for the corresponding idempotent.

If $\varepsilon' \geq \varepsilon$, we can understand $\mathbf{E}_{\varepsilon'}\cdot\mathcal{L}_\ell\cdot\mathbf{E}_\varepsilon$ as follows. Write $I_{\varepsilon',\varepsilon} = \{i_1,\dots,i_n\} = (\varepsilon'-\varepsilon)^{-1}(1)$, i.e., the set of indices i where $\varepsilon'_i > \varepsilon$. Then $\mathbf{E}_{\varepsilon'}\cdot\mathcal{L}_\ell\cdot\mathbf{E}_\varepsilon$ is generated by elements of the form

$$ \alpha_1\cdots\alpha_\ell\phi_{i_1}\cdots\phi_{i_n}, $$

where

(1) a_i is in $\mathbb{F}[\mathcal{U}_i, \mathcal{V}_i]$ if $\varepsilon'_i = 0$,

(2) a_i is in $\mathbb{F}[\mathcal{U}_i, \mathcal{V}_i, \mathcal{V}_i^{-1}]$ if $\varepsilon'_i = 1$,

(3) each ϕ_{i_j} is either σ_{i_j} or τ_{i_j}.

If $\mathbf{x}_1, \dots, \mathbf{x}_n$ is a free basis of $\mathcal{CFL}(L)$ over $\mathbb{F}[\mathcal{U}_1, \dots, \mathcal{U}_\ell, \mathcal{V}_1, \dots, \mathcal{V}_\ell]$, then we define the generators of $\mathcal{X}_\Lambda(L, \mathcal{A})^{\mathcal{L}_\ell}$ to be

$$\mathrm{Span}_{\mathbb{F}}(\mathbf{x}_1, \dots, \mathbf{x}_n) \otimes_{\mathbb{F}} \mathbf{E}_\ell.$$

In particular, if $\mathbf{x}$ is a basis element of $\mathcal{CFL}(L)$, we have a generator $\mathbf{x}^\varepsilon$ for each $\varepsilon \in \mathbb{E}_\ell$.

If $M \subseteq L$, write $\varepsilon(M) \in \mathbb{E}_\ell$ for the coordinate such that $\varepsilon(M)_i = 0$ if $L_i \not\subseteq M$ and $\varepsilon(M)_i = 1$ if $L_i \in M$.

Suppose that $\mathbf{x}$ is a basis element of $\mathcal{CFL}(L)$ and $M \subseteq L$. By Lemma 6.5, we may view $\mathbf{x}^{\varepsilon(M)}$ as an element of the group $\mathcal{C}_\Lambda(L, \mathcal{A}) = \prod_{\mathbf{s} \in \mathbb{H}(L)} \mathfrak{A}(\mathcal{H}^{L \setminus M}, \psi^M(\mathbf{s}))$. Suppose also that $\vec{N}$ is an oriented sublink of $L \setminus M$, and that $\Phi^{\vec{N}}(\mathbf{x}^{\varepsilon(M)})$ has a summand of $\mathbf{y}^{\varepsilon(M \cup N)} \cdot f$, where we view f as an element of the 2ℓ-variable polynomial ring, localized at the variables $\mathcal{V}_i$ such that $L_i \subseteq N$. We may naturally view f as being an element of

$$\mathbf{E}_{\varepsilon(M \cup N)} \cdot \mathcal{L}_\ell \cdot \mathbf{E}_{\varepsilon(M \cup N)}.$$

There is an algebra element

$$t^{\vec{N}}_{\varepsilon(M), \varepsilon(M \cup N)} \in \mathbf{E}_{\varepsilon(M \cup N)} \cdot \mathcal{L}_\ell \cdot \mathbf{E}_{\varepsilon(M)}$$

which is the tensor of σ_i for i such that $L_i \subseteq \vec{N}$ and L_i is oriented the same as L, τ_i for i such that $L_i \subseteq \vec{N}$ and L_i is oriented oppositely from L, and 1 for i such that $L_i \not\subseteq N$. With this notation, we declare $\delta^1(\mathbf{x}^{\varepsilon(M)})$ to have the summand

$$\mathbf{y}^{\varepsilon(M \cup N)} \otimes f \cdot t^{\vec{N}}_{\varepsilon(M), \varepsilon(M \cup N)}.$$

Lemma 8.7. *$\mathcal{X}_\Lambda(L, \mathcal{A})^{\mathcal{L}_\ell}$ is a type-*D *module.*

Proof. The proof is similar to Lemma 8.6. We can decompose the structure map δ^1 as

$$\delta^1 = \sum_{\vec{M} \subseteq L} \delta^1_{\vec{M}},$$

where $\delta^1_{\vec{M}}$ consists of all terms which are weighted by idempotent-preserving multiples of the algebra element $t^{\vec{M}}$.

The map $\delta^1_{\vec{M}}$ encodes the map $\Phi^{\vec{M}}$ on the surgery formula, in the sense that if $\Phi^{\vec{M}}(\mathbf{x})$ has a summand $a \cdot \mathbf{y}$ for some monomial a in the $\mathcal{U}_i$ and $\mathcal{V}_i^{\pm 1}$, then $\delta^1_{\vec{M}}$ has a summand $\mathbf{y} \otimes a t^{\vec{M}}$.

Let $\vec{N} \subseteq L$ be an oriented link, and consider the components of

$$((\mathrm{id}_{\mathcal{X}} \otimes \mu_2) \circ (\delta^1 \otimes \mathrm{id}_{\mathcal{L}_\ell}) \circ \delta^1)(\mathbf{x}^\varepsilon)$$

whose algebra elements are weighted by idempotent-preserving multiples of $t^{\vec{N}}$. These can be identified with

$$\sum_{\substack{\vec{M}_1 \cup \vec{M}_2 = \vec{N},\\ \vec{M}_1 \cap \vec{M}_2 = \emptyset}} (\mathbb{I}_{\mathcal{X}} \otimes \mu_2) \circ (\delta^1_{\vec{M}_2} \otimes \mathbb{I}_{\mathcal{K}}) \circ \delta^1_{\vec{M}_1}. \tag{8.3}$$

We consider a component of the above map corresponding to sublinks $\vec{M}_1$ and $\vec{M}_2$, applied to some generator $\mathbf{x}$. Suppose that $\delta^1_{\vec{M}_1}(\mathbf{x})$ contains a summand $\mathbf{y} \otimes at^{\vec{M}_1}$. Such a summand corresponds to a summand of $\mathbf{y} \cdot a$ in $\Phi^{\vec{M}_1}(\mathbf{x})$. Next, consider a summand $\mathbf{z} \otimes bt^{\vec{M}_2}$ in $\delta^1_{\vec{M}_2}(\mathbf{y})$. In the composition appearing in equation (8.3), there is a corresponding term of

$$\mathbf{z} \otimes bt^{\vec{M}_2}at^{\vec{M}_1} = \mathbf{z} \otimes b\phi^{\vec{M}_2}(a)t^{\vec{N}}.$$

(The second equality follows from the definition of the algebra $\mathcal{L}_\ell$.) Using Lemma 6.7, which states that $\Phi^{\vec{M}_2}(a\mathbf{y}) = \phi^{\vec{M}_2}(a) \cdot \Phi^{\vec{M}_2}(\mathbf{y})$, we see that there is a corresponding summand in $(\Phi^{\vec{M}_2} \circ \Phi^{\vec{M}_1})(\mathbf{x})$ of $\mathbf{z} \cdot b\phi^{\vec{M}_2}(a)$. Therefore, the fact equation (8.3) vanishes follows from the fact that

$$\sum_{\substack{\vec{M}_1 \cup \vec{M}_2 = \vec{N},\\ \vec{M}_1 \cap \vec{M}_2 = \emptyset}} \Phi^{\vec{M}_2} \circ \Phi^{\vec{M}_1} = 0,$$

which is proven in [32, Proposition 9.4]. ■

Given a collection of integers $\Lambda = (\lambda_1, \dots, \lambda_\ell)$, there is a type-A module

$${}_{\mathcal{L}_\ell}\mathcal{D}_\Lambda$$

defined by taking the external tensor product of the modules ${}_{\mathcal{K}}[\mathcal{D}_{\lambda_i}]$ from Section 8.2. There is also a right action of $\mathbb{F}[U_1, \dots, U_\ell]$, and $\mathcal{D}_\Lambda$ may be viewed as an AA-bimodule

$${}_{\mathcal{L}_\ell}[\mathcal{D}_\Lambda]_{\mathbb{F}[U_1,\dots,U_\ell]}.$$

The following is immediate from the definition of $\mathcal{X}_\Lambda(L, \mathcal{A})^{\mathcal{L}_\ell}$.

Proposition 8.8. *There is a chain isomorphism*

$$\mathcal{C}_\Lambda(L)_{\mathbb{F}[U_1,\dots,U_\ell]} \cong \mathcal{X}_\Lambda(L)^{\mathcal{L}_\ell} \boxtimes {}_{\mathcal{L}_\ell}[\mathcal{D}_{(0,\dots,0)}]_{\mathbb{F}[U_1,\dots,U_\ell]}.$$

We will also need to consider type-A and type-DA versions of the link surgery formula. An important special case is when we have a distinguished component $K_i \subseteq L$. We define a bimodule ${}_{\mathcal{K}}\mathcal{X}_\Lambda(L)^{\mathcal{L}_{\ell-1}}$, where the $\mathcal{K}$-input corresponds to K_i by setting

$$
{}_{\mathcal{K}}\mathcal{X}_\Lambda(L)^{\mathcal{L}_{\ell-1}} := \mathcal{X}_\Lambda(L)^{\mathcal{L}_\ell} \mathbin{\hat{\boxtimes}} {}_{\mathcal{K}|\mathcal{K}}[\mathbb{I}^{\Supset}],
$$

where the tensor product is taken along the algebra component corresponding to K_i. In the above equation, ${}_{\mathcal{K}|\mathcal{K}}[\mathbb{I}^{\Supset}]$ is the bimodule from Section 8.4. More generally, we may turn more algebra components from type-D to type-A by tensoring additional copies of ${}_{\mathcal{K}|\mathcal{K}}[\mathbb{I}^{\Supset}]$.