Chapter 9

σ-basic systems of Heegaard diagrams

In this chapter, we define the notion of a σ-*basic* system of Heegaard diagrams. The notion is a very small generalization of Manolescu and Ozsváth's *basic system*. These are collections of Heegaard diagrams which can be used to compute the link surgery formula. We use the small distinction in notation to disambiguate the notions, and to highlight the algebraic significance of these systems. For these systems, the maps $\Phi^{\vec{N}}$ vanish unless $\vec{N} = +K$ (i.e., a single component, oriented consistently with L), or $\vec{N}$ contains only negatively oriented components relative to L. We also describe how to construct a σ-basic system for the connected sum of two links.

9.1 Systems of arcs

In this section, we define the notion of a *system of arcs*, which is a necessary piece of auxiliary data used to define the link surgery formula.

Definition 9.1. Suppose that $(S^3, L, \mathbf{w}, \mathbf{z})$ is an oriented, multi-pointed link, which is link minimal (i.e., each component of L contains exactly one base point of $\mathbf{w}$, and one base point of $\mathbf{z}$).

(1) A *system of arcs* for L consists of a set $\mathcal{A} = \{\mathcal{A}_1, \dots, \mathcal{A}_\ell\}$ of pairwise disjoint arcs satisfying that $\partial \mathcal{A}_i = \mathcal{A}_i \cap L = \{w_i, z_i\}$.

(2) We say that an arc $\mathcal{A}_i$ is *beta-parallel* if $\mathcal{A}_i$ is isotopic to a push-off of the subarc of L_i which goes from z_i to w_i.

(3) We say that $\mathcal{A}_i$ is *alpha-parallel* if $\mathcal{A}_i$ is isotopic to a push-off of the subarc of L_i which goes from w_i to z_i.

Note that an arc $\mathcal{A}_i$ is beta-parallel if for any Heegaard diagram of $(Y, L, \mathbf{w}, \mathbf{z})$, $\mathcal{A}_i$ is isotopic to the subarc of $K_i \subseteq L$ which lies in the beta handlebody U_β. A similar remark holds for alpha-parallel arcs.

Given a link minimal $(L, \mathbf{w}, \mathbf{z})$ there are two distinguished systems of arcs, $\mathcal{A}_\alpha$ and $\mathcal{A}_\beta$, which are the ones which have only alpha-parallel arcs, or only beta-parallel arcs, respectively.

Remark 9.2. Manolescu and Ozsváth's notion of *good sets of trajectories* [32, Definition 8.26] corresponds to our notion of a system of alpha-parallel arcs.

9.2 σ-basic systems of Heegaard diagrams

In this section, we define the notion of a *σ-basic system of Heegaard diagrams*. Our notion is a small adaptation of Manolescu and Ozsváth's construction.

Definition 9.3. Suppose that $(L, \mathbf{w}, \mathbf{z})$ is a minimally pointed, n-component link in a 3-manifold Y, and suppose that $\mathcal{A} = (\mathcal{A}_1, \dots, \mathcal{A}_\ell)$ is a system of arcs for $(Y, L, \mathbf{w}, \mathbf{z})$. A *$\sigma$-basic system of Heegaard diagrams* for $(Y, L, \mathcal{A})$ consists of the following data:

(1) A tuple of positive integers $\mathbf{d} = (d_1, \dots, d_\ell)$ where $n = |L|$.

(2) A collection of Heegaard diagrams $\mathcal{H}(M) = (\mathscr{H}_\varepsilon)_{\varepsilon \in \mathbb{E}(\mathbf{d})}$. We assume all Heegaard diagrams have the same underlying Heegaard surface Σ, which we also assume contains the arcs $\mathcal{A}_1, \dots, \mathcal{A}_\ell$. We write $\mathscr{H} = (\Sigma, \boldsymbol{\alpha}_\varepsilon, \boldsymbol{\beta}_\varepsilon, \mathbf{w}_\varepsilon, \mathbf{z}_\varepsilon, \mathbf{p}_\varepsilon)$, where $\mathbf{w}_\varepsilon$ and $\mathbf{z}_\varepsilon$ are link base points and $\mathbf{p}_\varepsilon$ are free base points.

(3) For each component $i \in \{1, \dots, n\}$, we have a formal labeling of each of the intervals $[0, 1], \dots, [d_i - 1, d_i]$ as being either an *alpha incrementing interval*, a *beta incrementing interval* or a *surface isotopy interval*.

(4) A decomposition of each arc $\mathcal{A}_j$ into a concatenation of oriented subarcs $l_{i,1}, \dots, l_{i,j_i}$, as well as a choice of surface isotopies $\phi_{i,1}, \dots, \phi_{i,j_i}$. We assume $\phi_{i,j}$ is supported in a small neighborhood of $l_{i,j}$ and moves the initial endpoint of $l_{i,j}$ to the terminal endpoint.

We assume that the following conditions are satisfied:

(1) $\mathbf{w}_\varepsilon = \{w_i : \varepsilon_i = 0\}$ and $\mathbf{z}_\varepsilon = \{z_i : \varepsilon_i = 0\}$.

(2) Let $\mathscr{H}_\varepsilon = (\Sigma, \boldsymbol{\alpha}_\varepsilon, \boldsymbol{\beta}_\varepsilon, \mathbf{w}_\varepsilon, \mathbf{z}_\varepsilon, \mathbf{p}_\varepsilon)$ and let $e_i = (0, \dots, 1, \dots, 0) \in \{0, 1\}^n$ denote a length 1 vector (i.e., a direction). As described above, the interval $I = [\varepsilon_i, \varepsilon_i + e_i]$ is labeled as either an alpha incrementing interval, a beta incrementing interval, or a surface isotopy interval.

   (a) If I is an alpha incrementing interval, then $\boldsymbol{\beta}_\varepsilon = \boldsymbol{\beta}_{\varepsilon+e_i}$ while $\boldsymbol{\alpha}_{\varepsilon+e_i}$ and $\boldsymbol{\alpha}_\varepsilon$ differ. Furthermore, $\mathbf{p}_{\varepsilon+e_i} = \mathbf{p}_\varepsilon$ unless $\varepsilon_i = 0$, in which case $\mathbf{p}_{\varepsilon+e_i} = \mathbf{p}_\varepsilon \cup \{z_i\}$.

   (b) If I is a beta incrementing interval, then the same holds as above except with the alpha and beta curves interchanged.

   (c) If I is a surface isotopy interval, then $\mathscr{H}_{\varepsilon+e_i}$ is the image of $\mathscr{H}_\varepsilon$ under one of the surface isotopies $\phi_{i,j}$ described above. Similarly, $\mathbf{w}_{\varepsilon+e_i} = \mathbf{w}_\varepsilon$ and $\mathbf{z}_{\varepsilon+e_i} = \mathbf{z}_\varepsilon$. If $\varepsilon_i > 0$, then $\mathbf{p}_{\varepsilon+e_i} = \phi_{i,j}(\mathbf{p}_\varepsilon)$. If $\varepsilon_i = 0$, then $\mathbf{p}_{\varepsilon+e_i} = \phi_{i,j}(\mathbf{p}_\varepsilon \cup z_i)$.

(3) The faces on opposite sides of the box have the same Heegaard diagrams. In more detail, if $\varepsilon \in \mathbb{E}(\mathbf{d})$ and $\varepsilon_i = 0$, then

$$\boldsymbol{\alpha}_{\varepsilon+d_i e_i} = \boldsymbol{\alpha}_\varepsilon, \quad \boldsymbol{\beta}_{\varepsilon+d_i e_i} = \boldsymbol{\beta}_\varepsilon, \quad \mathbf{p}_{\varepsilon+d_i e_i} = \mathbf{p}_\varepsilon \cup \{w_i\}.$$

We illustrate an example of a σ-basic system in Figure 9.1.

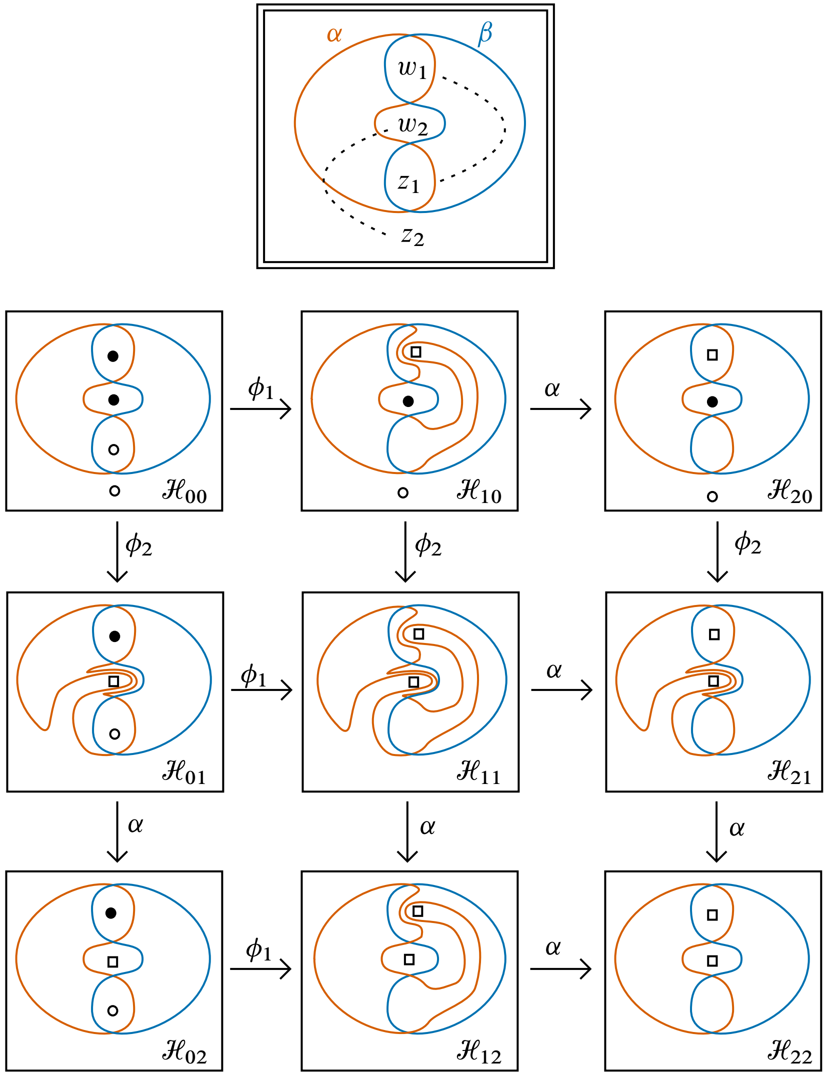

Figure 9.1. An example of a σ-basic system of Heegaard diagrams for the Hopf link. At the top, we show the underlying Heegaard link diagram of the Hopf link, with an arc system consisting of two beta-parallel arcs $\mathcal{A}_1$ and $\mathcal{A}_2$ (shown as dashed arcs). For this example, the subdivisions of $\mathcal{A}_1$ and $\mathcal{A}_2$ are trivial (i.e., consist of a single arc). The labels of the arrows indicate whether the corresponding subintervals are "alpha incrementing intervals", "beta incrementing intervals" or "surface isotopies intervals". We label the **w** base points as solid circles, the **z** base points as open circles, and the free base points as squares.

Remark 9.4. The terminology *σ-basic system* refers to the fact that when we use such systems of Heegaard diagrams to construct the link surgery formula, the resulting complexes have Φ^{+K_i} equal to inclusions for localization, and $\Phi^{\vec{M}} = 0$ if $\vec{M}$ has more than one component and contains a positively oriented component. In particular, using these diagrams we obtain models of the type-D modules $\mathcal{X}_\Lambda(L)^{\mathcal{L}\ell}$ where the components weighted by σ_i are very simple. Our definition is slightly more general than Manolescu and Ozsváth's notion of a basic system of Heegaard diagrams, which has the same algebraic properties with respect to the differential.

The following definition plays an important role in our proof of the connected sum formula for surgery hypercubes.

Definition 9.5. Suppose that $\mathcal{L}_\beta = (\boldsymbol{\beta}_\varepsilon, \theta_{\varepsilon,\varepsilon'})_{\varepsilon\in\mathbb{E}_n}$ is a weakly admissible hypercube of handleslide equivalent beta attaching curves on a pointed Heegaard surface $(\Sigma, \mathbf{w}, \mathbf{z}, \mathbf{p})$ (where $\mathbf{w}, \mathbf{z}$ are link base points and $\mathbf{p}$ are free base points). We say $\mathcal{L}_\beta$ is *algebraically rigid* if each $\mathbb{T}_{\beta_\varepsilon} \cap \mathbb{T}_{\beta_{\varepsilon'}}$ has $2^{g(\Sigma)+|\mathbf{w}|+|\mathbf{p}|-1}$ intersection points (the minimal possible number). We make a parallel definition for alpha hyperboxes. We say a hyper*box* of attaching curves is algebraically rigid if each sub-hypercube is algebraically rigid.

Note that a hypercube of attaching curves being algebraically rigid is a property of the attaching curves, and has nothing to do with the chains $\theta_{\varepsilon,\varepsilon'}$. Therefore, abusing notation slightly, we will say a σ-basic system of Heegaard diagrams $\mathcal{H}$ is algebraically rigid if it has the property that $\mathbb{T}_{\beta_\varepsilon} \cap \mathbb{T}_{\beta_{\varepsilon'}}$ has $2^{g(\Sigma)+|\mathbf{w}_\varepsilon|+|\mathbf{p}_\varepsilon|}$ intersection points whenever $|\varepsilon' - \varepsilon|_{L^\infty} \le 1$ and none of the intervals $[\varepsilon, \varepsilon + e_i]$ is a surface isotopy interval for i such that $\varepsilon_i < \varepsilon_i'$. We make the same assumption for the alpha curves.

9.3 Meridional σ-basic systems

In this section, we describe a particular family of σ-basic systems which are useful in our proof of the pairing system. Similar diagrams are ubiquitous in Heegaard Floer theory. For example, they are the diagrams which Ozsváth and Szabó use to prove the mapping cone formula [43].

Definition 9.6. We say that a σ-basic system of Heegaard diagrams $\mathcal{H}$ for (Y, L), with a system of arcs $\mathcal{A}$, is a *meridional σ-basic system* if the following hold:

(1) Each arc of $\mathcal{A}$ is either beta-parallel or alpha-parallel.

(2) The diagram $\mathcal{H}(\emptyset) = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{w}, \mathbf{z})$ has the following property. Suppose $K_i \in L$ is a component with corresponding arc $\mathcal{A}_i \in \mathcal{A}$ and base points w_i, z_i. Then $\mathcal{A}_i$ is either alpha-parallel or beta-parallel. If $\mathcal{A}_i$ is alpha-parallel, then $\mathcal{A}_i$ is disjoint from the alpha curves, and intersects a single beta curve β_i^s.

A similar assumption is made if $\mathcal{A}_i$ is beta-parallel. In particular, if $|L| = \ell$, there are n special alpha and beta circles, denoted α_i^s and β_j^s.

(3) In each Heegaard diagram of $\mathcal{H}$, there are similarly ℓ special alpha and beta curves, and the rest are designated as non-special. The non-special alpha curves of each subcube of $\mathcal{H}$ consist of small Hamiltonian translates of the non-special curves of $\mathscr{H}$.

(4) If $K_i \subseteq L$ is a component whose arc $\mathcal{A}_i$ is beta-parallel, then the component of $\Sigma \setminus \boldsymbol{\beta}$ which contains the base points of K_i is a punctured disk which contains a special beta circle β_i^s as two of its boundary components. Gluing the two β_i^s boundary components together, we obtain a punctured torus. We assume that K_i-axis direction of $\mathcal{H}$ realizes the isotopy of β_i^s in a loop around this punctured torus (handlesliding β_i^s over the boundary punctures). We make an analogous assumption for components with beta-parallel decoration.

Lemma 9.7. *Suppose L is an ℓ-component link in S^3 and $\mathcal{A}$ is a system of arcs for L such that each arc is either alpha-parallel or beta-parallel. Then there exists a link minimal, weakly admissible, algebraically rigid σ-basic system of Heegaard diagrams for* $(S^3, L, \mathcal{A})$.

Proof. Except for the claim about admissibility, existence of such a σ-basic system is clear since the construction is given in Definition 9.6.

Admissibility is verified as follows. We first verify admissibility for each hypercube of alpha and beta Lagrangians appearing in the construction. For concreteness, suppose that $\mathscr{L}_\alpha$ is a hypercube of alpha attaching curves appearing in the construction. For appropriately chosen Hamiltonian translates, admissibility of $\mathscr{L}_\alpha$ is equivalent to admissibility of the diagram where we delete the tuples of curves which appear only as small translates of each other, and surger Σ along these curves as well. The resulting Heegaard diagram is a union of tori, and admissibility is straightforward to arrange. See Figure 9.2.

Admissibility of each pair $(\mathscr{L}_\alpha, \mathscr{L}_\beta)$ may be arranged by winding one of the cubes (say $\mathscr{L}_\alpha$) relative to $\mathscr{L}_\beta$, following the procedure described in [39, Section 4.2.2] and [41, Section 3.4]. ∎

9.4 Basic systems for general systems of arcs

We now consider general systems of arcs. Generalizing Lemma 9.7, we prove the following.

Lemma 9.8. *Suppose that $\mathcal{A}$ is a system of arcs for $L \subseteq S^3$. Then there is an algebraically rigid σ-basic system of Heegaard diagrams for* $(L, \mathcal{A})$.

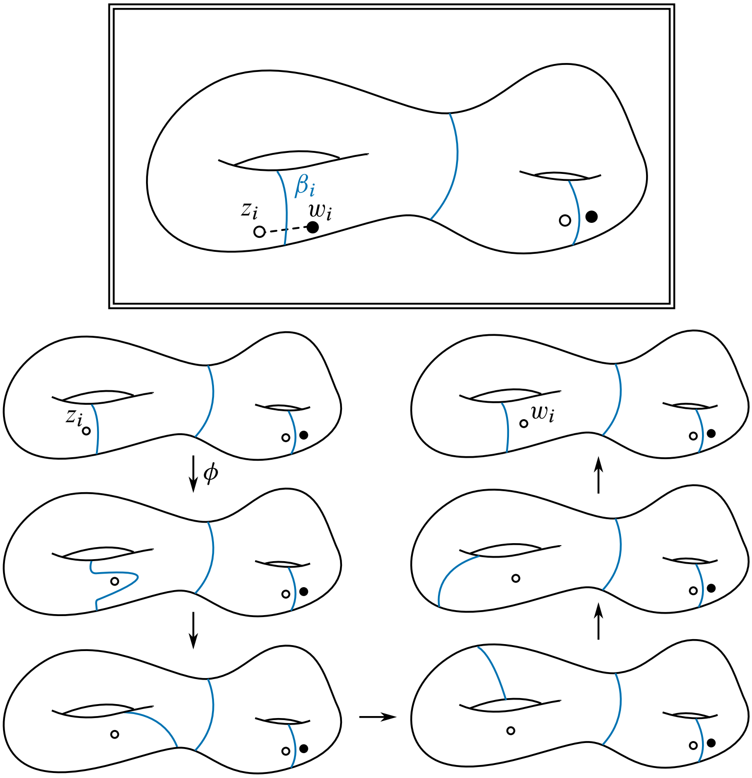

Figure 9.2. Some of the beta curves and a base point translation from Lemma 9.7. The first arrow is a base point translation, and the remainder are holomorphic polygon counting maps. The dashed arc on the top is c_i.

Proof. The proof is similar to the proof of Lemma 9.7. We begin with a minimally pointed Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{w}, \mathbf{z})$ for (Y, L) such that each arc $\mathcal{A}_i$ is embedded in Σ and such that the arcs $\mathcal{A}_i$ are pairwise disjoint. By performing an isotopy of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ supported in a neighborhood of the arcs $\mathcal{A}_i$, we may assume that each $\mathcal{A}_i$ is the concatenation of two arcs A_i^α and A_i^β such that A_i^α is disjoint from $\boldsymbol{\beta}$ and A_i^β is disjoint from $\boldsymbol{\alpha}$. We assume that $\partial A_i^\beta = \{z_i, x_i\}$ and $\partial A_i^\alpha = \{w_i, x_i\}$, for some point $x_i \in \Sigma$.

We now stabilize the Heegaard diagram $2|L|$ times, as shown in Figure 9.3. For each $\mathcal{A}_i$, we introduce two stabilizations. One of the stabilizations corresponds to attaching a 1-handle with feet near z_i and x_i. We introduce a new alpha and a new beta curve. The alpha curve runs parallel to A_i^β. The new beta curve, denoted β_i^s, is the belt-sphere of the 1-handle. Similarly, the other stabilization corresponds to attaching a 1-handle with feet near x_i and w_i. Here, the new beta curve runs parallel to A_i^α. The new alpha curve, α_i^s, is the belt sphere of the 1-handle.

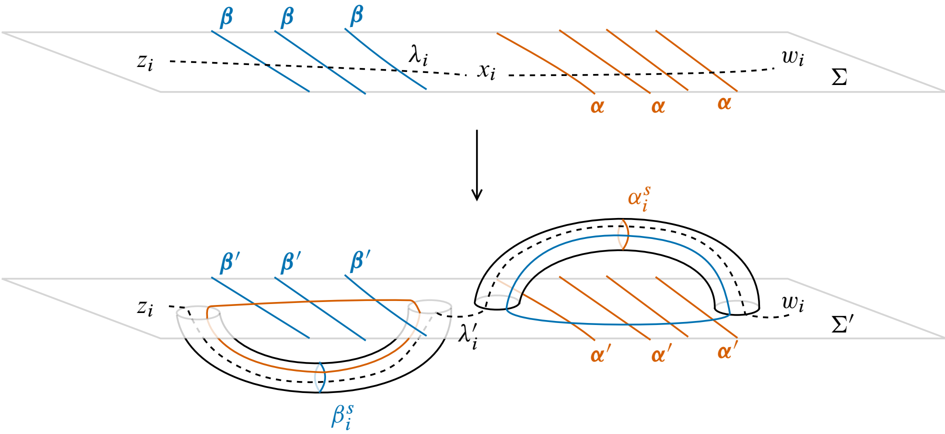

Figure 9.3. Top: the Heegaard surface $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{w}, \mathbf{z})$ and the arc $\mathcal{A}_i$ from Lemma 9.8. Bottom: the stabilized Heegaard surface $(\Sigma', \boldsymbol{\alpha}', \boldsymbol{\beta}', \mathbf{w}, \mathbf{z})$ and the arc $\mathcal{A}_i'$. Also shown are the special curves α_i^s and β_i^s.

We write $(\Sigma', \boldsymbol{\alpha}', \boldsymbol{\beta}', \mathbf{w}, \mathbf{z})$ for the stabilized Heegaard diagram. We handleslide the arcs $\mathcal{A}_i$ into the new stabilization tubes to obtain arcs $\mathcal{A}_i' \subseteq \Sigma'$. The arcs $\mathcal{A}_1', \dots, \mathcal{A}_\ell'$ are still isotopic to the arcs in $\mathcal{A}$.

By definition of a Heegaard link diagram, the base points $w_i, z_i \in \Sigma$ are contained in a single component $A_i \subseteq \Sigma \setminus \boldsymbol{\alpha}$ and also a single component $B_i \subseteq \Sigma \setminus \boldsymbol{\beta}$. Note that the point $x_i \in \Sigma$ is contained in A_i since there is a path A_i^{β} from x_i to z_i which is disjoint from $\boldsymbol{\alpha}$. Similarly, $x_i \in B_i$. Write A_i' and B_i' for the analogous components of $\Sigma' \setminus \boldsymbol{\alpha}'$ and $\Sigma' \setminus \boldsymbol{\beta}'$. Note that A_i' is obtained by removing four disks from A_i, and similarly for B_i'.

We define our σ-basic system of Heegaard diagrams as follows. In the i-th axis direction, we first move β_i^s around B_i by a sequence of handleslides across the curves of $\boldsymbol{\beta}$. This is similar to the case of meridional σ-basic systems in Lemma 9.7. We perform a sequence of isotopies and handleslides to move β_s^s to the other side of z_i, so that z_i may be moved to x_i without crossing any attaching curves. Write z_i' for x_i, thought of as a base point. The next step in the i-th axis direction is to move α_i^s around A_i so that it moves to the other side of z_i', allowing z_i' to be moved to w_i.

Note that since β_i^s moves only around B_i', and α_i^s moves only around A_i', the β_i^s may be moved around simultaneously (for different i) and similarly different α_i^s may be moved simultaneously. Additionally, the original curves $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are also unchanged in this process. In particular, we may build the different axis directions of the hypercube simultaneously to build the σ-basic system of Heegaard diagrams. By a similar argument to Lemma 9.7, we may choose the constituent hypercubes of attaching circles to be algebraically rigid and weakly admissible. ∎

9.5 Tensor products of hypercubes of attaching curves

If $\mathcal{L}_\beta$ is a hypercube of attaching curves on $(\Sigma, \mathbf{w}, \mathbf{z})$, and $\mathcal{L}_{\beta'}$ is a hypercube of attaching curves on $(\Sigma', \mathbf{w},', \mathbf{z}')$, of dimensions n and m, respectively, we now define an $(n+m)$-dimensional hypercube $\mathcal{L}_\beta \otimes \mathcal{L}_{\beta'}$ on $(\Sigma \sqcup \Sigma', \mathbf{w} \cup \mathbf{w}', \mathbf{z} \cup \mathbf{z}')$. We note that the construction of $\mathcal{L}_\beta \otimes \mathcal{L}_{\beta'}$ is essentially identical to Lipshitz, Ozsváth and Thurston's construction of a *connected sum of chain complexes of attaching circles* [26, Definition 3.40].

We begin with the underlying attaching curves of $\mathcal{L}_\beta \otimes \mathcal{L}_{\beta'}$, which we denote by $(\boldsymbol{\delta}_{(\varepsilon,\nu)})_{(\varepsilon,\nu)\in \mathbb{E}_n \times \mathbb{E}_m}$, where $n = \dim(\mathcal{L}_\beta)$ and $m = \dim(\mathcal{L}_{\beta'})$. We set $\boldsymbol{\delta}_{(\varepsilon,\nu)}$ to be a small translation of $\boldsymbol{\beta}_\varepsilon \cup \boldsymbol{\beta}'_\nu$. We assume the small translations are chosen so that the copies of $\boldsymbol{\beta}_\varepsilon$ from $\boldsymbol{\delta}_{(\varepsilon,\nu)}$ and $\boldsymbol{\delta}_{(\varepsilon,\nu')}$ intersect in $2^{g(\Sigma)+|\mathbf{w}|-1}$ points, and similarly for the translated curves on Σ'.

Before defining the morphisms of $\mathcal{L}_\beta \otimes \mathcal{L}_{\beta'}$, we begin with some helpful terminology.

Definition 9.9. If $\varepsilon, \varepsilon' \in \mathbb{E}_n$, we say that there is an *arrow* from ε to ε' if $\varepsilon < \varepsilon'$.

(1) We say that an arrow from $(\varepsilon, \nu) \in \mathbb{E}_n \times \mathbb{E}_m$ to (ε', ν') is *mixed* if $\varepsilon < \varepsilon'$ and $\nu < \nu'$.

(2) We say that an arrow from (ε, ν) to (ε', ν') is *non-mixed* if $\varepsilon = \varepsilon'$ or $\nu = \nu'$.

We now describe the morphisms of $\mathcal{L}_\beta \otimes \mathcal{L}_{\beta'}$. We define the chains in $\mathcal{L}_\beta \otimes \mathcal{L}_{\beta'}$ for each mixed arrow to be zero. For the arrow from (ε, ν) to (ε, ν'), where $\nu < \nu'$, we use the chain

$$\Theta^+_{\varepsilon,\varepsilon} \otimes \Theta_{\nu,\nu'},$$

where $\Theta^+_{\varepsilon,\varepsilon}$ denotes the top degree generator (recall that we have perturbed one copy of $\boldsymbol{\beta}_\varepsilon$ slightly in the definition), and $\Theta_{\nu,\nu'}$ denotes the chain from $\mathcal{L}_{\beta'}$. Similarly, for the arrow from (ε, ν) to (ε', ν), where $\varepsilon < \varepsilon'$, we use the chain

$$\Theta_{\varepsilon,\varepsilon'} \otimes \Theta^+_{\nu,\nu}.$$

Remark 9.10. We will only use the above construction in the case that $\mathcal{L}_\beta$ and $\mathcal{L}_{\beta'}$ are algebraically rigid. In this case, $\mathcal{L}_\beta \otimes \mathcal{L}_{\beta'}$ is also algebraically rigid, and the hypercube relations are automatic.

9.6 Basic systems and connected sums

In this section, we define a connected sum operation on σ-basic systems of Heegaard diagrams.

We begin by defining a connected sum operation for systems of arcs. Let L_1 and L_2 be links in S^3, with distinguished components K_1 and K_2, respectively. If $\mathcal{A}_1$

and $\mathcal{A}_2$ are two systems of arcs for L_1 and L_2, we can form their connected sum $\mathcal{A}_1\#\mathcal{A}_2$ by taking the connected sum of L_1 and L_2 at $w_1 \in L_1$ and $z_2 \in L_2$. We then concatenate the arcs for the components K_1 and K_2 to obtain the arc for $K_1\#K_2$. *A priori*, this operation is asymmetric between L_1 and L_2.

The most important special cases of this construction are the following:

(#-1) $\alpha\alpha$ *connected sums*: Suppose $\mathcal{A}_1$ and $\mathcal{A}_2$ are systems of arcs for L_1 and L_2, such that the arcs for K_1 and K_2 are both alpha-parallel. We form a system of arcs for $L_1\#L_2$ by using an alpha-parallel arc for $K_1\#K_2$, and using the remaining arcs without change.

(#-2) $\beta\beta$ *connected sums*: These are analogous to $\alpha\alpha$.

(#-3) $\alpha\beta$ *connected sums*: Suppose that $\mathcal{A}_1$ and $\mathcal{A}_2$ are systems of arcs for L_1 and L_2, such that the arc for K_1 is alpha-parallel while the arc for K_2 is beta-parallel. We form the system $\mathcal{A}_1\#\mathcal{A}_2$ by using the co-core of the connected sum band as the arc for $K_1\#K_2$.

These special cases are illustrated in Figure 9.4.

We now describe how to realize the above connected sum operations on the level of σ-basic systems of Heegaard diagrams. Suppose that we have σ-basic systems $\mathcal{H}_1$ and $\mathcal{H}_2$ for L_1 and L_2. We focus on the largest hyperbox of the system, since the smaller hyperboxes are determined by the compatibility condition with respect to inclusion.

We take the connected sum of the Heegaard surfaces Σ_1 and Σ_2 at $w_1 \in K_1$ and $z_2 \in K_2$, deleting those two base points, and leaving z_1 and w_2. See Figure 9.5.

We write $\mathcal{H}_1^{(0)}, \mathcal{H}_1^{(1)} \subseteq \mathcal{H}_1$ for the codimension 1 subboxes where the K_1-component is 0 (resp. where the K_1-component is maximal). Write $\mathcal{H}_2^{(0)}$ and $\mathcal{H}_2^{(1)}$ for the analogous subboxes of $\mathcal{H}_2$.

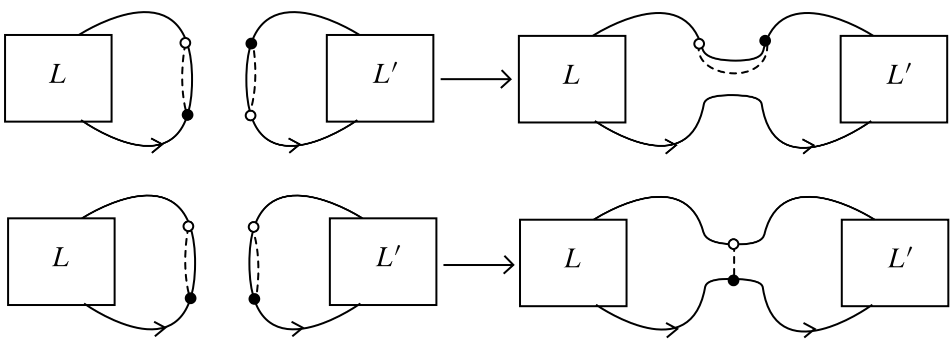

Figure 9.4. The connected sums of two alpha-parallel arcs (top row), and the connected sum of one alpha-parallel arc and one beta-parallel arc (bottom row).

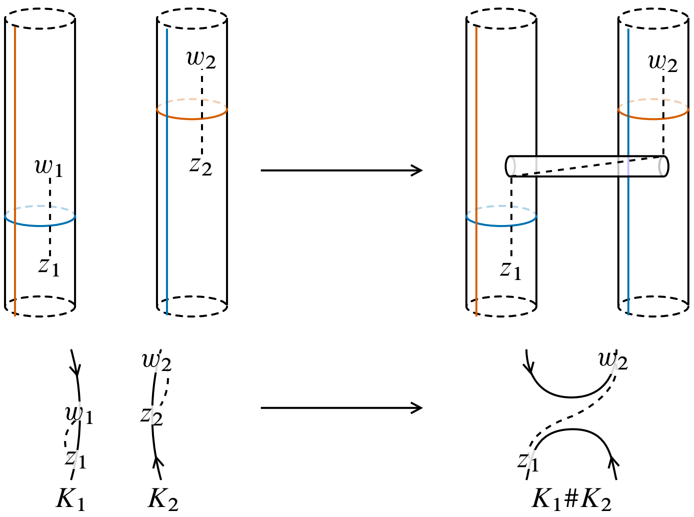

Figure 9.5. Forming the connected sum of two σ-basic systems of Heegaard diagrams. The dashed arcs denote the arcs in our σ-basic system. On the left, we begin with an alpha-parallel arc for K_1 and a beta-parallel arc for K_2. On the right, we have the arc for the connected sum.

We first construct a hyperbox of Heegaard diagrams on $\Sigma_1\#\Sigma_2$, which we denote $\mathcal{H}_1\#\mathcal{H}_2^{(0)}$. We build this hyperbox similarly to the tensor product procedure from Section 9.5. We build an analogous hyperbox of Heegaard diagrams $\mathcal{H}_1^{(1)} \otimes \mathcal{H}_2$ of dimension $(\ell_1 + \ell_2 - 1)$. Note that since these hypercubes take place on the connected sum $\Sigma_1\#\Sigma_2$, as opposed to the disjoint union, we may not be able to build the chains in the constituent hypercubes of attaching curves using the tensor product operation. Nonetheless, we may always use the curves so-constructed, and fill in the morphisms of these hypercubes using the standard filling construction of Manolescu–Ozsváth [32, Lemma 8.6].

Both $\mathcal{H}_1 \otimes \mathcal{H}_2^{(0)}$ and $\mathcal{H}_1^{(1)} \otimes \mathcal{H}_2$ share $\mathcal{H}_1^{(1)} \otimes \mathcal{H}_2^{(0)}$ as a codimension 1 subface. In particular, we may stack $\mathcal{H}_1 \otimes \mathcal{H}_2^{(0)}$ and $\mathcal{H}_1^{(1)} \otimes \mathcal{H}_2$. Hence, for the connected sum $L_1\#L_2$, we use

$$\mathcal{H}_1\#\mathcal{H}_2 := \mathrm{St}(\mathcal{H}_1 \otimes \mathcal{H}_2^{(0)}, \mathcal{H}_1^{(1)} \otimes \mathcal{H}_2), \tag{9.1}$$

where St denotes stacking hyperboxes. This construction yields a σ-basic system of Heegaard diagrams for $L_1\#L_2$, whose largest hyperbox is given by equation (9.1).

Remark 9.11. If $\mathcal{H}_1$ and $\mathcal{H}_2$ are algebraically rigid, then $\mathcal{H}_1\#\mathcal{H}_2$ is also algebraically rigid.

Remark 9.12. Note that $\mathrm{St}(\mathcal{H}_1^{(0)} \otimes \mathcal{H}_2, \mathcal{H}_1^{(1)} \otimes \mathcal{H}_2)$ does not naturally define a σ-basic system of Heegaard diagrams. This is because the Heegaard diagrams must encode an isotopy of the base point z_1 to the base point w_2 on the connected sum of the Heegaard surfaces. The stacking in equation (9.1) naturally encodes an isotopy (concatenating the isotopy from z_1 to w_1, the isotopy of the base point across the connected sum tube from w_1 to z_2, and the isotopy of z_2 to w_2). The alternate stacking does not naturally encode an isotopy of this sort.