Chapter 10

Hypercubes and disjoint unions

In this chapter, we consider hypercubes of attaching curves on disconnected Heegaard surfaces. Recall that in Section 9.5, we defined a tensor product operation on hypercubes of attaching curves. The goal of this chapter is to prove the following tensor product formula for pairing such hypercubes of attaching curves.

Proposition 10.1. *Suppose that $\mathcal{L}_\beta$ and $\mathcal{L}_{\beta'}$ are admissible hypercubes of handleslide equivalent beta attaching curves on $(\Sigma, \mathbf{w}, \mathbf{z})$ and $(\Sigma', \mathbf{w}', \mathbf{z}')$, respectively. Suppose additionally that $\mathcal{L}_\alpha$ and $\mathcal{L}_{\alpha'}$ are hypercubes of alpha attaching curves, satisfying the same assumptions. Suppose also that if $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are curves in $\mathcal{L}_\alpha$ and $\mathcal{L}_\beta$, then $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{w}, \mathbf{z})$ is the diagram of a link in a rational homology 3-sphere, and similarly for $\mathcal{L}_{\alpha'}$ and $\mathcal{L}_{\beta'}$.*

1. *If the small translates in the construction of $\mathcal{L}_\beta \otimes \mathcal{L}_{\beta'}$ are chosen suitably small, then $\mathcal{L}_\beta \otimes \mathcal{L}_{\beta'}$ is a hypercube of attaching curves. The same holds for $\mathcal{L}_\alpha \otimes \mathcal{L}_{\alpha'}$.*
2. *If the translations used in the constructions of $\mathcal{L}_\alpha \otimes \mathcal{L}_{\alpha'}$ and $\mathcal{L}_\beta \otimes \mathcal{L}_{\beta'}$ are suitably small, then the canonical map of vector spaces*

$$\begin{aligned}\mathbf{CF}^-_{J \sqcup J}(\Sigma \sqcup \Sigma', \mathcal{L}_\alpha \otimes \mathcal{L}_{\alpha'}, \mathcal{L}_\beta \otimes \mathcal{L}_{\beta'}, \mathbf{w} \cup \mathbf{w}', \mathbf{z} \cup \mathbf{z}') \\ \cong \mathbf{CF}^-_J(\Sigma, \mathcal{L}_\alpha, \mathcal{L}_\beta, \mathbf{w}, \mathbf{z}) \otimes_{\mathbb{F}} \mathbf{CF}^-_{J'}(\Sigma', \mathcal{L}_{\alpha'}, \mathcal{L}_{\beta'}, \mathbf{w}', \mathbf{z}'),\end{aligned}$$

   is a chain isomorphism.

The same statements hold if Σ and Σ' have free base points in addition to link base points.

Remark 10.2. In our proof, the assumption that $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{w}, \mathbf{z})$ is a diagram for a link in a rational homology 3-sphere is only used to simplify several statements about admissibility. For general three-manifolds, the statement above holds as long as we restrict to a finite set of Spin^c structures on the hypercubes and assume the diagrams satisfy a stronger version of admissibility (see [39, Definition 4.10]).

Our proof of Proposition 10.1 follows from the same line of reasoning as [26, Section 3.5], though we present it for the benefit of the reader.

10.1 Small translate theorems

We now review a preliminary technical result concerning Heegaard diagrams with repeated attaching curves. These are based on the results of [26, Section 3] and [12, Section 11].

Suppose $\mathcal{D} = (\Sigma, \boldsymbol{\alpha}_j, \dots, \boldsymbol{\alpha}_1, \boldsymbol{\beta}_1, \dots, \boldsymbol{\beta}_k, \mathbf{w}, \mathbf{z})$ is a multi-pointed Heegaard multi-diagram, the curves $\boldsymbol{\beta}_1, \dots, \boldsymbol{\beta}_k$ are pairwise handleslide equivalent, and the curves $\boldsymbol{\alpha}_1, \dots, \boldsymbol{\alpha}_j$ are pairwise handleslide equivalent. Let $\mathbb{I} = (i_1, \dots, i_j)$ and $\mathbb{I}' = (i_1', \dots, i_k')$ be tuples of positive integers. For $s \in \{1, \dots, j\}$, pick curves $\boldsymbol{\alpha}_s^1, \dots, \boldsymbol{\alpha}_s^{i_s}$ which are small Hamiltonian translates of $\boldsymbol{\alpha}_s$. Similarly, for $t \in \{1, \dots, k\}$, pick attaching curves $\boldsymbol{\beta}_t^1, \dots, \boldsymbol{\beta}_t^{i_t'}$ which are small translates of $\boldsymbol{\beta}_t$. We define the diagram

$$ {}_{\mathbb{I}}\mathcal{D}_{\mathbb{I}'} := (\Sigma, \boldsymbol{\alpha}_j^{i_j}, \dots, \boldsymbol{\alpha}_j^1, \dots, \boldsymbol{\alpha}_1^{i_1}, \dots, \boldsymbol{\alpha}_1^1, \boldsymbol{\beta}_1^1, \dots, \boldsymbol{\beta}_1^{i_1'}, \dots, \boldsymbol{\beta}_k^1, \dots, \boldsymbol{\beta}_k^{i_k'}, \mathbf{w}, \mathbf{z}). $$

We assume that each $\mathbb{T}_{\alpha_s^n} \cap \mathbb{T}_{\alpha_s^m}$ contains exactly $2^{g(\Sigma)+|\mathbf{w}|-1}$ points, for each s, n and m, and similarly for the beta-translates.

Lemma 10.3. *Suppose $\mathcal{D} = (\Sigma, \boldsymbol{\alpha}_j, \dots, \boldsymbol{\alpha}_1, \boldsymbol{\beta}_1, \dots, \boldsymbol{\beta}_k, \mathbf{w}, \mathbf{z})$ is a multi-diagram such that the $\boldsymbol{\alpha}_i$ are all pairwise handleslide equivalent, and the $\boldsymbol{\beta}_i$ are also all pairwise handleslide equivalent. Assume that $(\Sigma, \boldsymbol{\alpha}_1, \boldsymbol{\beta}_1, \mathbf{w}, \mathbf{z})$ represents a link in a rational homology 3-sphere. Assume that $\mathcal{D}$ is admissible for each complete collection of base points $\mathcal{W} \subseteq \mathbf{w} \cup \mathbf{z}$. Let ${}_{\mathbb{I}}\mathcal{D}_{\mathbb{I}'}$ be the diagram constructed above. Assume the translations are chosen so that ${}_{\mathbb{I}}\mathcal{D}_{\mathbb{I}'}$ is also weakly admissible. Suppose that $(J_y)_{y \in K_{\ell-1}}$ is a generically chosen family of almost complex structures for counting holomorphic ℓ-gons, where ℓ is the total number of attaching curves on ${}_{\mathbb{I}}\mathcal{D}_{\mathbb{I}'}$. Here, $K_{\ell-1}$ is Stasheff's associahedron on $\ell - 1$ inputs. Equivalently, $K_{\ell-1}$ is the moduli space of complex disks with ℓ boundary marked points, one of which is distinguished as the "output". If the translations in the construction are chosen suitably small, then the following holds: If ψ is a class of ℓ-gons representing $\mathfrak{s}$ such that the input of each $\mathbf{CF}^-(\boldsymbol{\alpha}_s^{i+1}, \boldsymbol{\alpha}_s^i)$ and $\mathbf{CF}^-(\boldsymbol{\beta}_t^i, \boldsymbol{\beta}_t^{i+1})$ is the top degree generator and ψ has a J_y-holomorphic representative for some $y \in K_{\ell-1}$, then*

$$ \mu(\psi) \geq \min(0, 3 - j - k). $$

Equivalently, if $S := \sum_{s=1}^{j} (i_s - 1) + \sum_{t=1}^{k} (i_t' - 1)$ denotes the number of special inputs from small translates, then

$$ \mu(\psi) \geq \min(0, 3 - \ell + S). $$

Remark 10.4. The condition that $(\Sigma, \boldsymbol{\alpha}_1, \boldsymbol{\beta}_1)$ represents a rational homology 3-sphere is to obtain finiteness in the number of classes that the polygon maps count. In this case, the existence of the Maslov gradings $\mathrm{gr}_{\mathcal{W}}$ for each complete collection

$\mathcal{W} \subseteq \mathbf{w} \cup \mathbf{z}$ will ensure that for each N, there are only finitely many nonnegative classes in a given

$$\pi_2(\Theta^{\alpha}_{n,n-1}, \dots, \Theta^{\alpha}_{2,1}, \mathbf{x}, \Theta^{\beta}_{1,2}, \dots, \Theta^{\beta}_{m-1,m}, \mathbf{y})$$

with Maslov index N. Similar arguments can be made for diagrams of general 3-manifolds, though one would need to require a stronger version of admissible (such as strong $\mathfrak{s}$-admissibility). See [39, Definition 4.10].

We now briefly sketch the proof of Lemma 10.3. We assume for simplicity of notation that $j = 1$ and $i_1 = 1$, so that there are only beta-translates. Consider a class of ℓ-gons ψ on ${}_{\mathbb{I}}\mathcal{D}_{\mathbb{I}'}$, such that for each i, t, the input from $\mathbb{T}_{\beta^i_t} \cap \mathbb{T}_{\beta^{i+1}_t}$ is the top degree generator. There is a well-defined approximating class ψ^{app} on $\mathcal{D}$, and $\mu(\psi) = \mu(\psi^{\text{app}})$ (at this step, one needs the special inputs to be the top degree generator; see [12, Lemma 11.3], which generalizes easily from triangles to ℓ-gons). For each i and j, we pick a sequence $\{\boldsymbol{\beta}^i_{t,n}\}_{n\in\mathbb{N}}$ such that as $n \to \infty$, the curves $\boldsymbol{\beta}^i_{t,n}$ approach $\boldsymbol{\beta}_t$ (in the sense that there are maps $\boldsymbol{\beta}^i_{t,n}$ is the time 1 translation by a Hamiltonian vector field $X_{H_{i,t,n}}$ for a sequence of functions $H_{i,t,n}$ on Σ which converge to 0 in the C^∞ topology). Write $\mathcal{D}'_n$ for the diagram obtained by replacing each $\boldsymbol{\beta}^i_t$ with $\boldsymbol{\beta}^i_{t,n}$. Given a sequence u_n of holomorphic representatives of ψ on $\mathcal{D}'_n$, we may extract a subsequence which converges to a representative of ψ^{app}, which is a $(k+1)$-gon. In particular, by applying transversality for $(k+1)$-gons, one obtains that

$$\mu(\psi) \geq \min(0, 3 - (k+1)).$$

As noted in Remark 10.4, we use the assumption that $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{w}, \mathbf{z})$ is a diagram for a rational homology sphere so that there are only finitely many classes of ℓ-gons of index $3 - \ell$ representing a given Spin^c structure, and we apply the above argument to each class.

We refer the reader to [26, Section 3] and [12, Section 11] for additional details.

There is an additional refinement of the above lemma for the case when $\mathcal{D} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{w}, \mathbf{z})$ is an ordinary Heegaard diagram, and we have only one translate $\boldsymbol{\beta}'$ of the curves $\boldsymbol{\beta}$. In this case, if $\boldsymbol{\beta}'$ are chosen to be suitably small Hamiltonian translates of $\boldsymbol{\beta}$, then there is a canonical nearest point map

$$\Phi_{\text{np}}\colon \mathbb{T}_\alpha \cap \mathbb{T}_\beta \to \mathbb{T}_\alpha \cap \mathbb{T}_{\beta'}.$$

Extending this map linear over the variables, we obtain a map

$$\Phi_{\text{np}}\colon \mathbf{CF}^-(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{w}, \mathbf{z}) \to \mathbf{CF}^-(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}', \mathbf{w}, \mathbf{z}).$$

Lemma 10.5. *Suppose that* $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{w}, \mathbf{z})$ *is a diagram for a link in a rational homology* 3*-sphere which is weakly admissible for each complete collection* $\mathcal{W} \subseteq \mathbf{w} \cup \mathbf{z}$*,* *and suppose that $\boldsymbol{\beta}'$ are suitably small translates of the curves $\boldsymbol{\beta}$. Then we have an equality of maps*

$$\Phi_{\mathrm{np}}(-) = f_{\alpha,\beta,\beta'}(-, \Theta^+_{\beta,\beta'}).$$

The proof in the case of $\widehat{\mathrm{CF}}$ was given by Lipshitz, Ozsváth and Thurston in [26, Lemma 3.38]. This was extended to $\mathbf{CF}^-$ (mostly by adapting the notation of [26]) in [12, Proposition 11.1].

10.2 Disconnected Heegaard surfaces

Suppose that $\mathcal{L}_\beta$ and $\mathcal{L}_{\beta'}$ are hypercubes of handleslide equivalent attaching curves on $(\Sigma, \mathbf{w}, \mathbf{z})$ and $(\Sigma', \mathbf{w}', \mathbf{z}')$, of dimension n and m, respectively. In this section, we prove part (1) of Proposition 10.1, i.e., for suitable choices of translates, the diagram $\mathcal{L}_\beta \otimes \mathcal{L}_{\beta'}$ is a hypercube of attaching curves (compare [26, Proposition 3.52]).

We begin with a technical result, from which we will derive the hypercube relations. In the following, we will write $\kappa = (\varepsilon, \nu)$ for points in $\mathbb{E}_{n+m} \cong \mathbb{E}_n \times \mathbb{E}_m$. Also, we write $(\boldsymbol{\delta}_\kappa)_{\kappa \in \mathbb{E}_{n+m}}$ for the attaching curves of $\mathbb{E}_{n+m}$. If $\kappa_1 < \cdots < \kappa_\ell$ is an increasing sequence of indices, we write $f_{\kappa_1,\dots,\kappa_\ell}$ for the holomorphic ℓ-gon map $f_{\delta_{\kappa_1},\dots,\delta_{\kappa_\ell}}$. We similarly write $f_{\varepsilon_1,\dots,\varepsilon_\ell}$ and $f_{\nu_1,\dots,\nu_\ell}$ for holomorphic polygon maps on Σ and Σ'. We write $\Theta_{\kappa_{i-1},\kappa_i}$ for the chains on $\mathcal{L}_\beta \otimes \mathcal{L}_{\beta'}$. We recall that these are given by

$$\Theta_{\kappa_i,\kappa_{i+1}} = \begin{cases} \Theta_{\varepsilon_i,\varepsilon_{i+1}} \otimes \Theta^+_{\nu_i} & \text{if } \nu_i = \nu_{i+1}, \\ \Theta^+_{\varepsilon_i} \otimes \Theta_{\nu_i,\nu_{i+1}} & \text{if } \varepsilon_i = \varepsilon_{i+1}, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 10.6. *Let $\mathcal{L}_\beta$ and $\mathcal{L}_{\beta'}$ are hypercubes of handleslide equivalent attaching curves on $(\Sigma, \mathbf{w}, \mathbf{z})$ and $(\Sigma', \mathbf{w}', \mathbf{z}')$, respectively, and let $\mathcal{L}_\beta \otimes \mathcal{L}_{\beta'}$ denote the hypercube on $(\Sigma \sqcup \Sigma', \mathbf{w} \cup \mathbf{w}', \mathbf{z} \cup \mathbf{z}')$ described in Section 9.5. Suppose that $\kappa_1 < \cdots < \kappa_\ell$ is an increasing sequence in $\mathbb{E}_{n+m}$, and let $\Theta_{\kappa_{i-1},\kappa_i}$ denote the chains in $\mathcal{L}_\beta \otimes \mathcal{L}_{\beta'}$. If $\ell \neq 3$, then*

$$\begin{aligned} &f_{\kappa_1,\dots,\kappa_\ell}(\Theta_{\kappa_1,\kappa_2}, \dots, \Theta_{\kappa_{\ell-1},\kappa_\ell}) \\ &= \begin{cases} f_{\varepsilon_1,\dots,\varepsilon_\ell}(\Theta_{\varepsilon_1,\varepsilon_2}, \dots, \Theta_{\varepsilon_{\ell-1},\varepsilon_\ell}) \otimes \Theta^+_{\nu_1,\nu_1} & \textit{if } \nu_1 = \nu_\ell, \\ \Theta^+_{\varepsilon_1,\varepsilon_1} \otimes f_{\nu_1,\dots,\nu_\ell}(\Theta_{\nu_1,\nu_2}, \dots, \Theta_{\nu_{\ell-1},\nu_\ell}) & \textit{if } \varepsilon_1 = \varepsilon_\ell, \\ 0 & \textit{otherwise}. \end{cases} \end{aligned} \tag{10.1}$$

When $\ell = 2$, we interpret f_{κ_1,κ_2} as the ordinary Floer differential. When $\ell = 3$, equation (10.1) holds when $\nu_1 = \nu_3$ or $\varepsilon_1 = \varepsilon_3$. If $\ell = 3$ and instead $\varepsilon_1 < \varepsilon_2 = \varepsilon_3$ and $\nu_1 = \nu_2 < \nu_3$, then

$$f_{\kappa_1,\kappa_2,\kappa_3}(\Theta_{\kappa_1,\kappa_2}, \Theta_{\kappa_2,\kappa_3}) = \Theta_{\varepsilon_1,\varepsilon_2} \otimes \Theta_{\nu_2,\nu_3},$$

where we identify complexes whose attaching curves are small approximations of each other. Similarly, if $\varepsilon_1 = \varepsilon_2 < \varepsilon_3$ *and* $\nu_1 < \nu_2 = \nu_3$ *we have*

$$f_{\kappa_1,\kappa_2,\kappa_3}(\Theta_{\kappa_1,\kappa_2}, \Theta_{\kappa_2,\kappa_3}) = \Theta_{\varepsilon_2,\varepsilon_3} \otimes \Theta_{\nu_1,\nu_2}.$$

Proof. Consider first the claim when $\ell = 2$ (i.e., for holomorphic disks). Suppose that $(\varepsilon_1, \nu_1) < (\varepsilon_2, \nu_2)$ and consider $\Theta_{\kappa_1,\kappa_2}$. By assumption, $\Theta_{\kappa_1,\kappa_2}$ is only non-trivial if $\varepsilon_1 = \varepsilon_2$ or $\nu_1 = \nu_2$, so assume for concreteness that $\varepsilon_1 < \varepsilon_2$ and $\nu_1 = \nu_2$. By definition, $\Theta_{\kappa_1,\kappa_2} = \Theta_{\varepsilon_1,\varepsilon_2} \otimes \Theta^+_{\nu_1,\nu_1}$. The differential on the disjoint union of two diagrams is clearly tensorial, so

$$\partial(\Theta_{\varepsilon_1,\varepsilon_2} \otimes \Theta^+_{\nu_1,\nu_1}) = \partial\Theta_{\varepsilon_1,\varepsilon_2} \otimes \Theta^+_{\nu_1,\nu_1} + \Theta_{\varepsilon_1,\varepsilon_2} \otimes \partial\Theta^+_{\nu_1,\nu_1} = \partial\Theta_{\varepsilon_1,\varepsilon_2} \otimes \Theta^+_{\nu_1,\nu_1},$$

proving the claim in this case. The case that $\ell = 3$ is similarly straightforward to verify.

We assume now that $\ell > 3$. The map $f_{\kappa_1,\dots,\kappa_\ell}$ counts holomorphic ℓ-gons of Maslov index $3 - \ell$. The holomorphic curves map into $(\Sigma \sqcup \Sigma') \times D_\ell$. To define the polygon counting map, we pick a family of almost complex structures $(\mathcal{J}_y)_{y \in K_{\ell-1}}$ on $(\Sigma \sqcup \Sigma') \times D_\ell$. Such a family of almost complex structures induces two families, $(J_y)_{y \in K_{\ell-1}}$ and $(J'_y)_{y \in K_{\ell-1}}$, on $\Sigma \times D_\ell$ and $\Sigma' \times D_\ell$, respectively.

The holomorphic polygon map $f_{\kappa_1,\dots,\kappa_\ell}$ on the disjoint union may be equivalently described by counting pairs (u, u') representing pairs of classes of ℓ-gons (ψ, ψ') satisfying

$$\mu(\psi) + \mu(\psi') = 3 - \ell, \tag{10.2}$$

which have the same almost complex structure parameter $y \in K_{\ell-1}$. In particular, we may naturally view the moduli space for a class (ψ, ψ') (as would be counted by the polygon map $f_{\kappa_1,\dots,\kappa_\ell}$) as the fibered product

$$\begin{aligned} &\bigcup_{y \in K_{\ell-1}} \mathcal{M}_{\mathcal{J}_y}(\psi, \psi') \times \{y\} \\ &\quad = \Big(\bigcup_{y \in K_{\ell-1}} \mathcal{M}_{J_y}(\psi) \times \{y\}\Big) \times_{\mathrm{ev}} \Big(\bigcup_{y \in K_{\ell-1}} \mathcal{M}_{J'_y}(\psi') \times \{y\}\Big), \end{aligned} \tag{10.3}$$

where ev is the evaluation map to $K_{\ell-1}$, which sends a pair (u, y) to the parameter y.

Consider the last line of equation (10.1). This equation concerns sequences $\kappa_1 < \cdots < \kappa_\ell$ which increment both the $\mathbb{E}_n$-coordinates and the $\mathbb{E}_m$-coordinates. Consider a class of ℓ-gons (ψ, ψ') on $(\Sigma \sqcup \Sigma') \times D_\ell$, potentially counted by the ℓ-gon map $f_{\kappa_1,\dots,\kappa_\ell}$. Let k be the number of inputs of ψ which do not increment the $\mathbb{E}_n$-coordinate. Let k' be the number of inputs of ψ' which do not increment the $\mathbb{E}_m$-coordinate.

By Lemma 10.3, if (ψ, ψ') has a holomorphic representative, we must have

$$\mu(\psi) \geq \min(3 - \ell + k, 0) \quad \text{and} \quad \mu(\psi') \geq \min(3 - \ell + k', 0), \tag{10.4}$$

because each input of ψ which does not increase the $\mathbb{E}_n$-coordinate contributes a top degree generator in a pair of translated curves as input. Since $\mu(\psi) + \mu(\psi') = 3 - \ell$, by assumption, for a holomorphic representative to exist we must have

$$\min(3 - \ell + k, 0) + \min(3 - \ell + k', 0) \leq 3 - \ell.$$

We add $2(\ell - 3)$ to the above to obtain

$$\min(k, \ell - 3) + \min(k', \ell - 3) \leq \ell - 3. \tag{10.5}$$

On the other hand, there are $\ell - 1$ total inputs to the map $f_{\kappa_1,\dots,\kappa_\ell}$, and each input increments either $\mathbb{E}_n$ or $\mathbb{E}_m$, but not both (by definition of $\mathcal{L}_\beta \otimes \mathcal{L}_{\beta'}$). Hence

$$k + k' = \ell - 1. \tag{10.6}$$

Combining the above equation with equation (10.5), we deduce that it is not possible that both $k \leq \ell - 3$ and $k' \leq \ell - 3$, since then equation (10.5) would imply $k + k' \leq \ell - 3$, which contracts equation (10.6). Hence, we assume without loss of generality that $k > \ell - 3$. Equation (10.5) implies that

$$\min(k', \ell - 3) \leq 0,$$

which implies that either $k' = 0$ or $\ell \in \{2, 3\}$. Since we assumed that $\ell > 3$, we conclude that $k' = 0$. Equation (10.6) implies that $k = \ell - 1$. Symmetrically, if instead we assumed $k' > \ell - 3$, we would have $(k, k') = (0, \ell - 1)$. This proves the last line of equation (10.1).

We now consider the first two lines of equation (10.1). These correspond to the cases that (k, k') is $(\ell - 1, 0)$ or $(0, \ell - 1)$. Consider the case that $(k, k') = (0, \ell - 1)$. In this case, the holomorphic curve count occurs on the diagram

$$(\Sigma, \boldsymbol{\beta}_{\varepsilon_1}, \dots, \boldsymbol{\beta}_{\varepsilon_\ell}, \mathbf{w}, \mathbf{z}) \sqcup (\Sigma', \boldsymbol{\beta}'_\nu, \dots, \boldsymbol{\beta}'_\nu, \mathbf{w}', \mathbf{z}'),$$

where the ℓ attaching curves on the right-hand diagram are small translates and we write ν for ν_1. Equation (10.4) implies in this case that $\mu(\psi) \geq 3 - \ell$ and $\mu(\psi') \geq 0$. Together with equation (10.2) we see that

$$\mu(\psi) = 3 - \ell \quad \text{and} \quad \mu(\psi') = 0. \tag{10.7}$$

If $\psi' \in \pi_2(\Theta^+_{\nu,\nu}, \dots, \Theta^+_{\nu,\nu}, \mathbf{z})$, the Maslov index formula for the grading implies that

$$\mu(\psi') = n_{\mathbf{w}'}(\psi') + \mathrm{gr}_{\mathbf{w}'}(\Theta^+_{\nu,\nu}, \mathbf{z}).$$

Since $\mu(\psi') = 0$, we conclude that $n_{\mathbf{w}'}(\psi') = 0$ and $\mathbf{z} = \Theta^+_{\nu,\nu}$.

For the main claim, it suffices to show that

$$\#\bigcup_{y\in K_{\ell-1}} \mathcal{M}_{J_y}(\psi) \equiv \sum_{\substack{\psi'\in\pi_2(\Theta^+_{\nu,\nu},\dots,\Theta^+_{\nu,\nu}),\\ \mu(\psi')=0,\\ y\in K_{\ell-1}}} \mathcal{M}_{\mathcal{J}_y}(\psi,\psi') \pmod 2.$$

Equation (10.7) implies that

$$\dim \bigcup_{y\in K_{\ell-1}} \mathcal{M}_{J_y}(\psi)\times\{y\} = 0$$

and

$$\dim \bigcup_{y\in K_{\ell-1}} \mathcal{M}_{J'_y}(\psi')\times\{y\} = \dim(K_{\ell-1}) = \ell-3.$$

By the fibered product equation in equation (10.3) and the above dimension counts, it therefore suffices to show that the map

$$\mathrm{ev}\colon \bigcup_{\substack{\psi'\in\pi_2(\Theta^+_{\nu,\nu},\dots,\Theta^+_{\nu,\nu}),\\ \mu(\psi')=0,\\ y\in K_{\ell-1}}} \mathcal{M}_{J_y}(\psi') \to K_{\ell-1}$$

has odd degree. (Compare [26, Lemma 3.50].) This is proven by considering the preimage of a path $\gamma\colon[0,1]\to K_{\ell-1}$, such that $\gamma(0)=y$, $\gamma(t)\in \operatorname{int} K_{\ell-1}$ for $t\in[0,1)$ and $\gamma(1)$ is a point in $\partial K_{\ell-1}$ which realizes a degeneration of a holomorphic ℓ-gon into $\ell-2$ triangles (i.e., a point in the lowest-dimensional boundary strata of $K_{\ell-1}$). The codimension 1 degenerations along the image of $(0,1)$ under γ consist of index 1 holomorphic disks breaking off. The holomorphic disks which break off at the input generator cancel in pairs, since each $\Theta^+_{\nu,\nu}$ is a cycle. There are no disks which bubble off in the output, since they would leave an index -1 ℓ-gon in $\pi_2(\Theta^+_{\nu,\nu},\dots,\Theta^+_{\nu,\nu},\mathbf{y})$, for some $\mathbf{y}$, as well as an index 1 disk $\phi\in\pi_2(\mathbf{y},\Theta^+_{\nu,\nu})$ with $n_{\mathbf{w}}(\phi)=0$. The existence of such a degeneration would imply that $\mathrm{gr}(\mathbf{y},\Theta^+_{\nu,\nu})=1$, which is impossible since there are no generators with grading higher than $\Theta^+_{\nu,\nu}$.

The cardinality of the limit at $t=1$ corresponds to the component of $\Theta^+_{\nu,\nu}$ in the $\ell-2$ fold composition of maps of the form $f_{\beta'_\nu,\beta'_\nu,\beta'_\nu}(\Theta^+_{\nu,\nu},-)$, applied to the element $\Theta^+_{\nu,\nu}$ (e.g., by the grading preserving invariance of the Heegaard Floer complex of connected sums of $S^1\times S^2$). Clearly this is 1, modulo 2. This establishes the first line of (10.1). The second line follows from the same reasoning, establishing equation (10.1) and completing the proof. ■

We now use the previous lemma to finish our proof that $\mathcal{L}_\beta\otimes\mathcal{L}_{\beta'}$ is a hypercube of attaching curves.

Proof of part (1) of Proposition 10.1. Suppose that $\kappa < \kappa'$ are points in $\mathbb{E}_{n+m}$. We wish to show the hypercube relations for $\mathcal{L}_\beta \otimes \mathcal{L}_{\beta'}$. Write $\kappa = (\varepsilon, \nu)$ and $\kappa' = (\varepsilon', \nu')$. There are two cases to consider:

(1) $\varepsilon = \varepsilon'$ or $\nu = \nu'$. I.e., the arrow (κ, κ') is *non-mixed*.

(2) $\varepsilon < \varepsilon'$ and $\nu < \nu'$. I.e., the arrow (κ, κ') is *mixed*.

In the case of a non-mixed arrow, Lemma 10.6 shows that the hypercube relations for $\mathcal{L}_\beta \otimes \mathcal{L}_{\beta'}$ follow immediately from the hypercube relations on $\mathcal{L}_\beta$ and $\mathcal{L}_{\beta'}$.

For a mixed arrow (κ, κ'), Lemma 10.6 implies that there are exactly two terms which contribute to the hypercube relation. These correspond to the two broken arrow sequences $(\kappa, \kappa_0, \kappa')$ where $\kappa_0 \in \{(\varepsilon, \nu'), (\varepsilon', \nu)\}$. By Lemma 10.6 both of these sequences contribute to $\Theta_{\varepsilon,\varepsilon'} \otimes \Theta_{\nu,\nu'}$, which cancel, so the hypercube relations are satisfied. ∎

We are now able to prove the remainder of Proposition 10.1.

Proof of part (2) of Proposition 10.1. The proof is in the same spirit as part (1) of the proposition. To simplify the notation, we will assume that $\mathcal{L}_\alpha$ and $\mathcal{L}_{\alpha'}$ are both 0-dimensional, and consist of ordinary sets of attaching circles $\boldsymbol{\alpha}$ and $\boldsymbol{\alpha}'$. If $(\varepsilon, \nu) < (\varepsilon', \nu')$ are points in $\mathbb{E}_{n+m}$, then the hypercube map from (ε, ν) to (ε', ν') in

$$\mathbf{CF}^-(\Sigma \sqcup \Sigma', \boldsymbol{\alpha} \cup \boldsymbol{\alpha}', \mathcal{L}_\beta \otimes \mathcal{L}_{\beta'}, \mathbf{w} \cup \mathbf{w}') \tag{10.8}$$

is obtained by summing over all increasing sequences

$$(\varepsilon, \nu) = (\varepsilon_1, \nu_1) < \cdots < (\varepsilon_\ell, \nu_\ell) = (\varepsilon', \nu'),$$

the holomorphic $(\ell + 1)$-gon map which has special inputs from $\mathcal{L}_\beta \otimes \mathcal{L}_{\beta'}$. We say such a sequence is *mixed* if $\varepsilon < \varepsilon'$ and $\nu < \nu'$. We claim that in the pairing of the two hypercubes of attaching curves, mixed sequences make trivial contribution. This is an approximation argument similar to part (1) of the proposition, as we now describe.

We recall that $\mathcal{L}_\beta \otimes \mathcal{L}_{\beta'}$ has no mixed arrows which are assigned a non-zero chain. Hence we may consider only broken arrow paths in $\mathbb{E}_{n+m}$ where each individual arrow increases exactly one of the $\mathbb{E}_n$ or $\mathbb{E}_m$ coordinates. For such a sequence of length $\ell \geq 1$, let k be the number of arrows which do not increase the $\mathbb{E}_n$ coordinate, and let k' be the number of arrows which do not increase the $\mathbb{E}_m$ coordinate. In our present situation, $k + k' = \ell - 1$. The contribution of this arrow path is a count of holomorphic $(\ell + 1)$-gons, with $\ell - 1$ special inputs from the hypercube $\mathcal{L}_\beta \otimes \mathcal{L}_{\beta'}$, and one input from $\mathbf{CF}^-(\boldsymbol{\alpha} \cup \boldsymbol{\alpha}', \boldsymbol{\beta}_\varepsilon \cup \boldsymbol{\beta}'_\nu)$. Suppose (ψ, ψ') is a homology class which could potentially contribute. By construction, $\mu(\psi) + \mu(\psi') = 2 - \ell$. Equation (10.4) adapts to show that

$$\mu(\psi) \geq \min(2 - \ell + k, 0) \quad \text{and} \quad \mu(\psi') \geq \min(2 - \ell + k', 0),$$

so

$$\min(2-\ell+k,0)+\min(2-\ell+k',0)\le 2-\ell,$$

and hence

$$\min(k,\ell-2)+\min(k',\ell-2)\le \ell-2.$$

Similar to the argument from part (1), the only allowable configurations are $(k,k')\in\{(\ell-1,0),(0,\ell-1)\}$. This implies that there are no mixed arrows in the hypercube of equation (10.8).

It remains to verify that the non-mixed arrows of equation (10.8) coincide with the tensor product differential. Arguing similarly to the proof of part (1), it suffices to show that if $\mathbf{x}\in\mathbb{T}_\alpha\cap\mathbb{T}_{\beta_\varepsilon}$, then the map

$$\mathrm{ev}\colon \bigcup_{\substack{\psi\in\pi_2(\mathbf{x},\Theta^+_{\varepsilon,\varepsilon},\dots,\Theta^+_{\varepsilon,\varepsilon},\mathbf{y}),\\ \mu(\psi)=0,\\ y\in K_\ell}} \mathcal{M}_{J_y}(\psi)\to K_\ell$$

is odd degree if $\mathbf{y}$ is the canonical nearest point to $\mathbf{x}$, $\mathbf{y}=\mathbf{x}_{\mathrm{np}}$, and is even degree otherwise. Similarly to part (1), we consider the preimage of a path $\gamma\colon[0,1]\to K_\ell$, which connects a generic $y\in\operatorname{int}K_\ell$ to a point in ∂K_ℓ of maximal codimension. The only possible generic degenerations on the interior consist of an index 1 disk breaking off. Disks breaking off at the $\Theta^+_{\varepsilon,\varepsilon}$ inputs cancel in pairs. Index 1 disks breaking off at the $\mathbf{x}$ input or the $\mathbf{y}$ output are impossible, since they leave an index -1 $(\ell+1)$-gon which has $\ell-1$ inputs equal to the top degree generator, whose existence would violate Lemma 10.3. Hence, we identify the preimage over y with the preimage of $\gamma(1)$. By the nearest point map argument of Lemma 10.5, the count is odd if and only if $\mathbf{y}=\mathbf{x}_{\mathrm{np}}$, completing the proof. ■