Chapter 17

Minimal models for the Hopf link surgery complex

In this chapter, we consider the link surgery complex for the Hopf link as a DA bimodule. We do this via a tensor product

$$ {}_{\mathcal{K}}\mathcal{H}_{\Lambda}^{\mathcal{K}} := \mathcal{H}_{\Lambda}^{\mathcal{K}\otimes\mathcal{K}} \hat{\boxtimes} {}_{\mathcal{K}|\mathcal{K}}[\mathbb{I}^{\supseteq}]. $$

In the above, just one algebra output of $\mathcal{H}_{\Lambda}^{\mathcal{K}\otimes\mathcal{K}}$ is input into ${}_{\mathcal{K}|\mathcal{K}}[\mathbb{I}^{\supseteq}]$.

We note that the number of generators in the complex $\mathcal{H}_{\Lambda}^{\mathcal{K}\otimes\mathcal{K}}$ can be reduced by homotopy equivalence. To this end, we define the maximal ideal $\mathfrak{m} \subseteq \mathcal{K}$, to be the one which is generated by $\mathcal{U}, \mathcal{V} \in \mathbf{I}_0 \cdot \mathcal{K} \cdot \mathbf{I}_0$, $\sigma, \tau \in \mathbf{I}_1 \cdot \mathcal{K} \cdot \mathbf{I}_0$ and $\mathcal{U} \in \mathbf{I}_1 \cdot \mathcal{K} \cdot \mathbf{I}_1$. We make the following definition (compare [6, page 476]).

Definition 17.1. We say that a type-DA bimodule ${}_{\mathcal{K}}\mathcal{X}^{\mathcal{K}}$ is *minimal* if $\delta_1^1 \colon \mathcal{X} \to \mathcal{X} \otimes \mathcal{K}/\mathfrak{m}$ is zero.

In the Heegaard Floer literature, one frequently uses the term *reduced* instead of minimal. In this chapter, we will explore a minimal model of the Hopf link complex ${}_{\mathcal{K}}\mathcal{H}_{\Lambda}^{\mathcal{K}}$, for which we will write ${}_{\mathcal{K}}\mathcal{Z}^{\mathcal{K}}$. We will also compute a minimal model of the complex ${}_{\mathcal{K}}\overline{\mathcal{H}}_{\Lambda}^{\mathcal{K}}$ obtained by using both alpha and beta-parallel arcs on the Hopf link.

17.1 DA-perspectives on the Hopf link complex

In this section, we describe how to view the Hopf link complex as a type-DA module, and we compute a minimal model. This perspective was first explored in [14].

Write $H = L_1 \cup L_2$ for the Hopf link. Write $P_1 = \mathbb{F}[\mathcal{U}_1, \mathcal{V}_1]$ and $P_2 = \mathbb{F}[\mathcal{U}_2, \mathcal{V}_2]$, where P_i contains the variables for L_i. Let us write $\mathcal{C}$ for $\mathcal{CFL}(H)$.

We may view $\mathcal{C}$ as a type-D module over P_2. We do this by viewing $\mathcal{C}^{P_2}$ as freely generated by $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ over P_1. The structure map δ^1 is shown below:

$$ \mathcal{C}^{P_2} \cong \begin{array}{ccc} \mathbf{a}[\mathcal{U}_1,\mathcal{V}_1] & \xleftarrow{1|\mathcal{U}_2} & \mathbf{b}[\mathcal{U}_1,\mathcal{V}_1] \\ \uparrow{\scriptstyle \mathcal{V}_1|1} & & \downarrow{\scriptstyle \mathcal{U}_1|1} \\ \mathbf{c}[\mathcal{U}_1,\mathcal{V}_1] & \xrightarrow{1|\mathcal{V}_2} & \mathbf{d}[\mathcal{U}_1,\mathcal{V}_1] \end{array} $$

An arrow from $\mathbf{x}$ to $\mathbf{y}$ labeled by $a|b$ indicates that $\delta^1(\mathbf{x})$ has a summand of $(a \cdot \mathbf{y}) \otimes b$.

As a type-D module, $\mathcal{C}^{P_2}$ is not minimal. We now describe a homotopy equivalent complex which is minimal. It is helpful to realize that the above description of $\mathcal{C}^{P_2}$

may be further decomposed as a box tensor product, as follows. Let $\mathcal{E}_2$ denote the exterior algebra on two generators,

$$\mathcal{E}_2 := \Lambda^*(\phi_2, \psi_2).$$

We consider the following type-DD bimodule ${}^{\mathcal{E}_2}\mathcal{K}^{P_2}$. As a vector space, $\mathcal{K} \cong \mathbb{F}$. The type-DD structure map is given by the formula

$$\delta^{1,1}(1) = \phi_2|1|\mathcal{U}_2 + \psi_2|1|\mathcal{V}_2.$$

We may define a type-A module $\mathcal{C}_{\mathcal{E}_2}$, generated freely over P_1 by **a**, **b**, **c** and **d**, with module structure given by the following diagram:

$$\mathcal{C}_{\mathcal{E}_2} = \begin{array}{ccc} \mathbf{a}[\mathcal{U}_1, \mathcal{V}_1] & \xleftarrow{\ \phi_2\ } & \mathbf{b}[\mathcal{U}_1, \mathcal{V}_1] \\ \uparrow{\scriptstyle \mathcal{V}_1} & & \downarrow{\scriptstyle \mathcal{U}_1} \\ \mathbf{c}[\mathcal{U}_1, \mathcal{V}_1] & \xrightarrow{\ \psi_2\ } & \mathbf{d}[\mathcal{U}_1, \mathcal{V}_1] \end{array}$$

In the above, solid arrows denote the m_2 actions of $\mathcal{E}_2$, while the dashed arrows denote the m_1 actions (i.e., internal differentials). To illustrate the notation, if $f \in \mathbb{F}[\mathcal{U}_1, \mathcal{V}_1]$, then

$$m_2(f \cdot \mathbf{b}, \phi_2) = f \cdot \mathbf{a} \quad \text{and} \quad m_1(f \cdot \mathbf{b}) = \mathcal{U}_1 \cdot f \cdot \mathbf{d}.$$

The action of ψ_2 is similar.

Clearly, there is an isomorphism of type-D modules

$$\mathcal{C}^{P_2} \cong \mathcal{C}_{\mathcal{E}_2} \boxtimes {}^{\mathcal{E}_2}\mathcal{K}^{P_2}.$$

In the above, we have forgotten about the natural action of P_1, given by ordinary multiplication. In fact, we may incorporate additionally this action to obtain bimodules ${}_{P_1}\mathcal{C}^{P_2}$ and ${}_{P_1}\mathcal{C}_{\mathcal{E}_2}$, such that

$${}_{P_1}\mathcal{C}^{P_2} \cong {}_{P_1}\mathcal{C}_{\mathcal{E}_2} \boxtimes {}^{\mathcal{E}_2}\mathcal{K}^{P_2}.$$

The module ${}_{P_1}\mathcal{C}^{P_2}$ has only δ_1^1 and δ_2^1 non-trivial, and ${}_{P_1}\mathcal{C}_{\mathcal{E}_2}$ has only $m_{0,1,0}$, $m_{1,1,0}$ and $m_{0,1,1}$ non-trivial. In the next section, we explore minimal models.

17.2 Minimal models for the Hopf link Floer complex

We apply the homological perturbation lemma to obtain minimal models of the Hopf link bimodules from the previous section. We will define a DA bimodule ${}_{P_1}Z^{P_2}$ which is homotopy equivalent to ${}_{P_1}\mathcal{C}^{P_2}$ and which has $\delta_1^1 = 0$. The techniques of this section formalize the construction in [14].

First, define the chain complex $\mathcal{C}_0$ by forgetting about the $\mathcal{E}_2$-action on $\mathcal{C}_{\mathcal{E}_2}$. Over $\mathbb{F}_2$, $\mathcal{C}_0$ is homotopy equivalent to its homology, which we view as the vector space

$$Z := \mathbf{a}[\mathcal{U}_1] \oplus \mathbf{d}[\mathcal{V}_1].$$

Here, $\mathbf{a}[\mathcal{U}_1]$ denotes a copy of $\mathbb{F}[\mathcal{U}_1]$, generated by $\mathbf{a}$, and similarly for $\mathbf{d}[\mathcal{V}_1]$. There are natural maps

$$i\colon Z \to \mathcal{C}_0, \quad \pi\colon \mathcal{C}_0 \to Z, \quad \text{and} \quad h\colon \mathcal{C}_0 \to \mathcal{C}_0$$

such that i and π are chain maps, $i \circ \pi = \mathrm{id} + [m_1, h]$, $\pi \circ i = \mathrm{id}$, $h \circ i = 0$, $\pi \circ h = 0$ and $h \circ h = 0$. The map i is given by

$$i(\mathcal{U}_1^n \mathbf{a}) = \mathcal{U}_1^n \mathbf{a} \quad \text{and} \quad i(\mathcal{V}_1^n \mathbf{d}) = \mathcal{V}_1^n \mathbf{d}.$$

The map π is given by

$$\pi(\mathcal{U}_1^n \mathcal{V}_1^m \mathbf{a}) = \begin{cases} \mathcal{U}_1^n \mathbf{a} & \text{if } m = 0, \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad \pi(\mathcal{U}_1^n \mathcal{V}_1^m \mathbf{d}) = \begin{cases} \mathcal{V}_1^m \mathbf{d} & \text{if } n = 0, \\ 0 & \text{otherwise.} \end{cases}$$

The map h is given by

$$h(\mathcal{U}_1^n \mathcal{V}_1^m \mathbf{a}) = \begin{cases} \mathcal{U}_1^n \mathcal{V}_1^{m-1} \mathbf{c} & \text{if } m \geq 1, \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad h(\mathcal{U}_1^n \mathcal{V}_1^m \mathbf{d}) = \begin{cases} \mathcal{U}_1^{n-1} \mathcal{V}_1^m \mathbf{b} & \text{if } n \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

The homological perturbation lemma for A_∞-modules, Lemma 4.1, endows Z with the structure of a right A_∞-module over the exterior algebra $\mathcal{E}_2$, for which we write $Z_{\mathcal{E}_2}$, which is A_∞-homotopy equivalent to $\mathcal{C}_{\mathcal{E}_2}$. In fact, the homotopy equivalence is given explicitly by the homological perturbation lemma. The maps i, π and h extend to A_∞-module morphisms i_*, π_* and h_*. We box these morphisms with the identity map on ${}^{\mathcal{E}_2}\mathcal{K}^{P_2}$ to obtain maps of type-D modules

$$\Pi^1\colon \mathcal{C}^{P_2} \to Z^{P_2}, \quad I^1\colon Z^{P_2} \to \mathcal{C}^{P_2}, \quad \text{and} \quad H^1\colon \mathcal{C}^{P_2} \to \mathcal{C}^{P_2}.$$

Lemma 17.2. *The morphisms I^1 and Π^1 are type-*D *homomorphisms (i.e., $\partial_{\mathrm{Mor}}(I^1) = 0$ and $\partial_{\mathrm{Mor}}(\Pi^1) = 0$). Furthermore,*

(1) $\Pi^1 \circ I^1 = \mathrm{id}$.

(2) $H^1 \circ H^1 = 0$.

(3) $H^1 \circ I^1 = 0$.

(4) $\Pi^1 \circ H^1 = 0$.

(5) $I^1 \circ \Pi^1 = \mathrm{id} + \partial_{\mathrm{Mor}}(H^1)$.

Proof. Lemma 4.1 (the homological perturbation lemma) implies that the stated formulas hold for i_*, h_* and π_*. Boxing with the identity map preserves these relations since the algebra is $\mathcal{E}_2$ (which is an associative algebra), so

$$(f_* \boxtimes \mathbb{I}_{\mathcal{K}}) \circ (g_* \boxtimes \mathbb{I}_{\mathcal{K}}) = ((f_* \circ g_*) \boxtimes \mathbb{I}_{\mathcal{K}}),$$

by [25, Remark 2.2.28]. Furthermore, boxing with the identity is a chain map

$$\partial_{\mathrm{Mor}}(f_* \boxtimes \mathbb{I}_{\mathcal{K}}) = (\partial_{\mathrm{Mor}}(f_*) \boxtimes \mathbb{I}_{\mathcal{K}}).$$

See [25, Lemma 2.3.3 (1)]. ∎

It is enlightening to compute concrete formulas for the morphisms Π^1, H^1 and I^1.

Lemma 17.3. *The type-*D *module maps* I^1, Π^1 *and* H^1 *are given by the following formulas:*

(1) I^1 *is given by* $I^1(\mathbf{x}) = \mathbf{x} \otimes 1$, *for* $\mathbf{x} \in Z$.

(2) Π^1 *vanishes on* $\mathrm{Span}(\mathbf{b}, \mathbf{c})$, *and satisfies*

$$\Pi^1(\mathcal{U}_1^i \mathcal{V}_1^j \mathbf{a}) = \begin{cases} \mathcal{U}_1^{i-j} \mathbf{a} \otimes \mathcal{U}_2^j \mathcal{V}_2^j & \text{if } i \geq j, \\ \mathcal{V}_1^{j-i-1} \mathbf{d} \otimes \mathcal{U}_2^i \mathcal{V}_2^{i+1} & \text{if } i < j, \end{cases}$$

$$\Pi^1(\mathcal{U}_1^i \mathcal{V}_1^j \mathbf{d}) = \begin{cases} \mathcal{U}_1^{i-j-1} \mathbf{a} \otimes \mathcal{U}_2^{j+1} \mathcal{V}_2^j & \text{if } i > j, \\ \mathcal{V}_1^{j-i} \mathbf{d} \otimes \mathcal{U}_2^i \mathcal{V}_2^i & \text{if } i \leq j. \end{cases}$$

(3) H^1 *vanishes on* $\mathrm{Span}(\mathbf{b}, \mathbf{c})$, *and maps* $\mathrm{Span}(\mathbf{a}, \mathbf{d})$ *to* $\mathrm{Span}(\mathbf{b}, \mathbf{c}) \otimes P_2$. *It is given by the formula*

$$H^1(\mathcal{U}_1^i \mathcal{V}_1^j \mathbf{a}) = \begin{cases} \mathcal{U}_1^i \mathcal{V}_1^{j-1} \mathbf{c} \otimes 1 + \mathcal{U}_1^{i-1} \mathcal{V}_1^{j-1} \mathbf{b} \otimes \mathcal{V}_2 + \cdots \\ \qquad + \mathcal{U}_1^{i-j} \mathbf{b} \otimes \mathcal{U}_2^{j-1} \mathcal{V}_2^j & \text{if } j \leq i, \\ \mathcal{U}_1^i \mathcal{V}_1^{j-1} \mathbf{c} \otimes 1 + \mathcal{U}_1^{i-1} \mathcal{V}_1^{j-1} \mathbf{b} \otimes \mathcal{V}_2 + \cdots \\ \qquad + \mathcal{V}_1^{j-i-1} \mathbf{c} \otimes \mathcal{U}_2^i \mathcal{V}_2^i & \text{if } j > i, \end{cases}$$

and

$$H^1(\mathcal{U}_1^i \mathcal{V}_1^j \mathbf{d}) = \begin{cases} \mathcal{U}_1^{i-1} \mathcal{V}_1^j \mathbf{b} \otimes 1 + \mathcal{U}_1^{i-1} \mathcal{V}_1^{j-1} \mathbf{c} \otimes \mathcal{U}_2 + \cdots \\ \qquad + \mathcal{U}_1^{i-j-1} \mathbf{b} \otimes \mathcal{U}_2^j \mathcal{V}_2^j & \text{if } i > j \\ \mathcal{U}_1^{i-1} \mathcal{V}_1^j \mathbf{b} \otimes 1 + \mathcal{U}_1^{i-1} \mathcal{V}_1^{j-1} \mathbf{c} \otimes \mathcal{U}_2 + \cdots \\ \qquad + \mathcal{V}_1^{j-i} \mathbf{c} \otimes \mathcal{U}_2^i \mathcal{V}_2^{i-1} & \text{if } i \leq j. \end{cases}$$

Proof. Lemma 4.1 gives a concrete formula for the A_∞-action on $Z_{\mathcal{E}_2}$. We will prove the first statement, and also compute $\Pi^1(\mathcal{U}_1^i \mathcal{V}_1^j \mathbf{a})$ when $i \geq j$ in order to illustrate the technique. We will leave the remaining cases to the reader.

Firstly, we begin with I^1. We are boxing with $\mathbb{I}_{\mathcal{K}}$, so suppose that $a_1|\cdots|a_n|1|b_n|\cdots|b_1$ is the output of repeated applications of $\delta^{1,1}$ on ${}^{\mathcal{E}_2}\mathcal{K}^{P_2}$. The homological perturbation lemma gives a recipe for $m_{n+1}^{Z}(\mathbf{x}, a_1, \dots, a_n)$. See Figure 4.1. The recipe is to include $\mathbf{x}$ into $\mathcal{C}$ via i. We then apply $m_2(-, a_1)$, then h, then $m_2(-, a_2)$, then h, and so forth, until one applies $m(-, a_n)$. Then we apply π. The algebra output is the product $b_n \cdots b_1$. The m_2 action of $\mathcal{E}_2$ on $\mathcal{C}$ vanishes on the image of i, so there are no contributions unless $n = 0$. The formula follows.

We now compute $\Pi^1(\mathcal{U}_1^i \mathcal{V}_1^j \mathbf{a})$ for $i \geq j$. We begin at $\mathcal{U}_1^i \mathcal{V}_1^j \mathbf{a} \in \mathcal{C}$. If $j = 0$, then $h(\mathcal{U}_1^i \mathcal{V}_1^j \mathbf{a}) = 0$, so our only option is to apply π, which gives $\mathcal{U}_1^i \mathbf{a}$ as claimed. If $j > 0$, the only option is to apply h to get $\mathcal{U}_1^i \mathcal{V}_1^{j-1} \mathbf{c}$. Then we apply $m_2(-, \psi_2)$ which gives $\mathcal{U}_1^i \mathcal{V}_1^{j-1} \mathbf{d} \otimes \mathcal{V}_2$. We then apply h and $m_2(-, \phi_2)$ and we get $\mathcal{U}_1^{i-1} \mathcal{V}_1^{j-1} \mathbf{a} \otimes \mathcal{U}_2 \mathcal{V}_2$. We repeat this procedure until we cannot apply h or m_2 anymore. In the case that $i \geq j$, the final term in this sequence will be $\mathcal{U}_1^{i-j} \mathbf{a} \otimes \mathcal{U}_2^j \mathcal{V}_2^j$. All of the remaining claims follow a similar analysis. ∎

Lemma 17.2 allows us to apply the homological perturbation lemma of DA bimodules, Lemma 4.2, which equips Z with a DA bimodule structure ${}_{P_1}Z^{P_2}$, and supplies morphisms of DA bimodules

$$\Pi_*^1 \colon {}_{P_1}\mathcal{C}^{P_2} \to {}_{P_1}Z^{P_2}, \quad I_*^1 \colon {}_{P_1}Z^{P_2} \to {}_{P_1}\mathcal{C}^{P_2}, \quad \text{and} \quad H_*^1 \colon {}_{P_1}\mathcal{C}^{P_2} \to {}_{P_1}\mathcal{C}^{P_2}$$

which satisfy relations identical to Lemma 17.2.

Lemma 17.4. *The bimodule ${}_{P_1}Z^{P_2}$ and the morphisms Π_*^1, I_*^1 and H_*^1 satisfy the following relations:*

1. *The structure maps δ_j^1 on ${}_{P_1}Z^{P_2}$ vanish if $j \neq 2$.*
2. *The maps Π_j^1 and H_j^1 vanish unless $j = 1$.*
3. *The map I_j^1 vanishes if $j > 2$.*
4. *The maps δ_*^1 and I_*^1 are strictly unital, i.e., they vanish if 1 is an input, except for δ_2^1, which satisfies $\delta_2^1(1 \otimes x) = x \otimes 1$. (Note that H_j^1 and Π_j^1 are trivially unital, since they are only non-trivial if $j = 1$.)*

Proof. The proofs of all statements are by explicit examination of the maps from the homological perturbation lemma.

We begin with the statements about δ_j^1. Consider first a sequence of algebra elements $(a_n, \dots, a_1)$ in P_1. The rule for computing δ_j^1 is to input via I^1, then apply m_2, then we apply pairs of H^1 followed by m_2 until we exhaust $(a_n, \dots, a_1)$, and then finally we apply Π_1. The algebra elements outputs are multiplied together by applying μ_2 repeatedly. However, applying m_2 does not move an elements position in $\mathcal{C}$ (i.e., $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ or $\mathbf{d}$). The map H^1 does change the generator, and it maps $\operatorname{Span}(\mathbf{a}, \mathbf{d})$ to $\operatorname{Span}(\mathbf{c}, \mathbf{b}) \otimes P_2$, and vanishes on $\operatorname{Span}(\mathbf{c}, \mathbf{b})$. If we apply another m_2, we remain in $\operatorname{Span}(\mathbf{c}, \mathbf{b})$. We note that Π^1 and H^1 both vanish on $\operatorname{Span}(\mathbf{c}, \mathbf{b})$. In particular, the only terms making non-trivial contribution are those with one δ_2^1, and no H^1 term. The same argument shows that Π_k^1 and H_k^1 vanish if $k > 1$.

To compute the map $I_{j+1}^1(a_j, \dots, a_1, \mathbf{x})$, the recipe is to first include $\mathbf{x}$ into $\mathcal{C} \otimes P_2$ via I^1. This includes $\mathbf{x}$ into $\mathrm{Span}(\mathbf{a}, \mathbf{d}) \otimes P_2$. We then apply $\delta_2^1(a_1, -)$. This preserves $\mathrm{Span}(\mathbf{a}, \mathbf{d}) \otimes P_2$. Next we apply H^1, which maps $\mathrm{Span}(\mathbf{a}, \mathbf{d}) \otimes P_2$ to $\mathrm{Span}(\mathbf{b}, \mathbf{c}) \otimes P_2$. Any further applications of $\delta_2^1(a_n, -)$ followed by H^1 would map to zero. Hence we may have I_2^1, but not I_j^1 for $j > 2$.

The final statement about being strictly unital follows from the homological perturbation lemma. ∎

It is helpful to explicitly compute δ_2^1 on ${}_{P_1}Z^{P_2}$. Since there is no δ_j^1 for $j > 2$ by Lemma 17.4, we compute only $\delta_2^1(\mathcal{U}_1, -)$ and $\delta_2^1(\mathcal{V}_1, -)$.

Lemma 17.5. *The map δ_2^1 on ${}_{P_1}Z^{P_2}$ satisfies the following:*

(1) $\delta_2^1(\mathcal{U}_1, \mathcal{U}_1^n \mathbf{a}) = \mathcal{U}_1^{n+1}\mathbf{a} \otimes 1$.

(2) $\delta_2^1(\mathcal{V}_1, \mathcal{U}_1^n \mathbf{a}) = \begin{cases} \mathcal{U}_1^{n-1}\mathbf{a} \otimes \mathcal{U}_2\mathcal{V}_2 & \text{if } n > 0, \\ \mathbf{d} \otimes \mathcal{V}_2 & \text{if } n = 0. \end{cases}$

(3) $\delta_2^1(\mathcal{V}_1, \mathcal{V}_1^m \mathbf{d}) = \mathcal{V}_1^{m+1}\mathbf{d} \otimes 1$.

(4) $\delta_2^1(\mathcal{U}_1, \mathcal{V}_1^m \mathbf{d}) = \begin{cases} \mathcal{V}_1^{m-1}\mathbf{d} \otimes \mathcal{U}_2\mathcal{V}_2 & \text{if } m > 0, \\ \mathbf{a} \otimes \mathcal{U}_2 & \text{if } m = 0. \end{cases}$

Finally, the complex $\mathcal{V}_1^{-1}\mathcal{C}$ will also be important to understand. Similarly to $\mathcal{C}$, the complex $\mathcal{V}_1^{-1}\mathcal{C}$ can be reduced in size via a homotopy equivalence. We note that $\mathcal{V}_1^{-1}\mathcal{C}$ is the following DA bimodule:

$$\mathcal{V}_1^{-1}\mathcal{C} \cong \begin{array}{ccc} \mathbf{a}[\mathcal{U}_1, \mathcal{V}_1, \mathcal{V}_1^{-1}] & \xleftarrow{1|\mathcal{U}_2} & \mathbf{b}[\mathcal{U}_1, \mathcal{V}_1, \mathcal{V}_1^{-1}] \\ \uparrow {\scriptstyle \mathcal{V}_1|1} & & \downarrow {\scriptstyle \mathcal{U}_1|1} \\ \mathbf{c}[\mathcal{U}_1, \mathcal{V}_1, \mathcal{V}_1^{-1}] & \xrightarrow{1|\mathcal{V}_2} & \mathbf{d}[\mathcal{U}_1, \mathcal{V}_1, \mathcal{V}_1^{-1}] \end{array}$$

In the above, all arrows denote δ_1^1. The actions of δ_2^1 are given by ordinary polynomial multiplication. Since $\mathcal{V}_1$ is invertible, the arrow labeled $\mathcal{V}_1|1$ and the two generators $\mathbf{a}$ and $\mathbf{c}$ may be completely canceled from the complex. Hence, the above is homotopy equivalent to a DA bimodule whose underlying type-D structure is shown below:

$$\mathbf{b}[\mathcal{U}_1, \mathcal{V}_1, \mathcal{V}_1^{-1}] \xrightarrow{\mathcal{U}_1|1} \mathbf{d}[\mathcal{U}_1, \mathcal{V}_1, \mathcal{V}_1^{-1}].$$

This complex above may be further reduced as a type-D structure, giving a minimal model, for which we write ${}_{\mathcal{V}_1^{-1}P_1}W^{P_2}$. As a vector space, $W \cong \mathbf{d}[\mathcal{V}_1, \mathcal{V}_1^{-1}]$. The same homological perturbation argument as before equips W with the bimodule structure ${}_{\mathcal{V}_1^{-1}P_1}W^{P_2}$ which has $\delta_j^1 = 0$ unless $j = 2$. For completeness, we record the actions.

Lemma 17.6. *The bimodule* ${}_{\mathcal{V}_1^{-1}P_1}W^{P_2}$ *has the following action:*

$$\delta_2^1(\mathcal{U}_1^i\mathcal{V}_1^j, \mathcal{V}_1^n\mathbf{d}) = \mathcal{V}_1^{j+n-i}\mathbf{d} \otimes \mathcal{U}_2^i\mathcal{V}_2^i.$$

We leave the proofs of Lemmas 17.5 and 17.6 to the reader.

17.3 Minimal models over $\mathcal{K}$

In this section, we describe a minimal model of the DA bimodule ${}_{\mathcal{K}_1}\mathcal{H}_{(\lambda_1,0)}^{\mathcal{K}_2}$. We will denote the minimal model by

$${}_{\mathcal{K}_1}\mathcal{Z}_{(\lambda_1,0)}^{\mathcal{K}_2}.$$

The existence of the minimal model follows from the homological perturbation lemma for hypercubes of DA bimodules. From the description of ${}_{\mathcal{K}_1}\mathcal{H}_{(\lambda_1,0)}^{\mathcal{K}_2}$ it is clear that there is a filtration by the cube $\mathbb{E}_2$. In Section 17.2, we described minimal models of $\mathcal{CFL}(H)^{\mathbb{F}[\mathcal{U}_2,\mathcal{V}_2]}$ and $\mathcal{V}_1^{-1}\mathcal{CFL}(H)^{\mathbb{F}[\mathcal{U}_2,\mathcal{V}_2]}$. Note that we may identify the underlying type-D module $\mathcal{H}_\varepsilon^{\mathcal{K}_2}$ with one of these type-D modules, for $\varepsilon \in \mathbb{E}_2$. Hence, the construction from Section 17.2 may be viewed as giving morphisms of type-D structures

$$\Pi_\varepsilon^1 \colon \mathcal{H}_\varepsilon^{\mathcal{K}_2} \to \mathcal{Z}_\varepsilon^{\mathcal{K}_2}, \quad I_\varepsilon^1 \colon \mathcal{Z}_\varepsilon^{\mathcal{K}_2} \to \mathcal{H}_\varepsilon^{\mathcal{K}_2}, \quad \text{and} \quad H_\varepsilon^1 \colon \mathcal{H}_\varepsilon^{\mathcal{K}_2} \to \mathcal{H}_\varepsilon^{\mathcal{K}_2}$$

which induce a homotopy equivalence of type-D structures, such that furthermore the algebraic assumptions of the homological perturbation lemma are satisfied. We obtain a homotopy equivalent hypercube of DA bimodules ${}_{\mathcal{K}_1}\mathcal{Z}_{(\lambda_1,0)}^{\mathcal{K}_2}$ by applying the homological perturbation lemma for hypercubes of DA bimodules, Lemma 4.5. We now schematically sketch the induced type-D structure map:

$$\delta_1^1 = \begin{array}{ccc} \mathcal{Z}_{0,0} & & \mathcal{Z}_{1,0} \\ \Big\downarrow {\scriptstyle L_2 f_1^1 + {}^{-L_2} f_1^1} & & \Big\downarrow {\scriptstyle L_2 g_1^1 + {}^{-L_2} g_1^1} \\ \mathcal{Z}_{0,1} & & \mathcal{Z}_{1,1} \end{array} \qquad \delta_2^1 = \begin{array}{c} m_2^1 \circlearrowright \mathcal{Z}_{0,0} \xrightarrow{{}^{L_1}p_2^1 + {}^{-L_1}p_2^1} \mathcal{Z}_{1,0} \circlearrowleft m_2^1 \\ \\ m_2^1 \circlearrowright \mathcal{Z}_{0,1} \xrightarrow{{}^{L_1}q_2^1 + {}^{-L_1}q_2^1} \mathcal{Z}_{1,1} \circlearrowleft m_2^1 \end{array}$$

$$\delta_3^1 = \begin{array}{ccc} \mathcal{Z}_{0,0} & & \mathcal{Z}_{1,0} \\ & \searrow {\scriptstyle {}^{-H}\omega_3^1} & \\ \mathcal{Z}_{0,1} & & \mathcal{Z}_{1,1} \end{array}$$

Proposition 17.7. *Give the negative Hopf link framing $(\lambda_1, 0)$. The structure maps of the minimal model ${}_{\mathcal{K}_1}\mathcal{Z}^{\mathcal{K}_2}_{(\lambda_1,0)}$ are as follows:*

(1) *The maps m_2^1 are the same as the δ_2^1 maps in Lemmas 17.5 and 17.6.*

(2) *The maps ${}^{L_1}p_2^1$ and ${}^{L_1}q_2^1$ are given by the same formulas as each other, as are ${}^{-L_1}p_2^1$ and ${}^{-L_1}q_2^1$. They are determined by the following formulas:*

(a) ${}^{L_1}p_2^1(\sigma_1, \mathcal{U}_1^i \mathbf{a}) = \mathcal{V}_1^{-i-1}\mathbf{d} \otimes \mathcal{U}_2^i \mathcal{V}_2^{i+1}$ *and* ${}^{L_1}p_2^1(\sigma_1, \mathcal{V}_1^j \mathbf{d}) = \mathcal{V}_1^j \mathbf{d} \otimes 1$.

(b) ${}^{-L_1}p_2^1(\tau_1, \mathcal{U}_1^i \mathbf{a}) = \mathcal{V}_1^{-i-1+\lambda_1}\mathbf{d} \otimes 1$ *and* ${}^{-L_1}p_2^1(\tau_1, \mathcal{V}_1^j \mathbf{d}) = \mathcal{V}_1^{j+\lambda_1}\mathbf{d} \otimes \mathcal{U}_2^{j+1}\mathcal{V}_2^j$.

(c) ${}^{L_1}p_2^1(\tau_1, -) = 0$ *and* ${}^{-L_1}p_2^1(\sigma_1, -) = 0$.

(3) *The maps for $\pm L_2$ are as follows:*

(a) ${}^{L_2}f_1^1(\mathcal{U}_1^i \mathbf{a}) = \mathcal{U}_1^i \mathbf{a} \otimes \sigma_2$ *and* ${}^{L_2}f_1^1(\mathcal{V}_1^j \mathbf{d}) = \mathcal{V}_1^j \mathbf{d} \otimes \sigma_2$.

(b) ${}^{-L_2}f_1^1(\mathcal{U}_1^i \mathbf{a}) = \mathcal{U}_1^{i+1}\mathbf{a} \otimes \tau_2$ *and*

$$ {}^{-L_2}f_1^1(\mathcal{V}_1^j \mathbf{d}) = \begin{cases} \mathbf{a} \otimes \mathcal{V}_2^{-1}\tau_2 & \text{if } j = 0, \\ \mathcal{V}_1^{j-1}\mathbf{d} \otimes \tau_2 & \text{if } j > 0. \end{cases} $$

(c) ${}^{L_2}g_1^1(\mathcal{V}_1^i \mathbf{d}) = \mathcal{V}_1^i \mathbf{d} \otimes \sigma_2$.

(d) ${}^{-L_2}g_1^1(\mathcal{V}_1^i \mathbf{d}) = \mathcal{V}_1^{i-1}\mathbf{d} \otimes \tau_2$.

(4) *The map ${}^{-H}\omega_3^1$ is determined by the relations*

$$ \begin{aligned} &{}^{-H}\omega_3^1(\tau_1, \mathcal{V}_1^m, \mathcal{U}_1^i \mathbf{a}) = \min(i+1, m)\mathcal{V}_1^{\lambda_1+m-i-2}\mathbf{d} \otimes \mathcal{U}_2^{m-1}\mathcal{V}_2^{m-1}\tau_2, \\ &{}^{-H}\omega_3^1(\tau_1, \mathcal{U}_1^n, \mathcal{U}_1^i \mathbf{a}) = 0, \\ &{}^{-H}\omega_3^1(\tau_1, \mathcal{U}_1^n, \mathcal{V}_1^j \mathbf{d}) = \min(n, j)\mathcal{V}_1^{j-n+\lambda_1-1}\mathbf{d} \otimes \mathcal{U}_2^{j-1}\mathcal{V}_2^{j-2}\tau_2, \\ &{}^{-H}\omega_3^1(\tau_1, \mathcal{V}_1^m, \mathcal{V}_1^j \mathbf{d}) = 0, \end{aligned} $$

and that ${}^{-H}\omega_3^1$ vanishes if an algebra input is a multiple of σ_1. The map ${}^{-H}\omega_3^1$ also vanishes on pairs of algebra elements with other configurations of idempotents.

Remark 17.8. We have not enumerated a complete list of the structure maps. Additional powers of $\mathcal{U}_1$ and $\mathcal{V}_1$ may be added to arguments in the above maps. There are no terms of δ_{j+1}^1 for $j > 2$, however. A more minimal list could also have been made by specifying the length 1 maps on only $\mathbf{a}$ and $\mathbf{d}$. As an example of several relations which are forced by the DA bimodule relations, we have

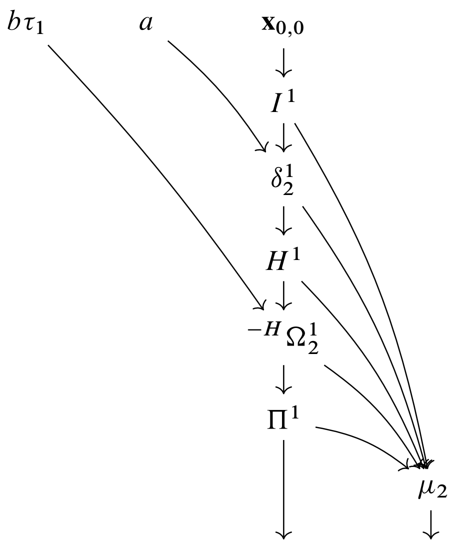
$$ {}^{-H}\omega_3^1(\tau_1, \mathcal{U}_1^i \mathcal{V}_1^j, \mathcal{U}_1^n \mathbf{a}) = {}^{-H}\omega_3^1(\tau, \mathcal{V}_1^j, \mathcal{U}_1^{i+n}\mathbf{a}) $$

and

$$ {}^{-H}\omega_3^1(\tau_1, \mathcal{U}_1^i \mathcal{V}_1^j, \mathcal{V}_1^n \mathbf{d}) = {}^{-H}\omega_3^1(\tau, \mathcal{U}_1^i, \mathcal{V}_1^{j+n}\mathbf{d}). $$

Proof. All of these computations are performed algorithmically using the homological perturbation lemma for hypercubes of DA bimodules, Lemma 4.5. For the first two sets of equations the computation is essentially straightforward. We first apply the map I_ε^1, then we apply a length 1 map of the cube (either with no algebra input, as for the maps labeled f_1^1 and g_1^1, or with an algebra input of σ_1 or τ_1, as for the maps p_2^1 and q_2^1). We leave these computations to the reader, as they are straightforward.

The map ${}^{-H}\omega_3^1$ is more interesting. There is exactly one configuration of a morphism graph which gives a non-trivial evaluation. This occurs for elements $\mathbf{x}_{0,0}$ in the idempotent $\varepsilon = (0,0)$. The structure graph is shown below:



(17.1)

In the above, the map ${}^{-H}\Omega_2^1$ denotes the component of δ_2^1 on $\mathcal{H}_\Lambda^{\mathcal{K}_1\otimes\mathcal{K}_2} \boxtimes {}_{\mathcal{K}_1|\mathcal{K}_1}[\mathbb{I}^{\supset}]$ contributed by the map component ${}^{-H}\Omega^1$ of δ^1 of $\mathcal{H}_\Lambda^{\mathcal{K}_1\otimes\mathcal{K}_2}$. Concretely, it is given by the formula

$$\Omega_2^1(b\tau_1,-) = \begin{cases} \mathcal{U}_1^i\mathcal{V}_1^j\mathbf{a}_{00} \longmapsto b\mathcal{V}_1^{\lambda_1-2}\phi^\tau(\mathcal{U}_1^i\mathcal{V}_1^j)\mathbf{c}_{11}\otimes\mathcal{V}_2^{-1}\tau_2 \\ \mathcal{U}_1^i\mathcal{V}_1^j\mathbf{b}_{00} \longmapsto 0 \\ \mathcal{U}_1^i\mathcal{V}_1^j\mathbf{c}_{00} \longmapsto b\mathcal{V}_1^{\lambda_1-1}\phi^\tau(\mathcal{U}_1^i\mathcal{V}_1^j)\mathbf{d}_{11}\otimes\tau_2 \\ \mathcal{U}_1^i\mathcal{V}_1^j\mathbf{d}_{00} \longmapsto b\mathcal{V}_1^{\lambda_1-1}\phi^\tau(\mathcal{U}_1^i\mathcal{V}_1^j)\mathbf{c}_{11}\otimes\mathcal{V}_2^{-2}\tau_2. \end{cases}$$

We focus on the case that $b = i_1$. Consider first the case that $\mathbf{x} = \mathcal{U}_1^i\mathbf{a}$. If $a = \mathcal{U}_1^n$, then $\delta_2^1(\mathcal{U}_1^n, \mathcal{U}_1^i\mathbf{a}) = \mathcal{U}_1^{i+n}\mathbf{a}\otimes 1$, however, H^1 vanishes on this element, so we conclude that

$${}^{-H}\omega_3^1(\tau_1, \mathcal{U}_1^n, \mathcal{U}_1^i\mathbf{a}) = 0.$$

The same argument implies that ${}^{-H}\omega_3^1(\tau_1, \mathcal{V}_1^m, \mathcal{V}_1^j\mathbf{d}) = 0$.

We now consider the case that $\mathbf{x} = \mathcal{U}_1^i\mathbf{a}$ and $a = \mathcal{V}_1^m$. In this case, $H^1(\mathcal{U}_1^i\mathcal{V}_1^m\mathbf{a})$ is a sum involving both $\mathbf{c}$ and $\mathbf{b}$. The next term in equation (17.1) is an application of ${}^{-H}\Omega_2^1$, which vanishes on multiples of $\mathbf{b}$. In particular, we only need to consider the terms of $H^1(\mathcal{U}_1^i\mathcal{V}_1^m\mathbf{a})$ involving $\mathbf{c}$. There are $\min(i+1,m)$ such terms. They are

$$\mathcal{U}_1^i\mathcal{V}_1^{m-1}\mathbf{c}\otimes 1+\mathcal{U}_1^{i-1}\mathcal{V}_1^{m-2}\mathbf{c}\otimes\mathcal{U}_2\mathcal{V}_2+\mathcal{U}_1^{i-2}\mathcal{V}_1^{m-3}\mathbf{c}\otimes\mathcal{U}_2^2\mathcal{V}_2^2+\cdots.$$

(The sum is over all terms in the sequence above where $\mathcal{U}_1$ and $\mathcal{V}_1$ both have nonnegative exponent.) We then apply ${}^{-H}\Omega_2^1$, Π^1, and then multiply the outgoing algebra elements by repeatedly applying μ_2. It is easy check that the application of ${}^{-H}\Omega_2^1$, Π^1, and μ_2 on each of the above summands coincide, so we will only consider their evaluation on $\mathcal{U}_1^i\mathcal{V}_1^{m-1}\mathbf{c}\otimes 1$. Applying ${}^{-H}\Omega_2^1$ gives

$$\mathcal{V}_1^{\lambda_1-1}\phi^\tau(\mathcal{U}_1^i\mathcal{V}_1^{m-1})\mathbf{d}\otimes\tau_2\otimes 1=\mathcal{U}_1^{m-1}\mathcal{V}_1^{2m+\lambda_1-3-i}\mathbf{d}\otimes\tau_2\otimes 1.$$

Applying Π^1 to the above generator, and then multiplying the algebra elements and the coefficient $\min(i+1,m)$ gives

$$\omega_3^1(\tau_1,\mathcal{V}_1^m,\mathcal{U}_1^i\mathbf{a})=\min(i+1,m)\mathcal{V}_1^{m-i-2+\lambda_1}\mathbf{d}\otimes\mathcal{U}_2^{m-1}\mathcal{V}_2^{m-1}\tau_2,$$

which is the stated formula.

We now consider ${}^{-H}\omega_3^1(\tau_1,\mathcal{U}_1^n,\mathcal{V}_1^j\mathbf{d})$. The terms of $H^1(\mathcal{U}_1^n\mathcal{V}_1^j\mathbf{d})$ which involve $\mathbf{c}$ are

$$\mathcal{U}_1^{n-1}\mathcal{V}_1^{j-1}\mathbf{c}\otimes\mathcal{U}_2+\mathcal{U}_1^{n-2}\mathcal{V}_1^{j-2}\mathbf{c}\otimes\mathcal{U}_2^2\mathcal{V}_2+\cdots.$$

As before, the sum contains all such elements in the sequence which have nonnegative powers of both $\mathcal{U}_1$ and $\mathcal{V}_1$. There are $\min(n,j)$ terms in this sum. As before, it is sufficient to evaluate ${}^{-H}\Omega_2^1$ and Π^1 only on the first term, and then multiply the result by $\min(n,j)$. Applying ${}^{-H}\Omega_2^1$, we obtain

$$\mathcal{V}_1^{\lambda_1-1}\phi^\tau(\mathcal{U}_1^{n-1}\mathcal{V}_1^{j-1})\mathbf{d}\otimes\tau_2\otimes\mathcal{U}_2=\mathcal{U}_1^{j-1}\mathcal{V}_1^{2j-2+\lambda_1-n}\mathbf{d}\otimes\tau_2\otimes\mathcal{U}_2.$$

Applying Π^1 and $\mathrm{id}\otimes\mu_2$, and multiplying by the coefficient $\min(n,j)$ gives

$$\omega_3^1(\tau_1,\mathcal{V}_1^m,\mathcal{U}_1^i\mathbf{a})=\min(n,j)\mathcal{V}_1^{j-1-n+\lambda_1}\mathbf{d}\otimes\mathcal{U}_2^{j-1}\mathcal{V}_2^{j-2}\tau_2,$$

which proves the statement. ∎

Remark 17.9. Recall that we write ${}_{\mathcal{K}_1}\overline{\mathcal{H}}_\Lambda^{\mathcal{K}_2}$ for the Hopf link complex obtained by using one alpha-parallel arc and one beta-parallel arc. The associated type-D module over $\mathcal{K}\otimes\mathcal{K}$ is computed in Proposition 16.7. Write ${}_{\mathcal{K}_1}\overline{\mathcal{Z}}_\Lambda^{\mathcal{K}_2}$ for the minimal model of the associated type-DA bimodule. We leave it to the reader to verify that ${}_{\mathcal{K}_1}\overline{\mathcal{Z}}_{(\lambda_1,0)}^{\mathcal{K}_2}$ has an identical description to ${}_{\mathcal{K}_1}\mathcal{Z}_{(\lambda_1,0)}^{\mathcal{K}_2}$ except that we omit the δ_3^1 term.

The DA bimodule structure maps of ${}_{\mathcal{K}_1}\overline{\mathcal{Z}}_{(0,0)}^{\mathcal{K}_2}$ are illustrated in Figure 17.1.

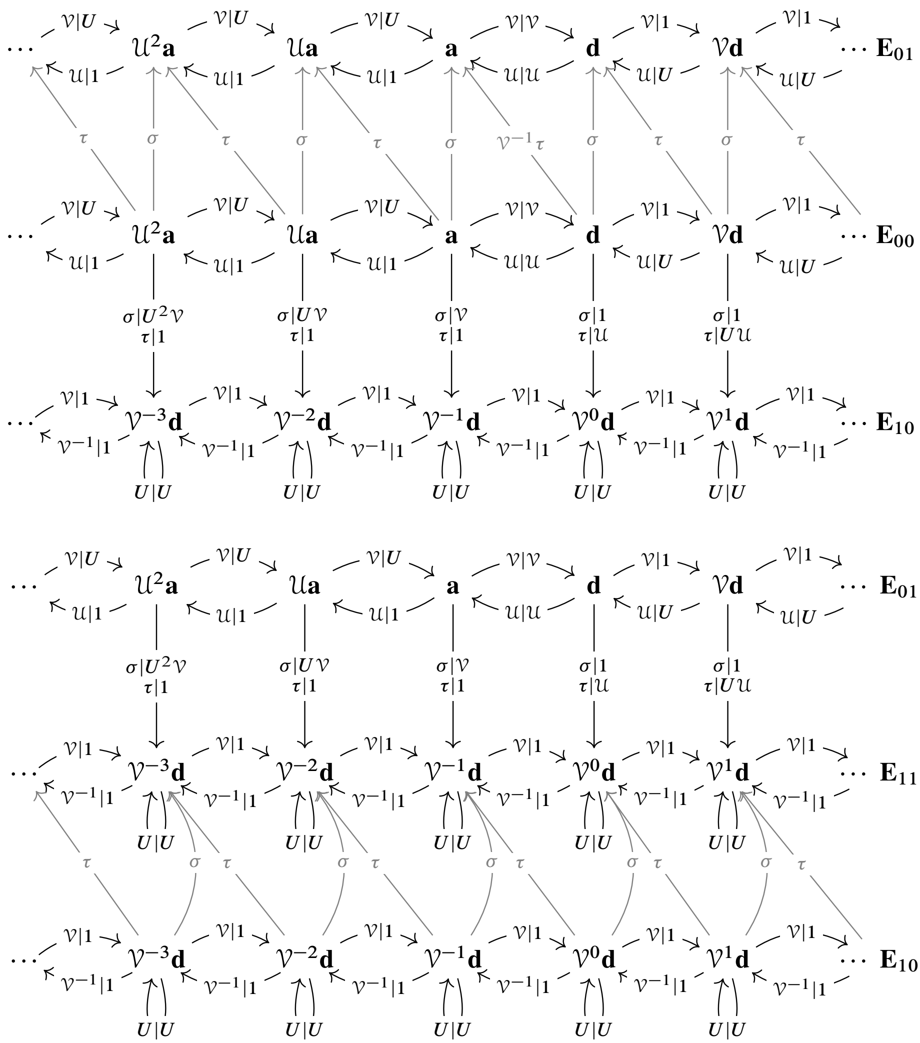


Figure 17.1. The DA bimodule of the negative Hopf link ${}_{\mathcal{K}}\overline{\mathcal{Z}}^{\mathcal{K}}_{(0,0)}$. The gray arrows denote δ^1_1. Subscripts on algebra elements indicating link components are omitted. This coincides with the bimodule ${}_{\mathcal{K}}\mathcal{Z}^{\mathcal{K}}_{(0,0)}$ except for the lack of the ω^1_3 differential.

We now prove that ${}_{\mathcal{K}_1}\mathcal{H}^{\mathcal{K}_2}_{(\lambda_1,0)}$ and ${}_{\mathcal{K}_1}\mathcal{Z}^{\mathcal{K}_2}_{(\lambda_1,0)}$ are both Alexander modules, and are furthermore homotopy equivalent.

Lemma 17.10. *The* DA *bimodule structure maps on* ${}_{\mathcal{K}_1}\mathcal{H}^{\mathcal{K}_2}_{(\lambda_1,0)}$ *and* ${}_{\mathcal{K}_1}\mathcal{Z}^{\mathcal{K}_2}_{(\lambda_1,0)}$ *are continuous.*

Furthermore, all of the maps involved in the homotopy equivalences described in Proposition 17.7 are continuous.

Proof. The DA bimodule structure maps on ${}_{\mathcal{K}_1}\mathcal{H}^{\mathcal{K}_2}_{(\lambda_1,0)}$ are continuous by virtue of the facts that the map δ^1 on $\mathcal{H}^{\mathcal{K}_1\otimes\mathcal{K}_2}_{(\lambda_1,0)}$ being continuous, since $\mathcal{H}_{(\lambda_1,0)}$ is a finitely generated vector space, and that ${}_{\mathcal{K}_1}\mathcal{H}^{\mathcal{K}_2}_{(\lambda_1,0)}$ is obtained by a tensor product of $\mathcal{H}^{\mathcal{K}_1\otimes\mathcal{K}_2}_{(\lambda_1,0)}$ with ${}_{\mathcal{K}_1|\mathcal{K}_1}[\mathbb{I}^{\supset}]$.

The structure maps on ${}_{\mathcal{K}_1}\mathcal{Z}^{\mathcal{K}_2}_{(\lambda_1,0)}$ and also the equivalence with ${}_{\mathcal{K}_1}\mathcal{H}^{\mathcal{K}_2}_{(\lambda_1,0)}$ are given by the homological perturbation lemma. These maps are finite compositions of the maps Π^1, H^1, I^1, as well as the map δ^1_2 of ${}_{\mathcal{K}_1}\mathcal{H}^{\mathcal{K}_2}_{(\lambda_1,0)}$. Hence, it is sufficient to show that each of these maps is continuous with respect to the appropriate topology. The map I^1 is obviously continuous, since it is given by $\mathbf{x} \mapsto \mathbf{x} \otimes 1$.

Consider the map Π^1 applied to elements in $\mathbf{I}_0 \cdot \mathcal{H}_{(\lambda_1,0)} \cdot \mathbf{I}_0$. This map is computed in Lemma 17.3. As an example, consider $\Pi^1(x)$ when $x = \mathscr{U}_1^i \mathscr{V}_1^j \mathbf{a}$ for $i \geq j \geq 0$. In this case,

$$\Pi^1(\mathscr{U}_1^i \mathscr{V}_1^j \mathbf{a}) = \mathscr{U}_1^{i-j} \mathbf{a} \otimes \mathscr{U}_2^j \mathscr{V}_2^j.$$

Given a finite set of $S \subseteq \mathbb{N}$ and some $n \in \mathbb{N}$, we wish to show that all but finitely many $\mathscr{U}_1^i \mathscr{V}_1^j \mathbf{a}$ are mapped into $\operatorname{Span}(\mathscr{U}_1^s \mathbf{a} \otimes \mathscr{U}_2^i \mathscr{V}_2^j : s \in \mathbb{N} \setminus S \text{ or } j \geq n)$. This is the case, since of course there are only finitely many $i, j \geq 0$ such that $0 \leq j \leq n$ and $i - j \in S$. A similar computation holds for the rest of $\mathbf{I}_0 \cdot \mathcal{H}_{(\lambda_1,0)} \cdot \mathbf{I}_0$, so Π^1 is continuous in these idempotents. Essentially the same argument applies for the other idempotents of $\mathcal{H}$.

The map H^1 is verified to be continuous by a very similar argument. ∎

17.4 Comparison with the Eftekhary–Hedden–Levine model

We now compare our Hopf link complex with the dual knot formulas of Hedden–Levine [11] and Eftekhary [4].

If $K \subseteq S^3$, we will write $\mathrm{EHL}_n(K)^{\mathbb{F}[\mathscr{U},\mathscr{V}]}$ for the complex described in the introduction of [11], which is a model for $\mathcal{CFL}(S^3_n(K), \mu)^{\mathbb{F}[\mathscr{U},\mathscr{V}]}$ where μ is a dual of K inside of the Dehn surgery.

Proposition 17.11. *If K is a knot in S^3, then there is a canonical isomorphism*

$$\mathrm{EHL}_n(K)^{\mathbb{F}[\mathscr{U},\mathscr{V}]} \cong \mathcal{X}_n(K)^{\mathcal{K}} \mathbin{\hat{\boxtimes}} {}_{\mathcal{K}}\mathcal{Z}^{\mathbb{F}[\mathscr{U},\mathscr{V}]}.$$

Proof sketch. Since we do not need this result for any later results, we will not spell out all details. Instead, we will sketch several important details from which the interested reader can easily work out the rest of the argument.

Both complexes are mapping cone complexes with similar structures. We will abbreviate our complex by $\mathbb{X}^{\mu}_n(K)$. We write

$$\mathrm{EHL}_n(K)^{\mathbb{F}[\mathscr{U},\mathscr{V}]} \cong \operatorname{Cone}(v' + h'_n \colon \mathbb{A}^-_{\mathrm{EHL}}(K) \to \mathbb{B}^-_{\mathrm{EHL}}(K))$$

and

$$\mathbb{X}^{\mu}_n(K)^{\mathbb{F}[\mathcal{U},\mathcal{V}]} \cong \operatorname{Cone}(v^{\mu} + h^{\mu}_n \colon \mathbb{A}^{\mu}(K) \to \mathbb{B}^{\mu}(K)).$$

We will only consider the claim when $n = 1$. Furthermore, we will only show that

$$\mathbb{A}^{-}_{\mathrm{EHL}}(K) \cong \mathbb{A}^{\mu}(K)$$

as (infinitely generated) type-D modules over $\mathbb{F}[\mathcal{U},\mathcal{V}]$. Most of the remaining details are straightforward extensions of the ideas we present.

We follow the description of $\mathrm{EHL}_{+1}(K)$ given by Hedden and Levine [11]. They focus on a version of the knot Floer complex which is denoted $\mathrm{CFK}^{\infty}(K)$. This takes the form of a free chain complex over $\mathbb{F}[U, U^{-1}]$ which is filtered by $\mathbb{Z} \oplus \mathbb{Z}$. The generators are of the form $[\mathbf{x}, i, j]$, where $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ and $A(\mathbf{x}) = j - i$. The variable U acts by $U \cdot [\mathbf{x}, i, j] = [\mathbf{x}, i-1, j-1]$. The components i and j are the two components of the $\mathbb{Z} \oplus \mathbb{Z}$-filtration. There is additionally a Maslov grading $\mathrm{gr}_{\mathbf{w}}([\mathbf{x}, i, j]) = \mathrm{gr}_{\mathbf{w}}(\mathbf{x}) + 2i$. Given such a filtered chain complex, we can recover the chain complex $\mathcal{CFK}(K)$ by replacing each $\mathbb{F}[U, U^{-1}]$ basis element $[\mathbf{x}, i, j]$ with a single generator $\mathbf{x}$, viewed as having Alexander grading $j - i$ and $\mathrm{gr}_{\mathbf{w}}$-grading $\mathrm{gr}_{\mathbf{w}}([\mathbf{x}, i, j]) - 2i$.

We will write $\mathbb{A}^{\infty}_{\mathrm{EHL}}(K)$ for the infinity version of the Hedden–Levine model. (We will later reformulate this to get the version which is a type-D module over $\mathbb{F}[\mathcal{U},\mathcal{V}]$.) By definition,

$$\mathbb{A}^{\infty}_{\mathrm{EHL}}(K) \cong \prod_{s \in \mathbb{Z}} A^{\infty}_s(K),$$

for some finitely generated $\mathbb{Z} \oplus \mathbb{Z}$-filtered complexes $A^{\infty}_s(K)$, as follows. For each $\mathbf{x}$ (an intersection point for a Heegaard diagram of K), the generators are of the form $[\mathbf{x}, i, j]$, where $A(\mathbf{x}) = j - i$, where $A(\mathbf{x})$ is Alexander grading from $\mathcal{CFK}(K)$. We will write $\mathbf{x}_{i,j}$ for $[\mathbf{x}, i, j]$. They describe two filtrations $\mathcal{I}$ and $\mathcal{J}$ on these generators given by the formulas

$$\mathcal{I}(\mathbf{x}_{i,j}) = \max(i, j - s), \quad \mathcal{J}(\mathbf{x}_{i,j}) = \max(i-1, j-s) + s.$$

Then the $\mathbb{Z} \oplus \mathbb{Z}$-filtration of $\mathbf{x}_{i,j}$ is $(\mathcal{I}(\mathbf{x}_{i,j}), \mathcal{J}(\mathbf{x}_{i,j}))$. They also define a Maslov grading

$$\mathrm{gr}_{\mathbf{w}}(\mathbf{x}_{i,j}) = \mathrm{gr}_{\mathbf{w}}(\mathbf{x}) + 2i + \frac{(2s-1)^2 - 1}{4}.$$

See [11, equations (1.5)–(1.10)], noting that we are setting $s = s_l$, $k = 1$ and $d = 1$ in their formulas.

We now rewrite $\mathcal{I}$ and $\mathcal{J}$ as follows:

$$\mathcal{I}(\mathbf{x}_{i,j}) = i + \max(0, j - i - s) \quad \text{and} \quad \mathcal{J}(\mathbf{x}_{i,j}) = \max(i - j + s - 1, 0) + j. \tag{17.2}$$

It is helpful to consider the two cases $A(\mathbf{x}) \geq s$ and $A(\mathbf{x}) < s$ separately. Recalling that $A(\mathbf{x}) = j - i$, we compute, using equation (17.2), that

$$(\mathcal{I}, \mathcal{J})(\mathbf{x}_{i,j}) = \begin{cases} (j - s, j) & \text{if } A(\mathbf{x}) \geq s, \\ (i, i + s - 1) & \text{if } A(\mathbf{x}) < s. \end{cases}$$

We now modify the above algebraic generators to get a type-D module over $\mathbb{F}[\mathcal{U}, \mathcal{V}]$. We do this using the same procedure as we described to go from $\mathit{CFK}^\infty(K)$ to $\mathcal{CFK}^-(K)$. We write $\mathbb{A}^-_{\mathrm{EHL}}(K) := \prod_{s \in \mathbb{Z}} A^-_s(K)$ for the type-D module constructed from $\mathbb{A}^\infty_{\mathrm{EHL}}(K)$ in this manner. For each generator $\mathbf{x}$ in $\mathcal{CFK}^-(K)$, we get a generator $\mathbf{x}'_s$ of $A^-_s(K)$ with Alexander grading

$$A(\mathbf{x}'_s) = \begin{cases} s & \text{if } A(\mathbf{x}) \geq s, \\ s - 1 & \text{if } A(\mathbf{x}) < s. \end{cases} \tag{17.3}$$

We have, additionally, that

$$\mathrm{gr}_{\mathbf{w}}(\mathbf{x}'_s) = \mathrm{gr}_{\mathbf{w}}(\mathbf{x}_{i,j}) - 2\mathcal{I}(\mathbf{x}_{i,j}) = \mathrm{gr}_{\mathbf{w}}(\mathbf{x}) + 2\min(0, s - A(\mathbf{x})) + \frac{(2s-1)^2 - 1}{2}. \tag{17.4}$$

We now consider the powers of $\mathcal{U}$ and $\mathcal{V}$ which appear in the differential. Recall that in general, if $\mathbf{x}$ and $\mathbf{y}$ are generators of a knot Floer complex $\mathcal{CFK}(K)$ (for a knot K in an integer homology 3-sphere) and there is a differential from $\mathbf{x}$ to $\mathbf{y}$ which is weighted by $\mathcal{U}^i \mathcal{V}^j$, then

$$j - i = A(\mathbf{x}) - A(\mathbf{y}) \quad \text{and} \quad \mathrm{gr}_{\mathbf{w}}(\mathbf{y}) = \mathrm{gr}_{\mathbf{w}}(\mathbf{x}) - 1 + 2i. \tag{17.5}$$

We now compare this with the type-D structure $\mathbb{A}^\mu(K)$. By construction, this type-D structure is the tensor product of $\mathcal{CFK}(K)^{\mathbb{F}[\mathcal{U},\mathcal{V}]}$ with the DA bimodule shown below:

$$\cdots\ \mathcal{U}^2\mathbf{a} \mathrel{\substack{\xrightarrow{\mathcal{V}|U} \\ \xleftarrow[\mathcal{U}|1]{}}} \mathcal{U}\mathbf{a} \mathrel{\substack{\xrightarrow{\mathcal{V}|U} \\ \xleftarrow[\mathcal{U}|1]{}}} \mathbf{a} \mathrel{\substack{\xrightarrow{\mathcal{V}|\mathcal{V}} \\ \xleftarrow[\mathcal{U}|\mathcal{U}]{}}} \mathbf{d} \mathrel{\substack{\xrightarrow{\mathcal{V}|1} \\ \xleftarrow[\mathcal{U}|U]{}}} \mathcal{V}\mathbf{d} \mathrel{\substack{\xrightarrow{\mathcal{V}|1} \\ \xleftarrow[\mathcal{U}|U]{}}} \mathcal{V}^2\mathbf{d}\ \cdots$$

We can give $\mathbb{A}^\mu(K)$ a similar description to $\mathbb{A}^-(K)$. The generators of $\mathbb{A}^\mu(K)$ are of the form $\mathbf{x} \otimes \mathbf{y}$, where $\mathbf{x}$ is a generator of $\mathcal{CFK}^-(K)$, and $\mathbf{y}$ is of the form $\mathcal{U}^i\mathbf{a}$ or $\mathcal{V}^i\mathbf{d}$. We define an Alexander grading on generators via the formula

$$A'(\mathcal{U}^i\mathbf{a}) = -i \quad \text{and} \quad A'(\mathcal{V}^i\mathbf{d}) = i + 1.$$

We then define $A^\mu_s(K) \subseteq \mathbb{A}^\mu(K)$ to be the $\mathbb{F}$ span of pairs $\mathbf{x} \otimes \mathbf{y}$ where

$$A(\mathbf{x}) + A'(\mathbf{y}) = s.$$

It is straightforward to see that $A_s^{\mu}(K)$ is a subcomplex of $\mathbb{A}^{\mu}$. We claim that there is an isomorphism of type-D modules

$$A_s^{\mu}(K)^{\mathbb{F}[\mathcal{U},\mathcal{V}]} \cong A_s^{-}(K)^{\mathbb{F}[\mathcal{U},\mathcal{V}]}.$$

Note that there is a canonical bijective correspondence between $\mathbb{F}[\mathcal{U},\mathcal{V}]$-generators of these complexes, since the generators of both complexes are bijectively identified with generators of $\mathcal{CFK}(K)$. Therefore, we have a canonical isomorphism of vector spaces between these modules. We claim that this isomorphism intertwines the differential. To see this, note that the differentials on both complexes are also identified with the ordinary differential of $\mathcal{CFK}(K)$, except with powers of $\mathcal{U}$ and $\mathcal{V}$ changed. Therefore, it suffices to show that the powers of $\mathcal{U}$ and $\mathcal{V}$ appearing in the two differentials coincide.

We consider a differential in $\mathcal{CFK}(K)$ from $\mathbf{x}$ to $\mathbf{y}$, weighted by $a \in \mathbb{F}[\mathcal{U},\mathcal{V}]$. We consider its induced weight in $A_s^{\mu}(K)$ and $A_s'(K)$. Suppose that in $A_s^{\mu}(K)$, it is weighted by an algebra element a^{μ}, and suppose that in $A_s'(K)$, it is weighted by an algebra element a'. Our goal is to show that $a^{\mu} = a'$.

Using equations (17.3) and (17.5), we observe that

$$A(a') = \begin{cases} 0 & \text{if } A(\mathbf{x}), A(\mathbf{y}) \geq s, \\ 0 & \text{if } A(\mathbf{x}), A(\mathbf{y}) < s, \\ 1 & \text{if } A(\mathbf{x}) \geq s \text{ and } A(\mathbf{y}) < s, \\ -1 & \text{if } A(\mathbf{x}) < s \text{ and } A(\mathbf{y}) \geq s. \end{cases} \tag{17.6}$$

Next, we compute, using equations (17.4) and (17.5), that

$$\mathrm{gr}_{\mathbf{w}}(a') = \mathrm{gr}_{\mathbf{w}}(\mathbf{x}) - \mathrm{gr}_{\mathbf{w}}(\mathbf{y}) - 1 + 2\min(0, s - A(\mathbf{x})) - 2\min(0, s - A(\mathbf{y})). \tag{17.7}$$

Note that $\mathrm{gr}_{\mathbf{w}}(a')$ and $A(a')$ uniquely determine a'.

Next, we compute a^{μ} and show that it is equal to a'. It is helpful to break the argument into four cases, parallel to equation (17.6). We will consider two of the four cases, and leave the rest to the reader.

We consider first the case that $A(\mathbf{x}), A(\mathbf{y}) \geq 0$. The corresponding generators of $A_s^{\mu}(K)$ are of the form

$$\mathbf{x} \otimes \mathcal{U}^{A(\mathbf{x})-s}\mathbf{a} \quad \text{and} \quad \mathbf{y} \otimes \mathcal{U}^{A(\mathbf{y})-s}\mathbf{a}.$$

We can write

$$a = U^{-\mathrm{gr}_{\mathbf{w}}(a)/2}V^{A(a)} = U^{-(\mathrm{gr}_{\mathbf{w}}(\mathbf{x})-\mathrm{gr}_{\mathbf{w}}(\mathbf{y})-1)/2}V^{A(\mathbf{x})-A(\mathbf{y})}.$$

Note that

$$\delta_2^1(a, \mathcal{U}^{A(\mathbf{x})-s}\mathbf{a}) = \mathcal{U}^{A(\mathbf{y})-s}\mathbf{a} \otimes U^{-(\mathrm{gr}_{\mathbf{w}}(\mathbf{x})-\mathrm{gr}_{\mathbf{w}}(\mathbf{y})-1)/2+A(\mathbf{x})-A(\mathbf{y})}.$$

(Some care must be taken to verify the above formula when $A(\mathbf{x}) - A(\mathbf{y})$ is negative, but the formula holds regardless of the sign of $A(\mathbf{x}) - A(\mathbf{y})$.) Hence,

$$a^\mu = U^{-(\mathrm{gr}_{\mathbf{w}}(\mathbf{x}) - \mathrm{gr}_{\mathbf{w}}(\mathbf{y}) - 1)/2 + A(\mathbf{x}) - A(\mathbf{y})}.$$

We observe that $A(a^\mu) = 0 = A(a')$. Also,

$$\mathrm{gr}_{\mathbf{w}}(a') = \mathrm{gr}_{\mathbf{w}}(\mathbf{x}) - \mathrm{gr}_{\mathbf{w}}(\mathbf{y}) - 1 + 2(A(\mathbf{y}) - A(\mathbf{x})),$$

which is the same as $\mathrm{gr}_{\mathbf{w}}(a^\mu)$.

We now consider the third case in equation (17.6), where $A(\mathbf{x}) \geq s$ and $A(\mathbf{y}) < s$. Since $A(\mathbf{x}) \geq s$, the corresponding generator of $A_s^\mu(K)$ is of the form $\mathbf{x} \otimes \mathcal{U}^{A(\mathbf{x})-s}\mathbf{a}$. Also, the generator of $A_s^\mu(K)$ corresponding to $\mathbf{y}$ will be of the form $\mathbf{y} \otimes \mathcal{V}^{s-A(\mathbf{y})-1}\mathbf{d}$. We write

$$a = U^{-(\mathrm{gr}_{\mathbf{w}}(\mathbf{x}) - \mathrm{gr}_{\mathbf{w}}(\mathbf{y}) - 1)/2}\mathcal{V}^{A(\mathbf{x}) - A(\mathbf{y})}.$$

In this case, we have $A(\mathbf{x}) - A(\mathbf{y}) > 0$. Therefore we can read a^μ as follows. Firstly, we have a factor of $U^{-(\mathrm{gr}_{\mathbf{w}}(\mathbf{x}) - \mathrm{gr}_{\mathbf{w}}(\mathbf{y}) - 1)/2}$. Then, we have an additional factor which is obtained by composing all of the right moving arrows from $\mathcal{U}^{A(\mathbf{x})-s}\mathbf{a}$ to $\mathcal{V}^{s-A(\mathbf{y})-1}\mathbf{d}$. This will give us factors of $U^{A(\mathbf{x})-s}\mathcal{V}$. Therefore,

$$a^\mu = U^{-(\mathrm{gr}_{\mathbf{w}}(\mathbf{x}) - \mathrm{gr}_{\mathbf{w}}(\mathbf{y}) - 1)/2 + A(\mathbf{x}) - s}\mathcal{V}.$$

This must coincide with a', since it has the same $(\mathrm{gr}_{\mathbf{w}}, A)$-bigrading by equations (17.6) and (17.7).

It remains to verify that $a' = a^\mu$ in the cases that $A(\mathbf{x}), A(\mathbf{y}) < s$ and when $A(\mathbf{x}) < s$ and $A(\mathbf{y}) \geq s$. The argument follows from the same line of reasoning as the two previously analyzed cases. We leave the details to the reader.

The analysis of $\mathbb{B}^\mu(K)$ and the maps v^μ and h_n^μ follows from similar, albeit somewhat tedious, reasoning, so we leave the details to the interested reader. ■