Chapter 15

The pair-of-pants bimodules

In this chapter, we describe several bimodules ${}_{\mathcal{K}|\mathcal{K}}W^{\mathcal{K}}_{\alpha\beta,\beta}$ and ${}_{\mathcal{K}|\mathcal{K}}W^{\mathcal{K}}_{\alpha\beta,\alpha}$. The bimodules are related by the transformer bimodule

$$ {}_{\mathcal{K}|\mathcal{K}}W^{\mathcal{K}}_{\alpha\beta,\beta} \mathbin{\hat{\boxtimes}} {}_{\mathcal{K}}\mathcal{T}^{\mathcal{K}} \simeq {}_{\mathcal{K}|\mathcal{K}}W^{\mathcal{K}}_{\alpha\beta,\beta}. $$

We call these the *pair-of-pants* bimodules. Other choices of α and β in the subscripts can also be constructed by tensoring in copies ${}_{\mathcal{K}}\mathcal{T}^{\mathcal{K}}$ to switch from α to β and vice-versa.

After defining these bimodules and proving several basic properties, we will prove the following.

Theorem 15.1. *Suppose that $L_1, L_2 \subseteq S^3$ are framed links with distinguished components $K_1 \subseteq L_1$ and $K_2 \subseteq L_2$. Write $n = |L_1|$ and $m = |L_2|$. Let $\mathcal{A}_{1,\alpha}$ and $\mathcal{A}_{2,\beta}$ be systems of arcs for L_1 and L_2. Suppose that the arc for K_1 is alpha-parallel and the arc for K_2 is beta-parallel. (The other arcs need not be alpha- or beta-parallel.) Let $\mathcal{A}_\alpha$ and $\mathcal{A}_\beta$ be the system of arcs for $L_1\#L_2$ which use an alpha-parallel (resp. beta-parallel) arc for K_1 (resp. K_2). Then*

$$ \begin{aligned} (\mathcal{X}_{\Lambda_1}(L_1,\mathcal{A}_{1,\alpha})^{\mathcal{K}\otimes\mathcal{L}_{n-1}}, \mathcal{X}_{\Lambda_1}(L_2,\mathcal{A}_{2,b})^{\mathcal{K}\otimes\mathcal{L}_{m-1}}) &\mathbin{\hat{\boxtimes}} {}_{\mathcal{K}|\mathcal{K}}W^{\mathcal{K}}_{\alpha\beta,\alpha} \\ \simeq \mathcal{X}_{\Lambda_1+\Lambda_2}(L_1\#L_2,\mathcal{A}_\alpha)^{\mathcal{L}_{n+m-1}} & \end{aligned} $$

and

$$ \begin{aligned} (\mathcal{X}_{\Lambda_1}(L_1,\mathcal{A}_{1,\alpha})^{\mathcal{K}\otimes\mathcal{L}_{n-1}}, \mathcal{X}_{\Lambda_1}(L_2,\mathcal{A}_{2,\beta})^{\mathcal{K}\otimes\mathcal{L}_{m-1}}) &\mathbin{\hat{\boxtimes}} {}_{\mathcal{K}|\mathcal{K}}W^{\mathcal{K}}_{\alpha\beta,\beta} \\ \simeq \mathcal{X}_{\Lambda_1+\Lambda_2}(L_1\#L_2,\mathcal{A}_\beta)^{\mathcal{L}_{n+m-1}}. & \end{aligned} $$

15.1 The $W_{\alpha\beta,\alpha}$ and $W_{\alpha\beta,\beta}$ bimodules

We now define the $W_{\alpha\beta,\alpha}$ and $W_{\alpha\beta,\beta}$ bimodules. We consider $W_{\alpha\beta,\beta}$ first. This bimodule has a δ_3^1 which is identical to the merge module M. Additionally, there is a δ_5^1, determined by the following formula:

$$ \delta_5^1(a|b, a'|b', 1|\tau, \tau|1, i_0) = i_1 \otimes \mathcal{V}^{-1}\partial_{\mathcal{U}}(ab)a'(\mathcal{U}\partial_{\mathcal{U}} + \mathcal{V}\partial_{\mathcal{V}})(b')\tau. \tag{15.1} $$

In the above, $a, b, a', b' \in \mathbf{I}_1 \cdot \mathcal{K} \cdot \mathbf{I}_1$.

The module $W_{\alpha\beta,\alpha}$ is similar, except it has

$$ \delta_5^1(a|b, a'|b', 1|\tau, \tau|1, i_0) = i_1 \otimes \mathcal{V}^{-1}\partial_{\mathcal{U}}(ab)b'(\mathcal{U}\partial_{\mathcal{U}} + \mathcal{V}\partial_{\mathcal{V}})(a')\tau. $$

Lemma 15.2. *The bimodules* ${}_{\mathcal{K}|\mathcal{K}}W^{\mathcal{K}}_{\alpha\beta,\alpha}$ *and* ${}_{\mathcal{K}|\mathcal{K}}W^{\mathcal{K}}_{\alpha\beta,\beta}$ *satisfy the* DA *bimodule structure relations. Furthermore, both are split Alexander bimodules when we give the incoming algebra factors the discrete partition.*

Proof. We consider the structure relations first. We take the perspective of mapping cone bimodules from Section 14.3. We begin by considering the merge module ${}_{\mathcal{K}|\mathcal{K}}M^{\mathcal{K}}$. We define bimodules ${}_{\mathcal{K}|\mathcal{K}}[\mathbf{m}_\varepsilon]^{\mathcal{K}}$ for $\varepsilon \in \{0, 1\}$ by

$$(\mathbf{I}_\varepsilon \otimes \mathbf{I}_\varepsilon) \cdot {}_{\mathcal{K}|\mathcal{K}}M^{\mathcal{K}} \cdot \mathbf{I}_\varepsilon.$$

The bimodule structure relations are easily seen to be satisfied. There are two DA bimodule morphisms

$$\mathcal{S}^1_*, \mathcal{T}^1_* \colon {}_{\mathcal{K}|\mathcal{K}}[\mathbf{m}_0]^{\mathcal{K}} \to {}_{\mathcal{K}|\mathcal{K}}[\mathbf{m}_1]^{\mathcal{K}}$$

as follows. We set $\mathcal{S}^1_j = 0$ if $j \neq 3$. We set $\mathcal{S}^1_3(1|\sigma, \sigma|1, i_0) = i_1 \otimes \sigma$. The map $\mathcal{T}^1_*$ is similar, but with τ replacing σ. The DA bimodule endomorphisms $\mathcal{S}^1_*$ and $\mathcal{T}^1_*$ are easily seen to be cycles. Furthermore,

$${}_{\mathcal{K}|\mathcal{K}}M^{\mathcal{K}} = \operatorname{Cone}\bigl(\mathcal{S}^1_* + \mathcal{T}^1_* \colon {}_{\mathcal{K}|\mathcal{K}}[\mathbf{m}_0]^{\mathcal{K}} \to {}_{\mathcal{K}|\mathcal{K}}[\mathbf{m}_1]^{\mathcal{K}}\bigr).$$

Note that we may form four DA bimodule endomorphisms ϕ^l_*, ϕ^r_*, ψ^l_* and ψ^r_* of ${}_{\mathcal{K}|\mathcal{K}}[\mathbf{m}_1]^{\mathcal{K}}$ by adapting the construction from Section 14.2. Here, ϕ^l_* differentiates with respect to $\mathscr{U}$ on the left factor of $\mathcal{K} \otimes \mathcal{K}$, while ϕ^r_* differentiates with respect to $\mathscr{U}$ on the right factor. We may also view these maps as endomorphisms of ${}_{\mathcal{K}|\mathcal{K}}[\mathbf{m}_1]^{\mathcal{K}}$ by boxing with the identity map on $[\mathbf{m}_1]$. With respect to the above notation, we observe that

$${}_{\mathcal{K}|\mathcal{K}}W^{\mathcal{K}}_{\alpha\beta,\alpha} = \operatorname{Cone}\Bigl([\mathbf{m}_0] \xrightarrow{\mathcal{S}^1_* + (\mathrm{id} + \mathscr{V}^{-1}(\phi^l_* + \phi^r_*) \circ (\mathscr{U}\phi^l_* + \mathscr{V}\psi^l_*)) \circ \mathcal{T}^1_*} [\mathbf{m}_1]\Bigr). \tag{15.2}$$

The map

$$(\mathrm{id} + \mathscr{V}^{-1}(\phi^l_* + \phi^r_*) \circ (\mathscr{U}\phi^l_* + \mathscr{V}\psi^r_*)) \circ \mathcal{T}^1_*$$

is a cycle since it is the composition of cycles. Similarly, $\mathcal{S}^1_*$ is a cycle by direct computation. Hence the DA bimodule relations are satisfied for ${}_{\mathcal{K}|\mathcal{K}}W^{\mathcal{K}}_{\alpha\beta,\alpha}$. The same argument works for ${}_{\mathcal{K}|\mathcal{K}}W^{\mathcal{K}}_{\alpha\beta,\beta}$.

Finally, the claim about split continuity is proven by an easy modification of the proof for the merge module, from Lemma 8.3. ■

15.2 Proof of Theorem 15.1

We now give a proof of Theorem 15.1.

Proof of Theorem 15.1. The proof is based on our formula for changing the path in the link surgery formula of Corollary 13.5. It is easier to analyze the surgery hypercubes, instead of their interpretation as type-D modules, so we focus on the statement in terms of hypercube complexes first.

Letting $K_1 \subseteq L_1$ and $K_2 \subseteq L_2$ be the special components, and suppose that $\mathcal{A}_{1,\alpha}$ and $\mathcal{A}_{2,\beta}$ are systems of arcs such that K_1 has an alpha-parallel arc and K_2 has a beta-parallel arc. We use the meridional σ-basic systems considered in Section 9.3, which we assume are algebraically rigid. These were constructed in Lemma 9.8. By Theorem 12.1, we know that

$$\begin{aligned}&\mathcal{C}_{\Lambda_1+\Lambda_2}(L_1\# L_2, \mathcal{A}_{1,\alpha}\#\mathcal{A}_{2,\beta})\\ &\qquad\cong \operatorname{Cone}\big(\mathcal{C}^0(L_1)\otimes\mathcal{C}^0(L_2)\xrightarrow{F^{K_1}|F^{K_2}+F^{-K_1}|F^{-K_2}}\mathcal{C}^1(L_1)\otimes\mathcal{C}^1(L_2)\big).\end{aligned}$$

(Recall that we are writing $F^{\pm K_i}$ for the sum of all surgery maps for oriented sublinks of L_i which contain $\pm K_i$.)

The above surgery hypercube is for the system of arcs $\mathcal{A}_{1,\alpha}\#\mathcal{A}_{2,\beta}$. The arc in $\mathcal{A}_{1,\alpha}\#\mathcal{A}_{2,\beta}$ for $K_1\# K_2$ is the co-core of the connected sum band. The present statement instead involves σ-basic systems of Heegaard diagrams for $L_1\# L_2$ which use an alpha- or beta-parallel arc on $K_1\# K_2$. If we want to change the arc to be alpha-parallel, we may stack the above hypercube for $\mathcal{C}_{\Lambda_1+\Lambda_2}(L_1\# L_2,\mathcal{A}_{1,\alpha}\#\mathcal{A}_{2,\beta})$ with the hypercube for moving the base point in a loop following K_2 in the subcube $\mathcal{C}^1(L_1)\otimes\mathcal{C}^1(L_2)$. We may compute this cube using Corollary 13.5. The effect on the resulting surgery hypercube is to replace $F^{-K_1}|F^{-K_2}$ with the expression

$$(\mathrm{id}+\mathscr{V}^{-1}\Phi_w\circ\mathcal{A}_{[K_2]})(F^{-K_1}|F^{-K_2})\colon \mathcal{C}^1(L_1\# L_2)\to\mathcal{C}^1(L_1\# L_2).$$

We are using Proposition 11.8 to identify $\mathcal{C}^\varepsilon(L_1\# L_2)\cong\mathcal{C}^\varepsilon(L_1)\otimes\mathcal{C}^\varepsilon(L_2)$, for $\varepsilon\in\{0,1\}$. The differential on each $\mathcal{C}^\varepsilon(L_1\# L_2)$ is the tensor product differential, i.e., $\partial_1\otimes\mathrm{id}+\mathrm{id}\otimes\partial_2$. In particular, the map Φ_w on $\mathcal{C}^\varepsilon(L_1\# L_2)$ is intertwined with the map

$$\Phi_{w_1}\otimes\mathrm{id}+\mathrm{id}\otimes\Phi_{w_2}$$

under the connected sum isomorphism $\mathcal{C}^1(L_1\# L_2)\cong\mathcal{C}^1(L_1)\otimes\mathcal{C}^1(L_2)$. Similarly, using the connected sum formulas for hypercubes in Propositions 10.1 and 11.8, it is straightforward to see that with respect to the isomorphism $\mathcal{C}^1(L_1\# L_2)\cong\mathcal{C}^1(L_1)\otimes\mathcal{C}^1(L_2)$ the map $\mathcal{A}_{[K_2]}$ is intertwined with $\mathrm{id}\otimes\mathcal{A}_{[K_2]}$. By Lemma 13.30 we see that

$$\mathrm{id}\otimes\mathcal{A}_{[K_2]}=\mathrm{id}\otimes(\mathscr{U}\Phi_{w_2}+\mathscr{V}\Psi_{z_2}),$$

as endomorphisms of $\mathcal{C}^\varepsilon(L_1)\otimes\mathcal{C}^\varepsilon(L_2)$. We conclude that

$$\Phi_w\circ\mathcal{A}_{[K_2]}=(\Phi_{w_1}\otimes\mathrm{id}+\mathrm{id}\otimes\Phi_{w_2})\circ(\mathrm{id}\otimes(\mathscr{U}\Phi_{w_2}+\mathscr{V}\Psi_{z_2})).$$

The addition of this term is encoded exactly by the extra δ_5^1 term in equation (15.1) (cf. Lemma 14.3), so the proof is complete. ∎

15.3 Relation with the AA-identity bimodule

We recall that the AA-identity bimodule ${}_{\mathcal{K}|\mathcal{K}}[\mathbb{I}^{\supset}]$ from Section 8.4 admits an extension to a bimodule

$$ {}_{\mathcal{K}|\mathcal{K}}[\mathbb{I}^{\supset}]_{\mathbb{F}[U]}. $$

It follows from Theorem 12.1 and Proposition 13.2 that we may use ${}_{\mathcal{K}|\mathcal{K}}[\mathbb{I}^{\supset}]$ to obtain a valid model of the link surgery formula when taking the connected sum of two links (i.e., the resulting hypercube has the correct module actions on the remaining knot components). It is not clear from this argument whether tensoring with ${}_{\mathcal{K}|\mathcal{K}}[\mathbb{I}^{\supset}]_{\mathbb{F}[U]}$ gives a valid model. In this section, we show that indeed tensoring with ${}_{\mathcal{K}|\mathcal{K}}[\mathbb{I}^{\supset}]_{\mathbb{F}[U]}$ produces a valid model, via the following algebraic result.

Proposition 15.3. *There is a homotopy equivalence*

$$ {}_{\mathcal{K}|\mathcal{K}}W^{\mathcal{K}}_{\alpha\beta,\alpha} \hat{\boxtimes} {}_{\mathcal{K}}[\mathcal{D}_0]_{\mathbb{F}[U]} \simeq {}_{\mathcal{K}|\mathcal{K}}[\mathbb{I}^{\supset}]_{\mathbb{F}[U]}. $$

The same holds if we replace $W_{\alpha\beta,\alpha}$ with $W_{\alpha\beta,\beta}$.

The above proposition will be verified in several steps.

Lemma 15.4. *The operations ϕ^1_* and ψ^1_* satisfy the following relations:*

1. *As morphisms from ${}_{\mathcal{K}}[\mathbf{i}_0]^{\mathcal{K}}$ to ${}_{\mathcal{K}}[\mathbf{i}_1]^{\mathcal{K}}$, we have*

$$ \phi^1_* \circ s^1_* \simeq s^1_* \circ \phi^1_* \quad \text{and} \quad \psi^1_* \circ s^1_* \simeq s^1_* \circ \psi^1_*. $$

Similarly,

$$ t^1_* \circ \phi^1_* \simeq \mathcal{V}^2 \psi^1_* \circ t^1_* \quad \text{and} \quad t^1_* \circ \psi^1_* \simeq \mathcal{V}^{-2} \phi^1_* \circ t^1_*. $$

2. *As morphisms from ${}_{\mathcal{K}|\mathcal{K}}[\mathbf{m}_0]^{\mathcal{K}}$ to ${}_{\mathcal{K}|\mathcal{K}}[\mathbf{m}_1]^{\mathcal{K}}$ we have*

$$ \phi^l_* \circ \mathcal{S}^1_* \simeq \mathcal{S}^1_* \circ \phi^l_* \quad \text{and} \quad \psi^l_* \circ \mathcal{S}^1_* \simeq \mathcal{S}^1_* \circ \psi^l_* $$

and

$$ \mathcal{T}^1_* \circ \phi^l_* \simeq \mathcal{V}^2 \psi^l_* \circ \mathcal{T}^1_* \quad \text{and} \quad \mathcal{T}^1_* \circ \psi^l_* \simeq \mathcal{V}^{-2} \phi^l_* \circ \mathcal{T}^1_*. $$

The same relations hold with ϕ^r_* and ψ^r_* in place of ϕ^l_* and ψ^l_*.

Proof. We begin with two algebraic relations on $\mathcal{K}$. If $f \in \mathbf{I}_1 \cdot \mathcal{K} \cdot \mathbf{I}_1$ and $g \in \mathbf{I}_0 \cdot \mathcal{K} \cdot \mathbf{I}_0$ are such that $\tau a = b\tau$ then

$$ \tau \partial_{\mathcal{U}}(a) = \mathcal{V}^2 \partial_{\mathcal{V}}(b)\tau \quad \text{and} \quad \tau \partial_{\mathcal{V}}(a) = \mathcal{V}^{-2} \partial_{\mathcal{U}}(b)\tau. $$

We verify the above equations. If $a = \mathcal{U}^i \mathcal{V}^j$, then $b = \phi^\tau(a) = \mathcal{U}^j \mathcal{V}^{2j-i}$. Then

$$ \tau \partial_{\mathcal{U}}(a) = \tau \partial_{\mathcal{U}}(\mathcal{U}^i \mathcal{V}^j) = \tau i \mathcal{U}^{i-1} \mathcal{V}^j = i \mathcal{U}^j \mathcal{V}^{2j-i+1} \tau = \mathcal{V}^2 \partial_{\mathcal{V}}(b)\tau $$

as claimed. Similarly,

$$ \tau \partial_{\mathcal{V}}(a) = \tau j \mathcal{U}^i \mathcal{V}^{j-1} = j \mathcal{U}^{j-1} \mathcal{V}^{2j-2-i} \tau = \mathcal{V}^{-2} \partial_{\mathcal{U}}(b)\tau. $$

We now consider the equation $\phi^1_* \circ s^1_* \simeq s^1_* \circ \phi^1_*$. We define a morphism $h^1_*\colon {}_{\mathcal{K}}[\mathbf{i}_0]^{\mathcal{K}} \to {}_{\mathcal{K}}[\mathbf{i}_1]^{\mathcal{K}}$ via the formula

$$h^1_2(f\sigma, 1) = 1 \otimes \partial_{\mathcal{U}}(f)\sigma.$$

One checks easily that $\partial_{\mathrm{Mor}}(h^1_*) = \phi^1_* \circ s^1_* + s^1_* \circ \phi^1_*$. The other relations with t^1_* and ϕ^1_* and ψ^1_* are proven similarly.

We now consider the relations for $\mathcal{S}^1_* \circ \phi^l_* \simeq \phi^l_* \circ \mathcal{S}^1_*$. Define $f^1_* := \mathcal{S}^1_* \circ \phi^l_*$ and note that f^1_* has f^1_4 non-trivial, and this map satisfies

$$f^1_4(\sigma|1, 1|\sigma, a|a', i_0) = i_1 \otimes \sigma\partial_{\mathcal{U}}(a)a'.$$

Note that more generally, $f^1_4(a\sigma|a', b|b'\sigma, c|c', i_0) = i_1 \otimes aa'bb'\sigma\partial_{\mathcal{U}}(c)c'$, though we omit the extra possible coefficients on the σ terms to simplify the notation. Define $g^1_* := \phi^l_* \circ \mathcal{S}^1_*$ and note that g^1_* has only g^1_4 non-trivial, and

$$g^1_4(a|a', \sigma|1, 1|\sigma, i_0) = i_1 \otimes \partial_{\mathcal{U}}(a)a'\sigma.$$

There is a third map of interest, k^1_*, which has

$$k^1_4(\sigma|1, a|a', 1|\sigma, i_0) = i_1 \otimes a'\sigma\partial_{\mathcal{U}}(a).$$

A homotopy h^1_* between f^1_* and k^1_* is given by

$$h^1_3(a\sigma|a', b|b'\sigma, i_0) = i_1 \otimes aa'b'\sigma\partial_{\mathcal{U}}(b).$$

A homotopy between k^1_* and g^1_* is constructed similarly. The other relations in the lemma statement are proven by minor modifications of the above argument. ■

Lemma 15.5. *There is a homotopy equivalence*

$${}_{\mathcal{K}|\mathcal{K}}W^{\mathcal{K}}_{\alpha\beta,\alpha} \mathbin{\hat{\boxtimes}} {}_{\mathcal{K}}[\mathscr{D}_0]_{\mathbb{F}[U]} \simeq ({}_{\mathcal{K}}\mathcal{T}^{\mathcal{K}}, {}_{\mathcal{K}}\mathcal{T}^{\mathcal{K}}) \mathbin{\hat{\boxtimes}} {}_{\mathcal{K}|\mathcal{K}}[\mathbb{I}^{\Supset}]_{\mathbb{F}[U]}.$$

The same holds if we replace $W_{\alpha\beta,\alpha}$ with $W_{\alpha\beta,\beta}$.

Remark 15.6. The above statement does not hold if we replace $W_{\alpha\beta,\alpha}$ with $W_{\alpha\alpha,\alpha}$, $W_{\alpha\alpha,\beta}$, $W_{\beta\beta,\alpha}$ or $W_{\beta\beta,\beta}$.

Proof of Lemma 15.5. The argument is similar to the proof of Lemma 14.8. The key is to observe that we may define a homotopy $\partial_{\mathcal{W}} = \mathscr{U}\partial_{\mathscr{U}} + \mathscr{V}\partial_{\mathscr{V}}$, and that $\partial_{\mathcal{W}}$ commutes with the action of $U = \mathscr{U}\mathscr{V}$.

In our present context, we want to view both modules in the statement as AA-bimodule mapping cones. For $\varepsilon \in \{0, 1\}$, write

$${}_{\mathcal{K}|\mathcal{K}}[\mathbf{md}_\varepsilon]_{\mathbb{F}[U]} := (\mathbf{I}_\varepsilon \otimes \mathbf{I}_\varepsilon) \cdot {}_{\mathcal{K}|\mathcal{K}}[\mathbb{I}^{\Supset}]_{\mathbb{F}[U]}.$$

Both of the bimodules in the statement may be viewed as a mapping cone of an AA-bimodule morphism from $[\mathbf{md}_0]$ to $[\mathbf{md}_1]$.

Similar to equation (15.2), we may view $W_{\alpha\beta,\alpha} \mathbin{\hat{\boxtimes}} \mathcal{D}_0$ as the mapping cone of

$$\mathcal{S}_* + (\mathrm{id} + \mathcal{V}^{-1}(\phi^l_* + \phi^r_*)(\mathcal{U}\phi^l_* + \mathcal{V}\psi^l_*)) \circ \mathcal{T}_*. \tag{15.3}$$

We observe that the morphism $H_{0,1,0}\colon [\mathbf{md}_1] \to [\mathbf{md}_1]$ given by $H_{0,1,0}(\mathbf{x}) = \partial_{\mathcal{W}}(\mathbf{x})$ satisfies

$$\partial_{\mathrm{Mor}}(H_{0,1,0}) = \mathcal{U}\phi^l_* + \mathcal{U}\phi^r_* + \mathcal{V}\psi^l_* + \mathcal{V}\psi^r_*.$$

In particular, we obtain that

$$\begin{aligned}
&\mathrm{id} + \mathcal{V}^{-1}(\phi^l_* + \phi^r_*)(\mathcal{U}\phi^l_* + \mathcal{V}\psi^l_*) \\
&\quad = \mathrm{id} + \mathcal{V}^{-1}\phi^l_*(\mathcal{U}\phi^l_* + \mathcal{V}\psi^l_*) + \mathcal{V}^{-1}\phi^r_*(\mathcal{U}\phi^l_* + \mathcal{V}\psi^l_*) \\
&\quad \simeq \mathrm{id} + \mathcal{V}^{-1}\phi^l_*(\mathcal{U}\phi^l_* + \mathcal{V}\psi^l_*) + \mathcal{V}^{-1}\phi^r_*(\mathcal{U}\phi^r_* + \mathcal{V}\psi^r_*) \\
&\quad \simeq \mathrm{id} + \phi^l_*\psi^l_* + \phi^r_*\psi^r_* \\
&\quad \simeq (\mathrm{id} + \phi^l_*\psi^l_*)(\mathrm{id} + \phi^r_*\psi^r_*).
\end{aligned}$$

In the last equivalence, we are using the fact that

$$\phi^l_*\psi^l_*\phi^r_*\psi^r_* \simeq 0$$

since $\phi^l_*\psi^l_* \simeq \mathcal{V}^{-1}\phi^l_*(\mathcal{U}\phi^l_* + \mathcal{V}\psi^l_*)$ (and similarly for $\phi^r_*\psi^r_*$), so

$$\phi^l_*\psi^l_*\phi^r_*\psi^r_* \simeq \mathcal{V}^{-2}\phi^l_*\phi^r_*(\mathcal{U}\phi^l_* + \mathcal{V}\psi^l_*)(\mathcal{U}\phi^r_* + \mathcal{V}\psi^r_*) \simeq \mathcal{V}^{-2}\phi^l_*\phi^r_*(\mathcal{U}\phi^l_* + \mathcal{V}\psi^l_*)^2 \simeq 0.$$

Hence, the map in equation (15.3) is homotopy equivalent to

$$\mathcal{S}_* + (\mathrm{id} + \phi^l_*\psi^l_*)(\mathrm{id} + \phi^r_*\psi^r_*)\mathcal{T}_*.$$

Arguing similarly to the proof of Lemma 15.4, one sees that the above map is homotopic to the map whose cone is

$$({}_{\mathcal{K}}\mathcal{T}^{\mathcal{K}}, {}_{\mathcal{K}}\mathcal{T}^{\mathcal{K}}) \mathbin{\hat{\boxtimes}} {}_{\mathcal{K}|\mathcal{K}}[\mathbb{I}^{\supset}]_{\mathbb{F}[U]},$$

completing the proof. ∎

Lemma 15.7. *There is a homotopy equivalence*

$$({}_{\mathcal{K}}\mathcal{T}^{\mathcal{K}}, {}_{\mathcal{K}}\mathcal{T}^{\mathcal{K}}) \mathbin{\hat{\boxtimes}} {}_{\mathcal{K}|\mathcal{K}}M^{\mathcal{K}} \mathbin{\hat{\boxtimes}} {}_{\mathcal{K}}\mathcal{T}^{\mathcal{K}} \simeq {}_{\mathcal{K}|\mathcal{K}}M^{\mathcal{K}}.$$

Proof. The proof is similar to the proof of Lemma 15.5. We observe first by computing algebraically using the techniques of Lemma 15.4 that $({}_{\mathcal{K}}\mathcal{T}^{\mathcal{K}}, {}_{\mathcal{K}}\mathcal{T}^{\mathcal{K}}) \mathbin{\hat{\boxtimes}} {}_{\mathcal{K}|\mathcal{K}}M^{\mathcal{K}} \mathbin{\hat{\boxtimes}} {}_{\mathcal{K}}\mathcal{T}^{\mathcal{K}}$ is homotopy equivalent to the mapping cone

$$[\mathbf{m}_0] \xrightarrow{\mathcal{S}^1_* + (\mathrm{id} + (\phi^l_* + \phi^r_*)(\psi^l_* + \psi^r_*))(\mathrm{id} + \phi^l_*\psi^l_*)(\mathrm{id} + \phi^r_*\psi^r_*)\mathcal{T}^1_*} [\mathbf{m}_1]. \tag{15.4}$$

One observes by direct computation that

$$(\mathrm{id} + (\phi^l_* + \phi^r_*)(\psi^l_* + \psi^r_*))(\mathrm{id} + \phi^l_*\psi^l_*)(\mathrm{id} + \phi^r_*\psi^r_*) \simeq (\mathrm{id} + \phi^l_*\psi^r_*)(\mathrm{id} + \psi^l_*\phi^r_*).$$

We define the mapping cone complexes

$${}_{\mathcal{K}|\mathcal{K}}X := \quad [\mathbf{m}_0] \xrightarrow{\mathcal{S}^1_* + (\mathrm{id}+\phi^l_*\psi^r_*)(\mathrm{id}+\psi^l_*\phi^r_*)\mathcal{T}^1_*} [\mathbf{m}_1],$$

and

$${}_{\mathcal{K}|\mathcal{K}}Y := \quad [\mathbf{m}_0] \xrightarrow{\mathcal{S}^1_* + (\mathrm{id}+\psi^l_*\phi^r_*)\mathcal{T}^1_*(\mathrm{id}+\phi^l_*\psi^r_*)} [\mathbf{m}_1].$$

We now claim that ${}_{\mathcal{K}|\mathcal{K}}X$ and ${}_{\mathcal{K}|\mathcal{K}}Y$ are homotopy equivalent. We construct a homotopy equivalence F by way of the following diagram:

$$\begin{tikzcd}
{}_{\mathcal{K}|\mathcal{K}}X^{\mathcal{K}} \arrow[d, "F"] \arrow[dr, phantom, "="] & {[\mathbf{m}_0]} \arrow[r, "\mathcal{S}^1_* + (\mathrm{id}+\phi^l_*\psi^r_*)(\mathrm{id}+\psi^l_*\phi^r_*)\mathcal{T}^1_*"] \arrow[d, "\mathrm{id}+\phi^l_*\psi^r_*"] \arrow[dr, dashed] & {[\mathbf{m}_1]} \arrow[d, "\mathrm{id}+\phi^l_*\psi^r_*"] \\
{}_{\mathcal{K}|\mathcal{K}}Y^{\mathcal{K}} & {[\mathbf{m}_0]} \arrow[r, "\mathcal{S}^1_* + (\mathrm{id}+\psi^l_*\phi^r_*)\mathcal{T}^1_*(\mathrm{id}+\phi^l_*\psi^r_*)"] & {[\mathbf{m}_1]}
\end{tikzcd}$$

The length 2 map above is a morphism which realizes the chain homotopies

$$(\mathrm{id} + \phi^l_*\psi^r_*)^2(\mathrm{id} + \psi^l_*\phi^r_*)\mathcal{T}^1_* \simeq (\mathrm{id} + \psi^l_*\phi^r_*)\mathcal{T}^1_*(\mathrm{id} + \phi^l_*\psi^r_*)^2,$$

and

$$[\mathrm{id} + \phi^l_*\psi^r_*, \mathcal{S}^1_*] \simeq 0.$$

Note that by a similar construction we can construct a map in the opposite direction:

$$\begin{tikzcd}
{}_{\mathcal{K}|\mathcal{K}}Y^{\mathcal{K}} \arrow[d, "G"] \arrow[dr, phantom, "="] & {[\mathbf{m}_0]} \arrow[r, "\mathcal{S}^1_* + (\mathrm{id}+\psi^l_*\phi^r_*)\mathcal{T}^1_*(\mathrm{id}+\phi^l_*\psi^r_*)"] \arrow[d, "\mathrm{id}+\phi^l_*\psi^r_*"] \arrow[dr, dashed] & {[\mathbf{m}_1]} \arrow[d, "\mathrm{id}+\phi^l_*\psi^r_*"] \\
{}_{\mathcal{K}|\mathcal{K}}X^{\mathcal{K}} & {[\mathbf{m}_0]} \arrow[r, "\mathcal{S}^1_* + (\mathrm{id}+\phi^l_*\psi^r_*)(\mathrm{id}+\psi^l_*\phi^r_*)\mathcal{T}^1_*"] & {[\mathbf{m}_1]}
\end{tikzcd}$$

Note that since $(\mathrm{id} + \phi^l_*\psi^r_*)^2 \simeq \mathrm{id}$ as a map from $[\mathbf{m}_i]$ to $[\mathbf{m}_i]$, it follows that $G \circ F$ is homotopic to a map $J\colon {}_{\mathcal{K}|\mathcal{K}}X^{\mathcal{K}} \to {}_{\mathcal{K}|\mathcal{K}}X^{\mathcal{K}}$ which takes the following form:

$$\begin{tikzcd}
{}_{\mathcal{K}|\mathcal{K}}X^{\mathcal{K}} \arrow[d, "J"] \arrow[dr, phantom, "="] & {[\mathbf{m}_0]} \arrow[r, "\mathcal{S}^1_* + (\mathrm{id}+\phi^l_*\psi^r_*)(\mathrm{id}+\psi^l_*\phi^r_*)\mathcal{T}^1_*"] \arrow[d, "\mathrm{id}"] \arrow[dr, dashed] & {[\mathbf{m}_1]} \arrow[d, "\mathrm{id}"] \\
{}_{\mathcal{K}|\mathcal{K}}X^{\mathcal{K}} & {[\mathbf{m}_0]} \arrow[r, "\mathcal{S}^1_* + (\mathrm{id}+\phi^l_*\psi^r_*)(\mathrm{id}+\psi^l_*\phi^r_*)\mathcal{T}^1_*"] & {[\mathbf{m}_1]}
\end{tikzcd}$$

Note that $J^2 = \mathrm{id}$. Therefore, the map F has a left homotopy inverse $J \circ G$ since $J \circ G \circ F \simeq \mathrm{id}$. One may modify the construction above to construct a right inverse to F by similar reasoning. By basic algebra, the left and right inverses of F must be homotopic. Hence F is a homotopy equivalence.

Finally, we use Lemma 15.4 to see that

$$(\mathrm{id} + \psi^l_* \phi^r_*)\mathcal{T}^1_*(\mathrm{id} + \phi^l_* \psi^r_*) \simeq (\mathrm{id} + \psi^l_* \phi^r_*)^2 \mathcal{T}^1_* \simeq \mathcal{T}^1_*.$$

It follows that the $({}_{\mathcal{K}}\mathcal{T}^{\mathcal{K}}, {}_{\mathcal{K}}\mathcal{T}^{\mathcal{K}}) \mathbin{\widehat{\boxtimes}} {}_{\mathcal{K}|\mathcal{K}}M^{\mathcal{K}} \boxtimes {}_{\mathcal{K}}\mathcal{T}^{\mathcal{K}}$ (shown in equation (15.4)) is homotopy equivalent to the mapping cone

$$\mathrm{Cone}(\mathcal{S}^1_* + \mathcal{T}^1_* \colon {}_{\mathcal{K}|\mathcal{K}}[\mathbf{m}_0]^{\mathcal{K}} \to {}_{\mathcal{K}|\mathcal{K}}[\mathbf{m}_1]^{\mathcal{K}})$$

which is the definition of ${}_{\mathcal{K}|\mathcal{K}}M^{\mathcal{K}}$. ■

Proposition 15.3 is now proven as follows. Lemma 15.5 implies that

$${}_{\mathcal{K}|\mathcal{K}}W^{\mathcal{K}}_{\alpha\beta,\alpha} \mathbin{\widehat{\boxtimes}} {}_{\mathcal{K}}[\mathcal{D}_0]_{\mathbb{F}[U]} \simeq ({}_{\mathcal{K}}\mathcal{T}^{\mathcal{K}}, {}_{\mathcal{K}}\mathcal{T}^{\mathcal{K}}) \mathbin{\widehat{\boxtimes}} {}_{\mathcal{K}|\mathcal{K}}M^{\mathcal{K}} \mathbin{\widehat{\boxtimes}} {}_{\mathcal{K}}[\mathcal{D}_0]_{\mathbb{F}[U]}.$$

Lemmas 14.7 and 15.7 imply that

$$({}_{\mathcal{K}}\mathcal{T}^{\mathcal{K}}, {}_{\mathcal{K}}\mathcal{T}^{\mathcal{K}}) \mathbin{\widehat{\boxtimes}} {}_{\mathcal{K}|\mathcal{K}}M^{\mathcal{K}} \mathbin{\widehat{\boxtimes}} {}_{\mathcal{K}}[\mathcal{D}_0]_{\mathbb{F}[U]} \simeq {}_{\mathcal{K}|\mathcal{K}}M^{\mathcal{K}} \mathbin{\widehat{\boxtimes}} {}_{\mathcal{K}}\mathcal{T}^{\mathcal{K}} \mathbin{\widehat{\boxtimes}} {}_{\mathcal{K}}[\mathcal{D}_0]_{\mathbb{F}[U]}.$$

Lemma 15.5 implies that the above is homotopy equivalent to

$${}_{\mathcal{K}|\mathcal{K}}M^{\mathcal{K}} \mathbin{\widehat{\boxtimes}} {}_{\mathcal{K}}[\mathcal{D}_0]_{\mathbb{F}[U]},$$

which is the definition of ${}_{\mathcal{K}|\mathcal{K}}[\mathbb{I}^{\Supset}]_{\mathbb{F}[U]}$.

15.4 Relation with the bordered invariant of $S^1 \times P$

In this section, we sketch why the bimodules ${}_{\mathcal{K}|\mathcal{K}}W^{\mathcal{K}}_{\alpha\beta,\beta}$ and ${}_{\mathcal{K}|\mathcal{K}}W^{\mathcal{K}}_{\alpha\beta,\alpha}$ can naturally be interpreted as the bordered invariants for $S^1 \times P$, where P denotes a 2-dimensional disk with 2 subdisks removed. In fact, this is a consequence of the connected sum formula and a computation from [3] of the invariant for the identity mapping cylinder over the 2-torus.

Proposition 15.8. *The bimodules* ${}_{\mathcal{K}|\mathcal{K}}W^{\mathcal{K}}_{\alpha\beta,\beta}$ *and* ${}_{\mathcal{K}|\mathcal{K}}W^{\mathcal{K}}_{\alpha\beta,\alpha}$ *are the bordered invariants for* $S^1 \times P$*, for various choices of arc systems on the boundary components.*

Remark 15.9. The arc systems on the boundary are compatible with the following gluing convention:

1. We glue alpha bordered boundary components to alpha bordered boundary components, and vice-versa.
2. ${}_{\mathcal{K}|\mathcal{K}}\mathbb{I}^{\Supset}$ has one alpha bordered boundary and one beta bordered boundary component.

Since we switch from type-D to type-A by tensoring with ${}_{\mathcal{K}|\mathcal{K}}\mathbb{I}^{\supseteq}$, this convention means that a type-A component of $\mathcal{K}$ is "alpha bordered" if the arc system is beta-parallel, whereas a type-D component of $\mathcal{K}$ is "alpha bordered" if the corresponding arc system is alpha-parallel.

Proof of Proposition 15.8. A surgery description of $S^1 \times P$ is given in Figure 15.1. We can view the 5-component link appearing in this surgery description as being obtained as the connected sum of two copies of a 3-component link J, as shown in Figure 15.1. (The link J is a connected sum of two Hopf links.) Write $\mathcal{A}_{\alpha,\beta}$ for an arc system on J where the top component is alpha-parallel, and the bottom component is beta-parallel. The middle component can have either alpha- or beta-parallel arc. Write ${}_{\mathcal{K}}\mathcal{X}_0(J, \mathcal{A}_{\alpha,\beta})^{\mathcal{K}}$ for the DA bimodule where the left action is the top-most component of J, and the right action is for the bottom-most component. The middle component is surgered out. (I.e., we take the type-D module for the 3-component link, tensor ${}_{\mathcal{K}}\mathcal{D}_0$ to the middle component, and tensor ${}_{\mathcal{K}|\mathcal{K}}\mathbb{I}^{\supseteq}$ to the top component.) It is proven in [3, Theorem 1.7] that

$$ {}_{\mathcal{K}}\mathcal{X}_0(J, \mathcal{A}_{\alpha,\beta})^{\mathcal{K}} \simeq {}_{\mathcal{K}}\mathcal{X}_0(J, \mathcal{A}_{\beta,\alpha})^{\mathcal{K}} \simeq {}_{\mathcal{K}}[\mathbb{I}]^{\mathcal{K}}. \tag{15.5} $$

(This computation is obtained by tensoring two copies of the Hopf link complex which we compute in Chapter 16 of the present memoir.) It follows from our connected sum formula that if ${}_{\mathcal{K}|\mathcal{K}}\mathcal{X}(L)^{\mathcal{K}}$ denotes the surgery module for the 5-component link describing $S^1 \times P$, then

$$ {}_{\mathcal{K}|\mathcal{K}}\mathcal{X}(L)^{\mathcal{K}} \simeq ({}_{\mathcal{K}}\mathcal{X}_0(J, \mathcal{A}_{\alpha,\beta})^{\mathcal{K}}, {}_{\mathcal{K}}\mathcal{X}_0(J, \mathcal{A}_{\beta,\alpha})^{\mathcal{K}}) \mathbin{\hat{\boxtimes}} {}_{\mathcal{K}|\mathcal{K}}W^{\mathcal{K}}_{\beta\alpha,\alpha}. $$

By equation (15.5), the above is homotopy equivalent to ${}_{\mathcal{K}|\mathcal{K}}W^{\mathcal{K}}_{\beta\alpha,\alpha}$, completing the proof. ■

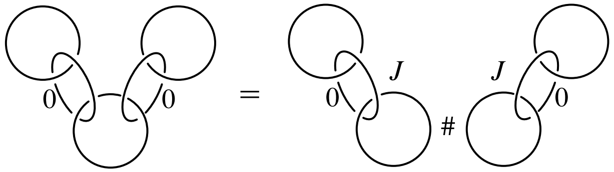

Figure 15.1. A Kirby diagram for $S^1 \times P$. The components marked 0 are surgered, while a tubular neighborhood of the unmarked components is removed. On the right, we describe the 5-component link as a connected sum of two copies of a 3-component link J.