## Chapter 13

# Arc systems and the link surgery formula

For any system of arcs $\mathcal{A}$ for a link $L$, the construction of Manolescu and Ozsváth gives a hypercube of a chain complex $\mathcal{C}_\Lambda(L, \mathcal{A})$, which is a module over $\mathbb{F}[[U_1, \dots, U_\ell]]$. When $\mathcal{A}$ consists only of beta-parallel arcs, their link surgery formula states that

$$H_*(\mathcal{C}_\Lambda(L, \mathcal{A})) \cong \mathbf{HF}^-(Y_\Lambda(L)).$$

When $\mathcal{A}$ consists only of arcs which are alpha-parallel or beta-parallel, their proof also adapts without substantial change to show the above isomorphism. However, for a general set of arcs, their work does not prove the isomorphism. In this chapter, we prove the following.

**Theorem 13.1.** *Suppose that $\mathcal{A}$ and $\mathcal{A}'$ are two systems of arcs for a framed link $L \subseteq S^3$. Suppose that $\mathcal{A}$ differs from $\mathcal{A}'$ by changing only the arc for the component $K_1 \subseteq L$. Then there is a homotopy equivalence of hypercubes*

$$\mathcal{C}_\Lambda(L, \mathcal{A}) \simeq \mathcal{C}_\Lambda(L, \mathcal{A}')$$

*which is equivariant over $\mathbb{F}[[U_2, \dots, U_\ell]]$.*

We note that our argument does not imply that the type-D modules $\mathcal{X}_\Lambda(L, \mathcal{A})^{\mathcal{L}}$ are homotopy equivalent for all choices of $\mathcal{A}$. Instead, our proof gives the following statement about the bordered link modules.

**Proposition 13.2.** *Let $\mathcal{A}$ and $\mathcal{A}'$ be two systems of arcs for $L \subseteq S^3$, and suppose that $\mathcal{A}$ differs from $\mathcal{A}'$ by changing only the arc for $K_1 \subseteq L$. Then there is a homotopy equivalence of type-*D *modules over $\mathcal{L}_{\ell-1}$:*

$$\mathcal{X}_\Lambda(L, \mathcal{A})^{\mathcal{L}_\ell} \hat{\boxtimes}_{\mathcal{K}_1} \mathcal{D}_0 \simeq \mathcal{X}_\Lambda(L, \mathcal{A}')^{\mathcal{L}_\ell} \hat{\boxtimes}_{\mathcal{K}_1} \mathcal{D}_0.$$

*In the above, $\mathcal{K}_1$ denotes the algebra factor corresponding to the component $K_1 \subseteq L$.*

## 13.1 Basepoint moving maps on hypercubes

Suppose that $\mathcal{L}_\alpha$ and $\mathcal{L}_\beta$ are two hypercubes of handleslide equivalent attaching curves on $(\Sigma, \mathbf{w}, \mathbf{z}, \mathbf{p})$. We suppose that $\gamma \subseteq \Sigma$ is an immersed loop, starting and ending at a free base point $p$. We now describe an endomorphism $\gamma_*$ of $\mathbf{CF}^-(\mathcal{L}_\alpha, \mathcal{L}_\beta)$.

We begin by picking a diffeomorphism of $\Sigma$, which corresponds to an isotopy which moves $p$ in a loop along $\gamma$, and is the identity outside of a small neighborhood of $\gamma$.

There is a canonical map

$$\phi_*\colon \mathbf{CF}^-(\mathcal{L}_\alpha, \mathcal{L}_\beta) \to \mathbf{CF}^-(\phi\mathcal{L}_\alpha, \phi\mathcal{L}_\beta)$$

obtained by pushing forward an intersection point tautologically under the diffeomorphism $\phi$, and extending equivariantly over $\mathcal{U}_i$, $\mathcal{V}_i$ and $U_j$. We now build a hyperbox of handleslide equivalent attaching curves

$$\phi\mathcal{L}_\alpha \longrightarrow \mathcal{L}'_\alpha \longrightarrow \mathcal{L}_\alpha. \tag{13.1}$$

We pick $\mathcal{L}'_\alpha$ by winding the curves of $\mathcal{L}_\alpha$ sufficiently so that $(\phi\mathcal{L}_\alpha, \mathcal{L}'_\alpha)$ and $(\mathcal{L}'_\alpha, \mathcal{L}_\alpha)$ are both weakly admissible. We use the standard filling procedure of Manolescu and Ozsváth [32, Lemma 8.6] to fill in the remaining chains of the above hyperbox.

By pairing the hyperbox in equation (13.1) with $\phi_*\mathcal{L}_\beta$ and compressing, we obtain a map

$$\Psi^{\phi\mathcal{L}_\beta}_{\phi\mathcal{L}_\alpha\to\mathcal{L}_\alpha}\colon \mathbf{CF}^-(\phi\mathcal{L}_\alpha, \phi\mathcal{L}_\beta) \to \mathbf{CF}^-(\mathcal{L}_\alpha, \phi\mathcal{L}_\beta).$$

A map $\Psi^{\phi\mathcal{L}_\beta\to\mathcal{L}_\beta}_{\mathcal{L}_\alpha}\colon \mathbf{CF}^-(\mathcal{L}_\alpha, \phi\mathcal{L}_\beta) \to \mathbf{CF}^-(\mathcal{L}_\alpha, \mathcal{L}_\beta)$ is defined similarly. We define the map $\gamma_*$ as a composition

$$\gamma_* := \Psi^{\phi\mathcal{L}_\beta\to\mathcal{L}_\beta}_{\mathcal{L}_\alpha} \circ \Psi^{\phi\mathcal{L}_\beta}_{\phi\mathcal{L}_\alpha\to\mathcal{L}_\alpha} \circ \phi_*.$$

It is not hard to see that the map $\gamma_*$ is well defined up to chain homotopies of hypercube morphisms by repeating the same procedure to build hypercubes of higher dimension.

**Remark 13.3.** The above construction may be packaged into a form of naturality for hypercubes. If $\mathcal{H} = (\mathcal{L}_\alpha, \mathcal{L}_\beta)$ and $\mathcal{H}' = (\mathcal{L}_{\alpha'}, \mathcal{L}_{\beta'})$ are two hypercubes of handleslide equivalent attaching curves, such that each set of curves in $\mathcal{L}_\alpha$ is handleslide equivalent to each set of curves in $\mathcal{L}_{\alpha'}$ and $\dim(\mathcal{L}_\alpha) = \dim(\mathcal{L}_{\alpha'})$, and similarly for $\mathcal{L}_\beta$ and $\mathcal{L}_{\beta'}$, then the above construction produces a map $\Psi_{\mathcal{H}\to\mathcal{H}'}\colon \mathbf{CF}^-(\mathcal{H}) \to \mathbf{CF}^-(\mathcal{H}')$. Furthermore,

$$\Psi_{\mathcal{H}'\to\mathcal{H}} \circ \Psi_{\mathcal{H}\to\mathcal{H}'} \simeq \mathrm{id}_{\mathbf{CF}^-(\mathcal{H})}. \tag{13.2}$$

Equation (13.2) is proven by using associativity to reduce to the case that $\mathcal{H}'$ consist of small translates of $\mathcal{L}_\alpha$ and $\mathcal{L}_\beta$, and then by either adapting Lipshitz's proof in the case of triangles [23] or by using the small translate theorems for holomorphic polygons in Lemma 10.3.

We now state our formula for the map $\gamma_*$. The statement is formally similar to the case of the ordinary Floer complexes [58, Theorem D]. Our formula involves two endomorphisms $\Phi_w$ and $\mathcal{A}_\gamma$ which are defined in Sections 13.3 and 13.2.

**Theorem 13.4.** *Suppose that $\mathscr{L}_\alpha$ and $\mathscr{L}_\beta$ are hypercubes of handleslide equivalent attaching curves on $(\Sigma, \mathbf{w}, \mathbf{z}, \mathbf{p})$. Let $p \in \mathbf{p}$ be a free base point and let $\gamma$ be a closed path on $\Sigma$, which is based at $p$. If $\gamma_*$ is the associated base point moving hypercube morphism, then*

$$\gamma_* \simeq \mathrm{id} + \Phi_p \circ \mathscr{A}_\gamma$$

*as morphisms of hypercubes.*

The proof of the above result will be given in Section 13.6. As a first application, we can compute the effect of changing one arc in a system of arcs for a link $L \subseteq S^3$.

**Corollary 13.5.** *Suppose that $L \subseteq S^3$ is a framed link and that $\mathcal{A}$ is a system of arcs for $L$. Let $\mathcal{H}$ be a $\sigma$-basic system of Heegaard diagrams for $(L, \mathcal{A})$. Let $\gamma$ be an embedded closed loop in $S^3$ which is based at some $z_i \in K_i \subseteq L$ but is otherwise disjoint from $L$ and $\mathcal{A}$. Assume $\gamma$ is contained in the Heegaard surface for $\Sigma$. Let $\mathcal{A}'$ be the system of arcs obtained by concatenating the arc for $K_i$ with $\gamma$. Let us view $\mathcal{C}_\Lambda(\mathcal{H})$ as*

$$\mathcal{C}_\Lambda(\mathcal{H}) \cong \mathrm{Cone}\Big(\mathcal{C}_0(\mathcal{H}) \xrightarrow{F^{K_i}+F^{-K_i}} \mathcal{C}_1(\mathcal{H})\Big), \tag{13.3}$$

*where $\mathcal{C}_i$ is a $|L|-1$-dimensional hypercube and $F^{-K_i}$ denotes the sum of all hypercube maps for oriented sublinks of $L$ which contain $-K_i$, and similarly for $F^{K_i}$. Then there is a $\sigma$-basic system of Heegaard diagrams $\mathcal{H}'$ for $(L, \mathcal{A}')$ such that*

$$\mathcal{C}_\Lambda(\mathcal{H}') \simeq \mathrm{Cone}\Big(\mathcal{C}_0(\mathcal{H}) \xrightarrow{F^{K_i}+(\mathrm{id}+\mathscr{V}_i^{-1}\mathscr{A}_\gamma\circ\Phi_{w_i})F^{-K_i}} \mathcal{C}_1(\mathcal{H})\Big), \tag{13.4}$$

*where $\mathscr{A}_\gamma$ and $\Phi_{w_i}$ are viewed as endomorphisms of $\mathcal{C}_1(\mathcal{H})$.*

We now sketch the proof of Theorem 13.1 using Corollary 13.5 and a few other basic results from later in this chapter.

*Proof of Theorem* 13.1. The map $\Phi_{w_i}$ will be introduced below in more detail in Section 13.3. According to Remark 13.11, below, the map $\Phi_{w_i}$ is null-homotopic as a hypercube endomorphism via the homotopy

$$\Phi_{w_i} = \partial_{\mathrm{Mor}}(\partial_{\mathscr{U}_i}) = \partial \circ \partial_{\mathscr{U}_i} + \partial_{\mathscr{U}_i} \circ \partial.$$

Here, $\partial$ denotes the hypercube differential on $\mathcal{C}_1(\mathcal{H})$, and $\partial_{\mathscr{U}_i}$ is the derivative with respect to $\mathscr{U}_i$ (i.e., $\partial_{\mathscr{U}_i}(\mathscr{U}_i^n \mathbf{x}) = n\mathscr{U}_i^{n-1}\mathbf{x}$), when $\mathbf{x}$ is a generator which has no $\mathscr{U}_i$ powers. See Remark 13.11 for more details on this null-homotopy.

The null-homotopy $\partial_{\mathscr{U}_i}$ of $\Phi_{w_i}$ gives a chain homotopy equivalence between the hypercubes in equations (13.3) and (13.4). The map $\partial_{\mathscr{U}_i}$ does not commute with $\mathscr{U}_i$ or $U_i = \mathscr{U}_i\mathscr{V}_i$, however, it will commute with $\mathscr{U}_j$ and $\mathscr{V}_j$ for $j \neq i$. In particular, the null-homotopy $\partial_{\mathscr{U}_i}$ cannot be used to induce a morphism between $\mathscr{X}_\Lambda(L, \mathcal{A})^{\mathscr{L}_\ell}$ and $\mathscr{X}_\Lambda(L, \mathcal{A}')^{\mathscr{L}_\ell}$. However, after tensoring with $\mathscr{D}_0$, the endomorphism $\mathbb{I} \otimes \partial_{\mathscr{U}_i}$ can be defined, so we may package the null-homotopy of $\Phi_{w_i}$ as a homotopy equivalence of type-D modules

$$\mathcal{X}_\Lambda(L,\mathscr{A})^{\mathcal{L}_\ell} \mathbin{\hat{\boxtimes}}_{\mathcal{K}} \mathcal{D}_0 \simeq \mathcal{X}_\Lambda(L,\mathscr{A}')^{\mathcal{L}_\ell} \mathbin{\hat{\boxtimes}}_{\mathcal{K}} \mathcal{D}_0$$

as in the statement of Proposition 13.2. ■

We now describe the proof of Corollary 13.5.

*Proof of Corollary* 13.5. One first builds the surgery hyperbox for sublinks of $-L$. We compress the direction for $K_i$. The resulting hyperbox takes the form

$$\mathrm{Cone}\big(\widetilde{\mathcal{C}}_0(\mathcal{H}) \xrightarrow{F^{-K_i}} \widetilde{\mathcal{C}}_1(\mathcal{H})\big),$$

where $\widetilde{\mathcal{C}}_j(\mathcal{H})$ compresses to $\mathcal{C}_j(\mathcal{H})$. To build the hyperbox for $\mathcal{H}'$, we stack the above hyperbox with the base point moving map hyperbox for $\gamma_*$ to form the following hyperbox:

$$\big(\widetilde{\mathcal{C}}_0(\mathcal{H}) \xrightarrow{F^{-K_i}} \widetilde{\mathcal{C}}_1(\mathcal{H}) \xrightarrow{\gamma_*} \widetilde{\mathcal{C}}_1(\mathcal{H})\big).$$

If $\widetilde{\mathcal{C}}_1(\mathcal{H})$ has dimension $n-1$ and size $\mathbf{d}$, the hyperbox for $\gamma_*$ has dimension $n$ and size $\mathbf{d}\times\{1\}$.

Theorem 13.4 implies that the above hyperbox has homotopy equivalent compression to the compression of

$$\big(\widetilde{\mathcal{C}}_0(\mathcal{H}) \xrightarrow{F^{-K_i}} \widetilde{\mathcal{C}}_1(\mathcal{H}) \xrightarrow{\mathscr{V}_i^{-1}\Phi_{w_i}} \widetilde{\mathcal{C}}_1(\mathcal{H}) \xrightarrow{\mathscr{A}_\gamma} \widetilde{\mathcal{C}}_1(\mathcal{H})\big).$$

The factor of $\mathscr{V}_i^{-1}$ in front of $\Phi_{w_i}$ is to make the map for sublinks of $L$ which contain $-K_i$ have the correct Alexander grading, since $\mathscr{A}_\gamma$ has $A_i$-Alexander grading 0 while $\Phi_{w_i}$ has $A_i$-Alexander grading 1. (Note that $\mathscr{A}_\gamma$ and $\Phi_{w_i}$ will not necessarily be homogeneous in the other Alexander gradings, since they are hypercube morphisms, and hence may increment the cube directions for other link components.) We recall that for these maps, the powers of $\mathscr{V}_i$ are determined by the overall Alexander grading change of the map. Compressing the above hyperbox gives the statement. ■

**Remark 13.6.** We observe that in the above corollary, there is some ambiguity in the definition of $\Phi_{w_i}$. In practice, we would like to compute $\Phi_{w_i}$ on the link surgery hypercube, as opposed computing $\Phi_{w_i}$ using a large hyperbox whose compression is the link surgery hypercube. It is an easy consequence of the Leibniz rule that $\Phi_{w_i}$ commutes with the compression operation, so either choice gives the same answer.

The following remark indicates some subtleties of the hypercube homology actions $\mathscr{A}_\gamma$ and the base point moving map formulas.

**Remark 13.7.** The statement of Corollary 13.5 is false if $\gamma$ is not disjoint from the curves $\mathcal{A}$ on the Heegaard diagram. An example is the genus 0 diagram for the Hopf link in Figure 16.1. In this case, composing with the base point moving map for moving one $w_i$ around one of the components will switch between two models of the Hopf link complex (for different arc systems) which we show are non-isomorphic type-D modules in Chapter 16. On the other hand, any loop on this Heegaard diagram bounds a disk since the surface is $S^3$, so Lemma 13.13 can be used to show that the homology action is null-homotopic. In this case, the hypotheses are not satisfied, since the curve $\gamma = K_2$ is not disjoint from the arc $\mathcal{A}_1$ for $K_1$.

## 13.2 Basepoint actions on hypercubes

In this section, we define the base point action on hypercubes. The base point actions are analogs of natural endomorphisms which appear on the ordinary Heegaard Floer complexes, especially in the context of base point moving maps and diffeomorphism maps. See [49, 53, 58], for example.

Suppose that $\mathcal{L}_\alpha$ and $\mathcal{L}_\beta$ are hypercubes of attaching curves on $(\Sigma, \mathbf{w}, \mathbf{z}, \mathbf{p})$. Here and throughout, we assume $\mathbf{w}$ and $\mathbf{z}$ are link base points and $\mathbf{p}$ are free base points. In this section, we will consider the Floer complex $\mathbf{CF}^-(\mathcal{L}_\alpha, \mathcal{L}_\beta)$ which is a free module over $\mathbb{F}[\mathscr{U}_1, \dots, \mathscr{U}_\ell, \mathscr{V}_1, \dots, \mathscr{V}_\ell, U_1, \dots, U_t]$, where $\mathscr{U}_j$ is the variable for $w_j \in \mathbf{w}$, $\mathscr{V}_j$ is the variable for $z_j \in \mathbf{z}$, and $U_j$ is the variable for $p_j \in \mathbf{p}$.

If $w_i \in \mathbf{w}$, we now describe an endomorphism

$$\Phi_{w_i} \colon \mathbf{CF}^-(\mathcal{L}_\alpha, \mathcal{L}_\beta) \to \mathbf{CF}^-(\mathcal{L}_\alpha, \mathcal{L}_\beta).$$

If $z_i \in \mathbf{z}$, or $p_i \in \mathbf{p}$, there will be analogous endomorphisms, denoted $\Psi_{z_i}$ and $\Phi_{p_i}$, all defined by essentially the same construction.

We define the map $\Phi_{w_i}$ by formally differentiating the hypercube differential $\partial$ on $\mathbf{CF}^-(\mathcal{L}_\alpha, \mathcal{L}_\beta)$ with respect to $\mathscr{U}_i$. More precisely, if $\mathbf{x}$ and $\mathbf{y}$ are intersection points $\mathbf{CF}^-(\mathcal{L}_\alpha, \mathcal{L}_\beta)$ and $\partial(\mathbf{x})$ has a summand of $\mathbf{a} \cdot \mathbf{y}$ for some $\mathbf{a} \in \mathbb{F}[\mathscr{U}_1, \mathscr{V}_1, \dots, \mathscr{U}_\ell, \mathscr{V}_\ell, U_1, \dots, U_t]$, then we define $\Phi_{w_i}(\mathbf{x})$ to have a summand of $\partial_{\mathscr{U}_i}(\mathbf{a}) \cdot \mathbf{y}$. We extend $\Phi_{w_i}$ to all of $\mathbf{CF}^-(\mathcal{L}_\alpha, \mathcal{L}_\beta)$ by declaring it to be $\mathbb{F}[\mathscr{U}_1, \mathscr{V}_1, \dots, \mathscr{U}_\ell, \mathscr{V}_\ell, U_1, \dots, U_t]$-equivariant.

**Lemma 13.8.** *The map $\Phi_{w_i}$ satisfies $\partial_{\mathrm{Mor}}(\Phi_{w_i}) = 0$, where $\partial_{\mathrm{Mor}}$ is the morphism differential for hypercube morphisms (see equation (2.3)).*

*Proof.* The proof is the same as in the setting of the ordinary Floer chain complexes. See, e.g., [53, Lemma 3.1]. This is obtained by taking the equation $\partial^2 = 0$, and differentiating with respect to $\mathscr{U}_i$. The Leibniz rule yields $\Phi_{w_i} \circ \partial + \partial \circ \Phi_{w_i} = 0$. ∎

**Remark 13.9.** In Lemma 13.8, we are implicitly using the fact that the differential commutes with the action of $\mathscr{U}_i$. We note that in the link surgery formula $\mathcal{C}_\Lambda(L)$, although the group is naturally a completion of a free-module over $\mathbb{F}[\mathscr{U}_i]$, the differential does not commute with $\mathscr{U}_i$ (cf. Lemma 6.7). However, if $w \in K$ and we restrict to a subcube of $\mathcal{C}_\Lambda(L)$ where the coordinate for $K$ is constant, then the differential does commute with $\mathscr{U}_i$ and the map $\Phi_{w_i}$ can be defined. This is the only case that we consider in the present memoir.

When $\mathcal{L}_\alpha$ and $\mathcal{L}_\beta$ are algebraically rigid, there is an alternate definition of $\Phi_{w_i}$, for which we write $\Phi^0_w$. The map $\Phi^0_{w_i}$ is defined as follows. Suppose that $\mathcal{L}_\alpha$ is $m$-dimensional and $\mathcal{L}_\beta$ is $n$-dimensional. Given $\varepsilon_1 < \cdots < \varepsilon_j$ in $\mathbb{E}_m$ and $\nu_1 < \cdots < \nu_k$ in $\mathbb{E}_n$, we define the *$w_i$-weighted polygon counting map*, $f^{w_i}_{\alpha_{\varepsilon_j},\dots,\alpha_{\varepsilon_1},\beta_{\nu_1},\dots,\beta_{\nu_k}}$, to count holomorphic polygons of index $3 - j - k$ with a multiplicative weight of

$$n_{w_i}(\psi)\mathscr{U}_1^{n_{w_1}(\psi)}\mathscr{V}_1^{n_{z_1}(\psi)}\cdots\mathscr{U}_\ell^{n_{w_\ell}(\psi)}\mathscr{V}_\ell^{n_{w_\ell}(\psi)}U_1^{n_{p_1}(\psi)}\cdots U_t^{n_{p_t}(\psi)}.$$

We define

$$(\phi_{w_i})^{\nu_1<\cdots<\nu_k}_{\varepsilon_1<\cdots<\varepsilon_j} \colon \mathbf{CF}^-(\boldsymbol{\alpha}_{\varepsilon_1},\boldsymbol{\beta}_{\nu_1}) \to \mathbf{CF}^-(\boldsymbol{\alpha}_{\varepsilon_j},\boldsymbol{\beta}_{\nu_k}) \tag{13.5}$$

via the formula

$$(\phi_{w_i})^{\nu_1<\cdots<\nu_k}_{\varepsilon_1<\cdots<\varepsilon_j}(\mathbf{x}) := \mathscr{U}_i^{-1} f^{w_i}_{\alpha_{\varepsilon_j},\dots,\alpha_{\varepsilon_1},\beta_{\nu_1},\dots,\beta_{\nu_k}}(\Theta_{\alpha_{\varepsilon_j},\alpha_{\varepsilon_{j-1}}},\dots,\Theta_{\alpha_{\varepsilon_2},\alpha_{\varepsilon_1}},\mathbf{x},$$
$$\Theta_{\beta_{\nu_1},\beta_{\nu_2}},\dots,\Theta_{\beta_{\nu_{k-1}},\beta_{\nu_k}}).$$

We define $\Phi^0_{w_i} \colon \mathbf{CF}^-(\mathcal{L}_\alpha,\mathcal{L}_\beta) \to \mathbf{CF}^-(\mathcal{L}_\alpha,\mathcal{L}_\beta)$ via the formula

$$\Phi^0_{w_i} := \sum_{\substack{\nu_1<\cdots<\nu_k,\\ \varepsilon_1<\cdots<\varepsilon_j}} (\phi_{w_i})^{\nu_1<\cdots<\nu_k}_{\varepsilon_1<\cdots<\varepsilon_j}.$$

**Lemma 13.10.** *If $\mathcal{L}_\alpha$ and $\mathcal{L}_\beta$ are algebraically rigid hypercubes of handleslide equivalent attaching curves, then $\Phi_{w_i} = \Phi^0_{w_i}$.*

*Proof.* For algebraically rigid hypercubes of attaching curves, the only special inputs for the hypercube differential are top degree intersection points, which are uniquely determined and have no $\mathscr{U}_i$-powers. In particular, the $\mathscr{U}_i$-power of an arrow in the differential of $\mathbf{CF}^-(\mathcal{L}_\alpha,\mathcal{L}_\beta)$ is determined entirely by the $n_{w_i}$ multiplicity of the corresponding holomorphic $\ell$-gon, which is exactly what the map $\Phi^0_{w_i}$ records. ■

**Remark 13.11.** Similar to the setting of the 3-manifold invariants, the map $\Phi_{w_i}$ is a null-homotopic hypercube endomorphism, though the natural homotopy does not commute with the action of $\mathscr{U}_i$. A similar comment holds for the endomorphisms $\Psi_{z_i}$ and $\Phi_{p_i}$, when $p_i$ is a free base point. As in [58, equation (14.34)], we have

$$\Phi_{w_i} = [\partial, \partial_{\mathscr{U}_i}] = \partial \circ \partial_{\mathscr{U}_i} + \partial_{\mathscr{U}_i} \circ \partial, \tag{13.6}$$

where $\partial$ denotes the hypercube differential and $\partial_{\mathscr{U}_i}$ is the map

$$\partial_{\mathscr{U}_i}(\mathscr{U}_i^k \mathbf{x}) = k\mathscr{U}_i^{k-1}\mathbf{x}$$

extended equivariantly over the actions of the other variables. Conceptually, we can think of equation (13.6) as an application of the Leibniz rule, in the following sense. If we think of $\Phi_{w_i}$ and $\partial$ as matrices with coefficients in $\mathbb{F}[\mathscr{U}_i]$ and a generator $\mathbf{x}$ as a column vector with entries in $\mathbb{F}[\mathscr{U}_i]$, then $\Phi_{w_i}(\mathbf{x})$ and $\partial(\mathbf{x})$ can be thought of as matrix products. The Leibniz rule for derivatives implies that

$$\partial_{\mathscr{U}_i}(\partial \mathbf{x}) = (\partial_{\mathscr{U}_i}(\partial))(\mathbf{x}) + \partial(\partial_{\mathscr{U}_i}(\mathbf{x})).$$

By definition $\partial_{\mathscr{U}_i}(\partial) = \Phi_{w_i}$, so rearranging the above equation yields the null-homotopy $\Phi_{w_i} = [\partial, \partial_{\mathscr{U}_i}]$.

## 13.3 Homology actions on hypercubes

We now recall the construction of various homological actions on hypercubes from [13, Section 6.2]. We will describe an action for both closed curves on $\Sigma$ and also arcs which have boundary on two base points $\{w_1, w_2\} \subseteq \mathbf{w}$.

In the case of closed curves, the construction is an adaptation of the original case [39, Section 4.2.5]. For arcs with boundary, the construction extends [36] and [58, Section 5].

We consider first the case of a closed curve $\gamma \subseteq \Sigma$. Suppose that $\mathscr{L}_\alpha$ and $\mathscr{L}_\beta$ are hypercubes of attaching curves on $(\Sigma, \mathbf{w}, \mathbf{z})$. We now describe an endomorphism

$$\mathscr{A}_\gamma \colon \mathbf{CF}^-(\mathscr{L}_\alpha, \mathscr{L}_\beta) \to \mathbf{CF}^-(\mathscr{L}_\alpha, \mathscr{L}_\beta)$$

as follows. Suppose

$$\psi \in \pi_2(\mathbf{x}_{\alpha_{\varepsilon_j},\alpha_{\varepsilon_{j-1}}}, \dots, \mathbf{x}_{\alpha_{\varepsilon_2},\alpha_{\varepsilon_1}}, \mathbf{x}, \mathbf{x}_{\beta_{\nu_1},\beta_{\nu_2}}, \dots, \mathbf{x}_{\beta_{\nu_{k-1}},\beta_{\nu_k}}, \mathbf{y}),$$

where each $\boldsymbol{\alpha}_{\varepsilon_i}$ and $\boldsymbol{\beta}_{\varepsilon_i}$ are curves in $\mathscr{L}_\alpha$ or $\mathscr{L}_\beta$, respectively, and each intersection point labeled $\mathbf{x}_{\gamma,\gamma'}$ is in $\mathbb{T}_\gamma \cap \mathbb{T}_{\gamma'}$. We define the 1-chain $\partial_\alpha(\psi) \subseteq \Sigma$ to be the boundary of $\psi$ which lies in $\boldsymbol{\alpha}_{\varepsilon_1} \cup \cdots \cup \boldsymbol{\alpha}_{\varepsilon_j}$. We define a quantity $a(\psi, \gamma) \in \mathbb{F}$ via the formula

$$a(\psi, \gamma) = \#(\partial_\alpha(\psi) \cap \gamma).$$

We now define an endomorphism

$$\mathscr{A}_\gamma \colon \mathbf{CF}^-(\mathscr{L}_\alpha, \mathscr{L}_\beta) \to \mathbf{CF}^-(\mathscr{L}_\alpha, \mathscr{L}_\beta).$$

The map counts holomorphic polygons as would be counted in the hypercube differential for $\mathbf{CF}^-(\mathscr{L}_\alpha, \mathscr{L}_\beta)$, except with an extra multiplicative weight of $a(\psi, \gamma)$.

**Lemma 13.12.** *Suppose that $\mathcal{L}_\alpha$ and $\mathcal{L}_\beta$ are hypercubes of attaching curves on $(\Sigma, \mathbf{w}, \mathbf{z}, \mathbf{p})$ and $\gamma$ is a closed curve on $\Sigma$. Then $\partial_{\mathrm{Mor}}(\mathcal{A}_\gamma) = 0$, where $\partial_{\mathrm{Mor}}$ is the morphism differential for hypercube morphisms.*

*Proof.* The proof that $\partial_{\mathrm{Mor}}(\mathcal{A}_\gamma) = 0$ follows from a Gromov compactness argument, similar to the proof of the hypercube relations for $\mathbf{CF}^-(\mathcal{L}_\alpha, \mathcal{L}_\beta)$.

Suppose that $\mathbf{x} \in \mathbb{T}_{\alpha_\varepsilon} \cap \mathbb{T}_{\beta_\nu}$ and $\mathbf{y} \in \mathbb{T}_{\alpha_{\varepsilon'}} \cap \mathbb{T}_{\beta_{\nu'}}$, for $\varepsilon \le \varepsilon'$ and $\nu \le \nu'$. We consider a class of polygons

$$\psi \in \pi_2(\mathbf{x}_{\alpha_{\varepsilon_j}, \alpha_{\varepsilon_{j-1}}}, \dots, \mathbf{x}_{\alpha_{\varepsilon_2}, \alpha_{\varepsilon_1}}, \mathbf{x}, \mathbf{x}_{\beta_{\nu_1}, \beta_{\nu_2}}, \dots, \mathbf{x}_{\beta_{\nu_{k-1}}, \beta_{\nu_k}}, \mathbf{y}),$$

where each $\mathbf{x}_{\alpha_{\varepsilon_k}, \alpha_{\varepsilon_{k-1}}}$ is an intersection point appearing as a summand of a chain from $\mathcal{L}_\alpha$, and similarly for the beta labeled intersection points. We suppose that

$$\mu(\psi) = 4 - j - k$$

so that $\mathcal{M}(\psi)$ is 1-dimensional. We consider the ends of the 1-dimensional moduli space $\mathcal{M}(\psi)$.

If $j + k > 2$, the ends of $\mathcal{M}(\psi)$ generically consist of $\psi$ breaking into a pair of holomorphic polygons (with at least 2 sides, each) representing homology classes $\psi_1, \psi_2$ satisfying $\psi = \psi_1 + \psi_2$. It is helpful to organize possible degenerations into three subtypes (up to reordering $\psi_1$ and $\psi_2$):

(e-1) $\psi_1$, $\psi_2$ both have boundary on both the alpha and beta curves.

(e-2) $\psi_1$ has boundary on both alpha and beta curves, while $\psi_2$ has boundary only on the alpha curves.

(e-3) $\psi_1$ has boundary on both alpha and beta curves, while $\psi_2$ has boundary only on the beta curves.

See Figure 13.1 for a schematic.

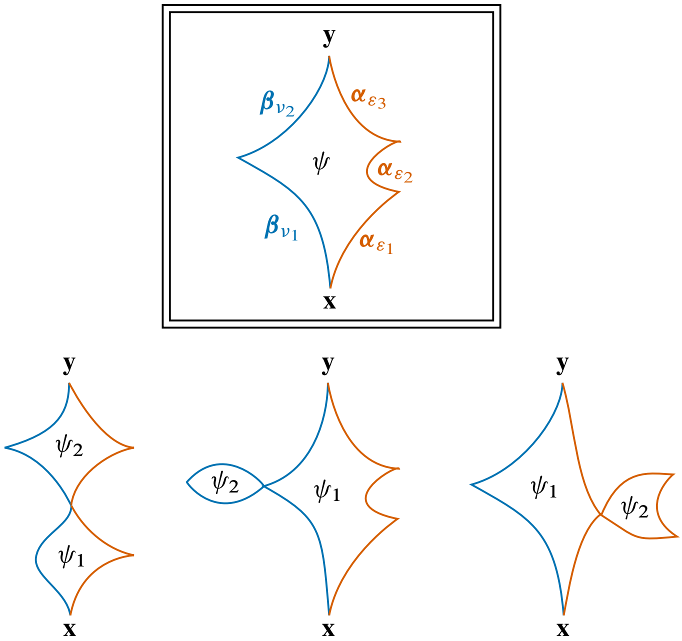
We consider the sum over all sequences of intersections points $\mathbf{x}_{\alpha_{\varepsilon_j}, \alpha_{\varepsilon_{j-1}}}, \dots, \mathbf{x}_{\alpha_{\varepsilon_2}, \alpha_{\varepsilon_1}}$ and $\mathbf{x}_{\beta_{\nu_1}, \beta_{\nu_2}}, \dots, \mathbf{x}_{\beta_{\nu_{k-1}}, \beta_{\nu_k}}$ from the corresponding chains of $\mathcal{L}_\alpha$ and $\mathcal{L}_\beta$, as well as classes $\psi$, as above, of

$$a(\psi, \gamma) \# \partial \mathcal{M}(\psi) \mathscr{U}_1^{n_{w_1}(\psi)} \mathscr{V}_1^{n_{z_1}(\psi)} \cdots \mathscr{U}_\ell^{n_{w_\ell}(\psi)} \mathscr{V}_\ell^{n_{z_\ell}(\psi)} U_1^{n_{p_1}(\psi)} \cdots U_t^{n_{p_t}(\psi)}.$$

(We also multiply by any variables $\mathscr{U}_i$ or $\mathscr{V}_i$ appearing in the corresponding chains of $\mathcal{L}_\alpha$ or $\mathcal{L}_\beta$, but we omit this from the above equation.) We observe that if $\psi = \psi_1 + \psi_2$, as above, then

$$a(\psi, \gamma) = a(\psi_1, \gamma) + a(\psi_2, \gamma).$$

Therefore, when summing over the ends of $\# \partial \mathcal{M}(\psi)$, instead of summing each end with weight $a(\psi, \gamma)$, we can instead sum each end twice, once with weight $a(\psi_1, \gamma)$ and once with weight $a(\psi_2, \gamma)$.



(e-1) (e-2) (e-3)

**Figure 13.1.** The degenerations (e-1), (e-2), and (e-3) from Lemma 13.12.

Ends (e-1), summed as above, contribute the $\mathbf{y}$ coefficient of

$$(\partial \circ \mathscr{A}_\gamma + \mathscr{A}_\gamma \circ \partial)(\mathbf{x}).$$

When weighted by $a(\psi_1, \gamma)$, we claim that the ends (e-2) and (e-3) cancel modulo 2 by the hypercube structure relation for $\mathcal{L}_\alpha$ and $\mathcal{L}_\beta$.

We will show that when weighted by $a(\psi_2, \gamma)$, the ends (e-2) and (e-3) cancel modulo 2. To establish this, we claim more generally that if $\psi_2$ is a class of polygons on a subdiagram of $(\Sigma, \mathcal{L}_\alpha)$ or $(\Sigma, \mathcal{L}_\beta)$, then

$$a(\psi_2, \gamma) = 0, \tag{13.7}$$

which clearly will imply the claim. If $\psi_2$ is a class of polygons on $(\Sigma, \mathcal{L}_\beta)$ this claim is trivial since the alpha boundary of $\psi_2$ is trivial. If $\psi_2$ is instead a class on a subdiagram of $(\Sigma, \mathcal{L}_\alpha)$, equation (13.7) follows from the fact that $\partial_\alpha(\psi_2) = \partial(D(\psi_2))$, where we are writing $D(\psi_2)$ for the *domain* of the class $\psi_2$. We recall that $D(\psi_2)$ is the formal integral combination of the components of $\Sigma \setminus \bigcup_{\boldsymbol{\alpha} \in \mathcal{L}_\alpha} \boldsymbol{\alpha}$ whose multiplicity at a point $p \in \Sigma \setminus \bigcup_{\boldsymbol{\alpha} \in \mathcal{L}_\alpha} \boldsymbol{\alpha}$ coincides with $n_p(\psi_2)$. Therefore,

$$a(\psi_2, \gamma) = \#\partial_\alpha(\psi_2) \cap \gamma = \#\partial(D(\psi_2)) \cap \gamma \equiv \#D(\psi_2) \cap \partial\gamma = 0, \tag{13.8}$$

since $\partial\gamma = \emptyset$. This establishes equation (13.7).

Finally, we consider the ends of the moduli spaces $\psi$, as above, when $k = j = 1$ (i.e., the $\psi$ is a class of disks). In this case, ends of the form (e-2) and (e-3) do not appear, though ends of the form (e-1) contribute the $\mathbf{y}$ coefficient of $(\mathscr{A}_\gamma \circ \partial + \partial \circ \mathscr{A}_\gamma)(\mathbf{x})$ as before. In this case, there is an additional type of end which appears when $\mathbf{x} = \mathbf{y}$:

(e-4) A constant holomorphic strip at $\mathbf{x}$, together with a Maslov index 2 boundary degeneration.

See [41, Section 5] for more on boundary degenerations in Heegaard Floer theory. We claim that boundary degenerations make trivial algebraic contribution to the weighted sum of the ends of the moduli spaces under consideration. This follows from the fact that if $\psi_2$ is a class of boundary degenerations, then the same argument as equation (13.8) establishes that $a(\psi_2, \gamma) = 0$.

Summing all ends as above, we conclude that $\partial_{\mathrm{Mor}}(\mathscr{A}_\gamma) = 0$. ■

There is a more streamlined presentation of the proof of the above lemma. It is helpful to view $\gamma$ itself as a type of Floer morphism from a set of attaching curves $\boldsymbol{\alpha}$ to itself. We refer to $\gamma$ as a *formal endomorphism* of $\boldsymbol{\alpha}$. We can define holomorphic polygon counts with $\gamma$ as an input as follows. If $\boldsymbol{\delta}_1, \dots, \boldsymbol{\delta}_n$ are attaching curves, we define

$$f_{\delta_1,\dots,\delta_j,\delta_j,\dots,\delta_n}(\Theta_{1,2}, \dots, \Theta_{j-1,j}, \gamma, \Theta_{j,j+1}, \dots, \Theta_{n-1,n})$$

to count holomorphic $n$-gons of Maslov index $3 - n$ which are weighted by a factor or $\# \partial_{\delta_j}(\psi) \cap \gamma$. If we formally set $\partial \gamma = 0$, then it is straightforward to adapt the argument from the proof of Lemma 13.12 to see that standard associativity relations hold if we allow for a formal endomorphism as an argument.

We can extend the above formalism by defining a formal endomorphism of hypercubes of attaching curves

$$F_\gamma \colon \mathscr{L}_\alpha \to \mathscr{L}_\alpha.$$

The morphism $L_\gamma$ has only length 1 components, all of which are $\gamma$. There are no higher length components. We will think of $F_\gamma$ as a morphism of twisted complexes (see Section 2.4).

Equation (13.7) immediately implies

$$\mu_1^{\mathrm{Tw}}(F_\gamma) = 0.$$

Furthermore, by definition

$$\mathscr{A}_\gamma = \mu_2^{\mathrm{Tw}}(F_\gamma, -).$$

The $A_\infty$-associativity conditions for the category of twisted complexes implies that

$$\partial_{\mathrm{Mor}}(\mathscr{A}_\gamma) := \mu_1^{\mathrm{Tw}}(\mu_2^{\mathrm{Tw}}(F_\gamma, -)) + \mu_2^{\mathrm{Tw}}(F_\gamma, \mu_1^{\mathrm{Tw}}(-)) = 0, \tag{13.9}$$

which is the statement of Lemma 13.12.

Another important property of the homology action is the following.

**Lemma 13.13.** *Suppose that $\mathcal{L}_\alpha$ and $\mathcal{L}_\beta$ are hypercubes of attaching curves on $(\Sigma, \mathbf{w}, \mathbf{z}, \mathbf{p})$ and that $\gamma$ is a closed 1-chain on $\Sigma$. Suppose that $C \subseteq \Sigma$ is an integral 2-chain such that $\partial C = \gamma + S_\alpha + S_\beta$ where $S_\alpha$ are closed 1-chains which are disjoint from all curves in $\mathcal{L}_\alpha$ and $S_\beta$ are closed 1-chains which are disjoint from all curves in $\mathcal{L}_\beta$. Then*

$$\mathcal{A}_\gamma \simeq 0$$

*as morphisms of hypercubes.*

*Proof.* We construct the following diagram to realize the chain homotopy:

$$\begin{tikzcd}
\mathbf{CF}^-(\mathcal{L}_\alpha, \mathcal{L}_\beta) \arrow[r, "\mathcal{A}_\gamma"] \arrow[d, "\mathrm{id}"] \arrow[dr, dashed, "H_C"] & \mathbf{CF}^-(\mathcal{L}_\alpha, \mathcal{L}_\beta) \arrow[d, "\mathrm{id}"] \\
\mathbf{CF}^-(\mathcal{L}_\alpha, \mathcal{L}_\beta) \arrow[r, "0"] & \mathbf{CF}^-(\mathcal{L}_\alpha, \mathcal{L}_\beta)
\end{tikzcd}$$

We define the map $H_C$ to have only length 2 chains in the above diagram (i.e., to increment the $\gamma$- and id-directions, but no other directions), and to send $\mathbf{x} \in \mathbf{CF}^-(\boldsymbol{\alpha}_\varepsilon, \boldsymbol{\beta}_\nu)$ to $n_{\mathbf{x}}(C) \cdot \mathbf{x} \in \mathbf{CF}^-(\boldsymbol{\alpha}_\varepsilon, \boldsymbol{\beta}_\nu)$, and to be equivariant in actions of the variables.

We claim that the hypercube relations are satisfied. To see this, it is sufficient to show that if

$$\varepsilon_1 < \cdots < \varepsilon_n \quad \text{and} \quad \nu_1 < \cdots < \nu_m$$

are increasing sequences in the cubes for $\mathcal{L}_\alpha$ and $\mathcal{L}_\beta$, respectively, then

$$H_C \circ D_{\varepsilon_1 < \cdots < \varepsilon_n}^{\nu_1 < \cdots < \nu_m} + D_{\varepsilon_1 < \cdots < \varepsilon_n}^{\nu_1 < \cdots < \nu_m} \circ H_C = (\mathcal{A}_\gamma)_{\varepsilon_1 < \cdots < \varepsilon_n}^{\nu_1 < \cdots < \nu_m}. \tag{13.10}$$

In the above, $D_{\varepsilon_1 < \cdots < \varepsilon_n}^{\nu_1 < \cdots < \nu_m}$ denotes the summands of the hypercube differential of $\mathbf{CF}^-(\mathcal{L}_\alpha, \mathcal{L}_\beta)$ which counts curves on $(\Sigma, \boldsymbol{\alpha}_{\varepsilon_n}, \dots, \boldsymbol{\alpha}_{\varepsilon_1}, \boldsymbol{\beta}_{\nu_1}, \dots, \boldsymbol{\beta}_{\nu_m})$.

In order to prove equation (13.10), we consider a class

$$\psi \in \pi_2(\Theta_{\varepsilon_n, \varepsilon_{n-1}}, \dots, \Theta_{\varepsilon_2, \varepsilon_1}, \mathbf{x}, \Theta_{\nu_1, \nu_2}, \dots, \Theta_{\nu_{m-1}, \nu_m}, \mathbf{y})$$

of $(n+m)$-gons, and consider $\partial_\alpha(\psi) \cap C \subseteq \Sigma$. Since $\partial_\alpha(\psi)$ is 1-dimensional and $C$ is 2-dimensional, the intersection $\partial_\alpha(\psi) \cap C$ is 1-dimensional. We compute $\partial(\partial_\alpha(\psi) \cap C)$ using the Leibniz rule for intersections, which yields

$$\begin{aligned}
\partial(\partial_\alpha(\psi) \cap C) &\equiv \#(\partial\partial_\alpha(\psi)) \cap C + \#\partial_\alpha(\psi) \cap \partial C \pmod 2 \\
&\equiv n_{\mathbf{x}}(C) + n_{\mathbf{y}}(C) + \#\partial_\alpha(\psi) \cap (\gamma + S_\alpha + S_\beta) \pmod 2.
\end{aligned}$$

We first note that $\partial_\alpha(\psi) \cap S_\alpha = \partial_\beta(\psi) \cap S_\beta = \emptyset$. Furthermore, $\partial_\alpha(\psi)$ and $\partial_\beta(\psi)$ are homologous via the 2-chain $D(\psi)$, so we also have $\#\partial_\alpha(\psi) \cap S_\beta \equiv 0$. Hence

$$n_{\mathbf{x}}(C) + n_{\mathbf{y}}(C) \equiv \#\partial_\alpha(\psi) \cap \gamma \pmod 2.$$

This implies equation (13.10), completing the proof. ∎

## 13.4 Relative homology actions and hypercubes

We now define an endomorphism $\mathscr{A}_\lambda$ for arcs $\lambda$ on $\Sigma$ which have boundary on two base points. We focus on the case that $\partial\lambda$ consists of two free base points, $p_i$ and $p_j$, to simplify the exposition.

In the case of the 3-manifold invariants, if $(Y, \mathbf{p})$ is a multi-pointed 3-manifold and $\lambda$ is an arc connecting two base points $p_i, p_j \in \mathbf{p}$, then there is an endomorphism

$$A_\lambda\colon \mathbf{CF}^-(Y, \mathbf{p}) \to \mathbf{CF}^-(Y, \mathbf{p})$$

which satisfies

$$\partial_{\mathrm{Mor}}(A_\lambda) = U_i + U_j.$$

See [58, Section 5].

We now describe how to extend the construction of this map into the setting of hypercubes. When $\mathcal{L}_\alpha$ and $\mathcal{L}_\beta$ are hypercubes of strongly equivalent attaching curves on $(\Sigma, \mathbf{w}, \mathbf{z}, \mathbf{p})$ and $\lambda$ connects two free base points $p_i$, $p_j$, we will construct a morphism of hypercubes

$$\mathscr{A}_\lambda\colon \mathbf{CF}^-(\mathcal{L}_\alpha, \mathcal{L}_\beta) \to \mathbf{CF}^-(\mathcal{L}_\alpha, \mathcal{L}_\beta)$$

which satisfies

$$\partial_{\mathrm{Mor}}(\mathscr{A}_\lambda) = U_i + U_j.$$

Similar to the case of closed curves $\gamma$, we will define

$$\mathscr{A}_\lambda = \mu_2^{\mathrm{Tw}}(F_\lambda, -)$$

for some formal endomorphism

$$F_\lambda\colon \mathcal{L}_\alpha \to \mathcal{L}_\alpha.$$

The map $F_\lambda$ will have length 1 components equal to the formal endomorphisms $\lambda\colon \boldsymbol{\alpha}_\varepsilon \to \boldsymbol{\alpha}_\varepsilon$. However, unlike for closed curves, we will need to add higher length arrows to the map $F_\lambda$. The main issue is that if $\psi$ is a class of $\ell$-gons on a subdiagram of $(\Sigma, \mathcal{L}_\alpha, \mathbf{w}, \mathbf{z}, \mathbf{p})$, then in contrast to equation (13.7) we have

$$a(\psi, \lambda) \equiv n_{p_i}(\psi) - n_{p_j}(\psi) \pmod 2, \tag{13.11}$$

which may be non-zero. Therefore the proof of Lemma 13.12 does not carry over without modification.

To construct $F_\lambda$ and understand its formal properties, we have to first expand our notation of a formal endomorphism slightly. If $\boldsymbol{\alpha}$ is a set of attaching curves, we will also consider formal morphisms of the form $a \cdot \mathbb{I}\colon \boldsymbol{\alpha} \to \boldsymbol{\alpha}$, where $a$ is a polynomial in the $\mathscr{U}_i$, $\mathscr{V}_i$ and $U_j$ variables. We define holomorphic polygon maps with $a \cdot \mathbb{I}$ as an input by declaring them to be $\mathcal{U}_i$, $\mathcal{V}_i$ and $U_i$-equivariant, and also strictly unital. That is, we declare

$$f_{\alpha_1,\dots,\alpha_j,\alpha_j,\dots,\alpha_n}(\mathbf{x}_1,\dots,a\cdot\mathbb{I},\dots,\mathbf{x}_{n-1})$$

to vanish unless $n=2$. We define

$$f_{\alpha_1,\alpha_2,\alpha_2}(\mathbf{x},a\cdot\mathbb{I})=a\cdot\mathbf{x}\quad\text{and}\quad f_{\alpha_1,\alpha_1,\alpha_2}(a\cdot\mathbb{I},\mathbf{x})=a\cdot\mathbf{x}.$$

We also declare $\partial(\mathbb{I})=0$.

We can extend the above construction to define a formal endomorphism $\mathbb{I}\colon\mathcal{L}_\alpha\to\mathcal{L}_\alpha$ for any hypercube of attaching curves $\mathcal{L}_\alpha$. The morphism $\mathbb{I}$ has only length 1 components, all of which are of the form $\mathbb{I}_{\alpha_\varepsilon}$.

We will construct $F_\lambda$ so that

$$\mu_1^{\mathrm{Tw}}(F_\lambda)=(U_i+U_j)\cdot\mathbb{I}.$$

Once we construct $F_\lambda$, we will define

$$A_\lambda(-):=\mu_2^{\mathrm{Tw}}(F_\lambda,-).$$

In the following lemma, we prove that it is always possible to construct the higher length chains of $F_\lambda$ when $\mathcal{L}_\alpha$ is a hypercube of handleslide equivalent curves.

**Lemma 13.14.** *Suppose that $\mathcal{L}_\alpha$ is a hypercube of handleslide equivalent attaching curves on $(\Sigma,\mathbf{w},\mathbf{z},\mathbf{p})$, and let $\lambda$ be an immersed arc on $\Sigma$.*

(1) *If $\partial\lambda=\{p_i,p_j\}$, where $p_i$, $p_j$ are free base points, then $L_\lambda$ may be constructed so that*
$$\mu_1^{\mathrm{Tw}}(F_\lambda)=U_i+U_j.$$

(2) *More generally, if $\partial\lambda=\{x_1,x_2\}\subseteq\mathbf{w}\cup\mathbf{z}\cup\mathbf{p}$, then $F_\lambda$ may be constructed so that*
$$\mu_1^{\mathrm{Tw}}(F_\lambda)=E^\alpha_{x_1}+E^\alpha_{x_2},$$
*where $E^\alpha_{x_i}$ denotes the product of all variables for base points in the component of $\Sigma\setminus\boldsymbol{\alpha}_\varepsilon$ which contains $x_i$. (Note that this is independent of $\varepsilon$, since $\mathcal{L}_\alpha$ consists of handleslide equivalent attaching curves.)*

**Remark 13.15.** (1) It follows from the associativity relations for $\mu_i^{\mathrm{Tw}}$ (as in equation (13.9)) that if $\mathcal{L}_\beta$ is another hypercube of attaching curves and $\partial\lambda=\{p_i,p_j\}$ consists of free base points, then

$$\partial_{\mathrm{Mor}}(\mathcal{A}_\lambda)=(U_i+U_j)\cdot\mathbb{I}$$

as an endomorphism of $\mathbf{CF}^-(\mathcal{L}_\alpha,\mathcal{L}_\beta)$. The second part of the lemma has a similar consequence.

(2) As an example of the second part of the lemma, if $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{w}, \mathbf{z})$ represents a link $L \subseteq Y$ for some $\boldsymbol{\alpha} \in \mathscr{L}_\alpha$ and $\boldsymbol{\beta} \in \mathscr{L}_\beta$, and $x_1 = w_i$ and $x_2 = w_j$ are two link base points on components $K_i$ and $K_j$ of $L$, then

$$\partial_{\mathrm{Mor}}(\mathscr{A}_\lambda) = (\mathscr{U}_i \mathscr{V}_i + \mathscr{U}_j \mathscr{V}_j) \cdot \mathrm{id}.$$

*Proof of Lemma* 13.14. We focus on the first part of the lemma, as the second part is not substantially different. The proof is described in [13, Section 6.2] when we set $U_i = U_j$, though we review it in detail since we need to prove additional properties of the maps $\mathscr{A}_\lambda$.

We begin by considering the length 2 arrows which increment the $\lambda$-direction. If $\boldsymbol{\alpha}_1$ is an immediate successor of $\boldsymbol{\alpha}_0$ in $\mathscr{L}_\alpha$, then we wish to construct a chain $\eta_{\alpha_1,\alpha_0}$ so that the diagram below is a hypercube (except for the length 1 relations along $\lambda$):

$$\begin{tikzcd}
\boldsymbol{\alpha}_0 \arrow[r, "\lambda"] \arrow[d, "\Theta_{\alpha_1,\alpha_0}"] \arrow[dr, dashed, "\eta_{\alpha_1,\alpha_0}"] & \boldsymbol{\alpha}_0 \arrow[d, "\Theta_{\alpha_1,\alpha_0}"] \\
\boldsymbol{\alpha}_1 \arrow[r, "\lambda"] & \boldsymbol{\alpha}_1
\end{tikzcd}$$

Concretely, this amounts to finding $\eta_{\alpha_1,\alpha_0} \in \mathbf{CF}^-(\Sigma, \boldsymbol{\alpha}_1, \boldsymbol{\alpha}_0)$ such that

$$\partial \eta_{\alpha_1,\alpha_0} = A_\lambda^{\alpha_0}(\Theta_{\alpha_1,\alpha_0}) + A_\lambda^{\alpha_1}(\Theta_{\alpha_1,\alpha_0})$$

Above, the map $A_\lambda^{\alpha_0}$ weights curves by $\#(\partial_{\alpha_0}(\phi) \cap \lambda)$ while $A_\lambda^{\alpha_1}$ weights curves by $\#(\partial_{\alpha_1}(\phi) \cap \lambda)$.

We recall that for the ordinary Floer complexes,

$$A_\lambda^{\alpha_0}(\Theta_{\alpha_1,\alpha_0}) + A_\lambda^{\alpha_1}(\Theta_{\alpha_1,\alpha_0}) = U_{p_i} \Phi_{p_i}(\Theta_{\alpha_1,\alpha_0}) + U_{p_j} \Phi_{p_j}(\Theta_{\alpha_1,\alpha_0}).$$

See [58, Lemma 5.7]. However, $\Phi_{p_i}(\Theta_{\alpha_1,\alpha_0})$ and $\Phi_{p_j}(\Theta_{\alpha_1,\alpha_0})$ are cycles of a Maslov $(\mathrm{gr}_{\mathbf{w} \cup \mathbf{p}}, \mathrm{gr}_{\mathbf{z} \cup \mathbf{p}})$-bigrading $(1,1)$ larger than the canonical generator of $\mathbf{HF}^-(\Sigma, \boldsymbol{\alpha}_1, \boldsymbol{\alpha}_0)$, and hence are each boundaries. We note that $\eta_{\alpha_1,\alpha_0}$ may be chosen to be in the submodule of $\mathbf{CF}^-(\Sigma, \boldsymbol{\alpha}_1, \boldsymbol{\alpha}_0)$ generated by the ideal $(U_{p_i}, U_{p_j})$.

We now define the length 3 morphisms of $F_\lambda$ which increase the $\lambda$-direction. Suppose that $\boldsymbol{\alpha}_{00}$ and $\boldsymbol{\alpha}_{11}$ differ by two coordinates of $\mathscr{L}_\alpha$. We consider the 3-dimensional cube spanned by $\boldsymbol{\alpha}_{00}, \boldsymbol{\alpha}_{10}, \boldsymbol{\alpha}_{01}, \boldsymbol{\alpha}_{11}$, extended into the $\lambda$-direction. All length 2 hypercube relations are satisfied. We sum the terms which would appear in the length 3 relation of this hypercube except for contribution of the length 3 arrow. Write

$$C_{\alpha_{11},\alpha_{00}} \in \mathbf{CF}^-(\boldsymbol{\alpha}_{11}, \boldsymbol{\alpha}_{00})$$

for this element. We observe three facts:

($C$-1) $C_{\alpha_{11},\alpha_{00}}$ is a cycle.

($C$-2) $C_{\alpha_{11},\alpha_{00}}$ has Maslov bigrading equal to that of the canonical top degree generator of $\mathbf{HF}^-(\boldsymbol{\alpha}_{11},\boldsymbol{\alpha}_{00})$.

($C$-3) $C_{\alpha_{11},\alpha_{00}}$ is in the submodule generated by the ideal $(U_{p_i},U_{p_j})$.

The first two equations are easily verified. To see the third, we observe that the length 3 relation is obtained by summing over all ways of taking a consecutive sequence of morphisms in the cube, and then inserting $\lambda$ between 2 terms or at the start or beginning and then applying the holomorphic polygon maps interpreted as above. As an example, the expression $C_{\alpha_{11},\alpha_{00}}$ contains the sum

$$\begin{aligned} &f_{\alpha_{11},\alpha_{01},\alpha_{00},\alpha_{00}}(\Theta_{\alpha_{11},\alpha_{01}},\Theta_{\alpha_{01},\alpha_{00}},\lambda)\\ &\quad+f_{\alpha_{11},\alpha_{01},\alpha_{01},\alpha_{00}}(\Theta_{\alpha_{11},\alpha_{01}},\lambda,\Theta_{\alpha_{01},\alpha_{00}})\\ &\quad+f_{\alpha_{11},\alpha_{11},\alpha_{01},\alpha_{00}}(\lambda,\Theta_{\alpha_{11},\alpha_{01}},\Theta_{\alpha_{01},\alpha_{00}}).\end{aligned}$$

The above expression may be collapsed into a count of holomorphic triangles which are weighted by $n_{p_i}(\psi)+n_{p_j}(\psi)$. In particular, the output lies in the submodule generated by $U_{p_i}$ and $U_{p_j}$. The same reasoning holds for the other terms in $C_{\alpha_{11},\alpha_{00}}$.

Further, since $\boldsymbol{\alpha}_{11}$ and $\boldsymbol{\alpha}_{00}$ are handleslide equivalent, the canonical generator of $\mathbf{HF}^-(\boldsymbol{\alpha}_{11},\boldsymbol{\alpha}_{00})$ has both the maximal $\mathrm{gr}_{\mathbf{w}\cup\mathbf{p}}$- and maximal $\mathrm{gr}_{\mathbf{z}\cup\mathbf{p}}$-grading of any non-zero element, and furthermore, the canonical element is the unique non-zero element of homology in this grading. Furthermore, it is straightforward to verify (e.g., by a model computation on a simple diagram) that the quotient map

$$\Pi\colon \mathbf{CF}^-(\boldsymbol{\alpha}_{11},\boldsymbol{\alpha}_{00})\to\mathbf{CF}^-(\boldsymbol{\alpha}_{11},\boldsymbol{\alpha}_{00})/(U_i,U_j) \tag{13.12}$$

is non-vanishing on the canonical generator of top degree. This map $\Pi$ vanishes on $C_{\alpha_{11},\alpha_{00}}$ because $C_{\alpha_{11},\alpha_{00}}$ is in the ideal $(U_i,U_j)$. We note that $C_{\alpha_{11},\alpha_{00}}$ has the same grading as the top degree element of $\mathbf{HF}^-(\boldsymbol{\alpha}_{11},\boldsymbol{\alpha}_{00})$, and hence $C_{\alpha_{11},\alpha_{00}}$, being zero on homology, must be a boundary in $\mathbf{CF}^-(\boldsymbol{\alpha}_{11},\boldsymbol{\alpha}_{00})$. We pick any homogeneously graded chain $\omega_{\alpha_{11},\alpha_{00}}$ such that $\partial\omega_{\alpha_{11},\alpha_{00}}=C_{\alpha_{11},\alpha_{00}}$ to be the length 3 map in our cube.

Filling in the higher-dimensional cubes proceeds in a similar, though slightly simpler manner. In this case, when we want to fill over an $n$-dimensional subcube for $n>3$, all of whose morphisms except for the longest morphism are defined, the corresponding element $C$ is a cycle of bidegree $(n-3,n-3)$ greater than the canonical generator, and hence will automatically be a cycle. ■

**Remark 13.16.** If one restricts to the case that $\mathcal{L}_\alpha$ is algebraically rigid, then we can take $F_\lambda$ to have only length 1 components, which consist of the formal endomorphisms $\lambda$. This is because if $\psi\in\pi_2(\Theta_{\alpha_{\varepsilon_n},\alpha_{\varepsilon_{n-1}}},\dots,\Theta_{\alpha_{\varepsilon_2},\alpha_{\varepsilon_1}},\mathbf{y})$ is a class of polygons with expected dimension 0 on $(\Sigma,\boldsymbol{\alpha}_{\varepsilon_n},\dots,\boldsymbol{\alpha}_{\varepsilon_1},\mathbf{w},\mathbf{z},\mathbf{p})$, then the Maslov grading formula in Heegaard Floer theory implies that

$$\mu(\psi)=\mathrm{gr}_{\mathbf{w}\cup\mathbf{p}}(\Theta_{\alpha_{\varepsilon_n},\alpha_{\varepsilon_1}},\mathbf{y})+2n_{\mathbf{w}\cup\mathbf{p}}(\psi).$$

Since $\mathscr{L}_\alpha$ is algebraically rigid, $\mathrm{gr}_{\mathbf{w}\cup\mathbf{p}}(\Theta_{\alpha_{\varepsilon_n},\alpha_{\varepsilon_1}}, \mathbf{y}) \geq 0$. It follows that, if $\mu(\psi) = 3 - n$ (so that $\mathcal{M}(\psi)$ is 0-dimensional) then there are no holomorphic representatives of $\psi$ if $n \geq 3$ and if $n = 2$ or $n = 3$, then we have $n_{\mathbf{w}\cup\mathbf{p}}(\psi) = 0$. In particular, $n_{p_i}(\psi) = n_{p_j}(\psi) = 0$ and hence

$$\#\partial_\alpha(\psi) \cap \lambda = 0$$

by equation (13.11). In particular, we can argue as in Lemma 13.12 to see that $\partial(F_\lambda) = (U_i + U_j) \cdot \mathbb{I}$, when $F_\lambda$ is defined with only length 1 morphisms. Therefore, when $\mathscr{L}_\alpha$ is algebraically rigid, we can define $\mathcal{A}_\lambda = \mu_2^{\mathrm{Tw}}(F_\lambda, -)$ (with $F_\lambda$ having no higher length arrows) just as we defined $\mathcal{A}_\gamma$ in Lemma 13.12, whereas in general we need to add higher length arrows to $F_\lambda$.

The following is a technical result which is useful when we prove properties about $\mathcal{A}_\lambda$ (cf. [13, Lemma 6.2]).

**Lemma 13.17.** *If $\partial\lambda = \{x_1, x_2\}$, where $x_i \in \mathbf{w} \cup \mathbf{z} \cup \mathbf{p}$, then all of the higher length morphisms of $F_\lambda$ may be chosen to be in the submodule spanned by $(E_{x_1}, E_{x_2})$.*

*Proof.* We focus on the case that $\partial\lambda = \{p_i, p_j\}$ are free base points, so $E_{x_1} = U_i$ and $E_{x_2} = U_j$, since the general case is no different. In the proof of Lemma 13.14, we observed that the length 2 components of $F_\lambda$ could be chosen to be in the submodule generated by the ideal $(U_i, U_j)$. Consider the length 3 arrows of $F_\lambda$ (the argument for higher length chains is essentially the same). Following the proof of Lemma 13.14, let $C_{\alpha_{11},\alpha_{00}}$ be the sum of the terms in the length 3 relation, excluding the term $\partial\omega_{\alpha_{11},\alpha_{00}}$, which has yet to be defined.

We observed in Lemma 13.14 that $C_{\alpha_{11},\alpha_{00}}$ was a boundary, and we picked some $\omega_{\alpha_{11},\alpha_{00}}$ in bigrading $(1, 1)$ higher than the canonical generator such that $\partial\omega_{\alpha_{11},\alpha_{00}} = C_{\alpha_{11},\alpha_{00}}$. Write

$$\omega_{\alpha_{11},\alpha_{00}} = U_i\omega_i + U_j\omega_j + \omega_0,$$

where $\omega_0$ has no powers of $U_i$ or $U_j$. Note that the image $\Pi(\omega)$ of $\omega_0$ in $\mathbf{CF}^-(\boldsymbol{\alpha}_{11}, \boldsymbol{\alpha}_{00})/(U_i, U_j)$ is a cycle because in claim (C-3) in the proof of Lemma 13.14 we showed that $C_{\alpha_1 1,\alpha_{00}}$ is in the submodule spanned by $(U_i, U_j)$. Further, $\omega$ has bigrading $(1, 1)$ higher than the top degree. Hence $\Pi(\omega_0)$ is a boundary, i.e., $\Pi(\omega_0) = \partial(\mathbf{x}_0)$. We lift $\mathbf{x}_0$ to $\mathbf{CF}^-(\boldsymbol{\alpha}_{11}, \boldsymbol{\alpha}_{00})$ to get a chain $\mathbf{x}$ of homogeneous bigrading $(1, 1)$ higher than $\omega_0$, so that

$$\partial\mathbf{x} = \omega_0 + U_i\mathbf{y}_i + U_j\mathbf{y}_j,$$

for some $\mathbf{y}_i, \mathbf{y}_j$. Note that

$$\partial(\omega_{\alpha_{11},\alpha_{00}} + \partial\mathbf{x}) = C_{\alpha_{11},\alpha_{00}}$$

and $\omega_{\alpha_{11},\alpha_{00}} + \partial \mathbf{x}$ is in the submodule spanned by $(U_i, U_j)$, so we use $\omega_{\alpha_{11},\alpha_{00}} + \partial \mathbf{x}$ in place of $\omega_{\alpha_{11},\alpha_{00}}$. The proof for chains of length $n > 3$ follows from essentially the same logic. ■

**Remark 13.18.** Applying essentially the same construction as above shows that $F_\lambda$ is well defined up to chain homotopy. If $F_\lambda$ and $F'_\lambda$ are two constructions of this morphism, then we can apply the filling construction to build a diagonal morphism $h$ in the following diagram:

$$\begin{tikzcd}
\mathscr{L}_\alpha \arrow[r, dashed, "F_\lambda"] \arrow[d, dashed, "\mathbb{I}"] \arrow[dr, dashed, "h"] & \mathscr{L}_\alpha \arrow[d, dashed, "\mathbb{I}"] \\
\mathscr{L}_\alpha \arrow[r, dashed, "F'_\lambda"] & \mathscr{L}_\alpha
\end{tikzcd}$$

which gives a chain homotopy of generalized endomorphisms

$$\partial_{\mathrm{Mor}}(h) = F_\lambda + F'_\lambda.$$

## 13.5 Free stabilization

In [58, Section 6], the author describes maps for adding and removing base points to the Heegaard Floer complexes. These are referred to as *free-stabilization* maps. One can also define a free-stabilization map on the level of hypercubes. These are described in [13, Section 7.1], and we recall the construction presently.

Suppose that $\mathscr{L}_\alpha$ and $\mathscr{L}_\beta$ are hypercubes of attaching curves on a surface $(\Sigma, \mathbf{w}, \mathbf{z}, \mathbf{p})$, and that $p$ is a point on $\Sigma \setminus (\mathbf{w} \cup \mathbf{z} \cup \mathbf{p})$, which is not contained in any curves of $\mathscr{L}_\alpha$ or $\mathscr{L}_\beta$. We define stabilized hypercubes $\mathscr{L}_\alpha^+$ and $\mathscr{L}_\beta^+$. We construct the attaching curves of $\mathscr{L}_\alpha^+$ and $\mathscr{L}_\beta^+$ as follows. For each $\boldsymbol{\alpha}$ in $\mathscr{L}_\alpha$, we adjoin a new component which consists of a small circle bounding a disk centered at $p$. We define $\mathscr{L}_\beta^+$ similarly. We translate these curves slightly to achieve admissibility. If $\varepsilon < \varepsilon'$ are points in the cube for $\mathscr{L}_\alpha$, and $\Theta_{\alpha_{\varepsilon'},\alpha_\varepsilon}$ is the corresponding chain of $\mathscr{L}_\alpha$, then we define the corresponding chain in $\mathscr{L}_\alpha^+$ to be $\Theta_{\alpha_{\varepsilon'},\alpha_\varepsilon} \otimes \theta^+$, where $\theta^+$ is the top graded generator of the new attaching curves centered at $p$. We make a similar definition for the chains of $\mathscr{L}_\beta^+$.

We define a map

$$\mathcal{S}_p^+ \colon \mathbf{CF}^-(\mathscr{L}_\alpha, \mathscr{L}_\beta) \to \mathbf{CF}^-(\mathscr{L}_\alpha^+, \mathscr{L}_\beta^+)$$

via the formula $\mathbf{x} \mapsto \mathbf{x} \otimes \theta^+$, extended equivariantly over the variables $\mathcal{U}_i$, $\mathcal{V}_i$ and $U_j$.

There is also a map $\mathcal{S}_p^-$ in the opposite direction, defined dually by the formula $\mathbf{x} \otimes \theta^+ \mapsto 0$ and $\mathbf{x} \otimes \theta^- \mapsto \mathbf{x}$, extended equivariantly over the variables $\mathcal{U}_i$, $\mathcal{V}_i$ and $U_j$.

**Lemma 13.19.** *The hypercube free-stabilization maps, $\mathcal{S}_p^+$ and $\mathcal{S}_p^-$, are chain maps.*

The proof follows from a standard index computation and gluing argument. See [13, Proposition 5.5] for a proof in the general case, and [58, Propositions 6.5 and Theorem 6.7] for the case of holomorphic disks and triangles.

The fact that $\mathcal{S}_p^+$ and $\mathcal{S}_p^-$ are chain maps of hypercubes may be interpreted as implying a form of naturality for the free-stabilization maps, similar to Remark 13.3. Suppose $\mathcal{H} = (\Sigma, \mathcal{L}_\alpha, \mathcal{L}_\beta)$ and $\mathcal{H}' = (\Sigma, \mathcal{L}_{\alpha'}, \mathcal{L}_{\beta'})$ are diagrams of hypercubes of attaching curves and that the curves of $\mathcal{L}_\alpha$ and $\mathcal{L}_{\alpha'}$ are pairwise handleslide equivalent, and similarly for $\mathcal{L}_\beta$ and $\mathcal{L}_{\beta'}$. Write $\mathcal{H}_p^+$ and $\mathcal{H}'^+_p$ for their stabilizations. Using the notation of Remark 13.3, we see that Lemma 13.19 implies the following (strict) commutation of the diagram:

$$\begin{tikzcd}
\mathbf{CF}^-(\mathcal{H}) \arrow[r, "\Psi_{\mathcal{H}\to\mathcal{H}'}"] \arrow[d, "\mathcal{S}_p^+"] & \mathbf{CF}^-(\mathcal{H}') \arrow[d, "\mathcal{S}_p^+"] \\
\mathbf{CF}^-(\mathcal{H}_p^+) \arrow[r, "\Psi_{\mathcal{H}_p^+\to\mathcal{H}'^+_p}"] & \mathbf{CF}^-(\mathcal{H}'^+_p)
\end{tikzcd}$$

Similar to the setting of the graph TQFT, we have the following relation between the free-stabilization maps and the base point actions from Section 13.2.

**Lemma 13.20.** *For appropriately degenerated almost complex structures, we have* $\mathcal{S}_p^+\mathcal{S}_p^- = \Phi_p$.

*Proof.* The argument is essentially the same as [13, Proposition 5.5], though we repeat the argument here for completeness. We consider a class of polygons

$$\begin{aligned}\psi\#\psi' \in \pi_2(&\Theta_{\alpha_n,\alpha_{n-1}} \times \theta^+, \dots, \Theta_{\alpha_2,\alpha_1} \times \theta^+, \mathbf{x}\times\theta, \Theta_{\beta_1,\beta_2}\times\theta^+, \dots,\\ &\Theta_{\beta_{m-1},\beta_m}\times\theta^+, \mathbf{y}\times\theta')\end{aligned}$$

which potentially contributes to the differential on the hypercube $\mathbf{CF}^-(\mathcal{L}_\alpha, \mathcal{L}_\beta)$. Here, $\theta, \theta' \in \{\theta^+, \theta^-\}$. Here, $\psi'$ has is a class in the stabilization region, and $\psi$ is a class on the unstabilized diagram. By a standard computation

$$\mu(\psi\#\psi') = \mu(\psi) + 2n_p(\psi') + \mathrm{gr}(\theta,\theta'). \tag{13.13}$$

See [58, equation 6.3]. The hypercube differential counts solutions with $\mu(\psi\#\psi') = 3 - m - n$.

In equation (13.13), the term $\mathrm{gr}(\theta,\theta')$ is in $\{-1, 0, 1\}$. For sufficiently stretched almost complex structures, we know that $\mu(\psi) \geq 3 - m - n$ if $m + n \geq 3$, by transversality. Therefore, it follows that when $m + n \geq 3$, we have

$$(\mu(\psi), \mathrm{gr}(\theta,\theta'), n_p(\psi')) \in \{(3-m-n, 0, 0), (4-m-n, -1, 0)\}.$$

In particular, $n_p(\psi) = 0$ so such curves do not contribute to $\Phi_p$ because $n_p(\psi') = 0$.

When $m+n=2$, it is possible for $\mu(\psi)=2-m-n=0$ and for $\psi$ to have a representative, though such curves must represent the constant class. The only possibility with $n_p(\psi')=1$ is when $\operatorname{gr}(\theta,\theta')=-1$, $\mu(\psi)=0$ and $n_p(\psi')=1$. These curves have domain equal to the bigon in the stabilization region which covers $p$. These are the only curves which are counted by the hypercube map $\Phi_p$, and we see that

$$\Phi_p=\mathcal{S}_p^+\circ\mathcal{S}_p^-. \qquad \blacksquare$$

The same neck-stretching argument as in Lemma 13.19 can be used to prove the following (cf. [58, Corollary 6.6]).

**Lemma 13.21.** *Suppose that $\lambda$ is an arc or a closed path which is disjoint from $p$. For an appropriately degenerated almost complex structure, we have*

$$[\mathcal{A}_\lambda,\mathcal{S}_p^\pm]=0.$$

We leave the proof of the above result to the reader, as it is very similar to Lemmas 13.19 and 13.20.

Similarly, we have the following analog of [58, Proposition 6.14], whose proof we also leave to the reader.

**Lemma 13.22.** *If $p$ and $p'$ are distinct free base points, then $[\mathcal{S}_p^{\circ_1},\mathcal{S}_{p'}^{\circ_2}]\simeq 0$, where $\circ_1,\circ_2\in\{+,-\}$.*

## 13.6 Algebraic relations and the diffeomorphism action

In this section, we prove some important algebraic relations involving the maps $\mathcal{A}_\lambda$ and $\mathcal{S}_p^+$. The relations we prove are analogs of the relations from the graph TQFT [58].

In the next lemma, we collect several first relations about the maps $\mathcal{A}_\lambda$ and $\mathcal{S}_p^\pm$.

**Lemma 13.23.** *Suppose that $\mathcal{L}_\alpha$ and $\mathcal{L}_\beta$ are hypercubes of handleslide equivalent attaching curves on $(\Sigma,\mathbf{w},\mathbf{z},\mathbf{p})$.*

(1) *If $p\in\mathbf{p}$, then $\Phi_p^2\simeq 0$. The same holds for base points in $\mathbf{w}\cup\mathbf{z}$.*

(2) *Suppose that $p,p'$ are distinct free base points, and that $\lambda$ is an arc on $(\Sigma,\mathbf{w},\mathbf{z},\mathbf{p})$ such that $\partial\lambda=\{p,p'\}$. Then*

$$[\mathcal{A}_\lambda,\Phi_p]\simeq[\mathcal{A}_\lambda,\Phi_{p'}]\simeq\mathrm{id}.$$

(3) *Suppose that $p\in\mathbf{p}$ is a free base point, and $p'\in\Sigma\setminus(\mathbf{w}\cup\mathbf{z}\cup\mathbf{p})$ is a point which is not contained in $\mathcal{L}_\alpha$ or $\mathcal{L}_\beta$. If $\lambda$ is an arc with $\partial\lambda=\{p,p'\}$, then*

$$\mathcal{S}_{p'}^-\mathcal{A}_\lambda\mathcal{S}_{p'}^+\simeq\mathrm{id}$$

*as endomorphisms of $\mathbf{CF}^-(\mathcal{L}_\alpha,\mathcal{L}_\beta,\mathbf{w},\mathbf{z},\mathbf{p})$. Here, we view $\mathcal{S}_{p'}^+$ as having codomain $\mathbf{CF}^-(\mathcal{L}_\alpha^+,\mathcal{L}_\beta^+,\mathbf{w},\mathbf{z},\mathbf{p}\cup\{p'\})/(U_p+U_{p'})$.*

*Proof.* The claim that $\Phi_p^2 \simeq 0$ follows from the argument in [49, Lemma 4.4]. Cf. [58, Lemma 14.8].

The proof of the second claim is essentially the same as in the case of the ordinary 3-manifold invariants [58, Lemma 14.6]. We differentiate the relation $[\partial, \mathcal{A}_\lambda] = (U_p + U_{p'}) \cdot \mathrm{id}$ from Lemma 13.14 with respect to the $U_p$ to obtain the relation $[\Phi_p, \mathcal{A}_\lambda] \simeq \mathrm{id}$. The same logic gives $[\Phi_{p'}, \mathcal{A}_\lambda] \simeq \mathrm{id}$.

The final claim that $\mathcal{S}_{p'}^- \mathcal{A}_\lambda \mathcal{S}_{p'}^+ \simeq \mathrm{id}$ is proven in [13, Proposition 7.2]. Compare [58, Lemma 7.10] for the case of the ordinary Heegaard Floer complexes. ■

The following lemma is a hypercube analog of [39, Proposition 4.17] and [58, Lemma 5.5].

**Lemma 13.24.** *Let $\mathcal{L}_\alpha$ and $\mathcal{L}_\beta$ be hypercubes of handleslide equivalent attaching curves on $(\Sigma, \mathbf{w}, \mathbf{z}, \mathbf{p})$. If $\gamma$ is a closed curve on $\Sigma$ then*

$$\mathcal{A}_\gamma^2 \simeq 0.$$

*If $\lambda$ is an arc which connects two distinct free base points $p_1, p_2 \in \mathbf{p}$, then*

$$\mathcal{A}_\lambda^2 \simeq U_1 \cdot \mathrm{id} \simeq U_2 \cdot \mathrm{id}.$$

*Proof.* We begin by considering the claim for $\mathcal{A}_\gamma$, when $\gamma$ is a closed curve. Our strategy will be to interpret the expression $\mu_2^{\mathrm{Tw}}(F_\gamma, F_\gamma)$ and show that

$$\mu_2^{\mathrm{Tw}}(F_\gamma, F_\gamma) = 0.$$

Assuming such a formalism can be established, the associativity relations for twisted complexes would show that

$$\mathcal{A}_\gamma \circ \mathcal{A}_\gamma = \mu_2^{\mathrm{Tw}}(F_\gamma, \mu_2^{\mathrm{Tw}}(F_\gamma, -)) = \partial_{\mathrm{Mor}}(\mu_3^{\mathrm{Tw}}(F_\gamma, F_\gamma, -)). \tag{13.14}$$

We now describe how to interpret $\mu_2^{\mathrm{Tw}}(F_\gamma, F_\gamma)$ and $\mu_3^{\mathrm{Tw}}(F_\gamma, F_\gamma, -)$. To define these, it suffices to explain how to define the holomorphic polygon counting maps when there are two inputs which are $\gamma$. Let $\boldsymbol{\delta}_1, \dots, \boldsymbol{\delta}_n$ be attaching curves on $(\Sigma, \mathbf{w}, \mathbf{z}, \mathbf{p})$. We define

$$f_{\delta_1,\dots,\delta_i,\delta_i,\dots,\delta_j,\delta_j,\dots,\delta_n}(\Theta_{1,2}, \dots, \gamma, \Theta_{i,i+1}, \dots, \Theta_{j-1,j}, \gamma, \dots, \Theta_{n-1,n})$$

(where the two $\gamma$'s are non-adjacent inputs) to count holomorphic $n$-gons of index $3-n$ with a multiplicative weight of

$$(\#\partial_{\delta_i}(\psi) \cap \gamma) \cdot (\#\partial_{\delta_j}(\psi) \cap \gamma).$$

We define

$$f_{\delta_1,\dots,\delta_i,\delta_i,\delta_i,\dots,\delta_n}(\Theta_{1,2}, \dots, \Theta_{i-1,i}, \gamma, \gamma, \Theta_{i,i+1}, \dots, \Theta_{n-1,n})$$

to count holomorphic triangles with weight $3 - n$ and multiplicative weight

$$\frac{(\#\partial_{\delta_i}(\psi) \cap \gamma + 1)(\#\partial_{\delta_i}(\psi) \cap \gamma)}{2}.$$

The $A_\infty$-associativity relations are not hard to verify if we interpret $\mu_1(\gamma) = 0$ and $\mu_2(\gamma, \gamma) = 0$. This allows us to interpret all of the expressions in equation (13.14). These numerical choices can be motivated by thinking of the homology action as counting curves with marked points along their alpha boundaries. See [29, Section 3].

To establish equation (13.14), using associativity for $\mu_n^{\mathrm{Tw}}$, it suffices to show that

$$\mu_2^{\mathrm{Tw}}(F_\gamma, F_\gamma) = 0. \tag{13.15}$$

To establish equation (13.15), we argue as follows. To streamline the argument, we define the quantities

$$a_i^\psi := \#(\partial_{\alpha_i}(\psi) \cap \gamma) \quad \text{and} \quad a_{i,i}^\psi := \frac{a_i^\psi(a_i^\psi + 1)}{2}.$$

The length 1 components of $\mu_2^{\mathrm{Tw}}(F_\gamma, F_\gamma)$ are equal to $\mu_2(\gamma, \gamma)$, which is 0 by definition. The higher length components count sums as in

$$f_{a_{\varepsilon_1},\dots,\alpha_{\varepsilon_i},\alpha_{\varepsilon_i},\dots,\alpha_{\varepsilon_j},\alpha_{\varepsilon_j},\dots,\alpha_{\varepsilon n}}(\Theta_{1,2}, \dots, \gamma, \dots, \gamma, \dots, \Theta_{n-1,n}).$$

Summing over all ways of inserting two $\gamma$ terms into the above expression, we see that a given class $\psi$ is counted with weight

$$\sum_{i<j} a_i^\psi a_j^\psi + \sum_{i=1}^n a_{ii}^\psi.$$

It is straightforward to see that the above quantity is equal, modulo 2, to

$$\frac{(\#\partial_\alpha(\psi) \cap \gamma)(\#\partial_\alpha(\psi) \cap \gamma + 1)}{2}.$$

By equation (13.7), $\#\partial_\alpha(\psi) \cap \gamma \equiv 0 \pmod 2$, so we conclude that $\mu_2^{\mathrm{Tw}}(F_\gamma, F_\gamma) = 0$.

We now move on to the case that $\partial\lambda = \{p_1, p_2\}$ and $p_1, p_2$ are two free base points. In this case, we will construct a formal endomorphism $h_{\lambda,\lambda} \colon \mathcal{L}_\alpha \to \mathcal{L}_\alpha$ such that

$$\mu_2^{\mathrm{Tw}}(F_\lambda, F_\lambda) + \mu_1^{\mathrm{Tw}}(h_{\lambda,\lambda}) = U_1 \cdot \mathbb{I}. \tag{13.16}$$

When defining holomorphic polygon counts with two $\lambda$ terms as inputs, we use the same convention as above, except we use

$$\mu_2(\lambda, \lambda) = U_j \cdot \mathbb{I}$$

and

$$\mu_1(\lambda) = (U_i + U_j) \cdot \mathbb{I}.$$

We return to the general question of verifying the associativity conditions on a diagram $(\Sigma, \boldsymbol{\delta}_1, \dots, \boldsymbol{\delta}_n, \mathbf{w}, \mathbf{z}, \mathbf{p})$. We remark that both our definition of $\mu_2(\lambda, \lambda)$ and the definition of our polygon maps when $\lambda$ appears twice are asymmetric between $U_i$ and $U_j$. Note also that the quantity $a_{ii}^{\psi}$ requires a choice of orientation on $\lambda$. We orient $\lambda$ so that $\#(\partial(\psi) \cap \lambda) = n_{p_1}(\psi) - n_{p_2}(\psi)$. This implies that if $\psi_1$ and $\psi_2$ are the nonnegative classes of index 2 $\boldsymbol{\delta}_j$-boundary degenerations which cover $p_1$ and $p_2$, respectively, then

$$a_{j,j}^{\psi_1} \equiv 1 \pmod 2 \qquad \text{and} \qquad a_{j,j}^{\psi_2} \equiv 0 \pmod 2. \tag{13.17}$$

Furthermore, if $\psi$ is any other nonnegative class of index 2 boundary degeneration, then

$$a_{j,j}^{\psi} = 0. \tag{13.18}$$

With these conventions, we now claim that the $A_\infty$-associativity relations hold for polygons with two $\lambda$ inputs. We consider the desired associativity relations applied to the tuple $(\Theta_{1,2}, \dots, \Theta_{i-1,i}, \lambda, \dots, \lambda, \Theta_{j,j+1}, \dots, \Theta_{n-1,n})$. There are two cases to consider: $i < j$ and $i = j$. Suppose that $\psi = \psi_1 * \psi_2$ is a class of $n$-gons. We consider first the case that $i < j$. Here, the associativity relations follow from the fact that

$$a_i^{\psi_1} a_j^{\psi_1} + a_i^{\psi_1} a_j^{\psi_2} + a_i^{\psi_2} a_j^{\psi_1} + a_i^{\psi_2} a_j^{\psi_2} \equiv a_i^{\psi_1 * \psi_2} a_j^{\psi_1 * \psi_2} \pmod 2.$$

Since the right-hand side is independent of the decomposition of $\psi = \psi_1 * \psi_2$, we may obtain the associativity relations in this case summing $a_i^{\psi} a_j^{\psi} \# \partial \mathcal{M}(\psi)$ (multiplied by the appropriate powers of $\mathcal{U}_i$, $\mathcal{V}_i$ and $U_j$) over $n$-gon classes $\psi$ of index $4 - n$. If $n = 2$, there are additional ends corresponding to boundary degenerations. Since we are considering the case that $i < j$, this corresponds to considering the associativity relations for $(\lambda, \Theta_{1,2}, \lambda)$. In this case, boundary degenerations make trivial contribution to the associativity relations since $a_1^{\psi} \cdot a_2^{\psi} = 0$ for any nonnegative, index 2 class of boundary degenerations $\psi$ (since $\psi$ will only have non-trivial boundary on one of the two sets of attaching curves).

We now consider the associativity relation in the case that $i = j$. We observe the relation

$$a_{i,i}^{\psi_1} + a_{i,i}^{\psi_2} + a_i^{\psi_1} a_i^{\psi_2} \equiv a_{i,i}^{\psi_1 * \psi_2} \pmod 2.$$

When $n > 2$, the associativity relations follow from the above relation by summing over the ends of moduli spaces of $n$-gons with Maslov index $4 - n$, with a weight of $a_{i,i}^{\psi}$. When $n = 2$, there are additional ends corresponding to boundary degenerations. These cases are relevant to the tuples $(\lambda, \lambda, \Theta_{1,2})$ and $(\Theta_{1,2}, \lambda, \lambda)$. Focus on $(\lambda, \lambda, \Theta_{1,2})$, since the other configuration is handled by a similar argument. The values of $a_{i,i}^{\psi}$ on index 2 boundary degenerations are described in equations (13.18) and (13.17). When weighted by $a^{\psi}_{1,1}$, the total weighted count of boundary degenerations is $U_1$, and this is contributed by the $\boldsymbol{\delta}_1$-degeneration covering $p_1$. Similarly, when weighted by $a^{\psi}_{2,2}$, the total count is also $U_1$, and this is contributed by the $\boldsymbol{\delta}_2$-degeneration covering $p_1$. Both of these are encoded by the relation $\mu_2(\lambda,\lambda) = U_1 \cdot \mathbb{I}$. This completes the proof of associativity.

We now construct $h_{\lambda,\lambda}$ by a filling procedure similar to our construction of the morphism $F_\lambda$. We assume that the morphisms $F_\lambda$ have already been constructed, as in Lemma 13.14. We think of $h_{\lambda,\lambda}$ as being the diagonal morphism in a diagram of the following shape:

$$\begin{tikzcd}
\mathcal{L}_\alpha \arrow[r, dashed, "F_\lambda"] \arrow[dr, dashed, "h_{\lambda,\lambda}"'] & \mathcal{L}_\alpha \arrow[d, dashed, "F_\lambda"] \\
 & \mathcal{L}_\alpha
\end{tikzcd}$$

As such, we say that the *length* of a component of $h_{\lambda,\lambda}$ from $\boldsymbol{\alpha}_\varepsilon$ to $\boldsymbol{\alpha}_{\varepsilon'}$ is $|\varepsilon'-\varepsilon|_{L^1}+2$.

We now construct the components of $h_{\lambda,\lambda}$ by induction on their length. We define the length 2 components of $h_{\lambda,\lambda}$ to vanish. We consider the problem of filling arrows of length $n > 2$ of $h_{\lambda,\lambda}$. We illustrate the case that $n = 3$ below:

$$\begin{tikzcd}
 & \boldsymbol{\alpha}_1 \arrow[dr, "\lambda"] \arrow[dd, "\Theta_{\alpha_2,\alpha_1}"] \arrow[dddr, dashed, "\eta_{\alpha_2,\alpha_1}"] & \\
\boldsymbol{\alpha}_1 \arrow[ur, "\lambda"] \arrow[dd, "\Theta_{\alpha_2,\alpha_1}"] \arrow[dr, dashed, "\eta_{\alpha_2,\alpha_1}"] \arrow[ddrr, dotted, "\omega_{\alpha_2,\alpha_1}"] & & \boldsymbol{\alpha}_1 \arrow[dd, "\Theta_{\alpha_2,\alpha_1}"] \\
 & \boldsymbol{\alpha}_2 \arrow[dr, "\lambda"] & \\
\boldsymbol{\alpha}_2 \arrow[ur, "\lambda"] & & \boldsymbol{\alpha}_2
\end{tikzcd}$$

We let $C_{\alpha_2,\alpha_1}$ denote the terms appearing in the length 3 relation, except for $\partial\omega_{\alpha_2,\alpha_1}$ (which has yet to be defined). We make two claims:

(1) $C_{\alpha_2,\alpha_1}$ is a cycle of bidegree $(1,1)$ less than the top degree of homology.

(2) If $F_\lambda$ is constructed as in Lemma 13.17 so that its components of length 2 or more lie in the submodule generated by the ideal $(U_1,U_2)$, then $C_{\alpha_2,\alpha_1}$ is also in the submodule generated by the ideal $(U_1,U_2)$.

The claim that $C_{\alpha_2,\alpha_1}$ is a cycle is an easy consequence of the associativity relations, interpreted above, which we leave to the reader.

The fact that $C_{\alpha_2,\alpha_1}$ may be assumed to be in the submodule spanned by $(U_1,U_2)$ may be seen as follows. By assumption each summand of $C_{\alpha_2,\alpha_1}$ which involves $\eta_{\alpha_2,\alpha_1}$ is in this submodule. The remaining terms involve counts of holomorphic disks of index 1 which have $\Theta_{\alpha_2,\alpha_1}$ as an input. There are multiple ways for such a holomorphic disk to contribute to $C_{\alpha_2,\alpha_1}$, depending on the locations of the two $\lambda$-inputs. We observe that the total count of a class $\phi \in \pi_2(\Theta_{\alpha_2,\alpha_1}, \mathbf{y})$ to $C_{\alpha_2,\alpha_1}$ is weighted by

$$a_1^\phi a_2^\phi + a_{1,1}^\phi + a_{2,2}^\phi.$$

It is straightforward to see that the above quantity is congruent modulo 2 to

$$\frac{(a_1^\phi + a_2^\phi)(a_1^\phi + a_2^\phi + 1)}{2}.$$

We observe that $a_1^\phi + a_2^\phi = \pm(n_{p_1}(\phi) - n_{p_2}(\phi))$, since $\partial D(\phi) = \partial_{\alpha_1}(\phi) + \partial_{\alpha_2}(\phi)$. In particular, if a disk $\phi$ has $n_{p_1}(\phi) = n_{p_2}(\phi) = 0$, then it is weighted by 0 in $C_{\alpha_2,\alpha_1}$. Hence $C_{\alpha_2,\alpha_1}$ is in the submodule spanned by $(U_1, U_2)$.

We have established that $C_{\alpha_2,\alpha_1}$ is a cycle in bigrading $(1, 1)$ less than the canonical top degree generator, and furthermore is in the span of $U_1$ and $U_2$. We observe that the map $\Pi$ in equation (13.12) for quotienting by $U_1$ and $U_2$ is a chain map which is injective on the subspace of homology which lies in the grading of the canonical generator and also in the subspace of grading $(1, 1)$ less than the top degree of $\mathbf{HF}^-(\boldsymbol{\alpha}_2, \boldsymbol{\alpha}_1)$. Hence, $C_{\alpha_2,\alpha_1} = \partial\omega_{\alpha_2,\alpha_1}$ for some $\omega_{\alpha_2,\alpha_1}$ in the top degree.

We now argue that $\omega_{\alpha_2,\alpha_1}$ may be taken to be in the submodule $(U_1, U_2)$. The proof is essentially the same as Lemma 13.17. The image $\Pi(\omega_{\alpha_2,\alpha_1})$ is a cycle, and by possibly adding a top degree generator of $\mathbf{HF}^-(\boldsymbol{\alpha}_2, \boldsymbol{\alpha}_1)$, we may assume that $[\Pi(\omega_{\alpha_2,\alpha_1})] = 0$. From here, the argument of Lemma 13.17 may be applied verbatim.

From here, we continue filling the higher length chains of $h_{\lambda,\lambda}$. The argument for the higher length chains is essentially identical to the argument for the length 3 chains, so we leave it to the reader. This verifies equation (13.16). The main claim follows from this and the associativity relations for morphisms of twisted complexes of attaching curves. ■

**Remark 13.25.** We now make an algebraic digression. We note that the map $\mathcal{A}_\lambda$ is in general only defined up to chain homotopy, and is not a cycle unless we set $U_1 = U_2$. Some care is required when composing maps which are not cycles. For example, if $f_1$, $f_2$, $g_1$, $g_2$ are in $\operatorname{End}(C)$, (but not necessarily chain maps) and $f_1 \simeq f_2$ and $g_1 \simeq g_2$, it is not necessarily the case that $f_1 \circ g_1 \simeq f_2 \circ g_2$. For the above lemma to be meaningful, it must be the case that changing $\mathcal{A}_\lambda$ by a chain homotopy also changes $\mathcal{A}_\lambda^2$ by a chain homotopy. We observe the following: If $f_1 \simeq f_2$ and $\partial_{\mathrm{Mor}}(f_1) = \partial_{\mathrm{Mor}}(f_2)$ is central in $\operatorname{End}(C)$, then each $f_i^2$ is a chain map and

$$f_1^2 \simeq f_2^2.$$

Similarly, if $f_1, f_2, g_1, g_2$ are maps so that $f_1 \simeq f_2$, $g_1 \simeq g_2$ and $\partial_{\mathrm{Mor}}(f_i)$ and $\partial_{\mathrm{Mor}}(g_i)$ are central in $\operatorname{End}(C)$, then $[f_i, g_i]$ are chain maps $(i = 1, 2)$ and

$$[f_1, g_1] \simeq [f_2, g_2].$$

To see that $f_1^2 \simeq f_2^2$, we let $j$ be a chain homotopy between $f_1$ and $f_2$, i.e., $\partial_{\mathrm{Mor}}(j) = f_1 + f_2$. Then

$$\partial_{\mathrm{Mor}}(f_1 \circ j + j \circ f_2) = [\partial_{\mathrm{Mor}}(f_1), j] + f_1^2 + f_2^2 = f_1^2 + f_2^2.$$

(Note here that $\partial_{\mathrm{Mor}}(f_1) = \partial_{\mathrm{Mor}}(f_2)$ since $f_1 \simeq f_2$.) Similarly, if $\partial_{\mathrm{Mor}}(j) = f_1 + f_2$ and $\partial_{\mathrm{Mor}}(h) = g_1 + g_2$, then

$$\begin{aligned}
&\partial_{\mathrm{Mor}}([j, g_1] + [f_2, h]) \\
&\quad = [f_1 + f_2, g_1] + [j, \partial_{\mathrm{Mor}}(g_1)] + [f_2, g_1 + g_2] + [\partial_{\mathrm{Mor}}(f_2), h] \\
&\quad = [f_1, g_1] + [f_2, g_2].
\end{aligned}$$

**Lemma 13.26.** *Suppose that $\mathcal{L}_\alpha$ and $\mathcal{L}_\beta$ are hypercubes of handleslide equivalent attaching curves on $(\Sigma, \mathbf{w}, \mathbf{z}, \mathbf{p})$, where $\mathbf{p}$ consists of free base points. Suppose that $\lambda_1$ and $\lambda_2$ are closed curves or arcs on $\Sigma$ with boundary in $\mathbf{p}$. Then*

$$[\mathscr{A}_{\lambda_1}, \mathscr{A}_{\lambda_2}] \simeq \sum_{p_i \in \partial\lambda_1 \cap \partial\lambda_2} U_i.$$

*Proof.* The proof is similar to the proof of Lemma 13.24. We are going to construct a formal endomorphism

$$h_{[\lambda_1,\lambda_2]}\colon \mathcal{L}_\alpha \to \mathcal{L}_\alpha$$

which satisfies

$$\mu_1^{\mathrm{Tw}}(h_{[\lambda_1,\lambda_2]}) = \mu_2^{\mathrm{Tw}}(F_{\lambda_1}, F_{\lambda_2}) + \mu_2^{\mathrm{Tw}}(F_{\lambda_2}, F_{\lambda_1}). \tag{13.19}$$

We think of constructing $h_{[\lambda_1,\lambda_2]}$ as corresponding to building a diagram of the form

$$\mathcal{L}_\alpha^{[\lambda_1,\lambda_2]} := \begin{tikzcd}
\mathcal{L}_\alpha \arrow[r, "\lambda_1"] \arrow[d, "\lambda_2"'] \arrow[dr, dashed, "h_{[\lambda_1,\lambda_2]}"] & \mathcal{L}_\alpha \arrow[d, "\lambda_2"] \\
\mathcal{L}_\alpha \arrow[r, "\lambda_1"] & \mathcal{L}_\alpha
\end{tikzcd}$$

which satisfies a suitable version of the hypercube relations.

Similar to our proof of Lemma 13.24, we need to describe how to interpret the holomorphic polygon counts where we formally have both $\lambda_1$ and $\lambda_2$ as an input.

We have already defined the holomorphic polygon maps when there is a single input of $\lambda_1$ or $\lambda_2$. When there are two inputs, we make the following definitions. Suppose $\boldsymbol{\delta}_1, \dots, \boldsymbol{\delta}_n$ is a collection of attaching curves on $(\Sigma, \mathbf{w}, \mathbf{z}, \mathbf{p})$. If $j < k$, $s, t \in \{1, 2\}$, and $s \neq t$, we define

$$f_{\delta_1,\dots,\delta_j,\delta_j,\dots,\delta_k,\delta_k,\dots,\delta_n}(\Theta_{1,2}, \dots, \lambda_s, \dots, \lambda_t, \dots, \Theta_{n-1,n})$$

to count polygons weighted by $\#(\partial_{\delta_j}(\psi)\cap\lambda_s)\#(\partial_{\delta_k}(\psi)\cap\lambda_t)$. Next, we define

$$f_{\delta_1,\dots,\delta_j,\delta_j,\delta_j,\dots,\gamma_n}(\Theta_{1,2},\dots,\lambda_2,\lambda_1,\dots,\Theta_{n-1,n})$$

to also count curves weighted by $\#(\partial_{\delta_j}(\psi)\cap\lambda_2)\#(\partial_{\delta_j}(\psi)\cap\lambda_1)$. Finally, we define

$$f_{\delta_1,\dots,\delta_j,\delta_j,\delta_j,\dots,\delta_n}(\Theta_{1,2},\dots,\lambda_1,\lambda_2,\dots,\Theta_{n-1,n})=0.$$

We claim that these satisfy a slightly restricted version of the $A_\infty$-associativity relation. Instead of showing an $A_\infty$-associativity relation on all tuples, we will show that associativity is satisfied on only some generators. We will show that associativity holds on tuples of the form

$$(\Theta_{1,2},\dots,\Theta_{j-1,j},\lambda_t,\dots,\lambda_s,\Theta_{k,k+1},\dots,\Theta_{n-1,n}) \tag{13.20}$$

whenever $j<k$ and $t\neq s$. Additionally, we will show that the associativity relations also hold on

$$\begin{aligned}&(\Theta_{1,2},\dots,\Theta_{j-1,j},\lambda_1,\lambda_2,\Theta_{j,j+1},\dots,\Theta_{n-1,n})\\&\quad+(\Theta_{1,2},\dots,\Theta_{j-1,j},\lambda_2,\lambda_1,\Theta_{j,j+1},\dots,\Theta_{n-1,n})\end{aligned} \tag{13.21}$$

as long as we make the following conventions: We assume the polygon maps are strictly unital with the $\lambda$-inputs; $\partial\lambda_i=0$; and $[\lambda_1,\lambda_2]=\sum_{p_i\in\partial\lambda_1\cap\partial\lambda_2}U_i$. This weaker notion of associativity is sufficient for the purposes of constructing $h_{[\lambda_1,\lambda_2]}$ and proving equation (13.19). Note that instead of the above approach, we could attempt to prove associativity for general tuples, but this would require defining the polygon maps when there are inputs such as $\lambda_1\cdot\lambda_2$, which is an unnecessary detour.

The proof of associativity, as described above, is broken into several cases. The first case is when $n>2$ and $j<k$. In this case, the generic degenerations of a 1-dimensional family of holomorphic $n$-gons will consist of pairs of a $p$-gon and a $q$-gon for $p+q=n+2$. Write $\phi$ and $\psi$ for the classes of the $p$- and $q$-gons, respectively. If $s\in\{1,2\}$, write $a_j^\phi$ for $\#(\partial_{\delta_j}(\phi)\cap\lambda_1)$ and $b_j^\phi$ for $\#(\partial_{\delta_j}(\phi)\cap\lambda_2)$. In this case, the associativity relation applied to tuples as in equation (13.20) follows from the additive relation

$$a_j^\phi b_k^\psi+a_j^\psi b_k^\phi+a_j^\phi b_k^\phi+a_j^\psi b_k^\psi=a_j^{\phi*\psi}b_k^{\phi*\psi}.$$

Associativity for tuples as in equation (13.21), when $j=k$ and $n>2$, follows from the similar additive relation

$$a_j^\phi b_j^\psi+a_j^\psi b_j^\phi+a_j^\phi b_j^\phi+a_j^\psi b_j^\psi=a_j^{\phi*\psi}b_j^{\phi*\psi}.$$

Lastly, in the case that $n=2$, there are additional ends corresponding to boundary degenerations. These ends contribute in the associativity relations for $(\Theta_{1,2},\lambda_1,\lambda_2)+$

$(\Theta_{1,2}, \lambda_2, \lambda_1)$. The relation $[\lambda_1, \lambda_2] = \sum_{p \in \partial \lambda_1 \cap \partial \lambda_2} U_p$ corresponds to the fact that $a_2^\phi b_2^\phi \equiv 1$ if $\phi$ is a nonnegative, index 2 class of $\mathcal{S}_2$-boundary degenerations which covers a point of $\partial \lambda_1 \cap \partial \lambda_2$. The associativity relations for $(\lambda_1, \lambda_2, \Theta_{1,2}) + (\lambda_2, \lambda_1, \Theta_{1,2})$ are similar. Finally, we consider the relations for $(\lambda_1, \Theta_{1,2}, \lambda_2)$. In this case, boundary degenerations do not make any contribution, since $a_1^\phi b_2^\phi \equiv 0$ for any boundary degeneration because a boundary degeneration has boundary on only one set of attaching curves.

Having described the necessary associativity relations, filling the higher length chains of $h_{[\lambda_1, \lambda_2]}$ and verifying equation (13.19) follows the same line of reasoning as in the proof of Lemma 13.24. We leave the remaining details of the construction to the reader. ■

We prove a final lemma.

**Lemma 13.27.** *Suppose that* $\lambda_1$ *and* $\lambda_2$ *are two* 1*-chains on* $(\Sigma, \mathbf{w}, \mathbf{z}, \mathbf{p})$*, such that* $\partial \lambda_1 = \{x_1, x_2\}$ *and* $\partial \lambda_2 = \{x_2, x_3\}$*. Then*

$$\mathcal{A}_{\lambda_1} + \mathcal{A}_{\lambda_2} \simeq \mathcal{A}_{\lambda_1 * \lambda_2}.$$

*Proof.* We observe first that for any class of curves $\phi$ which has boundary on $\boldsymbol{\alpha}_i$, we have

$$\#(\partial_{\alpha_i}(\phi) \cap (\lambda_1 + \lambda_2)) = \#(\partial_{\alpha_i}(\phi) \cap \lambda_1) + \#(\partial_{\alpha_i}(\phi) \cap \lambda_2).$$

In particular, we may take the formal endomorphism $F_{\lambda_2 * \lambda_1}$ to be $F_{\lambda_2} + F_{\lambda_1}$. With this definition, $\mathcal{A}_{\lambda_1 * \lambda_2} = \mathcal{A}_{\lambda_1} + \mathcal{A}_{\lambda_2}$. Since $F_{\lambda_1 * \lambda_2}$ is well defined up to chain homotopy by Remark 13.18, the claim in the statement follows for an arbitrary construction of $F_{\lambda_2 * \lambda_2}$. ■

We now prove a helpful formula for understanding base point moving maps on the link surgery hypercubes. See [58, Theorem 14.11] for an analogous result in the setting of the ordinary Heegaard Floer complexes.

**Lemma 13.28.** *Let* $\mathcal{H} = (\Sigma, \mathcal{L}_\alpha, \mathcal{L}_\beta, \mathbf{w}, \mathbf{z}, \mathbf{p})$*, where* $\mathcal{L}_\alpha$ *and* $\mathcal{L}_\beta$ *are hypercubes of handleslide equivalent attaching curves,* $\mathbf{w} \cup \mathbf{z}$ *are link base points and* $\mathbf{p}$ *are free base points. Suppose that* $p_1 \in \mathbf{p}$*, and that* $p_2 \in \Sigma \setminus \mathbf{p}$ *is in the complement of the curves in* $\mathcal{L}_\alpha$ *and* $\mathcal{L}_\beta$*. Suppose also that* $(\Sigma, \mathcal{L}_\alpha, \mathcal{L}_\beta, \mathbf{w}, \mathbf{z}, \mathbf{p})$ *and* $(\Sigma, \mathcal{L}_\alpha, \mathcal{L}_\beta, \{p_2\} \cup \mathbf{p} \setminus \{p_1\})$ *are both weakly admissible. Let* $\mathcal{H}_{p_i}^+$ *denote the diagram obtained by free-stabilizing* $\mathcal{H}$ *at* $p_i$*. If* $\lambda$ *is a path from* $p_1$ *to* $p_2$*, then*

$$\mathcal{S}_{p_1}^- \Psi_{\mathcal{H}_{p_2}^+ \to \mathcal{H}_{p_1}^+} \mathcal{A}_\lambda \mathcal{S}_{p_2}^+$$

*is chain homotopic to the diffeomorphism map which moves* $p_1$ *to* $p_2$ *along* $\lambda$*. In the above, we work on the complexes where we have identified* $U_{p_1}$ *and* $U_{p_2}$ *so that* $\mathcal{A}_\lambda$ *is a chain map. Also* $\Psi_{\mathcal{H}_{p_2}^+ \to \mathcal{H}_{p_1}^+}$ *denotes the map defined in Section* 13.1.

*Proof.* We give a slightly simpler proof than appeared in the setting of 3-manifold invariants in [58, Theorem 14.11], as follows. Let $q$ be a point very near to $p_2$. We will view $\lambda$ as the concatenation of a path $\lambda_1$ from $p_1$ to $q$, and a path $\lambda_2$ from $q$ to $p_2$. In the following, we drop the transition maps $\Psi_{\mathcal{H}_{p_2}^+\to\mathcal{H}_{p_1}^+}$ from the notation. We will write $U$ for

$$U = U_{p_1} = U_{p_2}.$$

Let $\phi_*$ denote the point pushing a diffeomorphism map which moves $p_2$ to $p_1$ along $\lambda$, composed with the naturality map. It is sufficient to show that

$$\mathrm{id} \simeq \phi_* \mathcal{S}_{p_1}^- \mathcal{A}_\lambda \mathcal{S}_{p_2}^+. \tag{13.22}$$

As a first step, we claim that the map $\mathcal{S}_{p_1}^- \mathcal{A}_\lambda \mathcal{S}_{p_2}^+$ is functorial under concatenation of arcs, i.e., that

$$\mathcal{S}_{p_1}^- \mathcal{A}_\lambda \mathcal{S}_{p_2}^+ \simeq \mathcal{S}_q^- \mathcal{A}_{\lambda_2} \mathcal{S}_{p_2}^+ \mathcal{S}_{p_1}^- \mathcal{A}_{\lambda_1} \mathcal{S}_q^+. \tag{13.23}$$

To establish equation (13.23), first note that we may commute free-stabilization maps with other terms by Lemmas 13.21 and 13.22 to obtain

$$\begin{aligned}&\mathcal{S}_q^- \mathcal{A}_{\lambda_2} \mathcal{S}_{p_2}^+ \mathcal{S}_{p_1}^- \mathcal{A}_{\lambda_1} \mathcal{S}_q^+\\ &\qquad\simeq \mathcal{S}_{p_1}^- \mathcal{S}_q^- \mathcal{A}_{\lambda_2} \mathcal{A}_{\lambda_1} \mathcal{S}_q^+ \mathcal{S}_{p_2}^+.\end{aligned}$$

Next, we use the relations $\mathcal{A}_{\lambda_i}^2 \simeq U$ and $\mathcal{A}_{\lambda_1+\lambda_2} = \mathcal{A}_{\lambda_1} + \mathcal{A}_{\lambda_2}$ from Lemmas 13.24 and 13.27 to see that

$$\mathcal{S}_{p_1}^- \mathcal{S}_q^- \mathcal{A}_{\lambda_2} \mathcal{A}_{\lambda_1} \mathcal{S}_q^+ \mathcal{S}_{p_2}^+ \simeq \mathcal{S}_{p_1}^- \mathcal{S}_q^- (\mathcal{A}_{\lambda_2+\lambda_1} \mathcal{A}_{\lambda_1} + U) \mathcal{S}_q^+ \mathcal{S}_{p_2}^+.$$

We use the relation $\mathcal{S}_q^- \mathcal{S}_q^+ = 0$ (which is immediate from the formulas for $\mathcal{S}_q^+$ and $\mathcal{S}_q^-$) and the fact that $\lambda_2 + \lambda_1 = \lambda$ to see that the above is homotopic to

$$\mathcal{S}_{p_1}^- \mathcal{S}_q^- \mathcal{A}_\lambda \mathcal{A}_{\lambda_1} \mathcal{S}_q^+ \mathcal{S}_{p_2}^+.$$

Next, we use the fact that $\lambda$ is disjoint from $q$ to commute $\mathcal{S}_q^-$ with $\mathcal{A}_\lambda$ using Lemma 13.21, and then we use the relation $\mathcal{S}_q^- \mathcal{A}_{\lambda_1} \mathcal{S}_q^+ \simeq \mathrm{id}$ from Lemma 13.23 to see that the above equation is homotopic to

$$\mathcal{S}_{p_1}^- \mathcal{A}_\lambda \mathcal{S}_{p_2}^+.$$

This establishes equation (13.23).

To obtain the main statement in equation (13.22), we combine equation (13.23) with the naturality of all of the other maps with respect to diffeomorphisms, via the following manipulation:

$$\begin{aligned}\phi_* \mathcal{S}_{p_1}^- \mathcal{A}_\lambda \mathcal{S}_{p_2}^+ &\simeq \phi_* \mathcal{S}_q^- \mathcal{A}_{\lambda_2} \mathcal{S}_{p_2}^+ \mathcal{S}_{p_1}^- \mathcal{A}_{\lambda_1} \mathcal{S}_q^+\\ &\simeq \mathcal{S}_q^- \mathcal{A}_{\phi(\lambda_2)} \phi_* \mathcal{S}_{p_2}^+ \mathcal{S}_{p_1}^- \mathcal{A}_{\lambda_1} \mathcal{S}_q^+\\ &\simeq \mathcal{S}_q^- \mathcal{A}_{\phi(\lambda_2)} \mathcal{S}_{p_1}^+ \mathcal{S}_{p_1}^- \mathcal{A}_{\lambda_1} \mathcal{S}_q^+.\end{aligned}$$

We may easily arrange that $\phi(\lambda_2)$ and $\lambda_1$ are isotopic on $\Sigma$ (relative to its boundary). Hence $\mathcal{A}_{\phi(\lambda_2)} \simeq \mathcal{A}_{\lambda_1}$ by Lemma 13.13. The expression is easily shown to be homotopic to the identity using the graph TQFT relations, as we briefly recall. We obtain from Lemma 13.20 that

$$\mathcal{S}_q^- \mathcal{A}_{\lambda_1} \mathcal{S}_{p_1}^+ \mathcal{S}_{p_1}^- \mathcal{A}_{\lambda_1} \mathcal{S}_q^+ \simeq \mathcal{S}_q^- \mathcal{A}_{\lambda_1} \Phi_{p_1} \mathcal{A}_{\lambda_1} \mathcal{S}_q^+.$$

The above expression is homotopic to $\mathcal{S}_q^- \mathcal{A}_{\lambda_1} \mathcal{A}_{\lambda_1} \Phi_{p_1} \mathcal{S}_q^+ + \mathcal{S}_q^- \mathcal{A}_{\lambda_1} \mathcal{S}_q^+$ by Lemma 13.23. By Lemma 13.24 we see that $\mathcal{S}_q^- \mathcal{A}_{\lambda_1}^2 \Phi_{p_1} \mathcal{S}_q^+ \simeq U \mathcal{S}_q^- \mathcal{S}_q^+ \Phi_{p_1} = 0$ and by Lemma 13.23 we have that $\mathcal{S}_q^- \mathcal{A}_{\lambda_1} \mathcal{S}_q^+ \simeq \mathrm{id}$. This completes the proof. ■

**Remark 13.29.** The above argument also applies to the setting of ordinary Heegaard Floer complexes to give a shorter proof of the normalization axiom of the graph TQFT [58, Theorem 14.11].

We now prove the main base point moving map result.

*Proof of Theorem* 13.4. The proof follows from the same manipulation as in the setting of the graph TQFT, see [58, Section 14.4]. We produce the argument for the reader. We decompose $\gamma$ into the concatenation of two arcs $\lambda_1, \lambda_2$, and view $\lambda_1$ as a path from $p$ to a new base point $q$, and $\lambda_2$ as a path from $q$ to $p$. In all of the Floer complexes we consider, we identify the variables for $p$ and $q$ with a single variable $U$. We compute

$$\begin{aligned}
\gamma_* &\simeq \mathcal{S}_q^- \mathcal{A}_{\lambda_2} \mathcal{S}_p^+ \mathcal{S}_p^- \mathcal{A}_{\lambda_1} \mathcal{S}_q^+ \\
&\simeq \mathcal{S}_q^- \mathcal{A}_{\lambda_2} \Phi_p \mathcal{A}_{\lambda_1} \mathcal{S}_q^+ \\
&\simeq \mathcal{S}_q^- A_{\lambda_1} \mathcal{S}_q^+ + \mathcal{S}_q^- \Phi_p \mathcal{A}_{\lambda_2} \mathcal{A}_{\lambda_1} \mathcal{S}_q^+ \\
&\simeq \mathrm{id} + \Phi_p \mathcal{S}_q^- (\mathcal{A}_{\lambda_1} + \mathcal{A}_\gamma) \mathcal{A}_{\lambda_1} \mathcal{S}_q^+ \\
&\simeq \mathrm{id} + \Phi_p \mathcal{S}_q^- U \mathcal{S}_q^+ + \Phi_p A_\gamma \mathcal{S}_q^- \mathcal{A}_{\lambda_1} \mathcal{S}_q^+ \\
&\simeq \mathrm{id} + \Phi_p \mathcal{A}_\gamma,
\end{aligned}$$

completing the proof. ■

## 13.7 Relating homology actions and base point actions

Suppose that $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{w}, \mathbf{z})$ is a Heegaard link diagram for $(Y, L)$ and that $K_i \subseteq L$ is a knot component which has base points $w_i \in \mathbf{w}$ and $z_i \in \mathbf{z}$. We may represent $K_i$ on $\Sigma$ as an immersed curve $[K_i]$ on $\Sigma$ which is the concatenation of two arcs $\lambda_\alpha$ and $\lambda_\beta$ which both have boundary $\{w_i, z_i\}$. The arc $\lambda_\alpha$ intersects only alpha curves while $\lambda_\beta$ intersects only beta curves. See Figure 13.2. If $\phi$ is a class of disks, then

$$a(\phi, \gamma) = n_{w_i}(\phi) - n_{z_i}(\phi) \pmod 2. \tag{13.24}$$

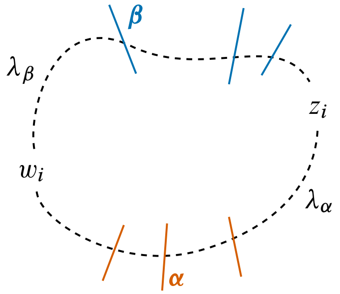


**Figure 13.2.** A knot shadow $K_i$ (dashed) decomposed as a concatenation of two subarcs $\lambda_\alpha$ and $\lambda_\beta$.

The map $A_{[K_i]}$ weights a holomorphic disk $\phi$ by $a(\phi, \gamma)$. On the other hand, we can define a map which counts holomorphic disks weighted by $n_{w_i}(\phi)$. This map is clearly equal to $\mathscr{U}_i \Phi_{w_i}$. Similarly, the map which counts disks weighted by $n_{z_i}(\phi)$ is equal to $\mathscr{V}_i \Psi_{z_i}$. Therefore, equation (13.24) translates into the equality

$$A_{[K_i]} = \mathscr{U}_i \Phi_{w_i} + \mathscr{V}_i \Psi_{z_i},$$

as endomorphisms of $\mathcal{CFL}(Y, L)$. Compare [55, Lemma 14.12]. In this section, we prove a hypercube version of the above statement.

We consider hypercubes of handleslide equivalent attaching curves $\mathcal{L}_\alpha$ and $\mathcal{L}_\beta$ on $(\Sigma, \mathbf{w}, \mathbf{z})$. We suppose that there is a pair of arcs $\lambda_\alpha, \lambda_\beta \subseteq \Sigma$ such that

$$\partial \lambda_\alpha = \partial \lambda_\beta = \{w_i, z_i\}, \quad K_i = \lambda_\alpha * \lambda_\beta,$$

and such that $\lambda_\alpha$ intersects only curves from $\mathcal{L}_\alpha$ while $\lambda_\beta$ intersects only curves from $\mathcal{L}_\beta$. Write $K_i$ for the concatenation of $\lambda_\alpha$ and $\lambda_\beta$. We recall from Section 13.3 that there is an endomorphism

$$\mathscr{A}_{[K_i]} \colon \mathbf{CF}^-(\mathcal{L}_\alpha, \mathcal{L}_\beta) \to \mathbf{CF}^-(\mathcal{L}_\alpha, \mathcal{L}_\beta).$$

Since $K_i$ is a closed curve on $\Sigma$, the map $\mathscr{A}_{[K_i]}$ counts holomorphic polygons (as would be counted by the ordinary hypercube differential) with a weight of $a(\phi, K_i)$, where $a(\phi, K_i)$ is the sum over $\#(\partial_{\alpha_\varepsilon} \phi \cap K)$ ranging over all $\boldsymbol{\alpha}_\varepsilon$ in $\mathcal{L}_\alpha$.

In Section 13.3, we constructed base point actions $\Phi_{w_i}$ and $\Psi_{z_i}$ on the hypercube $\mathbf{CF}^-(\mathcal{L}_\alpha, \mathcal{L}_\beta)$. The map $\Phi_{w_i}$ was defined as the formal differential of the hypercube differential with respect to $\mathscr{U}_i$. We note that in general powers of $\mathscr{U}_i$ are contributed by both the holomorphic curves in the hypercube differential, and also the powers of $\mathscr{U}_i$ present in the input chains of the polygon maps used to construct the hypercube. We defined a map $\Phi_{w_i}^0$ only taking into account the multiplicities of holomorphic curves at the base point $w_i$. We proved in Lemma 13.10 that if $\mathcal{L}_\alpha$ and $\mathcal{L}_\beta$ are algebraically rigid, then $\Phi_{w_i}^0 = \Phi_{w_i}$ and $\Psi_{w_i}^0 = \Psi_{w_i}$. In general, this is not the case. Nonetheless, we do have the following.

**Lemma 13.30.** *Suppose that* $\mathcal{L}_\alpha$ *and* $\mathcal{L}_\beta$ *are hypercubes of handleslide equivalent attaching curves on* $(\Sigma, \mathbf{w}, \mathbf{z})$*, and* $K_i \subseteq \Sigma$ *is a concatenation of two arcs* $\lambda_\alpha$ *and* $\lambda_\beta$*, as above, where* $\lambda_\alpha$ *intersects only the curves in* $\mathcal{L}_\alpha$*, and* $\mathcal{L}_\beta$ *intersects only the curves in* $\mathcal{L}_\beta$*. Then* $\mathcal{U}_i \Phi^0_{w_i} + \mathcal{V}_i \Phi^0_{z_i}$ *is a chain map (regardless of whether* $\mathcal{L}_\alpha$ *and* $\mathcal{L}_\beta$ *are algebraically rigid) and*

$$\mathcal{A}_{[K_i]} = \mathcal{U}_i \Phi^0_{w_i} + \mathcal{V}_i \Psi^0_{z_i} = \mathcal{U}_i \Phi_{w_i} + \mathcal{V}_i \Psi_{z_i},$$

*as hypercube morphisms.*

*Proof.* Since $a(\phi, K_i) \equiv n_{w_i}(\phi) - n_{z_i}(\phi) \pmod 2$ for any holomorphic polygon counted by the hypercube differential of $\mathbf{CF}^-(\mathcal{L}_\alpha, \mathcal{L}_\beta)$, we conclude that

$$\mathcal{A}_{[K_i]} = \mathcal{U}_i \Phi^0_{w_i} + \mathcal{V}_i \Psi^0_{z_i},$$

since the two sides count the same holomorphic curves, with the same weights. Since the left-hand side is a chain map by Lemma 13.12, the right-hand side must also be a chain map.

It remains to show that

$$\mathcal{U}_i \Phi^0_{w_i} + \mathcal{V}_i \Psi^0_{z_i} = \mathcal{U}_i \Phi_{w_i} + \mathcal{V}_i \Psi_{z_i}.$$

To see this, we observe that $\mathcal{U}_i \Phi_{w_i} + \mathcal{V}_i \Psi_{z_i}$ may be described as summing over sequences $\varepsilon_1 < \dots < \varepsilon_n$ and $\nu_1 < \dots < \nu_m$ the maps

$$\mathcal{U}_i \cdot (\phi_{w_i})^{\nu_1 < \dots < \nu_m}_{\varepsilon_1 < \dots < \varepsilon_n} + \mathcal{V}_i \cdot (\psi_{z_i})^{\nu_1 < \dots < \nu_m}_{\varepsilon_1 < \dots < \varepsilon_n} \tag{13.25}$$

(which count holomorphic curves weighted by their multiplicities over $w_i$ and $z_i$, respectively, as in equation (13.5)) and also summing expressions of the form

$$\mathbf{x} \mapsto f_{\alpha_{\varepsilon_n}, \dots, \alpha_{\nu_1}, \beta_{\nu_1}, \dots, \beta_{\nu_m}}(\Theta^\alpha_{n,n-1}, \dots, (\Theta^\alpha_{j+1,j})', \dots, \Theta^\alpha_{2,1}, \mathbf{x}, \Theta^\beta_{1,2}, \dots, \Theta^\beta_{m-1,m}), \tag{13.26}$$

as well as terms where the primed term is a beta-chain. In the above, we write $\Theta^\alpha_{j+1,j}$ for $\Theta_{\alpha_{\varepsilon_{j+1}}, \alpha_{\varepsilon_j}}$ and $(\Theta^\alpha_{j+1,j})'$ denotes $(\mathcal{U}_i \partial_{\mathcal{U}_i} + \mathcal{V}_i \partial_{\mathcal{V}_i})(\Theta^\alpha_{j+1,j})$, and similarly for the beta terms. That is, we sum over the holomorphic polygon maps where we apply $(\mathcal{U}_i \partial_{\mathcal{U}_i} + \mathcal{V}_i \partial_{\mathcal{V}_i})$ to exactly one of the inputs from $\mathcal{L}_\alpha$ or $\mathcal{L}_\beta$. The sum of maps in equation (13.25) is exactly the map $\mathcal{U}_i \Phi^0_{w_i} + \mathcal{V}_i \Psi^0_{z_i}$, so it suffices to show that the terms in equation (13.26) vanish. To see this, we will show that if $\Theta$ is any chain in $\mathcal{L}_\alpha$ or $\mathcal{L}_\beta$, then $(\mathcal{U}_i \partial_{\mathcal{U}_i} + \mathcal{V}_i \partial_{\mathcal{V}_i})(\Theta) = 0$. We argue as follows, focusing on $\mathcal{L}_\alpha$. We observe that on the diagram $(\Sigma, \mathcal{L}_\alpha, \mathbf{w}, \mathbf{z})$, the base points $w_i$ and $z_i$ are immediately adjacent (by virtue of the existence of $\lambda_\beta$, which is disjoint from the alpha curves). We note that each subdiagram $(\Sigma, \boldsymbol{\alpha}', \boldsymbol{\alpha}, \mathbf{w}, \mathbf{z})$ represents an unlink $\mathbb{U}$ in a connected sum of several copies of $S^1 \times S^2$. There is a $|\mathbb{U}|$-component Alexander grading on $\mathcal{CFL}(\mathbb{U})$, and the top Maslov bidegree generators lie in the Alexander grading

$(0,\dots,0)$. By construction, all of the chains of $\mathcal{L}_\alpha$ also lie in the Alexander grading $(0,\dots,0)$. Write $A_i$ for the component of the Alexander grading corresponding to $K_i$. Since $\lambda_\beta$ connects $w_i$ to $z_i$ and is disjoint from all curves of $\mathcal{L}_\alpha$, if $\mathbf{x} \in \mathbb{T}_{\alpha'} \cap \mathbb{T}_\alpha$, we have $A_i(\mathbf{x}) = 0$. In particular, any chain in $\mathcal{L}_\alpha$ must have equal powers of $\mathscr{U}_i$ and $\mathscr{V}_i$, i.e., it must be of the form $\mathscr{U}_i^N \mathscr{V}_i^N \cdot \mathbf{y}$, where $\mathbf{y}$ has no powers of $\mathscr{U}_i$ or $\mathscr{V}_i$. Clearly,

$$(\mathscr{U}_i \partial_{\mathscr{U}_i} + \mathscr{V}_i \partial_{\mathscr{V}_i})(\mathscr{U}_i^N \mathscr{V}_i^N \cdot \mathbf{y}) = 0,$$

as claimed. The same argument works for $\mathcal{L}_\beta$, so the proof is complete. ■

## 13.8 Properties of the $U$-action on link surgery complexes

We now explore some of the complexities of the surgery theorem when we allow arbitrary arcs for our $\sigma$-basic systems.

**Lemma 13.31.** *Suppose that $\mathcal{A}$ is a system of arcs for $L \subseteq S^3$, and suppose that the arcs for knot components $K_1, K_2 \subseteq L$ are alpha-parallel or beta-parallel (the case that one is alpha-parallel and the other is beta-parallel is allowed). Then the actions of $U_1$ and $U_2$ are homotopic on $\mathcal{X}_\Lambda(L,\mathcal{A})^{\mathcal{L}}$ in the sense that there is a type-*D *module morphism $H^1\colon \mathcal{X}_\Lambda(L,\mathcal{A})^{\mathcal{L}} \to \mathcal{X}_\Lambda(L,\mathcal{A})^{\mathcal{L}}$ such that*

$$\partial_{\mathrm{Mor}}(H^1) = \mathrm{id} \otimes (U_1 + U_2).$$

*In the above, $U_i$ denotes $\mathscr{U}_i\mathscr{V}_i$ and* $\mathrm{id} \otimes (U_1 + U_2)$ *means the map which sends* $\mathbf{x}$ *to* $\mathbf{x} \otimes (U_1 + U_2)$ *in all idempotents.*

*Proof.* Let $\mathcal{H}$ be a $\sigma$-basic system of Heegaard diagrams for $(L,\mathcal{A})$. We may assume that $\mathcal{H}$ is meridional for $K_1$ and $K_2$, in the sense of Section 9.5. We begin with the case that $K_1$ and $K_2$ are both alpha-parallel or both beta-parallel. For concreteness, assume that $K_1$ and $K_2$ are both alpha-parallel, so that the arcs of $\mathcal{A}$ are both disjoint from the alpha curves. We let $\lambda\colon [0,1] \to \Sigma$ denote a path from $z_1$ to $z_2$ which is disjoint from all of the arcs of $\mathcal{A}$.

Given hypercubes of alpha and beta Lagrangians $\mathcal{L}_\alpha$ and $\mathcal{L}_\beta$ on a pointed surface $(\Sigma,\mathbf{w},\mathbf{z})$, as well as an arc $\lambda$ connecting two points $x_1, x_2 \in \Sigma$, we obtain a map

$$\mathscr{A}_\lambda\colon \mathbf{CF}^-(\mathcal{L}_\alpha,\mathcal{L}_\beta) \to \mathbf{CF}^-(\mathcal{L}_\alpha,\mathcal{L}_\beta)$$

as in Section 13.3, satisfying

$$\partial_{\mathrm{Mor}}(\mathscr{A}_\lambda) = E_1^\alpha + E_2^\alpha,$$

where $E_i^\alpha$ denotes the product of the variables for the base points in the unique alpha degeneration containing $x_i$. Compare Lemma 13.14. We view the link surgery hypercube as a 2-dimensional hypercube by internalizing all axis directions except those of

$K_1$ and $K_2$. With this perspective, write:

$$\mathcal{C}_{\Lambda}(L) = \begin{tikzcd}
\mathcal{C}_{(0,0)} \arrow[r, "F^{K_1}+F^{-K_1}"] \arrow[d, "F^{K_2}+F^{-K_2}"] \arrow[dr, dashed, "F^{-K_1,-K_2}"] & \mathcal{C}_{(1,0)} \arrow[d, "F^{K_2}+F^{-K_2}"] \\
\mathcal{C}_{(0,1)} \arrow[r, "F^{K_1}+F^{-K_1}"] & \mathcal{C}_{(1,1)}
\end{tikzcd}$$

Write $\phi_1$ for the point-pushing diffeomorphism which moves $z_1$ to $w_1$, and write $\phi_2$ for the map which moves $z_2$ to $w_2$. Write

$$\lambda_{0,0} = \lambda, \quad \lambda_{1,0} = \phi_1(\lambda), \quad \lambda_{0,1} = \phi_2(\lambda), \quad \lambda_{1,1} = \phi_1(\phi_2(\lambda)).$$

We now adapt the above construction of $\mathcal{A}_\lambda$ to the setting of the surgery hypercube, which has maps corresponding to base point moving maps. We focus on a $\sigma$-basic system of Heegaard diagrams for $L$. We apply the aforementioned construction of the hypercube morphism $\mathcal{A}_\lambda$ to the hypercube consisting of all of the link surgery edge maps with negatively oriented components. We obtain a diagram of the following form, which is a hypercube except for the length 1 relations along arrows labeled $\mathcal{A}_{\lambda_\varepsilon}$ (which instead satisfy $[\partial, \mathcal{A}_{\lambda_\varepsilon}] = U_1 + U_2$):

$$\begin{tikzcd}
\mathcal{C}_{(0,0)} \arrow[rr, "F^{-K_1}"] \arrow[dd, "F^{-K_2}"] \arrow[dr, "\mathcal{A}_{\lambda_{0,0}}"] \arrow[drrr, dashed, "h^{-K_1}"] \arrow[dddr, dashed, "h^{-K_2}"] \arrow[ddrr, dashed, "F^{-K_1,-K_2}"] \arrow[dddrrr, dotted, "\omega^{-K_1,-K_2}"] & & \mathcal{C}_{(1,0)} \arrow[dd, "F^{-K_2}"] \arrow[dr, "\mathcal{A}_{\lambda_{1,0}}"] \arrow[dddr, dashed, "j^{-K_2}"] & \\
& \mathcal{C}_{(0,0)} \arrow[rr, "F^{-K_1}"] \arrow[dd, "F^{-K_2}"] \arrow[ddrr, dashed, "F^{-K_1,-K_2}"] & & \mathcal{C}_{(1,0)} \arrow[dd, "F^{-K_2}"] \\
\mathcal{C}_{(0,1)} \arrow[rr, "F^{-K_1}"] \arrow[dr, "\mathcal{A}_{\lambda_{0,1}}"] \arrow[drrr, dashed, "j^{-K_1}"] & & \mathcal{C}_{(1,1)} \arrow[dr, "\mathcal{A}_{\lambda_{1,1}}"] & \\
& \mathcal{C}_{(0,1)} \arrow[rr, "F^{-K_1}"] & & \mathcal{C}_{(1,1)}
\end{tikzcd} \tag{13.27}$$

We first claim that the maps labeled $\mathcal{A}_{\lambda_\varepsilon}$ in equation (13.27) are all equal. To see this, note that $\mathcal{A}_{\lambda_{1,0}}$ is obtained by concatenating $\lambda$ with the arc from $z_1$ to $w_1$ which is disjoint from the alpha curves. Therefore, $\#\partial_\alpha(\psi) \cap \lambda = \#\partial_\alpha(\psi) \cap \lambda_{1,0}$ for all classes of polygons $\psi$ used to define the complexes $\mathcal{C}_{(1,0)}$. Similar arguments apply to the other length 1 maps $\mathscr{A}_{\lambda_\varepsilon}$ so we conclude that the length 1 maps satisfy

$$\mathscr{A}_{\lambda_\varepsilon} = \mathscr{A}_{\lambda_{0,0}} \tag{13.28}$$

for all $\varepsilon \in \mathbb{E}_2$.

We now consider the length 2 maps, we claim that

$$h^{-K_i} = j^{-K_i} \quad \text{for } i = \{1,2\}. \tag{13.29}$$

The argument is similar to the argument for the length 1 maps. Namely, the maps $h^{-K_2}$ and $j^{-K_2}$ count the same holomorphic curves. Some of the curves contributing to $h^{-K_2}$ are counted with a weight of $a(\psi, \lambda)$ whereas the same curves are counted by $a(\psi, \phi_1(\lambda))$ for $j^{-K_2}$. Since $\phi_1(\lambda)$ is obtained by concatenating $\lambda$ with an arc which passes through only beta curves, we see that $a(\psi,\lambda) = a(\psi, \phi_1(\lambda))$, so these expressions are equal. We conclude that $h^{-K_i} = j^{-K_i}$.

We now define an endomorphism $\mathscr{A}_\lambda^{\mathbb{X}}$ of the link surgery hypercube $\mathscr{C}_\Lambda(L)$ to be the endomorphism induced by the components of the diagram in (13.27) which come out of the page (i.e., the $\mathscr{A}_{\lambda_\varepsilon}$-direction). By construction, the maps labeled $h^{-K_i}$ have the same equivariance properties as the link surgery maps $F^{-K_i}$, so the morphism $\mathscr{A}_\lambda^{\mathbb{X}}$ is induced by a morphism of type-D modules

$$\mathscr{A}_\lambda^{\mathcal{X},1} \colon \mathcal{X}_\Lambda(L, \mathcal{A})^{\mathcal{L}} \to \mathcal{X}_\Lambda(L, \mathcal{A})^{\mathcal{L}}.$$

(Note that outside of this lemma, we will write just $\mathscr{A}_\lambda$ or $\mathscr{A}_\lambda^1$ for the maps labeled $\mathscr{A}_\lambda^{\mathbb{X}}$ and $\mathscr{A}_\lambda^{\mathcal{X},1}$ above.)

The components labeled $\mathscr{A}_{\lambda_\varepsilon}$ preserve the idempotents for $K_1$ and $K_2$. The arrows labeled $-K_1$ are weighted in $H^1$ by multiples of the algebra element $\tau_1$. The arrows labeled $-K_2$ are weighted by multiples of $\tau_2$. The dotted arrow labeled $\omega^{-K_1,K_2}$ is weighted by a multiple of $\tau_1\tau_2$. None of the components of $\mathscr{A}_\lambda^{\mathcal{X},1}$ are weighted by algebra elements containing a factor of $\sigma_1$ or $\sigma_2$.

As a morphism from the back face of the diagram in (13.27) to the front face, the map $\mathscr{A}_\lambda^{\mathbb{X}}$ satisfies

$$\partial_{\mathrm{Mor}}(\mathscr{A}_\lambda^{\mathbb{X}}) = (U_1 + U_2) \cdot \mathrm{id}$$

by Remark 13.15. We claim that the same equation is satisfied if we view $\mathscr{A}_\lambda^{\mathbb{X}}$ as a morphism from $\mathscr{C}_\Lambda(L)$ to itself, and furthermore, that the type-D endomorphism $\mathscr{A}_\lambda^{\mathcal{X},1}$ satisfies $\partial_{\mathrm{Mor}}(\mathscr{A}_\lambda^{\mathcal{X},1}) = \mathrm{id} \otimes (U_1 + U_2)$ as an endomorphism of $\mathcal{X}_\Lambda(L)^{\mathcal{L}}$. Note that the hypercube differential in the diagram in (13.27) does not include the differentials $F^{K_1}$ or $F^{K_2}$. Since we are using $\sigma$-basic systems, these maps are identified with the maps induced by localization at $\mathcal{V}_1$ and $\mathcal{V}_2$. Once we include $F^{K_1}$ and $F^{K_2}$ into the differentials on the front and back faces, the fact that $\partial_{\mathrm{Mor}}(\mathscr{A}_\lambda^{\mathbb{X}})$ is still $(U_1 + U_2) \cdot \mathrm{id}$ follows immediately from equations (13.28) and (13.29). The same argument works for type-D morphisms. This completes the proof when both components are alpha-parallel.

We now consider the case that $K_1$ is alpha-parallel and $K_2$ is beta-parallel. In this case, we must modify the map $\mathscr{A}_\lambda^{\mathcal{X},1}$. Recall that $\mathscr{A}_\lambda^{\mathcal{X},1}$ was built from the map $\mathscr{A}_\lambda$. We instead replace this by a map $\mathscr{A}_\lambda^{\mathcal{X},1}$ built from $\mathscr{A}_\lambda + \mathscr{V}_2\Psi_{z_2}$.

If we view the complex as a 1-dimensional cone in the $K_2$-direction

$$\mathcal{C}_\Lambda(L,\mathscr{A}) = \operatorname{Cone}\big(\mathcal{C}_0 \xrightarrow{F^{K_2}+F^{-K_2}} \mathcal{C}_1\big),$$

then we can define endomorphisms $\mathscr{A}_\lambda + \mathscr{V}_2\Psi_{z_2}\colon \mathcal{C}_i \to \mathcal{C}_i$ similar to the case when $K_1$ and $K_2$ were both alpha-parallel. The map $\mathscr{A}_\lambda + \mathscr{V}_2\Psi_{z_2}\colon \mathcal{C}_0 \to \mathcal{C}_0$ consists of the components labeled $\mathscr{A}_{\lambda_{0,0}}$, $\mathscr{A}_{\lambda_{1,0}}$ and $h^{-K_1}$. As an endomorphism of $\mathcal{C}_1$, the map is similar. Since $K_1$ is alpha-parallel, we see that

$$\mathscr{A}_{\lambda_{0,0}} = \mathscr{A}_{\lambda_{1,0}} = \mathscr{A}_\lambda,$$

and so the same argument as before shows that

$$\partial_{\mathrm{Mor}}(\mathscr{A}_\lambda + \mathscr{V}_2\Psi_{z_2}) = (U_1 + U_2)\cdot \mathrm{id}$$

as endomorphisms of $\mathcal{C}_i$.

On the other hand, the point pushing map for $K_2$ moves $z_2$ to $w_2$ through exactly the alpha circles which intersect the trace of $K_2$ on the knot diagram. Hence

$$\mathscr{A}_{\lambda_{0,1}} = \mathscr{A}_{\lambda_{1,1}} = \mathscr{A}_\lambda + \mathscr{A}_{[K_2]}.$$

Here, $\mathscr{A}_{[K_2]}$ denotes the hypercube homology action for the trace of $K_2$ on the Heegaard diagram. We conclude that

$$F^{-K_2}\mathscr{A}_\lambda \simeq (\mathscr{A}_\lambda + \mathscr{A}_{[K_2]})F^{-K_2}, \tag{13.30}$$

as maps from $\mathcal{C}_0$ to $\mathcal{C}_1$. On the other hand, since $F^{-K_2}$ moves $z_2$ to $w_2$, we have

$$F^{-K_2}\mathscr{V}_2\Psi_{z_2} \simeq \mathscr{U}_2\Phi_{w_2}F^{-K_2}. \tag{13.31}$$

We recall from Lemma 13.30 that $\mathscr{A}_{[K_2]} \simeq \mathscr{U}_2\Phi_{w_2} + \mathscr{V}_2\Psi_{z_2}$, as endomorphisms of $\mathcal{C}_0$. Combining the relations in equations (13.30), (13.31) with this equation for $\mathscr{A}_{[K_2]}$, we obtain

$$F^{-K_2}(\mathscr{A}_\lambda + \mathscr{V}_2\Psi_{z_2}) \simeq (\mathscr{A}_\lambda + \mathscr{V}_2\Psi_{z_2})F^{-K_2},$$

as maps from $\mathcal{C}_0$ to $\mathcal{C}_1$. Note that since $F^{K_2}$ is the map for localizing at $\mathscr{V}_2$, we also have

$$F^{-K_2}(\mathscr{A}_\lambda + \mathscr{V}_2\Psi_{z_2}) = (\mathscr{A}_\lambda + \mathscr{V}_2\Psi_{z_2})F^{-K_2}.$$

Arranging the maps $\mathscr{A}_\lambda + \mathscr{V}_2\Psi_{z_2}$ (as maps from $\mathcal{C}_i$ to $\mathcal{C}_i$) as well as the above homotopies (which map $\mathcal{C}_0$ to $\mathcal{C}_1$) into an endomorphism $H$ of $\mathcal{C}_\Lambda(L)$, we see that $\partial_{\mathrm{Mor}}(H) = (U_1 + U_2)\cdot \mathrm{id}$. The same logic gives a type-D homotopy, so the proof is complete. ∎