

## Chapter 14

### The alpha-beta transformer

In this chapter, we describe a bimodule  ${}_{\mathcal{K}}\mathcal{T}^{\mathcal{K}}$ , which we call the *alpha-beta transformer bimodule*. We will prove the following.

**Theorem 14.1.** *Suppose that  $L \subseteq S^3$  is a framed link with a distinguished component  $K$ . Let  $\mathcal{A}$  be a system of arcs for  $L$  such that the arc for  $K$  is alpha-parallel. Let  $\mathcal{A}'$  be a system of arcs where  $K$  is beta-parallel, but otherwise the arcs are the same as  $\mathcal{A}$ . Then*

$$\mathcal{X}_{\Lambda}(L, \mathcal{A})^{\mathcal{L}} \simeq \mathcal{X}_{\Lambda}(L, \mathcal{A}')^{\mathcal{L}} \hat{\boxtimes} {}_{\mathcal{K}}\mathcal{T}^{\mathcal{K}}.$$

The same formula holds if we switch the roles of  $\mathcal{A}$  and  $\mathcal{A}'$ .

#### 14.1 The bimodule

The bimodule  $\mathcal{T}$  has rank 1 over  $\mathbf{I}$ , and has  $\delta_1^1 = 0$ . The map  $\delta_2^1$  is the unique  $\mathbf{I}$ -linear map which satisfies

$$\delta_2^1(a, 1) = 1 \otimes a,$$

for  $a \in \mathcal{K}$ . (This coincides with the identity bimodule.)

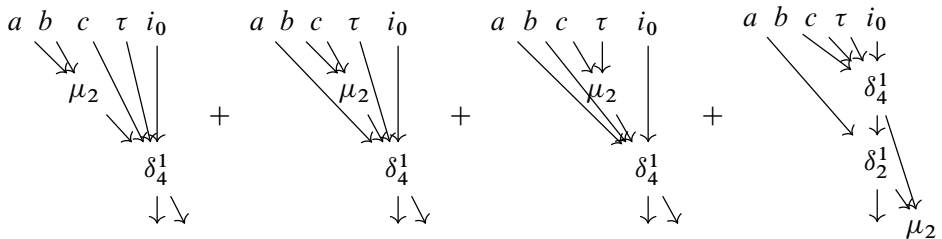
Additionally, there is a  $\delta_4^1$  term, which is non-vanishing on tuples of the form  $(a, b, f\tau, i_0)$  where  $f, a$  and  $b$  are concentrated in a single idempotent, and which vanishes on other elementary tensors. On such elements, we have the formula

$$\delta_4^1(a, b, f\tau, i_0) = i_1 \otimes (\partial_U(a) \cdot \partial_V(b) \cdot f\tau).$$

**Lemma 14.2.** *The structure maps on  ${}_{\mathcal{K}}\mathcal{T}^{\mathcal{K}}$  satisfy the DA bimodule relations.*

*Proof.* The only interesting structure relations to be verified are the ones with 5 inputs. There are only two non-vacuous relations to be verified, which are those for sequences of the form  $(a, b, c, \tau, i_0)$  and those of the form  $(a, b, \tau, c, i_0)$ .

The structure relations for sequences of the form  $(a, b, \tau, c, i_0)$  are straightforward, so we consider the tuple  $(a, b, c, \tau, i_0)$ . The structure diagrams with non-trivial contribution are the following:



The vanishing of the above equation follows from the algebraic relation

$$\partial_U(ab)\partial_V(c) + \partial_U(a)\partial_V(bc) + \partial_U(a)\partial_V(b)c + a\partial_U(b)\partial_V(c) = 0,$$

which is straightforward to verify. ■

In Section 14.3, we describe another perspective which makes the DA bimodule relations more straightforward to verify.

## 14.2 Basepoint actions as $A_\infty$ -morphisms

We now give a helpful reformulation of algebraic base point actions in terms of  $A_\infty$  morphisms. This framework helps us to understand the structure maps appearing in the alpha-to-beta transformer bimodule.

If  $C$  is a free  $\mathbb{F}[U]$ -module, then there is natural base point action  $\Phi_U: C \rightarrow C$ , defined as follows. We pick a free  $\mathbb{F}[U]$  basis  $B$  of  $C$ . If  $\mathbf{x} \in B$ , then we write  $\partial(\mathbf{x}) = \sum_{\mathbf{y} \in B} c_{\mathbf{y},\mathbf{x}} U^{n_{\mathbf{y},\mathbf{x}}} \cdot \mathbf{y}$ , where  $c_{\mathbf{y},\mathbf{x}} \in \mathbb{F}$  and  $n_{\mathbf{y},\mathbf{x}} \in \mathbb{N}$ . The map  $\Phi_U$  is defined via the formula

$$\Phi_U(\mathbf{x}) = \sum_{\mathbf{y} \in B} n_{\mathbf{y},\mathbf{x}} c_{\mathbf{y},\mathbf{x}} U^{n_{\mathbf{y},\mathbf{x}}-1} \cdot \mathbf{y}. \tag{14.1}$$

The induced map  $(\Phi_U)_*$  on  $H_*(C)$  is trivial (cf. Remark 13.11), though the map  $\Phi_U$  is often not null-homotopic through an  $\mathbb{F}[U]$ -equivariant homotopy.

Since  $C$  is free and the differential is  $\mathbb{F}[U]$ -equivariant, we may view  $C$  as corresponding to a type-D module  $C^{\mathbb{F}[U]}$ . Of course,

$$C^{\mathbb{F}[U]} \cong C^{\mathbb{F}[U]} \boxtimes_{\mathbb{F}[U][\mathbb{I}]} \mathbb{F}[U].$$

There is a DA bimodule endomorphism  $\phi_*^1$  of  $\mathbb{F}[U][\mathbb{I}]^{\mathbb{F}[U]}$  defined as follows. We set  $\phi_1^1 = 0$ , and we set

$$\phi_2^1(\alpha, 1) = 1 \otimes \partial_U(\alpha).$$

We set  $\phi_j^1 = 0$  if  $j > 2$ . The DA bimodule relations are straightforward to verify, and easily translated into the Leibniz rule for derivatives.

**Lemma 14.3.** *The map  $\Phi_U$  in equation (14.1), viewed as a type-D morphism*

$$\Phi_U: C^{\mathbb{F}[U]} \rightarrow C^{\mathbb{F}[U]},$$

*satisfies*

$$\Phi_U := \mathbb{I}_C \boxtimes \phi_*^1.$$

The proof is immediate.

We recall that the map  $\Phi_U$  satisfies  $\Phi_U^2 \simeq 0$ . See Lemma 13.23. It is helpful to note that this relation may be translated into a fact about the DA bimodule endomorphism  $\phi_*^1$ .

**Lemma 14.4.** *The DA bimodule morphism  $\phi_*^1$  satisfies*

$$\phi_*^1 \circ \phi_*^1 \simeq 0.$$

*Proof.* We define a DA bimodule endomorphism

$$H_*^1: \mathbb{F}[U][\mathbb{I}]^{\mathbb{F}[U]} \rightarrow \mathbb{F}[U][\mathbb{I}]^{\mathbb{F}[U]}$$

via the formula  $H_1^1 = 0$  and

$$H_2^1(U^n, 1) = 1 \otimes \frac{n(n-1)}{2} U^{n-2}.$$

The map  $\partial_{\text{Mor}}(H_*^1)$  has three summands, shown as graphs below:

$$\partial_{\text{Mor}}(H_*^1) = \begin{array}{c} \swarrow \downarrow \\ \mu_2 \\ \downarrow \\ \swarrow \downarrow \\ H_2^1 \\ \downarrow \searrow \\ \mu_2 \end{array} + \begin{array}{c} \swarrow \downarrow \\ H_2^1 \\ \swarrow \downarrow \\ \delta_2^1 \\ \downarrow \searrow \\ \mu_2 \end{array} + \begin{array}{c} \swarrow \downarrow \\ \delta_2^1 \\ \swarrow \downarrow \\ H_2^1 \\ \downarrow \searrow \\ \mu_2 \end{array}$$

Hence,

$$\begin{aligned} \partial_{\text{Mor}}(H_*^1)(U^n, U^m, 1) &= 1 \otimes \frac{n(n-1) + m(m-1) + (n+m)(n+m-1)}{2} U^{n+m-2} \\ &\equiv 1 \otimes nm U^{n+m-2} \pmod{2} \\ &= (\phi_* \circ \phi_*)_2^1(U^n, U^m, 1), \end{aligned}$$

completing the proof. ■

We will also need to consider analogous base point actions on complexes over the 2-variable polynomial ring  $\mathbb{F}[\mathcal{U}, \mathcal{V}]$ . In this case, we may define two DA bimodule endomorphisms of  $\mathbb{F}[\mathcal{U}, \mathcal{V}][\mathbb{I}]^{\mathbb{F}[\mathcal{U}, \mathcal{V}]}$ , denoted  $\phi_*^1$  and  $\psi_*^1$ , which differentiate with respect to  $\mathcal{U}$  and  $\mathcal{V}$ , respectively. Lemma 14.4 adapts to show that  $\phi_*^1 \circ \phi_*^1 \simeq 0$  and  $\psi_*^1 \circ \psi_*^1 \simeq 0$ . Additionally, we have the following.

**Lemma 14.5.** *As endomorphisms of  $\mathbb{F}[\mathcal{U}, \mathcal{V}][\mathbb{I}]^{\mathbb{F}[\mathcal{U}, \mathcal{V}]}$ , we have*

$$\phi_*^1 \circ \psi_*^1 \simeq \psi_*^1 \circ \phi_*^1.$$

*Proof.* We define a DA bimodule endomorphism  $H_*^1$  of  $\mathbb{F}[\mathcal{U}, \mathcal{V}][\mathbb{I}]^{\mathbb{F}[\mathcal{U}, \mathcal{V}]}$  whose only non-vanishing action is given by the formula

$$H_2(\alpha, 1) = 1 \otimes \partial_{\mathcal{U}, \mathcal{V}}^2(\alpha),$$

where  $\partial_{\mathcal{U}, \mathcal{V}}^2(a) = \partial_{\mathcal{U}}(\partial_{\mathcal{V}}(a))$ . We observe the equality

$$\partial_{\text{Mor}}(H_*^1) = \phi_*^1 \circ \psi_*^1 + \psi_*^1 \circ \phi_*^1$$

which follows from the easily verified relation

$$\partial_{\mathcal{U}, \mathcal{V}}^2(a \cdot b) + a \cdot \partial_{\mathcal{U}, \mathcal{V}}^2(b) + \partial_{\mathcal{U}, \mathcal{V}}^2(a) \cdot b = \partial_{\mathcal{U}}(a) \cdot \partial_{\mathcal{V}}(b) + \partial_{\mathcal{V}}(a) \cdot \partial_{\mathcal{U}}(b). \quad \blacksquare$$

### 14.3 The transformer as a mapping cone bimodule

In order to efficiently prove statements about the transformer bimodule, it is helpful to discuss some generalities about constructing bimodules. If  $f_*^1: \mathcal{A}M^{\mathcal{B}} \rightarrow \mathcal{A}N^{\mathcal{B}}$  is a morphism of DA bimodules such that  $\partial_{\text{Mor}}(f_*^1) = 0$ , then we may construct another bimodule  $\text{Cone}(f_*^1)$  which is also a DA bimodule. If  $f_*^1$  and  $g_*^1$  are two homotopic bimodule morphisms, then  $\text{Cone}(f_*^1)$  and  $\text{Cone}(g_*^1)$  are homotopy equivalent.

It is helpful to understand that the identity and transformer bimodules are mapping cone bimodules. For  $\varepsilon \in \{0, 1\}$ , write  $\mathcal{X}[\mathbf{i}_\varepsilon]^{\mathcal{K}}$ , denote  $\mathbf{I}_\varepsilon \cdot \mathcal{X}[\mathbb{I}]^{\mathcal{K}} \cdot \mathbf{I}_\varepsilon$ , with the restriction of the structure map from the identity bimodule  $\mathcal{X}[\mathbb{I}]^{\mathcal{K}}$ . There are two morphisms

$$s_*^1, t_*^1: \mathcal{X}[\mathbf{i}_0]^{\mathcal{K}} \rightarrow \mathcal{X}[\mathbf{i}_1]^{\mathcal{K}}$$

as follows. The map  $t_j^1 = 0$  for  $j \neq 2$ , and  $t_2^1(f\tau, i_0) = i_1 \otimes f\tau$  and  $t_2^1(f\sigma, i_0) = 0$ . The map  $s_*^1$  is similar, but with the roles of  $\sigma$  and  $\tau$  reversed. It is not hard to see that the DA bimodule morphisms  $s_*^1$  and  $t_*^1$  are cycles. Furthermore,

$$\mathcal{X}[\mathbb{I}]^{\mathcal{K}} = \text{Cone}(s_*^1 + t_*^1: \mathcal{X}[\mathbf{i}_0]^{\mathcal{K}} \rightarrow \mathcal{X}[\mathbf{i}_1]^{\mathcal{K}}).$$

The  $A_\infty$  base point actions  $\phi_*^1$  and  $\psi_*^1$  from Section 14.2 also induce endomorphisms of  $\mathcal{X}[\mathbf{i}_\varepsilon]^{\mathcal{K}}$  for  $\varepsilon \in \{0, 1\}$ . In particular, we observe that the transformer bimodule may be described as

$$\mathcal{X}\mathcal{T}^{\mathcal{K}} = \text{Cone}(s_*^1 + (\text{id} + \phi_*^1 \circ \psi_*^1) \circ t_*^1: \mathcal{X}[\mathbf{i}_0]^{\mathcal{K}} \rightarrow \mathcal{X}[\mathbf{i}_1]^{\mathcal{K}}).$$

**Remark 14.6.** The above perspective gives a more streamlined proof that  $\mathcal{X}\mathcal{T}^{\mathcal{K}}$  satisfies the DA bimodule structure relations. Indeed, it is sufficient to verify that  $\phi_*^1$ ,  $\psi_*^1$ ,  $s_*^1$  and  $t_*^1$  are cycles, which is easy to verify.

### 14.4 Properties of the transformer bimodule

In this section, we prove some important properties of the transformer bimodule.

We begin by showing that in analogy to Sarkar’s map for a Dehn twist [49], the transformer bimodule is involutive.

**Lemma 14.7.** *The transformer bimodule satisfies*

$$\mathcal{K}\mathcal{T}^{\mathcal{K}} \hat{\boxtimes} \mathcal{K}\mathcal{T}^{\mathcal{K}} \simeq \mathcal{K}[\mathbb{I}]^{\mathcal{K}}.$$

*Proof.* We compute that  $\mathcal{K}\mathcal{T}^{\mathcal{K}} \hat{\boxtimes} \mathcal{K}\mathcal{T}^{\mathcal{K}}$  is equal to the mapping cone bimodule

$$\text{Cone}(s_*^1 + (\text{id} + \phi_*^1 \circ \psi_*^1 \circ \phi_*^1 \circ \psi_*^1) \circ t_*^1: \mathcal{K}[\mathbf{i}_0]^{\mathcal{K}} \rightarrow \mathcal{K}[\mathbf{i}_1]^{\mathcal{K}}).$$

On the other hand,  $\phi_*^1 \circ \psi_*^1 \circ \phi_*^1 \circ \psi_*^1 \simeq 0$  by Lemmas 14.4 and 14.5. Hence,  $\mathcal{K}\mathcal{T}^{\mathcal{K}} \hat{\boxtimes} \mathcal{K}\mathcal{T}^{\mathcal{K}}$  is homotopy equivalent to  $\text{Cone}(s_*^1 + t_*^1)$ , which is  $\mathcal{K}[\mathbb{I}]^{\mathcal{K}}$ . ■

Next, we consider the tensor product of the transformer with the modules for a solid torus.

**Lemma 14.8.** *The transformer bimodule satisfies*

$$\mathcal{D}_\lambda^{\mathcal{K}} \hat{\boxtimes} \mathcal{K}\mathcal{T}^{\mathcal{K}} \simeq \mathcal{D}_\lambda^{\mathcal{K}} \quad \text{and} \quad \mathcal{K}\mathcal{T}^{\mathcal{K}} \hat{\boxtimes} \mathcal{K}\mathcal{D}_\lambda^{\mathbb{F}[U]} \simeq \mathcal{K}\mathcal{D}_\lambda^{\mathbb{F}[U]}.$$

*Proof.* The relation  $\mathcal{D}_\lambda^{\mathcal{K}} \hat{\boxtimes} \mathcal{K}\mathcal{T}^{\mathcal{K}} \simeq \mathcal{D}_\lambda^{\mathcal{K}}$  is trivial, so we focus on the relation  $\mathcal{K}\mathcal{T}^{\mathcal{K}} \hat{\boxtimes} \mathcal{K}\mathcal{D}_\lambda^{\mathbb{F}[U]} \simeq \mathcal{K}\mathcal{D}_\lambda^{\mathbb{F}[U]}$ . To simplify the notation, it is easier to construct a homotopy equivalence

$$\mathcal{K}\mathcal{T}^{\mathcal{K}} \hat{\boxtimes} \mathcal{K}[\mathcal{D}_\lambda]_{\mathbb{F}[U]} \simeq \mathcal{K}[\mathcal{D}_\lambda]_{\mathbb{F}[U]},$$

though the maps we write down can be easily adapted to give the statement about DA bimodules, since they are strictly  $\mathbb{F}[U]$ -equivariant.

We will view both bimodules as mapping cone bimodules, in the sense of Section 14.3. Define

$$\mathcal{K}[\mathbf{d}_\varepsilon]_{\mathbb{F}[U]} := \mathbf{I}_\varepsilon \cdot \mathcal{K}[\mathcal{D}_\lambda]_{\mathbb{F}[U]},$$

for  $\varepsilon \in \{0, 1\}$ . We observe that both  $\mathcal{K}\mathcal{T}^{\mathcal{K}} \hat{\boxtimes} \mathcal{K}[\mathcal{D}_\lambda]_{\mathbb{F}[U]}$  and  $\mathcal{K}[\mathcal{D}_\lambda]_{\mathbb{F}[U]}$  are the mapping cones of type-AA module maps from  $\mathbf{d}_0$  to  $\mathbf{d}_1$ . Indeed, there are type-AA module maps

$$\Sigma_*, T_*: \mathcal{K}[\mathbf{d}_0]_{\mathbb{F}[U]} \rightarrow \mathcal{K}[\mathbf{d}_1]_{\mathbb{F}[U]},$$

where  $\Sigma_{1,1,0}(f\sigma, \mathbf{x}) = m_2(f\sigma, \mathbf{x})$  (where  $m_2$  is the action from  $\mathcal{D}_\lambda$ ) and  $T_{1,1,0}(f\tau, \mathbf{x}) = m_2(f\tau, \mathbf{x})$ . Furthermore,  $\Sigma_{1,1,0}(f\tau, \mathbf{x}) = 0$  and  $T_{1,1,0}(f\sigma, \mathbf{x}) = 0$ . Also  $\Sigma_*$  and  $T_*$  vanish on other configurations of inputs. It is straightforward to verify that  $\Sigma_*$  and  $T_*$  are cycles. Additionally,

$$\mathcal{K}[\mathcal{D}_\lambda]_{\mathbb{F}[U]} = \text{Cone}(\Sigma_* + T_*: \mathcal{K}[\mathbf{d}_0]_{\mathbb{F}[U]} \rightarrow \mathcal{K}[\mathbf{d}_1]_{\mathbb{F}[U]}) \quad (14.2)$$

and

$$\mathcal{K}\mathcal{T}^{\mathcal{K}} \hat{\boxtimes} \mathcal{K}[\mathcal{D}_\lambda]_{\mathbb{F}[U]} = \text{Cone}(\Sigma_* + (\text{id} + \phi_* \circ \psi_*) \circ T_*: \mathcal{K}[\mathbf{d}_0]_{\mathbb{F}[U]} \rightarrow \mathcal{K}[\mathbf{d}_1]_{\mathbb{F}[U]}). \quad (14.3)$$

Here, we are writing  $\phi_*$  and  $\psi_*$  for  $\phi_*^1 \boxtimes \mathbb{I}_{\mathcal{D}_\lambda}$  and  $\psi_*^1 \boxtimes \mathbb{I}_{\mathcal{D}_\lambda}$ , where  $\phi_*^1$  and  $\psi_*^1$  are the endomorphisms of  $\mathcal{X}[\mathbf{i}_1]^{\mathcal{K}}$  described above.

We observe that

$$\phi_* \circ \psi_* \simeq \mathcal{V}^{-1} \phi_* \circ (\mathcal{U}\phi_* + \mathcal{V}\psi_*), \tag{14.4}$$

as endomorphisms of  $[\mathbf{d}_1]$ , since  $\phi_*^2 \simeq 0$  by Lemma 14.4.

We write  $\partial_{\mathcal{W}} = \mathcal{U}\partial_{\mathcal{U}} + \mathcal{V}\partial_{\mathcal{V}}$ . We define an endomorphism  $J_*$  of  $\mathcal{X}[\mathbf{d}_1]_{\mathbb{F}[U]}$  via the formula

$$J_{0,1,0}(\mathbf{x}) = \partial_{\mathcal{W}}(\mathbf{x}).$$

We set  $J_{i,1,j} = 0$  unless  $i = j = 0$ . Observe that  $J_{0,1,0}$  commutes with the right action of  $U$ . We claim that

$$\partial_{\text{Mor}}(J_*) = \mathcal{U}\phi_* + \mathcal{V}\psi_*.$$

To see that, we note that there are exactly two structure trees which contribute to  $\partial_{\text{Mor}}(J_*)$ . These are shown below:

$$\partial_{\text{Mor}}(J_*)_{1,1,0}(a, \mathbf{x}) = \begin{array}{ccc} a & \searrow & \mathbf{x} \\ & & \downarrow \\ & & \delta_2^1 \\ & & \downarrow \\ & & J_{0,1,0} \\ & & \downarrow \end{array} + \begin{array}{ccc} a & \searrow & \mathbf{x} \\ & & \downarrow \\ & & J_{0,1,0} \\ & & \downarrow \\ & & \delta_2^1 \\ & & \downarrow \end{array}$$

Hence,

$$\partial_{\text{Mor}}(J_*)_{1,1,0}(a, \mathbf{x}) = \partial_{\mathcal{W}}(a \cdot \mathbf{x}) + a \cdot \partial_{\mathcal{W}}(\mathbf{x}).$$

By the Leibniz rule we have that the above is  $\partial_{\mathcal{W}}(a) \cdot \mathbf{x}$ . Therefore,

$$\partial_{\text{Mor}}(J_*) = \mathcal{U}\phi_* + \mathcal{V}\psi_*,$$

as endomorphisms of  $[\mathbf{d}_1]$ . We conclude that  $\phi_* \circ \psi_* \simeq 0$  as endomorphisms of  $[\mathbf{d}_1]$  by equation (14.4). Hence the two mapping cones in equations (14.2) and (14.3) are homotopy equivalent. The proof is complete. ■

### 14.5 Proof of Theorem 14.1

We now prove Theorem 14.1, which states that changing from an alpha-parallel arc to a beta-parallel arc may be achieved by tensoring with the transformer bimodule  $\mathcal{X}\mathcal{T}^{\mathcal{K}}$ .

*Proof of Theorem 14.1.* We use our base point moving map formula from Corollary 13.5.

Let  $\mathcal{A}$  be an arc system for  $L \subseteq S^3$ , and assume that  $K_i \subseteq L$  is a component whose arc  $\mathcal{A}_i$  is beta-parallel. Let  $\mathcal{A}'_i$  denote an arc for  $K_i$  which is alpha-parallel, and let  $\mathcal{A}'$  denote the arc system which has  $\mathcal{A}'_i$  in place of  $\mathcal{A}$ .

We pick a Heegaard link diagram  $\mathcal{H} = (\Sigma, \alpha, \beta, \mathbf{w}, \mathbf{z})$  for  $(S^3, L)$  so that all of the arcs of  $\mathcal{A}$  are embedded in  $\Sigma$  and pairwise disjoint. We assume the arc  $\mathcal{A}_i$  is disjoint from  $\beta$ . The arc  $\mathcal{A}'_i$  may also be viewed as being embedded on  $\Sigma$ . With respect to this embedding,  $\mathcal{A}'_i$  is disjoint from  $\alpha$ . In principle,  $\mathcal{A}'_i$  might intersect some of the other arcs of  $\mathcal{A}$ . However, by stabilizing the Heegaard diagram  $(\Sigma, \alpha, \beta, \mathbf{w}, \mathbf{z})$ , we may assume that  $\mathcal{A}'_i$  is disjoint from all of the other arcs of  $\mathcal{A}$ .

We build a hyperbox of Heegaard diagrams using the diagram  $\mathcal{H}$  as the initial diagram. If  $K_i \subseteq L$ , then the  $K_i$ -direction of this hypercube involves the composition of a tautological diffeomorphism map to move  $z_i$  along  $\mathcal{A}_i$ , followed by a holomorphic polygon counting maps which change the alpha and beta curves back to their original position. We do this in each axis direction to build a  $\sigma$ -basic system used for the link surgery formula.

For this  $\sigma$ -basic system, we may change  $\mathcal{A}_i$  to  $\mathcal{A}'_i$  by composing the  $K_i$ -direction with a final base point moving hypercube which moves  $w_i$  in a loop along the curve  $[K_i] = \mathcal{A}'_i * \mathcal{A}_i^{-1}$ . Corollary 13.5 implies that the effect on the link surgery hypercube is to replace  $F^{-K_i}$  (the sum of the hypercube maps ranging over all sublinks of  $L$  containing  $-K_i$ ) with the map  $(\text{id} + \mathcal{V}_i^{-1} \mathcal{A}_{[K_i]} \circ \Phi_{w_i}) \circ F^{-K_i}$ .

The map  $(\text{id} + \mathcal{V}_i^{-1} \mathcal{A}_{[K_i]} \circ \Phi_{w_i})$  is viewed as an endomorphism of the  $(\ell - 1)$ -dimensional subcube of  $\mathcal{C}_\Lambda^1(L, \mathcal{A}) \subseteq \mathcal{C}_\Lambda(L, \mathcal{A})$  which has  $K_i$ -component 1. On this subcube, the arc  $\mathcal{A}'_i$  intersects only beta curves, and  $\mathcal{A}_i$  intersects only alpha curves. By Lemma 13.30, we have

$$\mathcal{A}_{[K_i]} \simeq \mathcal{U}_i \Phi_{w_i} + \mathcal{V}_i \Psi_{z_i},$$

as endomorphisms of  $\mathcal{C}_\Lambda^1(L, \mathcal{A})$ . Hence

$$\text{id} + \mathcal{V}_i^{-1} \Phi_{w_i} \mathcal{A}_{[K_i]} = \text{id} + \mathcal{V}_i^{-1} \Phi_{w_i} (\mathcal{U}_i \Phi_{w_i} + \mathcal{V}_i \Psi_{z_i}) \simeq \text{id} + \Phi_{w_i} \Psi_{z_i},$$

since  $\Phi_{w_i}^2 \simeq 0$ . In particular, it follows that if we write

$$\mathcal{C}_\Lambda(L, \mathcal{A}) \cong \text{Cone}(\mathcal{C}_\Lambda^0(L, \mathcal{A}) \xrightarrow{F^{K_i} + F^{-K_i}} \mathcal{C}_\Lambda^1(L, \mathcal{A})),$$

then  $\mathcal{C}_\Lambda(L, \mathcal{A}')$  is homotopic to

$$\text{Cone}(\mathcal{C}_\Lambda^0(L, \mathcal{A}) \xrightarrow{F^{K_i} + (\text{id} + \Phi_{w_i} \Psi_{z_i}) F^{-K_i}} \mathcal{C}_\Lambda^1(L, \mathcal{A})).$$

On the level of type-D structures over  $\mathcal{L}$ , we claim that the above complex corresponds to the tensor product of  $\mathcal{X}_\Lambda(L, \mathcal{A})^{\mathcal{L}}$  with the transformer bimodule  $\mathcal{X} \mathcal{T}^{\mathcal{X}}$ . To

see this, observe that the components of the box tensor differential of  $\mathcal{X}_\Lambda(L, \mathcal{A})^{\mathcal{L}} \hat{\boxtimes}$   $\mathcal{X}\mathcal{T}^{\mathcal{K}}$  which involve the map  $\delta_2^1$  of  $\mathcal{X}\mathcal{T}^{\mathcal{K}}$  contribute the internal differentials of  $\mathcal{C}_\Lambda^\varepsilon(L, \mathcal{A})$  (for  $\varepsilon \in \{0, 1\}$ ), as well as the maps  $F^{K_i}$  and  $F^{-K_i}$  to the mapping cone differential. The terms of the box tensor product differential which involve  $\delta_4^1$  of  $\mathcal{X}\mathcal{T}^{\mathcal{K}}$  contribute the term  $\Phi_{w_i} \Psi_{z_i} F^{-K_i}$  to the above mapping cone differential (compare Lemma 14.3). This completes the proof. ■