

Chapter 16

The link surgery complex of the Hopf link

In this chapter, we compute the link surgery complex of the Hopf link. Our main result is Proposition 16.1.

16.1 Computation of the surgery complex

In this section, we compute the surgery hypercube for the Hopf link. We use the following model of the Hopf link:

$$\mathcal{EFL}(H) \cong \begin{array}{ccc} \mathbf{a} \leftarrow u_2 - & \mathbf{b} & \\ \uparrow & & \downarrow \\ v_1 & & u_1 \\ | & & \downarrow \\ \mathbf{c} - v_2 \rightarrow & \mathbf{d} & \end{array} \quad (16.1)$$

This complex is realized by the diagram in Figure 16.1. This complex is for the *negative* Hopf link, i.e., the two components have linking number -1 . The complex of the positive Hopf link has a similar model. The techniques of this section may also be applied, essentially verbatim, to the positive Hopf link, though we focus on the negative one for concreteness.

The Alexander gradings $A = (A_1, A_2)$ and Maslov gradings $\text{gr} = (\text{gr}_{\mathbf{w}}, \text{gr}_{\mathbf{z}})$ of the generators are

$$\begin{aligned} A(\mathbf{a}) &= \left(-\frac{1}{2}, \frac{1}{2}\right), & \text{gr}(\mathbf{a}) &= (0, 0), \\ A(\mathbf{b}) &= \left(-\frac{1}{2}, -\frac{1}{2}\right), & \text{gr}(\mathbf{b}) &= (-1, 1), \\ A(\mathbf{c}) &= \left(\frac{1}{2}, \frac{1}{2}\right), & \text{gr}(\mathbf{c}) &= (1, -1), \\ A(\mathbf{d}) &= \left(\frac{1}{2}, -\frac{1}{2}\right), & \text{gr}(\mathbf{d}) &= (0, 0). \end{aligned}$$

Note that we are using the conventions where the top degree generator of $\text{HF}^-(S^3, \mathbf{w})$ is in Maslov grading 0. See Remark 6.3.

The main result of this section is the following result.

Proposition 16.1. *Suppose that $\Lambda = (\lambda_1, \lambda_2)$ is an integral framing on the negative Hopf link H . Let $\mathcal{A}_{\alpha\alpha}$ be a system of arcs for the Hopf link where both arcs are*

alpha-parallel. The link surgery hypercube $\mathcal{C}_\Lambda(H, \mathcal{A}_{\alpha\alpha})$ has the following maps (up to overall homotopy equivalence):

- (1) *The maps $\Phi_{H,L_2}^{L_1}$ and $\Phi_{L_1,\emptyset}^{L_1}$ are the canonical inclusions of localization. The maps $\Phi_{H,L_2}^{-L_1}$ and $\Phi_{L_1,\emptyset}^{-L_1}$ are both given by the following formula:*

$$\Phi^{-L_1} = \begin{array}{l} \mathbf{a} \longmapsto \mathcal{V}_1^{\lambda_1-1} \mathbf{d} \\ \mathbf{b} \longmapsto 0 \\ \mathbf{c} \longmapsto \mathcal{V}_1^{\lambda_1+1} \mathbf{b} + \mathcal{V}_1^{\lambda_1} \mathcal{U}_2 \mathbf{c} \\ \mathbf{d} \longmapsto \mathcal{V}_1^{\lambda_1} \mathcal{U}_2 \mathbf{d} \end{array}$$

- (2) *The maps $\Phi_{H,L_1}^{L_2}$ and $\Phi_{L_2,\emptyset}^{L_2}$ are the canonical inclusions of localization. The maps $\Phi_{H,L_1}^{-L_2}$ and $\Phi_{L_2,\emptyset}^{-L_2}$ are both given by the following formula:*

$$\Phi^{-L_2} = \begin{array}{l} \mathbf{a} \longmapsto \mathcal{U}_1 \mathcal{V}_2^{\lambda_2} \mathbf{a} \\ \mathbf{b} \longmapsto 0 \\ \mathbf{c} \longmapsto \mathcal{V}_2^{\lambda_2+1} \mathbf{b} + \mathcal{U}_1 \mathcal{V}_2^{\lambda_2} \mathbf{c} \\ \mathbf{d} \longmapsto \mathcal{V}_2^{\lambda_2-1} \mathbf{a} \end{array}$$

- (3) *The length 2 map $\Phi_{H,\emptyset}^{-H}$ is given by the following formula:*

$$\Phi_{H,\emptyset}^{-H} = \begin{array}{l} \mathbf{a} \longmapsto \mathcal{V}_1^{\lambda_1-2} \mathcal{V}_2^{\lambda_2-1} \mathbf{c} \\ \mathbf{b} \longmapsto 0 \\ \mathbf{c} \longmapsto \mathcal{V}_1^{\lambda_1-1} \mathcal{V}_2^{\lambda_2} \mathbf{d} \\ \mathbf{d} \longmapsto \mathcal{V}_1^{\lambda_1-1} \mathcal{V}_2^{\lambda_2-2} \mathbf{c} \end{array}$$

The length 2 maps for other orientations of the Hopf link vanish.

We now compute the link surgery hypercube maps for the Hopf link. For the purposes of computation, it is easier to compute the induced maps after quotienting variables, instead of after localizing. Since the link surgery hypercube maps are defined by computing holomorphic polygons on diagrams where we have deleted certain base points, this is easily seen to contain equivalent information as the ordinary surgery hypercube. Compare Lemma 6.5.

As an example, if $L_1 \subseteq H$, then the descent map from surgery hypercube takes the form

$$\Phi_{H,H \setminus L_1}^{-L_1} : \mathcal{CF}\mathcal{L}(H) \rightarrow \mathcal{V}_1^{-1} \cdot \mathcal{CF}\mathcal{L}(H).$$

The map $\Phi_{H,H\setminus L_1}^{-L_1}$ is completely determined by the choice of framing on H , and the induced homotopy equivalence on the level of quotients

$$\tilde{\Phi}_{H,H\setminus L_1}^{-L_1} : \mathcal{CFL}(H)/(\mathcal{U}_1 - 1) \rightarrow \mathcal{CFL}(H)/(\mathcal{V}_1 - 1).$$

In this section, we write $\tilde{\Phi}^{\vec{N}}$ for the descent map on the level of quotients of $\mathcal{CFL}(H)$.

Lemma 16.2. *Write L_1 and L_2 for the two components of the negative Hopf link. Up to overall equivalence, the maps for the Hopf link are as follows:*

(1) *The map*

$$\tilde{\Phi}_{H,L_2}^{-L_1} : \mathcal{CFL}(H)/(\mathcal{U}_1 - 1) \rightarrow \mathcal{CFL}(H)/(\mathcal{V}_1 - 1)$$

is given by the following formula:

$$\tilde{\Phi}_{H,L_2}^{-L_1} = \begin{array}{l} \mathbf{a} \longmapsto \mathbf{d} \\ \mathbf{b} \longmapsto 0 \\ \mathbf{c} \longmapsto \mathbf{b} + \mathcal{U}_2\mathbf{c} \\ \mathbf{d} \longmapsto \mathcal{U}_2\mathbf{d} \end{array}$$

extended $\mathbb{F}[\mathcal{U}_2, \mathcal{V}_2]$ -equivariantly, and sending \mathcal{V}_1 to \mathcal{U}_1 .

(2) *The map*

$$\tilde{\Phi}_{L_1,\emptyset}^{-L_1} : \mathcal{CFL}(H)/(\mathcal{U}_1 - 1, \mathcal{V}_2 - 1) \rightarrow \mathcal{CFL}(H)/(\mathcal{V}_1 - 1, \mathcal{V}_2 - 1)$$

is given by the same formula as $\tilde{\Phi}_{H,L_2}^{-L_1}$, except with \mathcal{V}_2 set to 1.

(3) *The map*

$$\tilde{\Phi}_{H,L_1}^{-L_2} : \mathcal{CFL}(H)/(\mathcal{U}_2 - 1) \rightarrow \mathcal{CFL}(H)/(\mathcal{V}_2 - 1)$$

is given by the following formula:

$$\tilde{\Phi}_{H,L_1}^{-L_2} = \begin{array}{l} \mathbf{a} \longmapsto \mathcal{U}_1\mathbf{a} \\ \mathbf{b} \longmapsto 0 \\ \mathbf{c} \longmapsto \mathbf{b} + \mathcal{U}_1\mathbf{c} \\ \mathbf{d} \longmapsto \mathbf{a} \end{array}$$

extended equivariantly over $\mathbb{F}[\mathcal{U}_1, \mathcal{V}_1]$, and sending \mathcal{V}_2 to \mathcal{U}_2 .

(4) *The map*

$$\tilde{\Phi}_{L_2,\emptyset}^{-L_2} : \mathcal{CFL}(H)/(\mathcal{V}_1 - 1, \mathcal{U}_2 - 1) \rightarrow \mathcal{CFL}(H)/(\mathcal{V}_1 - 1, \mathcal{V}_2 - 1)$$

is given by the same formula as $\tilde{\Phi}_{H,L_1}^{-L_2}$, except with \mathcal{V}_1 set to 1.

(5) *The length 2 map*

$$\tilde{\Phi}_{H,\emptyset}^{-H} : \mathcal{CFL}(H)/(\mathcal{U}_1 - 1, \mathcal{U}_2 - 1) \rightarrow \mathcal{CFL}(H)/(\mathcal{V}_1 - 1, \mathcal{V}_2 - 1)$$

is given by the following formula:

$$\tilde{\Phi}_{H,\emptyset}^{-H} = \begin{array}{l} \mathbf{a} \longmapsto \mathbf{c} \\ \mathbf{b} \longmapsto 0 \\ \mathbf{c} \longmapsto \mathbf{d} \\ \mathbf{d} \longmapsto \mathbf{c} \end{array}$$

extended to send \mathcal{V}_1 to \mathcal{U}_1 and \mathcal{V}_2 to \mathcal{U}_2 .

(6) *The length 1 maps with positively oriented components, e.g., $\tilde{\Phi}_{H,L_2}^{L_1}$, are the canonical quotient maps. The length 2 maps vanish for any orientation other than $-H$.*

Remark 16.3. As we will see in the proof of the above lemma, all of the maps in the surgery hypercube for H are uniquely determined by the chain complex $\mathcal{CFL}(H)$, except for the length 2 map $\tilde{\Phi}_{H,L_2}^{L_1}$, and up to homotopy there are two possible choices. We will use additional algebraic restrictions on the link surgery hypercube to determine which choice of length 2 map is correct.

Proof of Lemma 16.2. We focus first on the length 1 maps. Consider the following complexes:

$$\mathcal{C}/(\mathcal{U}_1 - 1) = \begin{array}{ccc} \mathbf{a} \leftarrow \mathcal{U}_2 - \mathbf{b} & & \\ \uparrow & & | \\ \mathcal{V}_1 & & \mathbf{1} \\ | & & \downarrow \\ \mathbf{c} - \mathcal{V}_2 \rightarrow \mathbf{d} & & \end{array} \quad \text{and} \quad \mathcal{C}/(\mathcal{V}_1 - 1) = \begin{array}{ccc} \mathbf{a} \leftarrow \mathcal{U}_2 - \mathbf{b} & & \\ \uparrow & & | \\ \mathbf{1} & & \mathcal{U}_1 \\ | & & \downarrow \\ \mathbf{c} - \mathcal{V}_2 \rightarrow \mathbf{d} & & \end{array}$$

The map $\tilde{\Phi}_{H,L_2}^{-L_1}$ is a homotopy equivalence between these two complexes, which is $\mathbb{F}[\mathcal{U}_2, \mathcal{V}_2]$ -equivariant, and sends \mathcal{V}_1 to \mathcal{U}_1 . We perform the following changes of basis:

$$\mathcal{C}/(\mathcal{U}_1 - 1) \cong \begin{array}{ccc} \mathbf{a} & \mathbf{b} & \\ \uparrow & | & \\ \mathcal{V}_1 + \mathcal{U}_2 \mathcal{V}_2 & \mathbf{1} & \\ | & \downarrow & \\ \mathbf{c} + \mathcal{V}_2 \mathbf{b} & \mathbf{d} + \mathcal{U}_2 \mathbf{a} & \end{array} \quad \text{and} \quad \mathcal{C}/(\mathcal{V}_1 - 1) \cong \begin{array}{ccc} \mathbf{a} + \mathcal{V}_2 \mathbf{d} & \mathbf{b} + \mathcal{U}_2 \mathbf{c} & \\ \uparrow & | & \\ \mathbf{1} & \mathcal{U}_1 + \mathcal{U}_2 \mathcal{V}_2 & \\ | & \downarrow & \\ \mathbf{c} & \mathbf{d} & \end{array} \tag{16.2}$$

We may view each of the above complexes as the knot Floer complex for an unknot, with an extra free base point. In the following, we write $\text{gr}_{\mathbf{w}}$ for the grading induced by the complete collection of base points consisting of the \mathbf{w} -type link base points (i.e., the ones where the unknot intersects the Heegaard surface negatively), and all the free base points. We write $\text{gr}_{\mathbf{z}}$ for the grading induced by the complete collection

consisting of the \mathbf{z} -type base points and the free base points. We may easily compute the $(\text{gr}_w, \text{gr}_z)$ -gradings of all of the quotiented complexes, viewed in this manner:

$$\begin{array}{ll}
 \mathcal{C}/(\mathcal{U}_1 - 1) : & \begin{array}{l} \mathbf{a} \xrightarrow{(\text{gr}_w, \text{gr}_z)} (0, 0) \\ \mathbf{b} \longrightarrow (-1, 1) \\ \mathbf{c} \longrightarrow (-1, -1) \\ \mathbf{d} \longrightarrow (-2, 0) \end{array} & \mathcal{C}/(\mathcal{V}_1 - 1) : & \begin{array}{l} \mathbf{a} \longrightarrow (0, -2) \\ \mathbf{b} \longrightarrow (-1, -1) \\ \mathbf{c} \longrightarrow (1, -1) \\ \mathbf{d} \longrightarrow (0, 0) \end{array} \\
 \\
 \mathcal{C}/(\mathcal{U}_2 - 1) : & \begin{array}{l} \mathbf{a} \longrightarrow (-2, 0) \\ \mathbf{b} \longrightarrow (-1, 1) \\ \mathbf{c} \longrightarrow (-1, -1) \\ \mathbf{d} \longrightarrow (0, 0) \end{array} & \mathcal{C}/(\mathcal{V}_2 - 1) : & \begin{array}{l} \mathbf{a} \longrightarrow (0, 0), \\ \mathbf{b} \longrightarrow (-1, -1) \\ \mathbf{c} \longrightarrow (1, -1) \\ \mathbf{d} \longrightarrow (0, -2) \end{array}
 \end{array}$$

The homotopy equivalence $\tilde{\Phi}_{H,L_2}^{-L_1}$ is grading preserving with respect to the above gradings, since it is the map for changing Heegaard diagrams. It is straightforward to see from equation (16.2) that there is a unique choice of a non-zero chain map from $\mathcal{C}/(\mathcal{U}_1 - 1)$ to $\mathcal{C}/(\mathcal{V}_1 - 1)$ which is $(\text{gr}_w, \text{gr}_z)$ -grading preserving, is $\mathbb{F}[\mathcal{U}_2, \mathcal{V}_2]$ -equivariant, and sends \mathcal{V}_1 to \mathcal{U}_1 . (Note here we are using the $(\text{gr}_w, \text{gr}_z)$ -bigrading obtained by viewing $\mathcal{C}/(\mathcal{U}_1 - 1)$ and $\mathcal{C}/(\mathcal{V}_1 - 1)$ as the complex of an unknot, shown above; we are not using the gradings naively inherited from the Hopf link.) This map is the homotopy equivalence $\tilde{\Phi}_{H,L_2}^{-L_1}$ in the map in the statement. By a symmetric argument, one obtains the formula for $\tilde{\Phi}_{H,L_1}^{-L_2}$ in the statement.

Next, by the definition of a σ -basic system, the map $\tilde{\Phi}_{L_1, \emptyset}^{-L_1}$ is obtained by quotienting the domain and range of $\tilde{\Phi}_{H,L_2}^{-L_1}$ by $\mathcal{V}_2 - 1$. The maps $\tilde{\Phi}_{H,L_1}^{-L_2}$ and $\tilde{\Phi}_{L_2, \emptyset}^{-L_2}$ are related by a similar relation.

We now consider the diagonal map of the cube. Firstly, since we are working with a basis system, the length 2 map vanishes except for the orientation $-H$ (i.e., all components oriented negatively). In our present notation, this is the map

$$\tilde{\Phi}_{H, \emptyset}^{-H} : \mathcal{C}/(\mathcal{U}_1 - 1, \mathcal{U}_2 - 1) \rightarrow \mathcal{C}/(\mathcal{V}_1 - 1, \mathcal{V}_2 - 1).$$

We define the map

$$E := \tilde{\Phi}_{L_1, \emptyset}^{-L_1} \circ \tilde{\Phi}_{H,L_1}^{-L_2} + \tilde{\Phi}_{L_2, \emptyset}^{-L_2} \circ \tilde{\Phi}_{H,L_2}^{-L_1}. \tag{16.3}$$

Using our computations of the length 1 maps of the cube, we compute E to be

$$E = \begin{array}{l} \mathbf{a} \longrightarrow \mathbf{a} + \mathbf{d} \\ \mathbf{b} \longrightarrow 0 \\ \mathbf{c} \longrightarrow (\mathcal{U}_1 + \mathcal{U}_2)\mathbf{c} \\ \mathbf{d} \longrightarrow \mathbf{a} + \mathbf{d}, \end{array}$$

extended linearly by sending \mathcal{V}_1 to \mathcal{U}_1 and \mathcal{V}_2 to \mathcal{U}_2 .

Up to an overall chain homotopy intertwining $\mathbb{F}[\mathcal{V}_1, \mathcal{V}_2]$ and $\mathbb{F}[\mathcal{U}_1, \mathcal{U}_2]$, there are two choices for a graded null-homotopy of the map E . These are the maps

$$H_1 := \begin{array}{l} \mathbf{a} \longmapsto \mathbf{c} \\ \mathbf{b} \longmapsto 0 \\ \mathbf{c} \longmapsto 0 \\ \mathbf{d} \longmapsto \mathbf{c}, \end{array} \quad \text{and} \quad H_2 := \begin{array}{l} \mathbf{a} \longmapsto \mathbf{c} \\ \mathbf{b} \longmapsto 0 \\ \mathbf{c} \longmapsto \mathbf{d} \\ \mathbf{d} \longmapsto \mathbf{c}. \end{array}$$

To prove this, we first observe that H_1 and H_2 are easily checked to be valid homotopies. Any other valid homotopy will differ from H_1 by a $+1$ graded chain map from $\mathcal{C}/(\mathcal{U}_1 - 1, \mathcal{U}_2 - 1)$ to $\mathcal{C}/(\mathcal{V}_1 - 1, \mathcal{V}_2 - 1)$. However, it is easily checked that up to chain homotopy, there is exactly one non-zero $+1$ graded map from $\mathcal{C}/(\mathcal{U}_1 - 1, \mathcal{U}_2 - 1)$ to $\mathcal{C}/(\mathcal{V}_1 - 1, \mathcal{V}_2 - 1)$ which sends \mathcal{V}_i to \mathcal{U}_i . This is the map which sends \mathbf{c} to \mathbf{d} , sends \mathcal{V}_i to \mathcal{U}_i , and vanishes on other generators. Adding this map to H_1 yields H_2 . Hence, H_1 and H_2 are the only possible choices, up to further homotopy. We prove in the subsequent Lemma 16.4 that H_1 is not a valid choice when we use the arc system $\mathcal{A}_{\alpha\alpha}$, and hence the length 2 map of the surgery hypercube is H_2 . ■

We let $(S^2, \alpha, \beta, \{w_1, w_2\}, \{z_1, z_2\})$ be the diagram shown in Figure 16.1, and we have $p \in \alpha$ for the point shown therein. We define a map A_p by counting disks weighted by the difference in multiplicities on the two sides of the point p . Note that the map A_p may be identified with the relative homology map considered in Section 13.4 if we pick a path λ which connects a base point on L_1 to a base point on L_2 and satisfies $\lambda \cap \alpha = \{p\}$.

The map A_p is defined on $\mathcal{CF}\mathcal{L}(H)$, and is easily computed by counting holomorphic bigons to be given by the formula

$$A_p = \begin{array}{ccc} \mathbf{a} & \xrightarrow{\mathcal{V}_2} & \mathbf{b} \\ \downarrow \mathcal{U}_1 & & \uparrow \mathcal{V}_1 \\ \mathbf{c} & \xleftrightarrow[\mathcal{U}_2]{} & \mathbf{d} \end{array} = \begin{array}{l} \mathbf{a} \longmapsto \mathcal{V}_2\mathbf{b} + \mathcal{U}_1\mathbf{c} \\ \mathbf{b} \longmapsto 0 \\ \mathbf{c} \longmapsto \mathcal{V}_2\mathbf{d} \\ \mathbf{d} \longmapsto \mathcal{U}_2\mathbf{c} + \mathcal{V}_1\mathbf{b}. \end{array}$$

The map A_p has an extension to hypercubes, similar to the construction described in Section 13.4. We can describe this by considering a formal endomorphism

$$F_p: \alpha \rightarrow \alpha,$$

and setting the hypercube action \mathcal{A}_p to be $\mu_2^{\text{Tw}}(F_p, -)$.

As in Lemma 13.31 we can extend the above construction to give an endomorphism \mathcal{A}_p of the link surgery formula when we are using a σ -basic system of Heegaard diagrams for H where the arc system consists of only alpha-parallel arcs. In

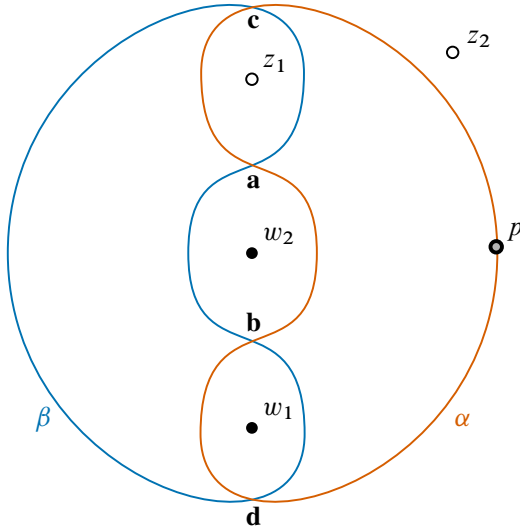


Figure 16.1. A diagram for the Hopf link with a special point $p \in \alpha$.

this case, the point-pushing diffeomorphisms never change α . We consider this hypercube on the 2-dimensional hypercube used to define all of the link surgery structure maps for negatively oriented sublinks of H . With these considerations in mind, we obtain a diagram of the following form:

$$\begin{array}{ccccc}
 \mathcal{C}/(\mathcal{U}_1 - 1, \mathcal{U}_2 - 1) & \xrightarrow{\tilde{\Phi}^{-L_1}} & \mathcal{C}/(\mathcal{V}_1 - 1, \mathcal{U}_2 - 1) & & \\
 \downarrow \tilde{\Phi}^{-L_2} & \searrow A_p & \downarrow h_p^{-L_1} & \swarrow A_p & \\
 & \mathcal{C}/(\mathcal{U}_1 - 1, \mathcal{U}_2 - 1) & \xrightarrow{\tilde{\Phi}^{-L_1}} & \mathcal{C}/(\mathcal{V}_1 - 1, \mathcal{U}_2 - 1) & \\
 \downarrow \tilde{\Phi}^{-L_2} & \searrow H & \downarrow \tilde{\Phi}^{-L_2} & \swarrow H & \downarrow \tilde{\Phi}^{-L_2} \\
 \mathcal{C}/(\mathcal{U}_1 - 1, \mathcal{V}_2 - 1) & \xrightarrow{\tilde{\Phi}^{-L_1}} & \mathcal{C}/(\mathcal{V}_1 - 1, \mathcal{V}_2 - 1) & & \\
 \downarrow A_p & \searrow h_p^{-L_2} & \downarrow \omega_p & \swarrow h_p^{-L_2} & \downarrow A_p \\
 \mathcal{C}/(\mathcal{U}_1 - 1, \mathcal{V}_2 - 1) & \xrightarrow{\tilde{\Phi}^{-L_1}} & \mathcal{C}/(\mathcal{V}_1 - 1, \mathcal{V}_2 - 1) & & \\
 & \downarrow h_p^{-L_1} & & & \\
 & \mathcal{C}/(\mathcal{U}_1 - 1, \mathcal{V}_2 - 1) & \xrightarrow{\tilde{\Phi}^{-L_1}} & \mathcal{C}/(\mathcal{V}_1 - 1, \mathcal{V}_2 - 1) &
 \end{array}
 \tag{16.4}$$

In the above diagram, both maps labeled $h_p^{-L_2}$ are induced by a single map from $\mathcal{C}/(\mathcal{U}_2 - 1)$ to $\mathcal{C}/(\mathcal{V}_2 - 1)$. The map labeled $h_p^{-L_1}$ is similar.

The diagram in equation (16.4) does not satisfy the hypercube relations. Instead, if we view it as $\text{Cone}(\mathcal{A}_p)$, where \mathcal{A}_p is a morphism from the back face to the front face, then

$$[\partial, \mathcal{A}_p] = (\mathcal{U}_1 \mathcal{V}_1 + \mathcal{U}_2 \mathcal{V}_2) \cdot \text{id}.$$

We now analyze the maps labeled $h_p^{-L_i}$. Consider first $h_p^{-L_2}$. Both instances of this map are induced by a single map from $\mathcal{C}/(\mathcal{U}_2 - 1)$ to $\mathcal{C}/(\mathcal{V}_2 - 1)$, for which we also write $h_p^{-L_2}$. This map preserves the Maslov bigradings, if we view $\mathcal{C}/(\mathcal{U}_2 - 1)$ and $\mathcal{C}/(\mathcal{V}_2 - 1)$ as the link Floer complexes for an unknot with an extra free base point. Extending the observation from the proof of Lemma 16.2 that we used to compute the length 1 maps, there were exactly two grading preserving chain maps from $\mathcal{C}/(\mathcal{U}_2 - 1)$ to $\mathcal{C}/(\mathcal{V}_2 - 1)$ which are $\mathbb{F}[\mathcal{U}_1, \mathcal{V}_1]$ -equivariant, and send \mathcal{V}_2 to \mathcal{U}_2 : the zero map, and the map labeled $\tilde{\Phi}^{-L_2}$. This observation constrains $h_p^{-L_2}$ to be one of two maps, which differ by the map $\tilde{\Phi}^{-L_2}$. The same comment applies for $h_p^{-L_1}$.

We compute that directly

$$[\tilde{\Phi}^{-L_1}, A_p] = 0.$$

Hence, by the above observation, we conclude that

$$h_p^{-L_1} = s \cdot \tilde{\Phi}^{-L_1} \tag{16.5}$$

for some $s \in \mathbb{F}$.

Similarly, we compute that

$$[\tilde{\Phi}^{-L_2}, A_p] = \begin{array}{l} \mathbf{a} \longmapsto 0 \\ \mathbf{b} \longmapsto 0 \\ \mathbf{c} \longmapsto \mathcal{U}_2 \mathbf{a} + \mathcal{U}_1 \mathbf{d} \\ \mathbf{d} \longmapsto 0. \end{array}$$

Hence,

$$h_p^{-L_2} = (\mathbf{c} \mapsto \mathbf{b}) + t \cdot \tilde{\Phi}^{-L_2}, \tag{16.6}$$

for some $t \in \mathbb{F}$.

Using the above observations, we are now able to compute the length 2 map of the link surgery hypercube of the Hopf link.

Lemma 16.4. *The length 3 hypercube relation in equation (16.4) is satisfiable with H_2 , but not with H_1 . Hence, H_2 is the length 2 map of the Hopf link surgery hypercube.*

Proof. For $i \in \{1, 2\}$, let C_i denote the sum of compositions of maps in (16.4) which would appear in the length 3 relation, except for $[\partial, \omega_p]$. Here, C_i is computed with the map $H = H_i$. The hypercube relations are satisfiable with H_i if and only if $C_i \simeq 0$.

We have determined the maps $h_p^{-L_2}$ and $h_p^{-L_1}$ up to a minor ambiguity, as shown in equations (16.5) and (16.6). Note that changing the value of s in $h_p^{-L_1}$ changes C_i

by addition of the map E from (16.3), which we already know to be null-homotopic. Changing t by adding 1 has the same effect. Hence, it is sufficient to consider the case where $s = t = 0$.

One easily computes

$$C_1 = [h_p^{-L_2}, \tilde{\Phi}^{-L_1}] + [H_1, A_p] = \begin{array}{l} \mathbf{a} \longmapsto \mathbf{d} \\ \mathbf{b} \longmapsto 0 \\ \mathbf{c} \longmapsto \mathbf{b} + \mathcal{U}_2\mathbf{c} \\ \mathbf{d} \longmapsto \mathbf{d}, \end{array}$$

which is not null-homotopic. Hence, H_1 is not a valid choice. It is easily checked that in fact

$$[h_p^{-L_2}, \tilde{\Phi}^{-L_1}] + [H_2, A_p] = 0.$$

Hence, the diagonal map must be H_2 . ■

Remark 16.5. View the maps H_1 and H_2 as being $\mathbb{F}[U_1, U_2]$ -equivariant, where U_i acts by \mathcal{V}_i on the domain, and \mathcal{U}_i on the codomain. The maps H_1 and H_2 are homotopic over $\mathbb{F}[U_1]$ or $\mathbb{F}[U_2]$, but not over $\mathbb{F}[U_1, U_2]$. If we wanted to compute the Heegaard Floer homology of surgeries on the Hopf link, either H_1 or H_2 would give the same answer. Compare [30, Section 5.2]. We will see in Section 16.3 that using H_1 to build the link surgery hypercube gives $\mathcal{C}_\Lambda(H, \mathcal{A}_{\alpha\beta})$ where $\mathcal{A}_{\alpha\beta}$ is a system of arcs where one arc is alpha-parallel and the other is beta-parallel.

Since we will be taking tensor products, we need to know the length 2 map over the ring $\mathbb{F}[U_1, U_2]$, and there are algebraically two distinct models.

It remains to describe the proof of Proposition 16.1.

Proof of Proposition 16.1. The proof is obtained by filling in powers of \mathcal{V}_i in the maps from Lemma 16.2 according to Alexander grading changes. We leave this mostly to the reader, but we verify the claim for the map $\Phi_{H,L_2}^{-L_1}$. This map sends Alexander grading \mathbf{s} to $\mathbf{s} + (\lambda_1, -1)$. Furthermore, it is $\mathbb{F}[\mathcal{U}_2, \mathcal{V}_2]$ -equivariant, and is T -equivariant with respect to $\mathbb{F}[\mathcal{U}_1, \mathcal{V}_1]$. As an example, the equivariance properties and the computation of the quotient map in Lemma 16.2 constrain $\Psi_{H,L_2}^{-L_1}(\mathbf{a}) = \mathcal{V}_1^N \mathbf{d}$ for some N . Since $A(\mathbf{a}) = (-\frac{1}{2}, \frac{1}{2})$ and $A(\mathbf{d}) = (\frac{1}{2}, -\frac{1}{2})$, we conclude that $N = \lambda_1 - 1$. Similar arguments hold for all of the other components. ■

16.2 The Hopf link complex as a type-D module

We now describe the type-D module over $\mathcal{L}_2 = \mathcal{K} \otimes \mathcal{K}$ for the Hopf link complex. We write $\mathcal{H}_\Lambda^{\mathcal{K} \otimes \mathcal{K}}$ for the type-D module for the negative Hopf link with an alpha-parallel system of arcs. We recall that by definition, for each $\varepsilon \in \mathbb{E}_2$, we define $\mathcal{H} \cdot \mathbf{I}_\varepsilon$

to have the same generators over \mathbb{F} as a free $\mathbb{F}[\mathcal{U}_1, \mathcal{V}_1, \mathcal{U}_2, \mathcal{V}_2]$ -basis of $\mathcal{CFL}(H)$. Hence, we write

$$\mathcal{H}_\Lambda^{\mathcal{K} \otimes \mathcal{K}} \cdot \mathbf{I}_\varepsilon = \text{Span}_{\mathbb{F}}(\mathbf{a}_\varepsilon, \mathbf{b}_\varepsilon, \mathbf{c}_\varepsilon, \mathbf{d}_\varepsilon).$$

It is helpful to write \mathcal{H}_ε for $\mathcal{H}_\Lambda \cdot \mathbf{I}_\varepsilon$. With this notation, we may decompose \mathcal{H}_Λ as

$$\mathcal{H}_\Lambda = \begin{array}{cc} \mathcal{H}_{0,0} & \mathcal{H}_{1,0} \\ \mathcal{H}_{0,1} & \mathcal{H}_{1,1} \end{array}$$

We now describe the structure maps.

Lemma 16.6. *Let $\mathcal{H}_\Lambda^{\mathcal{K} \otimes \mathcal{K}}$ be the type-D module associated to the negative Hopf link with an alpha-parallel system of arcs. The type-D module $\mathcal{H}_\Lambda^{\mathcal{K} \otimes \mathcal{K}}$ is as follows:*

$$\begin{array}{ccccc} \partial^1 \curvearrowright & \mathcal{H}_{0,0} & \xrightarrow{L_1P^1 + ^{-}L_1P^1} & \mathcal{H}_{1,0} & \curvearrowleft \partial^1 \\ & \downarrow & \dashrightarrow & \downarrow & \\ & L_2F^1 + ^{-}L_2F^1 & -H \Omega^1 & L_2G^1 + ^{-}L_2G^1 & \\ & \downarrow & & \downarrow & \\ \partial^1 \curvearrowright & \mathcal{H}_{0,1} & \xrightarrow{L_1Q^1 + ^{-}L_1Q^1} & \mathcal{H}_{1,1} & \curvearrowleft \partial^1 \end{array}$$

where the maps are as follows:

- (1) The map ∂^1 is the ordinary differential of the Hopf link complex. See equation (16.1).
- (2) The maps denoted L_1P^1 , $^{-}L_1P^1$, L_1Q^1 and $^{-}L_1Q^1$ correspond to the length 1 surgery maps for $\pm L_1$. The maps L_1P^1 and L_1Q^1 share the same formula as each other, and the maps $^{-}L_1P^1$ and $^{-}L_1Q^1$ share the same formula as each other. They are given by the following formulas:

$$(a) \quad \begin{array}{l} \mathbf{a} \longmapsto \mathbf{a} \otimes \sigma_1 \\ L_1P^1 = \mathbf{b} \longmapsto \mathbf{b} \otimes \sigma_1 \\ \mathbf{c} \longmapsto \mathbf{c} \otimes \sigma_1 \\ \mathbf{d} \longmapsto \mathbf{d} \otimes \sigma_1. \end{array}$$

$$(b) \quad \begin{array}{l} \mathbf{a} \longmapsto \mathbf{d} \otimes \mathcal{V}_1^{\lambda_1 - 1} \tau_1 \\ ^{-}L_1P^1 = \mathbf{b} \longmapsto 0 \\ \mathbf{c} \longmapsto \mathbf{b} \otimes \mathcal{V}_1^{\lambda_1 + 1} \tau_1 + \mathbf{c} \otimes \mathcal{U}_2 \mathcal{V}_1^{\lambda_1} \tau_1 \\ \mathbf{d} \longmapsto \mathbf{d} \otimes \mathcal{U}_2 \mathcal{V}_1^{\lambda_1} \tau_1. \end{array}$$

(3) The maps L_2F^1 and $-L_2F^1$ are the surgery maps for the component L_2 , with L_1 unreduced. They are given by the following formulas:

$$\begin{aligned}
 \text{(a)} \quad & \mathbf{a} \longmapsto \mathbf{a} \otimes \sigma_2 \\
 L_2F^1 = & \mathbf{b} \longmapsto \mathbf{b} \otimes \sigma_2 \\
 & \mathbf{c} \longmapsto \mathbf{c} \otimes \sigma_2 \\
 & \mathbf{d} \longmapsto \mathbf{d} \otimes \sigma_2. \\
 \\
 \text{(b)} \quad & \mathbf{a} \longmapsto \mathbf{a} \otimes \mathcal{U}_1 \mathcal{V}_2^{\lambda_2} \tau_2 \\
 -L_2F^1 = & \mathbf{b} \longmapsto 0 \\
 & \mathbf{c} \longmapsto \mathbf{b} \otimes \mathcal{V}_2^{\lambda_2+1} \tau_2 + \mathbf{c} \otimes \mathcal{U}_1 \mathcal{V}_2^{\lambda_2} \tau_2 \\
 & \mathbf{d} \longmapsto \mathbf{a} \otimes \mathcal{V}_2^{\lambda_2-1} \tau_2.
 \end{aligned}$$

The maps L_2G^1 and $-L_2G^1$ are the surgery maps L_2 , with L_1 already reduced. They are given by the same formulas.

(4) The map $-H\Omega^1$ is the length 2 surgery map for $-H = -L_1 \cup -L_2$. This map takes the following form:

$$\begin{aligned}
 -H\Omega^1 = & \mathbf{a} \longmapsto \mathbf{c} \otimes \mathcal{V}_1^{\lambda_1-2} \mathcal{V}_2^{\lambda_2-1} \tau_1 \tau_2 \\
 & \mathbf{b} \longmapsto 0 \\
 & \mathbf{c} \longmapsto \mathbf{d} \otimes \mathcal{V}_1^{\lambda_1-1} \mathcal{V}_2^{\lambda_2} \tau_1 \tau_2 \\
 & \mathbf{d} \longmapsto \mathbf{c} \otimes \mathcal{V}_1^{\lambda_1-1} \mathcal{V}_2^{\lambda_2-2} \tau_1 \tau_2.
 \end{aligned}$$

We leave the verification of the above lemma to the reader as it is mostly a restatement of Proposition 16.1 using the notation from Section 8.6.

16.3 The Hopf link with arc system $\mathcal{A}_{\alpha\beta}$

In this section, we compute the type-D module of the Hopf link complex when we use an arc system such that L_1 is alpha-parallel, and L_2 is beta-parallel. We write $\overline{\mathcal{H}}_{\Lambda}^{\mathcal{K} \otimes \mathcal{K}}$ for the associated type-D module.

Proposition 16.7. *The type-D module $\overline{\mathcal{H}}_{\Lambda}^{\mathcal{K} \otimes \mathcal{K}}$ is identical to the module $\mathcal{H}_{\Lambda}^{\mathcal{K} \otimes \mathcal{K}}$, except that we delete the term $\mathbf{c} \mapsto \mathbf{d} \otimes \mathcal{V}_1^{\lambda_1-1} \mathcal{V}_2^{\lambda_2} \tau_1 \tau_2$ from δ^1 .*

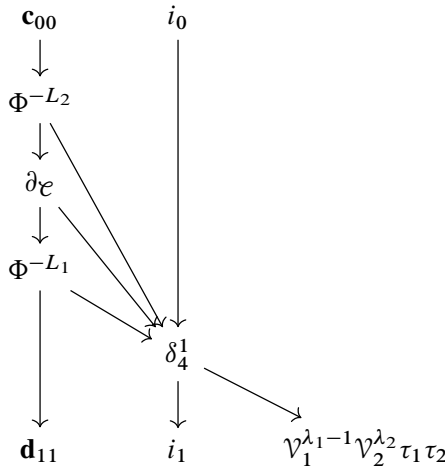
Proof. We may use Theorem 14.1 and the transformer bimodule to change one of the arcs from alpha-parallel to beta-parallel. Accordingly, we have that

$$\overline{\mathcal{H}}_{\Lambda}^{\mathcal{K}_1 \otimes \mathcal{K}_2} = \mathcal{H}_{\Lambda}^{\mathcal{K}_1 \otimes \mathcal{K}_2} \boxtimes \mathcal{K}_2 \mathcal{T} \mathcal{K}_2.$$

We claim that the above tensor product description of $\overline{\mathcal{H}}_{\Lambda}^{\mathcal{K}_1 \otimes \mathcal{K}_2}$ gives exactly the module structure described in the statement (i.e., with structure maps as in Lemma 16.6, except with the additional length 2 term $\mathbf{c} \mapsto \mathbf{d} \otimes \mathcal{V}_1^{\lambda_1-1} \mathcal{V}_2^{\lambda_2} \tau_1 \tau_2$). This is verified by a direct computation. We compute $\mathcal{H}_{\Lambda}^{\mathcal{K}_1 \otimes \mathcal{K}_2} \widehat{\boxtimes} \mathcal{K}_2 \mathcal{T}^{\mathcal{K}_2}$. The module $\mathcal{K}_2 \mathcal{T}^{\mathcal{K}_2}$ has a structure map term of the form $\delta_2^1(a, 1) = 1 \otimes a$, for all a . These contribute all of the differentials from $\mathcal{H}_{\Lambda}^{\mathcal{K}_1 \otimes \mathcal{K}_2}$ to $\overline{\mathcal{H}}_{\Lambda}^{\mathcal{K}_1 \otimes \mathcal{K}_2}$. Additionally, $\mathcal{K}_2 \mathcal{T}^{\mathcal{K}_2}$ has a δ_4^1 term, which is given by

$$\delta_4^1(a, b, c\tau, i_0) = i_1 \otimes \partial_{\mathcal{U}}(a) \partial_{\mathcal{V}}(b) c\tau.$$

The structure map diagram which adds the extra length 2 differential is the following:



In the above, Φ^{-L_2} , ∂e and Φ^{-L_1} denote specific summands of the structure map δ^1 on $\mathcal{H}_{\Lambda}^{\mathcal{K}_1 \otimes \mathcal{K}_2}$. The specific sequence of generators contributing to the above diagram is the following:

$$\begin{aligned} \mathbf{c}_{00} &\xrightarrow{\Phi^{-L_2}} \mathbf{c}_{01} \otimes \mathcal{U}_1 \mathcal{V}_2^{\lambda_2} \tau_2 \xrightarrow{\partial e} \mathbf{d}_{01} \otimes \mathcal{V}_2 \otimes \mathcal{U}_1 \mathcal{V}_2^{\lambda_2} \tau_2 \\ &\xrightarrow{\Phi^{-L_1}} \mathbf{d}_{11} \otimes \mathcal{U}_2 \mathcal{V}_1^{\lambda_1} \tau_1 \otimes \mathcal{V}_2 \otimes \mathcal{U}_1 \mathcal{V}_2^{\lambda_2} \tau_2. \end{aligned}$$

(Here, we omit all terms which evaluate to zero under $\mathbb{I} \otimes \delta_4^1$.) Applying δ_4^1 , we obtain the extra term

$$\mathbf{c}_{00} \mapsto \mathbf{d}_{11} \otimes \mathcal{V}_1^{\lambda_1} \tau_1 \mathcal{U}_1 \mathcal{V}_2^{\lambda_2} \tau_2 = \mathbf{d}_{11} \otimes \mathcal{V}_1^{\lambda_1-1} \mathcal{V}_2^{\lambda_2} \tau_1 \tau_2,$$

as claimed in the statement. Note that when evaluating δ_2^1 and δ_4^1 in the tensor product, we are using the extension of scalars construction from Section 3.5 to view $\mathcal{K}_1 \mathcal{T}^{\mathcal{K}_1}$ as a DA bimodule over $(\mathcal{K}_1 \otimes \mathcal{K}_2, \mathcal{K}_1 \otimes \mathcal{K}_2)$. ■

Remark 16.8. In our proof of Proposition 16.1, we observed that all of the maps in the Hopf link complex were uniquely determined except for the length 2 map, and there were two choices, up to overall homotopy. We wrote H_1 and H_2 for these maps. We proved that H_2 was the correct choice when the system of arcs was alpha-parallel. We observe that the above model of $\overline{\mathcal{H}}_{\Lambda}^{\mathcal{L}_2}$ corresponds exactly to the other choice of homotopy.