Chapter 11

Hypercubes and connected sums

In this chapter, we prove a connected sum formula for hypercubes of attaching curves. We begin with some preliminary definitions before stating our result in Proposition 11.8.

Definition 11.1. A *Heegaard surface with matched link base points* $(\Sigma, \mathbf{w}, \mathbf{z}, \mathbf{p})$ consists of a surface with a finite collection of points $\mathbf{w} \cup \mathbf{z} \cup \mathbf{p}$, which is equipped with a matching function

$$m\colon \mathbf{w} \to \mathbf{z},$$

which is a bijection. We call $\mathbf{w} \cup \mathbf{z}$ the *link base points*, and $\mathbf{p}$ the *free base points*.

Definition 11.2. Suppose that $(\Sigma, \mathbf{w}, \mathbf{z}, \mathbf{p})$ and $(\Sigma', \mathbf{w}', \mathbf{z}', \mathbf{p}')$ are two Heegaard surfaces with matched link base points, as in Definition 11.1. We say that $(\Sigma \# \Sigma', \mathbf{w}'', \mathbf{z}'', \mathbf{p}'')$ is formed by an *admissible connected sum* if it is formed by one of the two following procedures (possibly with the roles of Σ and Σ' reversed).

(1) (Connected sum at a free base point.) The connected sum is taken at a free base point $p \in \mathbf{p}$, and at some point $x' \in \Sigma' \setminus (\mathbf{w}' \cup \mathbf{z}')$. The base points are

$$\mathbf{w}'' = \mathbf{w} \cup \mathbf{w}', \quad \mathbf{z}'' = \mathbf{z} \cup \mathbf{z}', \quad \text{and} \quad \mathbf{p}'' = (\mathbf{p} \setminus \{p\}) \cup \mathbf{p}'.$$

(2) (Connected sum at link base points.) The connected sum is taken at two link base points $w \in \mathbf{w}$ and $z' \in \mathbf{z}'$. The base points are

$$\mathbf{w}'' = (\mathbf{w} \setminus \{w\}) \cup \mathbf{w}', \quad \mathbf{z}'' = \mathbf{z} \cup (\mathbf{z}' \setminus \{z'\}), \quad \text{and} \quad \mathbf{p}'' = \mathbf{p} \cup \mathbf{p}'.$$

Suppose that we form an admissible connected sum $(\Sigma \# \Sigma', \mathbf{w}'', \mathbf{z}'', \mathbf{p}'')$ from the pointed surfaces $(\Sigma, \mathbf{w}, \mathbf{z}, \mathbf{p})$ and $(\Sigma', \mathbf{w}', \mathbf{z}', \mathbf{p}')$. Suppose $\mathcal{L}_\beta$ and $\mathcal{L}_{\beta'}$ are hypercubes of handleslide equivalent attaching curves on Σ and Σ', respectively. We may construct a hypercube-shaped diagram $\mathcal{L}_\beta \otimes \mathcal{L}_{\beta'}$ on $(\Sigma \# \Sigma', \mathbf{w}'', \mathbf{z}'', \mathbf{p}'')$ via the same procedure as on the disjoint union of the two diagrams in Section 9.5. We do not claim that the hypercube relations are satisfied, however we have the following remark.

Remark 11.3. If $\mathcal{L}_\beta$ and $\mathcal{L}_{\beta'}$ are algebraically rigid, then the top degree chains coincide on disjoint unions of diagrams and on admissible connected sums, justifying the use of the notation $\mathcal{L}_\beta \otimes \mathcal{L}_{\beta'}$ for both situations. Also, if $\mathcal{L}_\beta$ and $\mathcal{L}_{\beta'}$ are algebraically rigid, then $\mathcal{L}_\beta \otimes \mathcal{L}_{\beta'}$ is automatically a hypercube of attaching curves.

To state our connected sum theorem for hypercubes, we need an additional condition on our hypercubes of attaching curves.

Definition 11.4. Suppose that $\mathcal{L}_\alpha$ is an algebraically rigid hypercube of handleslide equivalent attaching curves on $(\Sigma, \mathbf{w}, \mathbf{z}, \mathbf{p})$, and x is a point in the complement of $\Sigma \setminus \bigcup_{\alpha \in \mathcal{L}_\alpha} \boldsymbol{\alpha}$. We say that x is *base point-esque* for $\mathcal{L}_\alpha$ if for every composable sequence $\Theta_{\nu_j,\nu_{j-1}}, \dots, \Theta_{\nu_2,\nu_1}$ of chains in $\mathcal{L}_\alpha$, and every nonnegative class $\psi \in \pi_2(\Theta_{\nu_j,\nu_{j-1}}, \dots, \Theta_{\nu_2,\nu_1}, \mathbf{x})$, we have

$$\mu(\psi) \geq 2n_x(\psi).$$

(Note $\mathbf{x}$ is the outgoing intersection point in the class ψ.) We make a similar definition for hypercubes of algebraically rigid beta hypercubes.

Lemma 11.5. *If $\mathcal{L}_\alpha$ is an algebraically rigid hypercube of handleslide equivalent attaching curves on $(\Sigma, \mathbf{w}, \mathbf{z}, \mathbf{p})$, and $x \in \mathbf{w} \cup \mathbf{z} \cup \mathbf{p}$, then x is base point-esque for $\mathcal{L}_\alpha$.*

Proof. Assume for concreteness that $x \in \mathbf{w}$. If $\psi \in \pi_2(\Theta_{\nu_j,\nu_{j-1}}, \dots, \Theta_{\nu_2,\nu_1}, \mathbf{x})$ is a class, then the relation between the Maslov index and the absolute grading on the Heegaard Floer homology (see [40, Section 7]) implies

$$\mu(\psi) = 2n_{\mathbf{w}\cup\mathbf{p}}(\psi) + \mathrm{gr}_{\mathbf{w}\cup\mathbf{p}}(\Theta_{\nu_j,\nu_{j-1}}) + \cdots + \mathrm{gr}_{\mathbf{w}\cup\mathbf{p}}(\Theta_{\nu_2,\nu_1}) - \mathrm{gr}_{\mathbf{w}\cup\mathbf{p}}(\mathbf{x}),$$

where $\mathrm{gr}_{\mathbf{w}\cup\mathbf{p}}$ is the absolute grading, normalized so that the top degree cycle of

$$\mathbf{HF}^-(\Sigma, \boldsymbol{\alpha}_{\nu_i}, \boldsymbol{\alpha}_{\nu_{i-1}}, \mathbf{w}, \mathbf{z}, \mathbf{p})$$

has grading 0. Since $\mathcal{L}_\alpha$ is algebraically rigid, we also know that $\mathrm{gr}_{\mathbf{w}\cup\mathbf{p}}(\mathbf{x}) \leq 0$ and $\mathrm{gr}_{\mathbf{w}\cup\mathbf{p}}(\Theta_{\nu_{i+1},\nu_i}) = 0$ for all i and hence

$$\mu(\psi) \geq 2n_{\mathbf{w}\cup\mathbf{p}}(\psi).$$

If ψ is a nonnegative class, we have $n_{\mathbf{w}\cup\mathbf{p}}(\psi) \geq n_x(\psi)$, so we see that x is base point-esque for $\mathcal{L}_\alpha$. ■

Remark 11.6. Suppose

$$\mathcal{L}_\alpha = \left(\boldsymbol{\alpha} \xrightarrow{\Theta^+_{\alpha',\alpha}} \boldsymbol{\alpha}'\right),$$

where $\boldsymbol{\alpha}'$ is obtained by performing a Hamiltonian isotopy to $\boldsymbol{\alpha}$ which crosses x, then x is not base point-esque for $\mathcal{L}_\alpha$ because there is a bigon of index 1 which covers x once. See Figure 11.1.

Suppose that $\mathcal{L}_\alpha$ is a hypercube of handleslide equivalent attaching curves on $(\Sigma, \mathbf{w}, \mathbf{z}, \mathbf{p})$ and $x \in \Sigma$ is in the complement of each curve in $\mathcal{L}_\alpha$. If $\boldsymbol{\alpha}$ is in $\mathcal{L}_\alpha$, write E_x^α for the product of the variables of the base points in the component of $\Sigma \setminus \boldsymbol{\alpha}$ containing x.

Lemma 11.7. *Let $\mathcal{L}_\alpha$ be a hypercube of handleslide equivalent attaching curves on $(\Sigma, \mathbf{w}, \mathbf{z}, \mathbf{p})$ and suppose that x is base point-esque for $\mathcal{L}_\alpha$. Then E_x^α is independent of $\boldsymbol{\alpha} \in \mathcal{L}_\alpha$.*

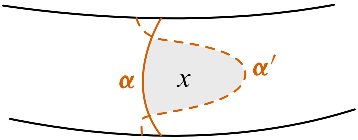

Figure 11.1. An example of a point x which is not base point-esque from Remark 11.6.

Proof. Let $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2 \in \mathscr{L}_\alpha$. Consider the endomorphism A_x of $\mathbf{CF}^-(\boldsymbol{\alpha}_2, \boldsymbol{\alpha}_1)$ which counts holomorphic disks of index 1, which are given a multiplicative factor of $n_x(\phi)$. It is straightforward to see that $[\partial, A_x] = (E_x^{\alpha_1} + E_x^{\alpha_2}) \cdot \mathrm{id}$. On the other hand, if $\Theta_{\alpha_2,\alpha_1}$ is a cycle generating the top degree of homology, then $\partial(\Theta_{\alpha_2,\alpha_1}) = 0$ since $\Theta_{\alpha_2,\alpha_1}$ is a cycle and $A_x(\Theta_{\alpha_2,\alpha_1}) = 0$ since x is base point-esque for $\mathscr{L}_\alpha$ so there are no Maslov index 1 and nonnegative homology classes of disks in any $\pi_2(\Theta_{\alpha_2,\alpha_1}, \mathbf{x})$ with multiplicity 1 on x. Since $\mathbf{CF}^-(\boldsymbol{\alpha}_2, \boldsymbol{\alpha}_1)$ is free and $0 = [\partial, A_x](\Theta_{\alpha_2,\alpha_1}) = \Theta_{\alpha_2,\alpha_1} \cdot (E_x^{\alpha_1} + E_x^{\alpha_2})$, it follows that $E_x^{\alpha_1} + E_x^{\alpha_2} = 0$, completing the proof. ■

The main result of this chapter is the following.

Proposition 11.8. *Suppose that $\mathscr{L}_\alpha$ and $\mathscr{L}_\beta$ are algebraically rigid hypercubes of handleslide equivalent attaching curves on $(\Sigma, \mathbf{w}, \mathbf{z}, \mathbf{p})$ and $\mathscr{L}_{\alpha'}$ and $\mathscr{L}_{\beta'}$ are algebraically rigid hypercubes of handleslide equivalent attaching curves on $(\Sigma', \mathbf{w}', \mathbf{z}', \mathbf{p}')$. Suppose that we form $(\Sigma \# \Sigma', \mathbf{w}'', \mathbf{z}'', \mathbf{p}'')$ by an admissible connected sum of $(\Sigma, \mathbf{w}, \mathbf{z}, \mathbf{p})$ and $(\Sigma', \mathbf{w}', \mathbf{z}', \mathbf{p}')$ at points $x \in \Sigma$ and $x' \in \Sigma'$. Form the hypercubes $\mathscr{L}_\alpha \otimes \mathscr{L}_{\alpha'}$ and $\mathscr{L}_\beta \otimes \mathscr{L}_{\beta'}$ as described above. If x is base point-esque for $\mathscr{L}_\beta$ and x' is base point-esque for $\mathscr{L}_{\alpha'}$, then there is a natural homotopy equivalence of hypercubes*

$$\begin{aligned}&\mathbf{CF}^-(\Sigma \# \Sigma', \mathscr{L}_\alpha \otimes \mathscr{L}_{\alpha'}, \mathscr{L}_\beta \otimes \mathscr{L}_{\beta'}, \mathbf{w}'', \mathbf{z}'', \mathbf{p}'')\\ &\qquad \simeq \mathbf{CF}^-(\Sigma, \mathscr{L}_\alpha, \mathscr{L}_\beta, \mathbf{w}, \mathbf{z}, \mathbf{p}) \otimes_R \mathbf{CF}^-(\Sigma', \mathscr{L}_{\alpha'}, \mathscr{L}_{\beta'}, \mathbf{w}', \mathbf{z}', \mathbf{p}').\end{aligned}$$

Also, R is as follows:

1. *If the connected sum is taken along a link component, then $R = \mathbb{F}[\mathscr{U}, \mathscr{V}]$ is the ring for the link components along which the connected sum is taken.*
2. *If the connected sum is taken at a free base point $x = p_i \in \mathbf{p}$, then the tensor product is taken over $\mathbb{F}[U]$, where we have U act by U_i on the left factor, and $E_{x'}^{\alpha'}$ on the right factor. (Cf. Lemma 11.7.)*
3. *If the connected sum is taken at a free base point $x' = p_i' \in \mathbf{p}'$ in $\mathbf{w}' \cup \mathbf{z}'$, the tensor product is taken over $\mathbb{F}[U]$, where we have U act by U_i on the right factor and E_x^β on the left factor.*

Unlike in Chapter 10, the homotopy equivalence of Proposition 11.8 is not given by the identity map on the level of groups. Instead, it involves counting holomorphic curves.

Remark 11.9. If instead x is base point-esque for $\mathcal{L}_\alpha$ and x' is base point-esque for $\mathcal{L}_{\beta'}$, then there is also a homotopy equivalence of hypercubes realizing a tensor product formula. Note that in the case that if x and x' are base point-esque for all of the hypercubes of Lagrangians, then there are two distinct maps of hypercubes realizing the homotopy equivalence. These are referred to as $\Phi^{\bullet<\circ'}$ and $\Phi^{\circ'<\bullet}$ below.

11.1 Cylindrical boundary degenerations

In this section, we recall some important facts about boundary degenerations.

Definition 11.10. Suppose $\boldsymbol{\beta}$ is a set of attaching curves on $(\Sigma, \mathbf{w})$, and $\mathbf{x} \in \mathbb{T}_\beta$. A *cylindrical beta boundary degeneration* at $\mathbf{x}$ consists of a tuple (u, S, j) such that (S, j) is a Riemann surface and

$$u\colon (S, \partial S) \to (\Sigma \times [0, \infty) \times \mathbb{R}, \boldsymbol{\beta} \times \{0\} \times \mathbb{R})$$

is a map satisfying the following:

(1) u is (j, J)-holomorphic.

(2) u is proper.

(3) For each $t \in \mathbb{R}$ and $i \in \{1, \dots, g(\Sigma) + |\mathbf{w}| - 1\}$, the set $u^{-1}(\beta_i \times \{0\} \times \{t\})$ consists of a single point.

(4) u has finite energy.

(5) $\pi_{\mathbb{H}} \circ u$ is non-constant on each component of S, where $\mathbb{H} = [0, \infty) \times \mathbb{R}$.

(6) S has a collection of $n = g(\Sigma) + |\mathbf{w}| - 1$ boundary punctures $p_1, \dots, p_n$. If $\mathbf{x} = (x_1, \dots, x_n)$, where $x_i \in \beta_i$, then

$$\lim_{z \to p_i} (\pi_{\mathbb{H}} \circ u)(z) = \infty \quad \text{and} \quad \lim_{z \to p_i} (\pi_\Sigma \circ u)(z) = x_i.$$

Cylindrical alpha boundary degenerations are defined by analogy. Of fundamental importance is the mod 2 count of boundary degenerations.

Proposition 11.11. *Suppose $\boldsymbol{\beta}$ is a set of attaching curves on $(\Sigma, \mathbf{w})$, and B is a Maslov index 2 class of boundary degenerations. For an appropriate choice of almost complex structures on $\Sigma \times [0, \infty) \times \mathbb{R}$, the moduli space of boundary degenerations $\mathcal{N}(B, \mathbf{x})$ is transversely cut out. Furthermore, the parametrized moduli space*

$$\bigcup_{\mathbf{x} \in \mathbb{T}_\beta} \mathcal{N}(B, \mathbf{x}) \times \{\mathbf{x}\}$$

is also transversely cut out. Furthermore,

$$\#\mathcal{N}(B, \mathbf{x}) / \operatorname{Aut}(\mathbb{H}) \equiv 1.$$

The case when $|\mathbf{w}| > 1$ is proven by Ozsváth and Szabó [41, Theorem 5.5]. Ozsváth and Szabó also proved that if $|\mathbf{w}| = 1$ there are generically no boundary degenerations for split almost complex structures on $\Sigma \times [0, \infty) \times \mathbb{R}$. However, for their boundary degenerations, the parametrized moduli space $\bigcup_{\mathbf{x}\in\mathbb{T}_\beta} \mathcal{N}(B, \mathbf{x}) \times \{\mathbf{x}\}$ might in general be non-empty, and there are always broken boundary degenerations at which transversality is not achieved (e.g., the union of a closed curve representing Σ and a constant disk at $\mathbf{x}$). This is sufficient for showing that $\partial^2 = 0$, but is not sufficient for our purposes. The $|\mathbf{w}| = 1$ case was revisited in [12, Section 7.6], where it was shown that for appropriately generic choices of almost complex structures, the count of Proposition 11.11 holds.

11.2 The 0-dimensional case

In this section, we present the proof of Proposition 11.8 in the case that $\mathcal{L}_\beta$ and $\mathcal{L}_{\beta'}$ are both 0-dimensional (i.e., we recover the standard connected sum formula of Ozsváth and Szabó [38]). The analytic details we present here will be the basis of the higher-dimensional cases of Proposition 11.8, which we consider in the subsequent section. Our argument is inspired by work of Ozsváth and Szabó in the bordered setting [45]. See also [12, Section 19.4].

Suppose $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{w}, \mathbf{z})$ and $\mathcal{H}' = (\Sigma', \boldsymbol{\alpha}', \boldsymbol{\beta}', \mathbf{w}', \mathbf{z}')$ are multi-pointed diagrams, and suppose that J and J' are almost complex structures on $\Sigma \times [0, 1] \times \mathbb{R}$ and $\Sigma' \times [0, 1] \times \mathbb{R}$, respectively. The notation $J \wedge J'$ means the data of the pair J and J' together with distinguished connected sum points, $x \in \Sigma$ and $x' \in \Sigma'$. We assume that the connected sum is admissible, in the sense of Definition 11.2.

Remark 11.12. In Proposition 11.8, we additionally had the assumption that x and x' were base point-esque for some of the hypercubes of attaching curves. In the present case, $\mathcal{L}_\alpha$, $\mathcal{L}_{\alpha'}$, $\mathcal{L}_\beta$ and $\mathcal{L}_{\beta'}$ are all 0-dimensional, and this condition is automatic.

For notational simplicity, we focus on the case where we have just one $\mathscr{U}$ variable, and one $\mathscr{V}$ variable, no free base points, and we are doing a connected sum of link components. The first step is to do a neck-stretching degeneration along the connected sum tube. The result of this degeneration is a chain complex, freely generated over the ground ring $R = \mathbb{F}[\mathscr{U}, \mathscr{V}]$ by pairs $\mathbf{x} \times \mathbf{x}'$, where $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ and $\mathbf{x}' \in \mathbb{T}_{\alpha'} \cap \mathbb{T}_{\beta'}$. The differential is given by the formula

$$\partial_{J\wedge J'}(\mathbf{x} \times \mathbf{x}') = \sum_{\substack{\mathbf{y}\times\mathbf{y}'\in(\mathbb{T}_\alpha\cap\mathbb{T}_\beta)\times(\mathbb{T}_{\alpha'}\cap\mathbb{T}_{\beta'}),\\ \phi\in\pi_2(\mathbf{x},\mathbf{y}),\\ \phi'\in\pi_2(\mathbf{x}',\mathbf{y}'),\\ n_x(\phi)=n_{x'}(\phi'),\\ \mu(\phi)+\mu(\phi')-2n_x(\phi)=1}} \#(\mathcal{M}\mathcal{M}(\phi, \phi')/\mathbb{R}) \cdot \mathscr{U}^{n_{\mathbf{w}''}(\phi+\phi')} \mathscr{V}^{n_{\mathbf{z}''}(\phi+\phi')} \cdot \mathbf{y} \times \mathbf{y}', \tag{11.1}$$

where $\mathcal{M}\mathcal{M}(\phi,\phi')$ is the *perfectly matched moduli space*, consisting of pairs (u,u') satisfying the following matching condition. In the following, we write $(S^{\mathbf{q}}, j)$ and $(T^{\mathbf{q}'}, j')$ for the sources curves of u and u', where (S,j) and (T,j') are Riemann surfaces, and $\mathbf{q}$ and $\mathbf{q}'$ are ordered collections of $n_x(\phi)=n_{x'}(\phi)$ marked points, for which we write $\mathbf{q}=\{q_1,\dots,q_n\}$ and $\mathbf{q}'=\{q_1,\dots,q_n'\}$. We define

$$\begin{aligned}\mathcal{M}\mathcal{M}_{J\wedge J'}(\phi,\phi') := \big\{(u,u') : {}& u \text{ is } J\text{-holomorphic},\\ & u' \text{ is } J'\text{-holomorphic},\\ & (\pi_\Sigma\circ u)(q_i)=x, (\pi_{\Sigma'}\circ u')(q_i')=x',\\ & (\pi_{\mathbb{D}}\circ u)(q_i)=(\pi_{\mathbb{D}}\circ u')(q_i')\big\}.\end{aligned} \tag{11.2}$$

It is not hard to see that this gives a chain complex, which we denote by $\mathbf{CF}^-_{J\wedge J'}(\mathcal{H},\mathcal{H}')$. (Compare [12, Section 19.4].) The construction is inspired by the matched moduli spaces which appear in [27, Section 9.1].

If I is a (non-singular) almost complex structure on $\Sigma\#\Sigma'\times[0,1]\times\mathbb{R}$, then one may define a chain homotopy equivalence

$$\mathbf{CF}^-_I(\mathcal{H}\#\mathcal{H}')\simeq \mathbf{CF}^-_{J\wedge J'}(\mathcal{H},\mathcal{H}')$$

by counting index 0 curves for a non-cylindrical almost complex structure on $\Sigma\#\Sigma'\times[0,1]\times\mathbb{R}$ which interpolates an ordinary cylindrical almost complex structure on $\Sigma\#\Sigma'\times[0,1]\times(-\infty,t_1]$, and a degenerate almost complex structure (i.e., one with infinite neck length) on $\Sigma\wedge\Sigma'\times[0,1]\times[t_2,\infty)$, for some $t_1\ll 0$ and $t_2\gg 0$. See [12, Section 19.4] for more details.

We now describe a chain homotopy equivalence

$$\Phi^{\bullet<\circ'}\colon \mathbf{CF}^-_{J\wedge J'}(\mathcal{H},\mathcal{H}')\to \mathbf{CF}^-_J(\mathcal{H})\otimes_R \mathbf{CF}^-_{J'}(\mathcal{H}').$$

If $t_0\in\mathbb{R}$, write

$$\lambda_{t_0} := [0,1]\times\{t_0\}\subseteq[0,1]\times\mathbb{R}.$$

Let us write

$$s\colon \Sigma\times[0,1]\times\mathbb{R}\to[0,1] \quad\text{and}\quad t\colon \Sigma\times[0,1]\times\mathbb{R}\to\mathbb{R}$$

for the projection maps.

Definition 11.13. Suppose $t_0\in\mathbb{R}$. A $(\Phi,\bullet<\circ',t_0)$*-matched* $J\wedge J'$*-holomorphic curve pair* consists of a pair of marked $J\wedge J'$ holomorphic disks (u,u'), equipped with the following data:

(1) A partition of the marked points of u (resp. u') into three sets, $\mathbf{S}$, $\mathbf{C}$ and $\mathbf{N}$ (resp. $\mathbf{S}'$, $\mathbf{C}'$ and $\mathbf{N}'$).

(2) A bijection $\phi\colon\mathbf{S}\to\mathbf{S}'$.

We assume the following are satisfied:

(1) If q_i is a marked point of u, then $(\pi_\Sigma \circ u)(q_i) = x$, and similarly for all marked points of u'.

(2) If $q \in \mathbf{S}$ and $q' \in \mathbf{S}'$ and $\phi(q) = q'$, then

$$(t \circ u)(q) = (t \circ u')(q') < 0 \quad \text{and} \quad (s \circ u)(q) = (s \circ u')(q').$$

(3) If $q \in \mathbf{N}$ then $(t \circ u)(q) > t_0$. The analogous statement holds for $q' \in \mathbf{N}'$.

(4) $|\mathbf{C}| = |\mathbf{C}'|$. Furthermore, if $q \in \mathbf{C}$ then $(t \circ u)(q) = t_0$, and similarly for the marked points of $\mathbf{C}'$. The marked points of $\mathbf{C}$ and $\mathbf{C}'$ alternate between those of $\mathbf{C}$ and those of $\mathbf{C}'$ along λ_{t_0}. Finally, the left-most marked point along this line is contained in $\mathbf{C}$.

See Figure 11.2 for a schematic of a $(\Phi, \bullet < \circ', t_0)$-matched curve pair.

If (ϕ, ϕ') are a pair of homology classes, M is a pair of marked source curves for ϕ and ϕ', and $t_0 \in \mathbb{R}$, we write $\mathcal{M}\mathcal{M}^{\Phi,\bullet<\circ'}(\phi, \phi', M, t_0)$ for the moduli space of $(\Phi, \bullet < \circ', t_0)$-matched disks representing ϕ and ϕ'. We consider the parametrized moduli space

$$\mathcal{M}\mathcal{M}^{\Phi,\bullet<\circ'}(\phi, \phi', M) = \bigcup_{t_0 \in \mathbb{R}} \mathcal{M}\mathcal{M}^{\Phi,\bullet<\circ'}(\phi, \phi', M, t_0) \times \{t_0\}.$$

If t_0 is fixed, the expected dimension of $\mathcal{M}\mathcal{M}^{\Phi,\bullet<\circ'}(\phi, \phi', M, t_0)$ is given by the formula

$$\operatorname{ind}(\phi, \phi', M) = \mu(\phi) + \mu(\phi') - 2(|\mathbf{S}| + |\mathbf{C}|).$$

The map $\Phi^{\bullet<\circ'}$ counts pairs of curves (u, u') in $\mathcal{M}\mathcal{M}^{\Phi,\bullet<\circ'}(\phi, \phi', M)/\mathbb{R}$, ranging over tuples with $\operatorname{ind}(\phi, \phi', M) = 0$. Equivalently, we can think of $\Phi^{\bullet<\circ'}$ as counting elements of the non-parametrized moduli spaces $\mathcal{M}\mathcal{M}^{\Phi,\bullet<\circ'}(\phi, \phi', M, t_0)$ for some fixed t_0. (The parametrized perspective becomes helpful later.)

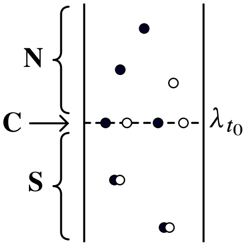

Figure 11.2. The projection to $[0, 1] \times \mathbb{R}$ of marked points of a $(\Phi, \bullet < \circ', t_0)$-matched holomorphic curve pair. Solid dots indicate the marked points of u, while open dots are the marked points of u'. The map $\Psi^{\bullet<\circ'}$ counts similar configurations.

Next, we explain the $\mathscr{U}$ and $\mathscr{V}$ weights of a curve. The base points not involved in the connected sum contribute powers of variables as normal. The marked points in $\mathbf{S} \cup \mathbf{S}' \cup \mathbf{C} \cup \mathbf{C}'$ contribute algebra elements which are the same as would be counted on the connected sum of the two diagrams, where we have removed any base points used for the connected sum. For the marked points in $\mathbf{N}$ and $\mathbf{N}'$, the algebraic contribution is the same as on the disjoint union of the two diagrams, where we treat the punctures as base points.

Concretely, if the connected sum is formed along a link component, and $x = w \in \mathbf{w}$ and $x' = z' \in \mathbf{z}'$, then the algebra element contributed by the marked points would be

$$\mathscr{U}^{|\mathbf{N}|}\mathscr{V}^{|\mathbf{N}'|}. \tag{11.3}$$

(There would be an additional contribution from the other base points not involved in the connected sum.) If instead the connected sum is formed at a free base point $x = w \in \Sigma$ and some non-base point $x' \in \Sigma'$, then the algebra contribution from the marked points would be $\mathscr{U}^{|\mathbf{N}|}$.

We define a similar map

$$\Psi^{\bullet<\circ'} \colon \mathbf{CF}^-_J(\mathcal{H}) \otimes_R \mathbf{CF}^-_{J'}(\mathcal{H}') \to \mathbf{CF}^-_{J\wedge J'}(\mathcal{H}, \mathcal{H}')$$

by counting holomorphic curves with similar conditions, but with the roles of the $\mathbf{S}$ and $\mathbf{N}$ labeled marked points switched.

Remark 11.14. Constant disks are counted by both $\Phi^{\bullet<\circ'}$ and $\Psi^{\bullet<\circ'}$.

Lemma 11.15. *The maps $\Phi^{\bullet<\circ'}$ and $\Psi^{\bullet<\circ'}$ are chain maps.*

Proof. We focus on $\Phi^{\bullet<\circ'}$ since the argument for $\Psi^{\bullet<\circ'}$ is not substantially different. Our proof is modeled on work of Ozsváth and Szabó in the setting of bordered knot Floer homology [45]. The proof is to count the ends of $(\Phi, \bullet < \circ', t_0)$-matched moduli spaces for triples (ϕ, ϕ', M) with $\operatorname{ind}(\phi, \phi', M) = 1$. Modulo the $\mathbb{R}$-action on the parametrized moduli space, such moduli spaces are 1-dimensional. In the present situation, it is sufficient to fix $t_0 = 0$, and consider only $(\Phi, \bullet < \circ', 0)$-matched holomorphic curves. The ends are constrained generically to the following configurations:

- (Φ-1) Two paired marked points in $\mathbf{S}$ and $\mathbf{S}'$ may collide with the line λ_0, away from the marked points in $\mathbf{C}$ and $\mathbf{C}'$.
- (Φ-2) A pair of punctures in $\mathbf{C}$ and $\mathbf{C}'$ may collide along λ_0.
- (Φ-3) A puncture of $\mathbf{N}$ or $\mathbf{N}'$ may collide with λ_0 (in the complement of $\mathbf{C}$ and $\mathbf{C}'$). There are two subcases:
  - (a) After the degeneration, the marked points along λ_0 do not alternate between those of u and u'.
  - (b) After the degeneration, the marked points along λ_0 do alternate between those of u and u'.

(Φ-4) A puncture of $\mathbf{C}$ may degenerate into an index 2 beta boundary degeneration at height $t = 0$.

(Φ-5) A puncture of $\mathbf{C}'$ may degenerate into an index 2 alpha boundary degeneration at height $t = 0$.

(Φ-6) Strip breaking may occur, leaving a $(\Phi, \bullet < \circ', 0)$-matched disk of index 0 as well as a holomorphic disk of index 1 in either positive or negative direction. There are two subcases:

  (a) The index 1 holomorphic disk degenerates towards $-\infty$, and is perfectly-matched.

  (b) The index 1 holomorphic disk degenerates towards $+\infty$, and has trivial matching (i.e., the projections of all marked points to $[0, 1] \times \mathbb{R}$ are distinct).

Most of these ends appear in canceling pairs, and the rest correspond to the relation $[\partial, \Phi^{\bullet<\circ'}] = 0$, as we describe presently.

The ends (Φ-1) cancel with the ends (Φ-2).

In an end of type (Φ-3a), there are two adjacent marked points along λ_0 which are both from u or both from u'. Such a curve appears twice in the boundary of the moduli spaces. See Figure 11.3.

The ends of type (Φ-3b) cancel with the end of type (Φ-4) and (Φ-5). Here we are using the mod 2 count of boundary degenerations from Proposition 11.11. Note also the algebra contributions coincide for the two canceling degenerations of type (Φ-3b) and type (Φ-4) or (Φ-5) (cf. equation (11.3)). If the connected sum is taken along a link component, each boundary degeneration which forms along λ_{t_0} will contain one puncture from $\mathbf{C} \cup \mathbf{C}'$ and also one link base point. The degeneration which cancels this boundary degeneration formation consists of a $\mathbf{N}$ or $\mathbf{N}'$ puncture colliding with λ_{t_0}. The algebra weight from the $\mathbf{N}$ or $\mathbf{N}'$ puncture coincides with the weight of the base point which is lost in the boundary degeneration. The case of a connected sum at a free base point is similar.

The remaining ends are (Φ-6), which correspond exactly to the commutator $[\partial, \Phi^{\bullet<\circ'}]$. Summing all ends, we conclude that

$$\partial \circ \Phi^{\bullet<\circ'} + \Phi^{\bullet<\circ'} \circ \partial = 0,$$

completing the proof. ∎

Lemma 11.16. *The maps $\Phi^{\bullet<\circ'}$ and $\Psi^{\bullet<\circ'}$ are homotopy inverses.*

Proof. We define two maps $H_\wedge$ and $H_\sqcup$, which we prove satisfy

$$\mathrm{id} + \Psi^{\bullet<\circ'} \circ \Phi^{\bullet<\circ'} = [\partial, H_\wedge] \quad \text{and} \quad \mathrm{id} + \Phi^{\bullet<\circ'} \circ \Psi^{\bullet<\circ'} = [\partial, H_\sqcup]. \tag{11.4}$$

We focus on the map $H_\wedge$, since $H_\sqcup$ is constructed by a straightforward modification.

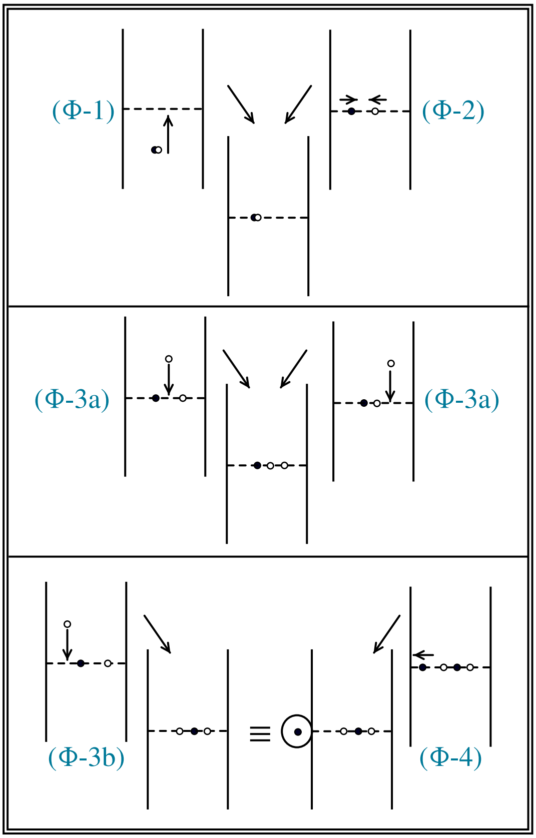

Figure 11.3. Some cancellations in the proof that $\Phi^{\bullet<\circ'}$ and $\Psi^{\bullet<\circ'}$ are chain maps.

The map $H_\wedge$ counts certain curve pairs (u,u') where u has five collections of marked points, **NN**, **NC**, **M**, **SC** and **SS**, and u' has five collections of marked points **NN**′, **NC**′, **M**′, **SC**′ and **SS**′.

Definition 11.17. Suppose that $t_0 < t_1$ are real numbers. We say a pair of marked holomorphic strips (u,u') is $(H_\wedge, t_0, t_1)$-*matched* if the following are satisfied:

1. The marked points of **NN** and **NN**′ are perfectly matched. Furthermore, their projection to $\mathbb{R}$ lies above t_1.
2. The marked points in **NC** and **NC**′ both project to λ_{t_1}. Furthermore, $|\mathbf{NC}| = |\mathbf{NC}'|$. As one travels along λ_{t_1}, the marked points alternate between $|\mathbf{NC}|$ and $|\mathbf{NC}'|$, and the left-most marked point is from **NC**, while the right-most is from **NC**′.
3. The marked points of **M** and **M**′ have no matching condition. Their projection to $[0,1]\times\mathbb{R}$ lies between λ_{t_0} and λ_{t_1}.

(4) The marked points **SC** and **SC**$'$ satisfy the same conditions as **NC** and **NC**$'$, except that they project to λ_{t_0}.

(5) The marked points in **SS** and **SS**$'$ are perfectly matched, and their projection to $\mathbb{R}$ lies below t_0.

See Figure 11.4 for a schematic.

If (ϕ, ϕ') are two homology classes of disks, equipped with decorated sources and matching data M, and $t_0 < t_1$ are fixed, then the expected dimension of the moduli space of curves which are $(H_\wedge, t_0, t_1)$-matched and represent (ϕ, ϕ', M) is given by

$$\operatorname{ind}(\phi, \phi', M) = \mu(\phi) + \mu(\phi') - 2|\mathbf{SS}| - 2|\mathbf{NN}| - 2|\mathbf{SC}| - 2|\mathbf{NC}|.$$

Write $\mathcal{MM}(\phi, \phi', M)$ for this moduli space. The parametrized moduli space

$$\mathcal{MM}(\phi, \phi', M) := \bigcup_{t_0<t_1} \mathcal{MM}(\phi, \phi', M, t_0, t_1) \times \{(t_0, t_1)\}$$

has a free $\mathbb{R}$-action, corresponding to overall translation (acting diagonally on t_0 and t_1).

Further, the map $H_\wedge$ counts $\mathcal{MM}(\phi, \phi', M)/\mathbb{R}$ for triples (ϕ, ϕ', M) satisfying $\operatorname{ind}(\phi, \phi', M) = -1$. In this case, the expected dimension of $\mathcal{MM}(\phi, \phi', M)$ is 1.

To verify equation (11.4), we count the ends of the parametrized moduli spaces $\mathcal{MM}(\phi, \phi', M)/\mathbb{R}$ where $\operatorname{ind}(\phi, \phi', M) = 0$ (so that $\dim \mathcal{MM}(\phi, \phi', M)/\mathbb{R} = 1$).

There are ends which occur at $0 < t_1 - t_0 < \infty$. These are the obvious analogs of the ends in the case of one special line (Φ-1), ... ,(Φ-5). They cancel by an identical proof to the case of one special line.

It remains to analyze the curves which appear as $t_1 - t_0 \to 0$ or $t_1 - t_0 \to \infty$. The ends which appear as $t_1 - t_0 \to \infty$ are easy to analyze: they correspond exactly to the composition $\Psi^{\bullet<\circ'} \circ \Phi^{\bullet<\circ'}$. We now focus on the ends which appear as $t_1 - t_0 \to 0$. These correspond to the lines λ_{t_0} and λ_{t_1} colliding to form a single line, for which

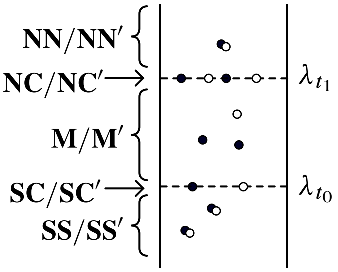

Figure 11.4. A $(H_\wedge, t_0, t_1)$-matched curve pair. On the left side, we indicate the regions where the marked points **NN**, **NC**, **M**, **SC**, and **SS** are sent.

we write $\lambda_{t'}$. The limiting curves live in a moduli space of pairs (u, u') where the points of **SS** are perfectly matched with **SS**$'$ and **NN** are perfectly matched to **NN**$'$, and the rest of the marked points are matched to $\lambda_{t'}$. For fixed t', this moduli space has expected dimension

$$\mu(\phi) + \mu(\phi') - 2|\mathbf{SS}| - 2|\mathbf{NN}| - 2|\mathbf{SC}| - 2|\mathbf{NC}| - |\mathbf{M}| - |\mathbf{M}'|.$$

By assumption, the above quantity is $-|\mathbf{M}| - |\mathbf{M}'|$. The moduli space which is parametrized over all t' has dimension $-|\mathbf{M}| - |\mathbf{M}'| + 1$, but since it also has a free $\mathbb{R}$ action, we conclude that $-|\mathbf{M}| - |\mathbf{M}'| \geq 0$, so $|\mathbf{M}| = |\mathbf{M}'| = 0$.

There are two remaining cases for the curves appearing as $t_1 - t_0 \to 0$:

($H_\wedge$-1) $|\mathbf{SC}| = |\mathbf{NC}| = |\mathbf{SC}'| = |\mathbf{NC}'| = 0$.

($H_\wedge$-2) At least one of $|\mathbf{SC}|$, $|\mathbf{NC}|$, $|\mathbf{SC}'|$, $|\mathbf{NC}'|$ is non-empty.

For the ends ($H_\wedge$-1), there are no marked points along the special line and the limiting curve pair (u, u') is perfectly matched, similar to the curves counted by $\partial_{J \wedge J'}$ in equation (11.1), except (u, u') has index 0 (i.e., the expected dimension, ignoring the special line, is 0). Using transversality and expected dimension counts, we see that u and u' must both represent the constant class. These ends contribute $\mathrm{id}_{\mathbf{CF}^-_{J\wedge J'}(\mathcal{H},\mathcal{H}')}$ to the left equation of (11.4).

We now consider the ends labeled ($H_\wedge$-2). We claim that these appear in canceling pairs. Indeed, each such end appears with even multiplicity, corresponding to switching the roles of **NC** and **SC**, and switching **NC**$'$ and **SC**$'$ (i.e., having the special lines pass through each other). See Figure 11.5.

The construction of the homotopy $H_\cup$ is defined via the obvious modification, and a similar analysis goes through to establish the right equation of (11.4). ■

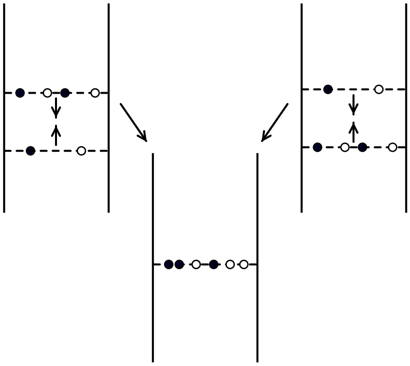

Figure 11.5. The cancellation of the ends labeled ($H_\wedge$-2).

11.3 Connected sums and hypercubes

We now extend the work of the previous section to handle the more general case of hypercubes of dimension greater than 0. We focus on the case that $\mathscr{L}_\alpha$ and $\mathscr{L}_{\alpha'}$ are 0-dimensional, i.e., $\mathscr{L}_\alpha = \boldsymbol{\alpha}$ and $\mathscr{L}_{\alpha'} = \boldsymbol{\alpha}'$, while $\mathscr{L}_\beta$ and $\mathscr{L}_{\beta'}$ are of arbitrary dimension.

Similar to the setting of ordinary Floer complexes in Section 11.2, it is straightforward to construct a neck-stretching homotopy equivalence

$$\mathbf{CF}^-_J(\Sigma\#\Sigma', \boldsymbol{\alpha}\cup\boldsymbol{\alpha}', \mathscr{L}_\beta\otimes\mathscr{L}_{\beta'}) \simeq \mathbf{CF}^-_{J\wedge J'}(\Sigma\wedge\Sigma', \boldsymbol{\alpha}\cup\boldsymbol{\alpha}', \mathscr{L}_\beta\otimes\mathscr{L}_{\beta'}).$$

The construction is essentially the same as the construction of a homotopy equivalence of hypercubes for changing the almost complex structure. Compare [12, Section 14.2].

The goal of this section is to describe a homotopy equivalence of hypercubes

$$\Phi^{\bullet<\circ'}\colon \mathbf{CF}^-_{J\wedge J'}(\Sigma\wedge\Sigma', \boldsymbol{\alpha}\cup\boldsymbol{\alpha}', \mathscr{L}_\beta\otimes\mathscr{L}_{\beta'}) \to \mathbf{CF}^-_{J\sqcup J'}(\Sigma\sqcup\Sigma', \boldsymbol{\alpha}\cup\boldsymbol{\alpha}', \mathscr{L}_\beta\otimes\mathscr{L}_{\beta'})$$

under certain restrictions on $\mathscr{L}_\beta$ and $\mathscr{L}_{\beta'}$. In the domain of $\Phi^{\bullet<\circ'}$, the curves counted are perfectly-matched. In codomain, the curves have trivial matching (i.e., they are counted in the normal way for disconnected Heegaard surfaces, as in Section 10.2). The map $\Phi^{\bullet<\circ'}$ is an extension of the map defined in Section 11.2, which we describe momentarily. Note that by Proposition 10.1, the codomain of $\Phi^{\bullet<\circ'}$ is homotopy equivalent to the tensor product of $\mathbf{CF}^-_J(\Sigma, \boldsymbol{\alpha}, \mathscr{L}_\beta)$ and $\mathbf{CF}^-_{J'}(\Sigma', \boldsymbol{\alpha}', \mathscr{L}_{\beta'})$, if the hypercube $\mathscr{L}_\beta\otimes\mathscr{L}_{\beta'}$ is constructed with sufficiently small Hamiltonian translations.

Similar to the case of connected sums, we will define the map $\Phi^{\bullet<\circ'}$ to count holomorphic polygons with a special line.

To simplify the exposition and notation, it is helpful to establish a few conventions. Firstly, we recall that the holomorphic ℓ-gon maps require a choice of a family of almost complex structures $(J_y)_{y\in K_{\ell-1}}$. The space $K_{\ell-1}$ may be identified with the moduli space of disks with ℓ marked points along its boundary. For the sake of notation, it is more convenient to consider the moduli space of $\ell-2$ marked points along the line

$$\{0\}\times\mathbb{R}\subseteq[0,1]\times\mathbb{R}.$$

Overall translation by the $\mathbb{R}$-action gives equivalent elements in the moduli space. For our purposes, it is actually more convenient to not initially quotient by this action, and consider a family of almost complex structures $(J_{y,\tau})_{(y,\tau)\in K_{\ell-1}\times\mathbb{R}}$. Let T_r denote translation of the $\mathbb{R}$ component of $\Sigma\times[0,1]\times\mathbb{R}$ by r units. We require that

$$J_{y,\tau} = (T_r^*)(J_{y,\tau+r}),$$

for each $y\in K_{\ell-1}$, and $\tau, r\in\mathbb{R}$. Note that this does *not* imply that each $J_{y,\tau}$ is translation invariant under the $\mathbb{R}$-action on $[0,1]\times\mathbb{R}$.

With this convention, the normal ℓ-gon counting maps can be viewed as counting the 1-dimensional parametrized moduli spaces

$$\bigcup_{(y,\tau)\in K_{\ell-1}\times\mathbb{R}} \mathcal{M}_{J_y}(\psi)\times\{(y,\tau)\},$$

modulo their free $\mathbb{R}$-action.

If $t_0\in\mathbb{R}$, the notion of a $(\Phi,\bullet<\circ',t_0)$-matched pair of $(\ell+1)$-gons is easily adapted from Definition 11.13. (Note that to define the map $\Phi^{\bullet<\circ'}$, the indexing is somewhat more natural if we count $(\ell+1)$-gons than ℓ-gons.) Here, t_0 denotes the height of the special line, where the marked points $\mathbf{C}$ and $\mathbf{C}'$ are matched. If (ψ,ψ',M) is a triple consisting of two homology classes of $(\ell+1)$-gons, and M is a pair of decorated source curves and matching data, and $(y,\tau)\in K_\ell\times\mathbb{R}$ and $t_0\in\mathbb{R}$ is fixed, then the expected dimension of the moduli space of $(\Phi,\bullet<\circ',t_0)$-matched $(\ell+1)$-gons, denoted $\mathcal{M}\mathcal{M}^{\Phi,\bullet<\circ'}_{J_{y,\tau}}(\psi,\psi',M,t_0)$, is given by

$$\operatorname{ind}(\psi,\psi',M)=\mu(\psi)+\mu(\psi')-2|\mathbf{S}|-2|\mathbf{C}|.$$

This is because the moduli space with $J_{y,\tau}$ fixed but with no puncture constraints has expected dimension $\mu(\psi)+\mu(\psi')$, and the perfect matching below t_0 and the matching along the special line give a codimension $2(|\mathbf{S}|+|\mathbf{C}|)$ constraint.

If $J=(J_{(y,\tau)})_{(y,\tau)\in K_\ell\times\mathbb{R}}$ is a family, we define the parametrized moduli space

$$\mathcal{M}\mathcal{M}^{\Phi,\bullet<\circ'}_{J}(\psi,\psi',M)=\bigcup_{\substack{(y,\tau)\in K_\ell\times\mathbb{R},\\ t_0\in\mathbb{R}}} \mathcal{M}\mathcal{M}^{\Phi,\bullet<\circ'}_{J_{(y,\tau)}}(\psi,\psi',M,t_0)\times\{(y,\tau,t_0)\},$$

which has expected dimension

$$e\dim \mathcal{M}\mathcal{M}^{\Phi,\bullet<\circ'}_{J}(\psi,\psi',M)=\operatorname{ind}(\psi,\psi',M)+\ell, \tag{11.5}$$

and has a free $\mathbb{R}$-action.

We now define our hypercube morphism $\Phi^{\bullet<\circ'}$. If $\kappa\le\kappa'$ and $\kappa,\kappa'\in\mathbb{E}_{n+m}$, we define the component $\Phi^{\bullet<\circ'}_{\kappa\to\kappa'}$ of $\Phi^{\bullet<\circ'}$ as follows. If $\kappa=\kappa'=(\varepsilon,\nu)$, then the map

$$\Phi^{\bullet<\circ'}_{\kappa\to\kappa'}\colon \mathbf{CF}^-_{J\wedge J'}(\Sigma\wedge\Sigma',\boldsymbol{\alpha}\cup\boldsymbol{\alpha}',\boldsymbol{\beta}_\varepsilon\cup\boldsymbol{\beta}'_\nu)\to\mathbf{CF}^-_{J\sqcup J'}(\Sigma\sqcup\Sigma',\boldsymbol{\alpha}\cup\boldsymbol{\alpha}',\boldsymbol{\beta}_\varepsilon\cup\boldsymbol{\beta}'_\nu)$$

is the tensor product map defined in Section 11.2.

If $\kappa_1<\cdots<\kappa_\ell$ is an increasing sequence of points in $\mathbb{E}_{n+m}$, then we write $\kappa_j=(\varepsilon_j,\nu_j)$ and we define a map

$$\begin{aligned}\varphi^{\bullet<\circ'}_{\kappa_1<\cdots<\kappa_\ell}\colon &\mathbf{CF}^-_{J\wedge J'}(\Sigma\wedge\Sigma',\boldsymbol{\alpha}\cup\boldsymbol{\alpha}',\boldsymbol{\beta}_{\varepsilon_1}\cup\boldsymbol{\beta}'_{\nu_1})\\ &\to\mathbf{CF}^-_{J\sqcup J'}(\Sigma\wedge\Sigma',\boldsymbol{\alpha}\cup\boldsymbol{\alpha}',\boldsymbol{\beta}_{\varepsilon_\ell}\cup\boldsymbol{\beta}'_{\nu_\ell})\end{aligned}$$

by counting $(\Phi,\bullet<\circ',t_0)$-matched holomorphic $(\ell+1)$-gons which have $\operatorname{ind}(\psi,\psi',M)=1-\ell$, and all of whose special inputs come from $\mathcal{L}_\beta\otimes\mathcal{L}_{\beta'}$. By equation (11.5) the parametrized moduli spaces of such families have expected dimension 1, and also have a free $\mathbb{R}$-action.

If $\kappa < \kappa'$, then we define

$$\Phi^{\bullet<\circ'}_{\kappa\to\kappa'} := \sum_{\kappa=\kappa_1<\cdots<\kappa_\ell=\kappa'} \varphi_{\kappa_1<\cdots<\kappa_\ell}.$$

We may also define a map

$$\Psi^{\bullet<\circ'}\colon \mathbf{CF}^-_{J\sqcup J'}(\Sigma\sqcup\Sigma',\boldsymbol{\alpha}\cup\boldsymbol{\alpha}',\mathcal{L}_\beta\otimes\mathcal{L}_{\beta'})\to\mathbf{CF}^-_{J\wedge J'}(\Sigma\wedge\Sigma',\boldsymbol{\alpha}\cup\boldsymbol{\alpha}',\mathcal{L}_\beta\otimes\mathcal{L}_{\beta'})$$

in the opposite direction via the obvious modification.

Lemma 11.18. *Suppose that $\mathcal{L}_\beta$ and $\mathcal{L}_{\beta'}$ are hypercubes of beta attaching curves on Σ and Σ', respectively, and $x \in \Sigma$ and $x' \in \Sigma'$ are choices of connected sum points. Suppose the following are satisfied:*

1. *$\mathcal{L}_\beta$ and $\mathcal{L}_{\beta'}$ are algebraically rigid.*
2. *x is base point-esque for $\mathcal{L}_\beta$.*

The following are the generic codimension 1 degenerations in the parametrized moduli spaces of $(\Phi,\bullet<\circ',t_0)$-matched $(\ell+1)$-gon curve pairs on $(\Sigma\wedge\Sigma',\boldsymbol{\alpha}\cup\boldsymbol{\alpha}',\mathcal{L}_\beta\otimes\mathcal{L}_{\beta'})$ representing classes with matching data satisfying $\operatorname{ind}(\psi,\psi',M)=2-\ell$ (i.e., of expected dimension 1 after quotienting by the free $\mathbb{R}$-action), where the special beta inputs are from the hypercube $\mathcal{L}_\beta\otimes\mathcal{L}_{\beta'}$:

(Φ^ℓ-1) *Two paired marked points in $\mathbf{S}$ and $\mathbf{S}'$ may collide with the special line (away from the marked points $\mathbf{C}$ and $\mathbf{C}'$). None of the punctures in $\mathbf{P}\cup\mathbf{P}'$ have the same height as the special line.*

(Φ^ℓ-2) *A pair of punctures of $\mathbf{C}$ and $\mathbf{C}'$ may collide along λ_{t_0} (and the $\mathbf{P}$ and $\mathbf{P}'$ punctures have different heights than λ_{t_0}).*

(Φ^ℓ-3) *A puncture of $\mathbf{N}$ or $\mathbf{N}'$ may collide with λ_{t_0} (in the complement of $\mathbf{C}$ and $\mathbf{C}'$, and with different height than $\mathbf{P}$ and $\mathbf{P}'$).*

(Φ^ℓ-4) *A puncture of $\mathbf{C}$ may degenerate into an index 2 beta boundary degeneration or alpha boundary degeneration. The height λ_{t_0} is different from any puncture in $\mathbf{P}$ or $\mathbf{P}'$.*

(Φ^ℓ-5) *The holomorphic pair (u,u') may break into two $\mathbb{R}$-levels, exactly one of which contains the special line.*

(Φ^ℓ-6) *Degenerations may occur which involve the punctures $\mathbf{P}$ and $\mathbf{P}'$. These are constrained to the following:*

   (a) *An index 1 disk pair may bubble at two punctures of $\mathbf{P}\cup\mathbf{P}'$. If it degenerates above the special line, then the matching is trivial. If it degenerates below the special line, the matching is perfect.*

(b) *Four punctures of* $\mathbf{P} \cup \mathbf{P}'$ *collide and bubble off a pair of index* 0 *holomorphic triangles. If they form below the special line, the matching is perfect. If they occur above the special line, the matching is trivial.*

See Figure 11.6 *for a schematic of the above degenerations.*

Proof. The claim that these degenerations have codimension 1 is clear. We claim that there are no further degenerations. There are two points which require explanation:

(P-1) There are generically no degenerations involving the punctures $\mathbf{P} \cup \mathbf{P}'$ which leave a puncture along $\{0\} \times \mathbb{R}$ of the same height as the special line λ_{t_0}.

(P-2) The generic degenerations involving the punctures $\mathbf{P}$ or $\mathbf{P}'$ along $\{0\} \times \mathbb{R}$ (which necessarily occur at heights other than that of the special line by (P-1)) are constrained to those listed in (Φ^ℓ-6). In particular, they are constrained only to index 1 disk bubbling and index 0 triangle degenerations. Such curves are perfectly matched if they occur below the special line, and trivially matched if they occur above the special line. Degenerations of pairs of j-gons at these punctures are prohibited for $j > 3$.

See Figure 11.7 for a schematic of a degeneration which we claim is non-generic.

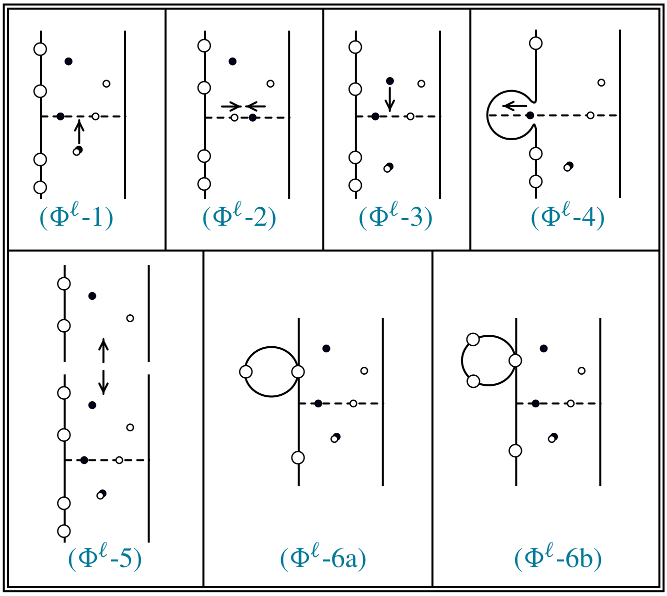

Figure 11.6. The generic degenerations of $(\Phi, \bullet < \circ', t_0)$-matched holomorphic ℓ-gon pairs. The large boundary dots denote punctures of $\mathbf{P}$ and $\mathbf{P}'$.

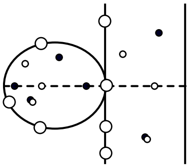

Figure 11.7. A putative example of a degeneration of type (P-1). Big dots are boundary punctures. Small dots are marked points. The left disk is identified with $\mathbb{H}$.

We now argue that degenerations (P-1) are non-generic. Consider a pair (u, u') of broken curves obtained as the limit of a 1-parameter family of $(\Phi, \bullet < \circ', t_0)$-matched $(\ell + 1)$-gons, representing a pair of classes (ψ, ψ'), all of whose special inputs are the top degree generators. Assume that each of u and u' consist of a j-gon and a k-gon for

$$j + k = \ell + 3. \tag{11.6}$$

Such a degeneration corresponds to the codimension 1 strata of K_ℓ. Our argument extends easily to further degenerations into a tree of curves, corresponding to the higher codimension strata of K_ℓ, though we focus on the codimension 1 strata to simplify the notation.

Suppose that u degenerates into a pair u_r and u_l, where u_r is a map into $\Sigma \times [0, 1] \times \mathbb{R}$, and we view u_l as a map into $\Sigma \times \mathbb{H}$ where $\mathbb{H} = [0, \infty) \times \mathbb{R}$. Here, u_r and u_l have additional boundary punctures, and we think of u_r as a holomorphic k-gon, and we think of u_l as a holomorphic j-gon. We assume that u' similarly consists of a pair u'_l and u'_r, which are holomorphic j-gons and k-gons.

We assume for the sake of the argument that $j, k \geq 2$. Note that we automatically have $k \geq 2$, since there are the punctures at $\pm\infty$. The case that $j = 1$ corresponds to a boundary degeneration. In this case, we may fill in the boundary puncture for the curves u_r and u'_r and obtain an $(\ell + 1)$-gon. We leave it to the reader to analyze the case of boundary degenerations by adapting the index arguments we present.

In the limit, write $\mathbf{N}_l, \mathbf{N}_r$, (resp. $\mathbf{C}_l, \mathbf{C}_r$; resp. $\mathbf{S}_l, \mathbf{S}_r$) for the marked points of u_l and u_r that lie above (resp. on; resp. below) the special line. Write $\mathbf{N}'_l$ (and so forth) for the analogous marked points of u'_l and u'_r. Here, we use the same designation of $\mathbf{N}$, $\mathbf{C}$ and $\mathbf{S}$ as from the 1-parameter family (so in principle, it is possible for an $\mathbf{N}$-marked point to lie on the special line, though it will follow from our argument that these degenerations occur in too high of codimension to be generic).

By construction of the 1-parameter family relevant to the degeneration, we have

$$\mu(\psi) + \mu(\psi') + \ell - 2 - 2(|\mathbf{C}| + |\mathbf{S}|) = 0. \tag{11.7}$$

For these families, the parametrized moduli space has expected dimension 2 by equation (11.5), and also has a free $\mathbb{R}$-action. We note that $|\mathbf{N}| = |\mathbf{N}_l| + |\mathbf{N}_r|$, and similarly for the other collections of marked points.

We may view the pair (u_l, u_l') as determining a map to $(\Sigma \sqcup \Sigma') \times [0,\infty) \times \mathbb{R}$. Excluding the puncture at ∞, there are $j-1$ punctures along $\{0\} \times \mathbb{R}$. We normalize the limit so that the special line coincides with $[0,\infty) \times \{0\}$. There is a 1-parameter family of conformal automorphisms of the half plane which preserve this line. They are multiplication by positive real numbers.

We claim that

$$\mu(\psi_l) + \mu(\psi_l') + j - 2 - 2|\mathbf{S}_l| - \max(|\mathbf{C}_l| + |\mathbf{C}_l'|, 1) \geq 0. \tag{11.8}$$

The reasoning for the above equation is as follows. We may view the limiting curves as being elements of a moduli space of holomorphic curves which is parametrized by the moduli space of $j-1$ marked points along $\{0\} \times \mathbb{R}$. This is a $(j-1)$-dimensional moduli space, however, there is a free action by $\mathbb{R} \times (0,\infty)$. The $\mathbb{R}$-factor translates in the i-direction, and the $(0,\infty)$-factor scales by a real positive number. If $|\mathbf{C}_l| + |\mathbf{C}_l'| = 0$, then both of these actions preserve the moduli space of curves which are parametrized over the moduli space of $j-1$ points along $\{0\} \times \mathbb{R}$, and normalized so that the special line is $\{0\} \times [0,\infty)$. This parametrized moduli space has expected dimension $\mu(\psi_l) + \mu(\psi_l') + j - 1 - 2|\mathbf{S}_l|$. Hence, $\mu(\psi_l) + \mu(\psi_l') + j - 3 - 2|\mathbf{S}_l| \geq 0$ if this moduli space is non-empty, as claimed. If $|\mathbf{C}_l| + |\mathbf{C}_l'| \geq 1$, then the $\mathbb{R}$-action which translates in the i-direction no longer preserves the matched moduli space, but the $(0,\infty)$-action does preserve the matched moduli space, so instead we only obtain

$$\mu(\psi_l) + \mu(\psi_l') + j - 2 - 2|\mathbf{S}_l| - |\mathbf{C}_l| - |\mathbf{C}_l'| \geq 0,$$

as claimed.

Similarly, (u_r, u_r') can be viewed as living in a moduli space of k-gons where the marked points $\mathbf{C}_r$ and $\mathbf{C}_r'$ lie on the line λ_{t_0}, the marked points $\mathbf{S}_r$ and $\mathbf{S}_r'$ are perfectly matched, and also that there is a boundary puncture which has the same height as the special line. Since the expected dimension of this moduli space must be nonnegative if it admits a representative, we conclude that

$$\mu(\psi_r) + \mu(\psi_r') + k - 3 - 2|\mathbf{S}_r| - |\mathbf{C}_r| - |\mathbf{C}_r'| \geq 0. \tag{11.9}$$

We may rearrange equations (11.7) and (11.6) to obtain

$$\begin{aligned} 0 &= \mu(\psi) + \mu(\psi') + \ell - 2 - 2(|\mathbf{C}| + |\mathbf{S}|) \\ &= \mu(\psi_l) + \mu(\psi_l') + j - 2 - |\mathbf{C}_l| - |\mathbf{C}_l'| - 2|\mathbf{S}_l| \\ &\quad + \mu(\psi_r) + \mu(\psi_r') + k - 3 - |\mathbf{C}_r| - |\mathbf{C}_r'| - 2|\mathbf{S}_r|. \end{aligned} \tag{11.10}$$

Next, since $\mathcal{L}_\beta$ is algebraically rigid and x is base point-esque for $\mathcal{L}_\beta$, we obtain

$$\mu(\psi_l) \geq 2n_p(\psi_l) = 2|\mathbf{C}_l| + 2|\mathbf{S}_l| + 2|\mathbf{N}_l|. \tag{11.11}$$

See Definition 11.4. In particular, we obtain that

$$\begin{aligned} &\mu(\psi_l) + \mu(\psi_l') + j - 2 - |\mathbf{C}_l| - |\mathbf{C}_l'| - 2|\mathbf{S}_l| \\ &\quad \geq \mu(\psi_l') + j - 2 + |\mathbf{C}_l| - |\mathbf{C}_l'| + 2|\mathbf{N}_l|. \end{aligned} \tag{11.12}$$

Since we are performing the $\bullet < \circ'$ degeneration, we know also that

$$|\mathbf{C}_l| \geq |\mathbf{C}_l'| \geq |\mathbf{C}_l| - 1, \tag{11.13}$$

since the left-most puncture along the special line will be in $\mathbf{C}_l$. By transversality for ordinary j-gons, we know that

$$\mu(\psi_l') + j - 2 \geq 1. \tag{11.14}$$

In particular, from equations (11.12) and (11.14) we obtain that

$$\mu(\psi_l) + \mu(\psi_l') + j - 2 - |\mathbf{C}_l| - |\mathbf{C}_l'| - 2|\mathbf{S}_l| \geq 1. \tag{11.15}$$

Equations (11.8) and (11.15) imply that equation (11.10) is the sum of two non-negative integers, one of which is at least 1. Hence the equation is never satisfiable. This implies claim (P-1).

It remains to consider degenerations into polygons where the special line does not occur at the same height as one of the punctures in $\mathbf{P} \cup \mathbf{P}'$. In such cases, $|\mathbf{C}_l| = |\mathbf{C}_l'| = 0$. Furthermore, equations (11.8) and (11.9) adapt to show that

$$\begin{aligned} \mu(\psi_l) + \mu(\psi_l') + j - 3 - 2|\mathbf{S}_l| \geq 0, \\ \mu(\psi_r) + \mu(\psi_r') + k - 2 - 2|\mathbf{S}_r| - |\mathbf{C}_r| - |\mathbf{C}_r'| \geq 0. \end{aligned} \tag{11.16}$$

Arguing similarly to equation (11.10), we see that both of the two inequalities above are equalities.

We consider first the case that (u_l, u_l') from below the special line. In this case, we observe that $\psi_l \# \psi_l'$ may be viewed as a class on $\Sigma \# \Sigma'$ since they have the same multiplicity at the connected sum point. The excision principle for the index (i.e., deleting disks at the connected sum points) implies that

$$\mu(\psi_l \# \psi_l') = \mu(\psi_l) + \mu(\psi_l') - 2|\mathbf{S}_l|.$$

By the absolute grading formula, if $\mathbf{z}$ is the outgoing intersection point of $\psi_l \# \psi_l'$, we have

$$\mu(\psi_l \# \psi_l') = n_{\mathbf{w}''}(\psi_l \# \psi_l') + \mathrm{gr}(\Theta^+, \mathbf{z}),$$

since the inputs of $\psi_l \# \psi_l'$ consist of only the top degree intersection points. We conclude that $\mu(\psi_l \# \psi_l') \geq 0$. The first line of equation (11.16) (now known to be an equality) now implies that $j - 3 \leq 0$. The only possibilities are $j = 2, 3$, as claimed. Degenerations of j-gons above the special line are handled via an essentially identical argument. ■

Lemma 11.19. *Under the assumptions in Lemma 11.18, the maps $\Phi^{\bullet < \circ'}$ and $\Psi^{\bullet < \circ'}$ are chain maps of hypercubes.*

Proof. We focus on the map $\Phi^{\bullet < \circ'}$, since the analysis for $\Psi^{\bullet < \circ'}$ is nearly identical. Suppose $\kappa \leq \kappa' \in \mathbb{E}_{n+m}$, and consider the hypercube relations for $\mathrm{Cone}(\Phi^{\bullet < \circ'})$. If $\kappa = \kappa'$, then the hypercube relations follow from Lemma 11.15, so assume $\kappa < \kappa'$.

Suppose that $\kappa = \kappa_1 < \cdots < \kappa_\ell = \kappa'$. We count the ends of 1-dimensional families of $(\Phi, \bullet < \circ', t_0)$-matched $\ell + 1$-gons, which have inputs from $\mathcal{L}_\beta \otimes \mathcal{L}_{\beta'}$. The codimension 1 degenerations are analyzed in Lemma 11.18. Many of these ends have a similar cancellation pattern as in Lemma 11.15. For example, the ends (Φ^ℓ-1) and (Φ^ℓ-2) cancel modulo 0. Similarly, each curve satisfying (Φ^ℓ-3) appears twice in the boundary of the modulo space, unless the limiting marked points alternate between Σ and Σ', in which case the curves cancel the degenerations in (Φ^ℓ-4). The curves appearing in (Φ^ℓ-6) cancel modulo 2 because of the hypercube relations for $\mathcal{L}_\beta \otimes \mathcal{L}_{\beta'}$ (on both $\Sigma \sqcup \Sigma'$ and $\Sigma \wedge \Sigma'$).

The remaining curves are those which appear in the degenerations labeled (Φ^ℓ-5). These ends correspond exactly to the hypercube relations for the mapping cone of $\Phi^{\bullet < \circ'}$. This verifies that $\Phi^{\bullet < \circ'}$ is a chain map. ■

Remark 11.20. In the above, we assumed that $\mathcal{L}_\alpha \otimes \mathcal{L}_{\alpha'}$ was 0-dimensional. In the case that both $\mathcal{L}_\alpha \otimes \mathcal{L}_{\alpha'}$ and $\mathcal{L}_\beta \otimes \mathcal{L}_{\beta'}$ have positive dimension, the argument is essentially the same. The only additional subtlety concerns for which hypercubes x and x' are base point-esque for. In the above, we used that x was base point-esque for $\mathcal{L}_\beta$ to obtain equations (11.11) and (11.13) for degenerations along $\{0\} \times \mathbb{R}$. To obtain the analogous bounds for degenerations along $\{1\} \times \mathbb{R}$ we instead would need p' to be base point-esque for $\mathcal{L}_{\alpha'}$, as in the statement of Proposition 11.8.

Lemma 11.21. *Under the assumptions in Lemma 11.18, the maps $\Phi^{\bullet < \circ'}$ and $\Psi^{\bullet < \circ'}$ are homotopy inverses of each other.*

Proof. The proof is similar to the proof of Lemma 11.16. Indeed, one can define $(H_\wedge, t_0)$-matched holomorphic $(\ell + 1)$-gons by analogy to the case of disks. Counting such curves gives a morphism of hypercubes

$$H_\wedge \colon \mathbf{CF}^-(\Sigma \wedge \Sigma', \boldsymbol{\alpha} \cup \boldsymbol{\alpha}', \mathcal{L}_\beta \otimes \mathcal{L}_{\beta'}) \to \mathbf{CF}^-(\Sigma \wedge \Sigma', \boldsymbol{\alpha} \cup \boldsymbol{\alpha}', \mathcal{L}_\beta \otimes \mathcal{L}_{\beta'}).$$

By a straightforward adaptation of Lemma 11.18 to the case where there are two special lines, and by adapting the argument from Lemma 11.16 to handle the limiting curves appearing as the special lines collide, we obtain exactly that the following diagram is an $(n+m+2)$-dimensional hypercube of chain complexes:

$$\begin{tikzcd}
\mathbf{CF}^{-}(\Sigma \wedge \Sigma', \boldsymbol{\alpha} \cup \boldsymbol{\alpha}', \mathcal{L}_{\beta} \otimes \mathcal{L}_{\beta'}) \arrow[r, "\Phi^{\bullet<\circ'}"] \arrow[d, "\mathrm{id}"] \arrow[dr, dashed, "H_{\wedge}"] & \mathbf{CF}^{-}(\Sigma \sqcup \Sigma', \boldsymbol{\alpha} \cup \boldsymbol{\alpha}', \mathcal{L}_{\beta} \otimes \mathcal{L}_{\beta'}) \arrow[d, "\Psi^{\bullet<\circ'}"] \\
\mathbf{CF}^{-}(\Sigma \wedge \Sigma', \boldsymbol{\alpha} \cup \boldsymbol{\alpha}', \mathcal{L}_{\beta} \otimes \mathcal{L}_{\beta'}) \arrow[r, "\mathrm{id}"] & \mathbf{CF}^{-}(\Sigma \wedge \Sigma', \boldsymbol{\alpha} \cup \boldsymbol{\alpha}', \mathcal{L}_{\beta} \otimes \mathcal{L}_{\beta'})
\end{tikzcd}$$

The above diagram realizes the relation

$$\Psi^{\bullet<\circ'} \circ \Phi^{\bullet<\circ'} + \mathrm{id} = [\partial, H_{\wedge}],$$

completing the proof. A homotopy equivalence in the other direction is defined by the obvious adaptation. ∎